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Two-Dimensional Measures of Accuracy in Navigational Systems

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16. Abstract

Two-dimensional measures generally used to depict the accuracy of radiolocation and navigation systems are described in this report. Application to the NAVSTAR Global Positioning System (GPS) is considered, with a number of geometric illustrations.

The 2 drms measure and the Circular Error Probable (CEP) are two common two-dimensional measures of accuracy. The 2 drms measure is obtained by establishing an orthogonal set of horizontal axes, computing the standard deviations with respect to the two axes, taking the root-sum-square of the two, and multiplying by two. Although simple to compute, the 2 drms probability level is not fixed. The CEP measure is the radius of the smallest circle which contains 50 percent of the positional measurements. Another measure, R95, is defined as the radius of the smallest circle which contains 95 percent of the positional measurements. The associated probability is fixed at 95 percent, but R95 is also more difficult to compute. Computer programs were developed to adapt the error measures, which apply to a particular position and time, to the NAVSTAR/GPS system over the U.S. for 24 hours. Based on 100 meter nominal accuracy, the error measures are computed and compared; East-West and North-South errors are correlated. Computations show that GPS is more accurate in the East-West direction. The probability level associated with the 2 drms measure is also determined.

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PREFACE

The two-dimensional measures generally used to depict the accuracy of radiolocation and navigation systems are described in detail in this report. They are compared with each other and with linear accuracy measures, with particular application to the NAVSTAR Global Positioning System (NAVSTAR/GPS). A number of geometric illustrations are presented to illuminate the basic ideas.

The two most common two-dimensional measures of accuracy are the 2 drms measure and the Circular Error Probable (CEP). The 2 drms measure is obtained by establishing an orthogonal set of horizontal axes, computing the standard deviation of errors with respect to the two axes, taking the root-sum-square of the two, and multiplying by two. This measure has the advantage of being simpler to compute. It has the disadvantage that the probability level associated with the measure is not fixed, and is somewhere between 95.4 percent and 98.2 percent. The CEP measure is defined as the radius of the (smallest) circle which contains 50 percent of the positional measurements.

In this report another measure is introduced similar to the CEP measure, namely R95. This is defined as the radius of the (smallest) circle which contains 95 percent of the positional measurements. It differs somewhat from 2 drms. This measure has the advantage that the associated probability level is fixed (at 95 percent), but it has the disadvantage of being more difficult to compute.

Computer programs were developed to adapt the various error measures, which apply strictly to a particular position and particular time, to the NAVSTAR/GPS system over the United States for a 24-hour period. Based on the 100 meter nominal accuracy anticipated for civil use, the various error measures are computed and compared. East-West and North-South errors turn out to be correlated, and the resulting computations show that the GPS provides better accuracy in the East-West direction than in the North-South direction. The probability level associated with the 2 drms accuracy measure is also computed.

I wish to thank Dr. Rudolph Kalafus, head of the Satellite Navigation Group, who described the problem of the different measures in use, cited the difficulties in defining accuracy measures that apply to different radiolocation and navigation systems, and provided guidance for this study. Dr. Kalafus's ideas are integrated into this report especially in the summary (Section 1.2).

Thanks to David Scull of the Research and Special Programs Administration, who supported these efforts and encouraged the addressing, and hopefully the clarification, of the issues involved in assessing the accuracy of navigation and positioning systems.

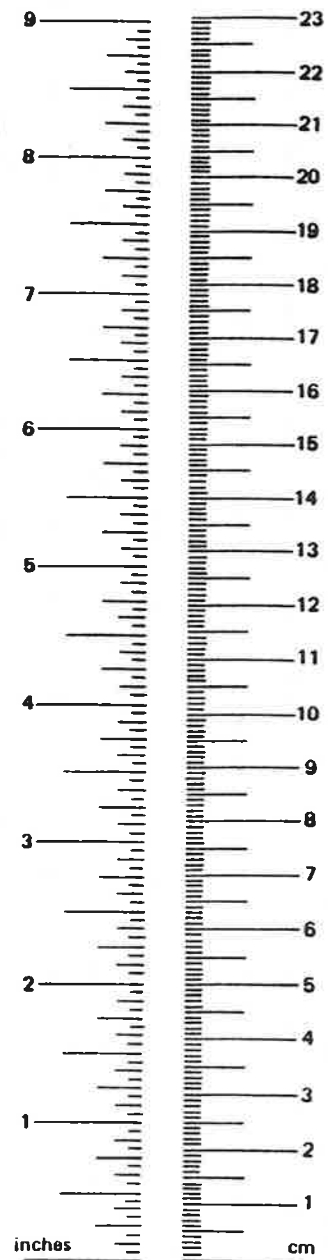
I wish to thank Shirish Dandekar for his programming effort at an earlier stage of this project. Thanks to Sian Steward, Jonathan Cheney, Elisabeth Marden and Kathy First of DYNATREND for assistance in editing and preparing the manuscript for publication. Thanks to Janis Vilcans, of the Satellite Navigation Group, for stimulating discussions on various conceptual aspects.

METRIC CONVERSION FACTORS

Approximate Conversions to Metric Measures

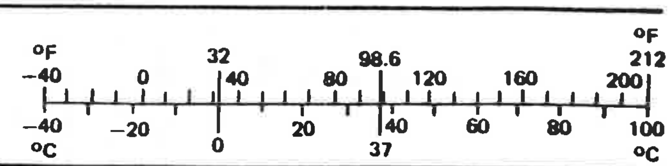
Symbol	When You Know	Multiply by	To Find	Symbol
		LENGTH		
in	inches	*2.5	centimeters	cm
ft	feet	30	centimeters	cm
yd	yards	0.9	meters	m
mi	miles	1.6	kilometers	km
		AREA		
in^2	square inches	6.5	square centimeters	cm^2
ft^2	square feet	0.09	square meters	m^2
yd^2	square yards	0.8	square meters	m^2
mi^2	square miles	2.6	square kilometers	km^2
	acres	0.4	hectares	ha
		MASS (weight)		
oz	ounces	28	grams	g
lb	pounds	0.45	kilograms	kg
	short tons (2000 lb)	0.9	tonnes	t
		VOLUME		
tsp	teaspoons	5	milliliters	ml
Tbsp	tablespoons	15	milliliters	ml
fl oz	fluid ounces	30	milliliters	ml
c	cups	0.24	liters	l
pt	pints	0.47	liters	l
qt	quarts	0.95	liters	l
gal	gallons	3.8	liters	l
ft^3	cubic feet	0.03	cubic meters	m^3
yd^3	cubic yards	0.76	cubic meters	m^3
		TEMPERATURE (exact)		
°F	Fahrenheit temperature	5/9 (after subtracting 32)	Celsius temperature	°C

*1 in. = 2.54 cm (exactly). For other exact conversions and more detail tables see NBS Misc. Publ. 286. Units of Weight and Measures. Price $2.25 SD Catalog No. C13 10 286.



Approximate Conversions from Metric Measures

Symbol	When You Know	Multiply by	To Find	Symbol
		LENGTH		
mm	millimeters	0.04	inches	in
cm	centimeters	0.4	inches	in
m	meters	3.3	feet	ft
m	meters	1.1	yards	yd
km	kilometers	0.6	miles	mi
		AREA		
cm^2	square centimeters	0.16	square inches	in^2
m^2	square meters	1.2	square yards	yd^2
km^2	square kilometers	0.4	square miles	mi^2
ha	hectares (10,000 m^2)	2.5	acres	
		MASS (weight)		
g	grams	0.035	ounces	oz
kg	kilograms	2.2	pounds	lb
t	tonnes (1000 kg)	1.1	short tons	
		VOLUME		
ml	milliliters	0.03	fluid ounces	fl oz
l	liters	2.1	pints	pt
l	liters	1.06	quarts	qt
l	liters	0.26	gallons	gal
m^3	cubic meters	36	cubic feet	ft^3
m^3	cubic meters	1.3	cubic yards	yd^3
		TEMPERATURE (exact)		
°C	Celsius temperature	9/5 (then add 32)	Fahrenheit temperature	°F



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1. INTRODUCTION

1.1 PURPOSE AND SCOPE

In this study we examine the two-dimensional measures of accuracy in navigation systems, in particular the Global Positioning System (GPS); we examine in turn the linear, elliptical, and circular measures of error. In the course of the discussion, the works of previous authors are integrated, certain errors are corrected, and ambiguities are resolved, while some topics are subjected to further development. A number of geometric illustrations are presented to help illuminate the basic ideas. The relationship between circular error measures and drms measures is discussed. The advantages and drawbacks of various measures (for example, 2 drms and R95 (radius of 95 percent probability circle)) in relation to different navigation systems are discussed. Computer programs are developed and used to obtain the distribution of accuracy measures for NAVSTAR/GPS over the CONUS (coterminous United States) and over a 24-hour period. The distributions calculated include the following: HDOP, XDOP (East- West), YDOP (North-South), the ellipticity parameter σ_y / σ_x distribution, distribution of the error ellipse orientation, distribution of R95 (Radius of circle containing 95 percent of position measures), distribution of CEP (Radius of 50 percent containment of position measures). When appropriate, the median, the mean values, and the standard deviations of the quantities are evaluated.

A method is presented to obtain a single 95 percent probability circle for the entire CONUS over a 24-hour period. In Section 7 are summarized (for NAVSTAR/GPS, 21 satellite system without altimeter, with selective availability imposed, all-in-view strategy) the average performance values (median and mean values) of the various measures of interest. Calculations are also made for best-set-of-four satellites strategy and the results are summarized in Table 1 along with the results of all-in-view strategy.

1.2 SUMMARY

1.2.1 Meaning of Accuracy

In order to describe the accuracy of a navigation system without lengthy discussion, it is highly desirable to have a limited number of figures to cite, preferably one. The Federal Radionavigation Plan (DOD/DOT, 1984)(1) uses three: predictable (geodetic, or absolute), repeatable, and relative. Predictable accuracy relates to the uncertainty between the charted and observed locations; repeatable accuracy gives a measure of the ability to return to the same location at a later time, and relative refers to the consistency of readings taken at the same time between receivers placed at the same place.

However, even three measures don't begin to circumscribe the issues encountered when trying to apply definitions to a particular application. First, platform dynamics are crucial: a receiver on a maneuvering vehicle may only achieve 50 meters accuracy from the same system that a stationary surveying instrument may obtain 1 meter accuracy. The time available for a measurement is a crucial factor. Seasonal time may introduce variations: day-night or winter- summer variations may be considerable. Second, the errors are made up of bias, or long-term errors, and noise, or short-term errors; the distinction between them depends not on the navigation system, but the user mission. Different applications assign different importances to bias and noise errors. Surveying applications involving the relative placement of monuments are not sensitive to bias errors, which can be removed. The leader of a bombing run, however, can take small comfort in the fact that the bombs all landed within 50 feet of each other (noise error) if they all missed the target by half a mile (bias). As a consequence, a meaningful measure of accuracy has to be tailored to the application at hand.

1.2.2 Accuracy Measures

To further complicate the situation, there are several different measures of accuracy that are employed to describe the capabilities of different systems. While in one-dimensional systems, the standard deviation and rms measures are most often used (they are identical if biases are removed), in two-dimensional systems there are many different measures. The ones that will be addressed in this paper are the 2drms, circular error probable (CEP), and an extension of the CEP measure, referred to herein as R95.

CEP is the radius of the smallest circle, centered at the true position point, that encompasses 50 percent of the measurements.

R95 is similarly defined, except a percentile of 95 percent is used instead of 50 percent.

2drms is twice the root-mean-square (rms) radial error. For a set of measurements, it is found by computing the rms of the distances from the true position to the measured positions.

It is important to note that all three measures involve only the radial errors, and do not involve any reference to coordinate axes. Thus by definition the measures are invariant to the rotation of axes used to describe the errors.

In this report it will be shown that each measure has certain properties that render it useful in some situations, and inappropriate in others. Three situations are considered: (1) analysis of measurement data, (2) estimation of errors using Monte Carlo techniques, and (3) estimation of errors using GDOP analysis.

2drms advantages:

- straightforward method to analyze data
- readily applied to Monte Carlo method
- highly compatible with GDOP analysis

2drms disadvantages:

- not tied to specific probability level
- can be skewed by large errors

CEP/R95 advantages:

- straightforward method to analyze data
- readily applied to Monte Carlo method
- tied to specific probability levels
- not skewed by large errors

CEP/R95 disadvantages:

- not compatible with GDOP analysis

It should also be noted that the definitions involve no assumptions of zero mean processes or normally distributed random variables. The zero mean property becomes important when relating these measures to standard deviations of timing errors, and to GDOP analyses. The normal distribution property comes into play when attempting to describe the relationships between the various measures. The significance of the assumption of normal distributions is treated at length by Mertikos, et al (1985).(2)

1.2.3 Elliptical Error Distributions

Two-dimensional plots of errors associated with navigation systems frequently have elliptical shapes. This is caused by two factors: (1) two lines of position (LOPs), each associated with constant range to a station or with range difference to two stations, are not perpendicular, and (2) the signal strength variation between the stations cause different range (or range difference) errors for the two LOPs. One can define a major and minor axis of the ellipse; errors are then greatest along the major axis, and smallest along the minor axis. For example, if two LOPs intersected at an angle of 30 degrees, the major axis of the error ellipse would be perpendicular to the bisector of the 30-degree angle (see Figure 1-1).

An elliptical error distribution would result if the ranging errors of a navigation system were normally distributed and if the two conditions mentioned above were met. The elliptical error distribution has the following properties:

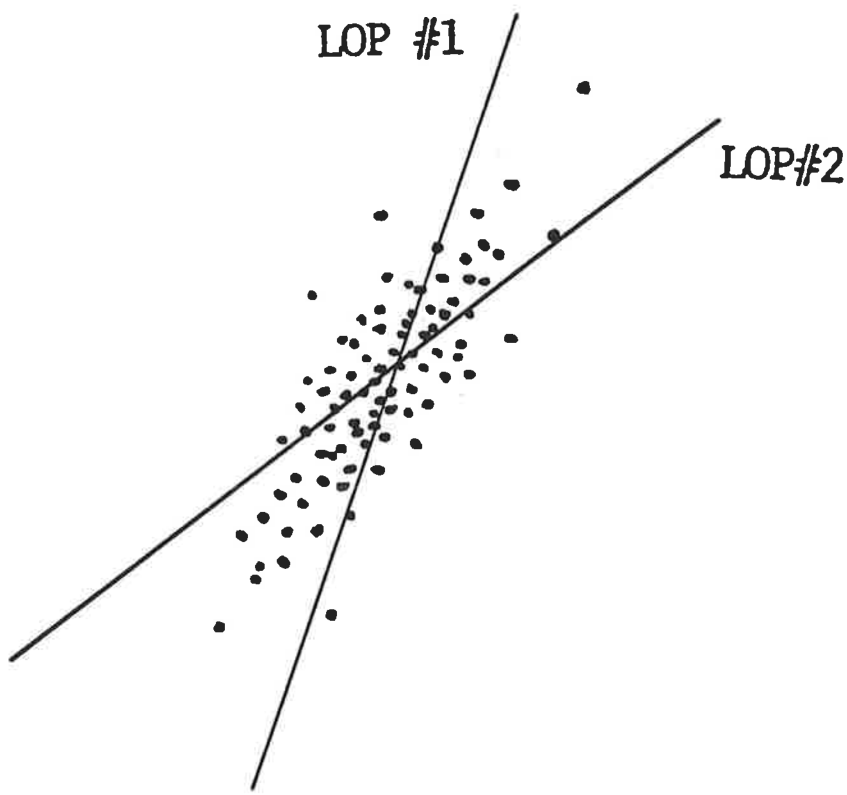


FIGURE 1-1. ERROR ELLIPSE ASSOCIATED WITH SKEWED LOPS

1. Constant probability density contours are ellipses with a fixed ellipticity. "Ellipticity ratio" is used here to denote the ratio of minor to major axis of an error ellipse.
2. Ellipticity ratio variations cause changes in the relationships between CEP, R95, and 2drms accuracy measures.
3. x and y errors are correlated except when the x and y axes correspond to the major and minor axes of the error ellipse.

Under certain conditions, which are described in the following sections, positional errors do have elliptical distributions, which makes it possible to relate the different accuracy measures. In general, however, the distributions of positional errors approximate but do not equal elliptical distributions.

1.2.4 The 2drms Accuracy Measure

This measure is usually written as "2drms", which often leads people who come upon the term for the first time to incorrectly assume it refers to "2- dimensional rms". Actually, it simply means "twice the drms measure", where d_{rms} is the rms radial error. Applied to a set of measurements, the 2drms accuracy is defined by

$$2drms = 2d_{rms}$$

$$= 2\sqrt{\left[\sum (x_n^2 + y_n^2)/N\right]} \tag{1-1}$$

where x_n and y_n are the x and y error components of a measurement with respect to a set of orthogonal axes. If the means of the error components are known to be zero, or if the bias of the measurements can be independently determined or removed, the data can be treated as a zero mean process. For a given set of measurements, the 2drms measure will be minimized when the origin of the x-y axis is placed at the center of gravity of the measurements.

If the timing errors of a navigation system are normally distributed, then a user at a given location will observe an elliptical error distribution over a period of time; this would hold true for a system involving fixed transmitting stations like LORAN-C or Omega. For GPS, however, the satellites are in motion, and the LOPs are changing; in fact as a satellite sets or rises, there will be a discontinuity in the LOPs. As a consequence, an elliptical error distribution would be unlikely to result.

An alternative technique would be to place a number of receivers very close to each other, and take simultaneous measurements. This would provide an ensemble average of measurements taken at the same time. By taking an ensemble of simultaneous receiver measurements, the error distributions would be elliptical, but when combining measurements over a period of time or over a region, they would not. To see this, try combining the errors taken at times 6 or more hours apart, where different satellites are visible to the user. The combined distribution would be the sum of two elliptical distributions, each with different orientations and amplitudes. The resulting distribution would be binodal, not elliptical at all. Intuitively one might anticipate that measurements over a period of 24 hours would tend to approach an elliptical, perhaps even circular distribution, but this is not justified.

In spite of the lack of ellipticity of the error distribution, the 2drms accuracy associated with a set of measurements can simply be computed by summing over all the radial errors recorded at different positions and times, using equation 1-1.

To estimate the 2drms accuracy of a system over a period of time and over a service coverage region, rather than use measurements, one can use either a Monte Carlo method, or a GDOP analysis. In the Monte Carlo method, a set of range errors to be associated with the stations or satellites is generated using a random number generator and a set of rules on the error distribution (usually a normal, zero-mean distribution, but not necessarily). The geometry of the system network as seen at the user location is used to compute a positional error in two or three dimensions. Enough range error sets are used to provide a given confidence level, and the 2drms accuracy estimate is found by summing over the resulting radial errors, using equation 1-1 as with a set of real measurements.

In a GDOP analysis, the timing errors (or pseudorange errors) are assumed to be normally distributed, with zero means. Assuming a standard deviation in pseudorange of σ_R and a standard deviation of radial error (RSS Horizontal Position Solution Error) of σ_H, the horizontal dilution-of-precision (HDOP) is given by:

$$
\begin{aligned}
HDOP &= \sigma_H / \sigma_R \\
&= (1/\sigma_R) \sqrt{\left[\sigma_x^{\,2} + \sigma_y^{\,2}\right]} \\
&= \sqrt{\left[XDOP^2 + YDOP^2\right]} \qquad (1\text{-}2)
\end{aligned}
$$

The standard deviation of radial error is equal to the d_{rms} measure for zero- mean processes, so that the 2drms accuracy can be estimated by:

$$2drms = 2\, HDOP \bullet \sigma_R \qquad (1\text{-}3)$$

But HDOP is only defined for a particular place and a particular time, so in order to define a 2drms accuracy that holds over a service coverage region and over a stated time period, equation 1-2 must be generalized. If $HDOP_m$ is the HDOP at a particular point and time, it can be shown that the variance of the radial error over the collection of M points is given by:

$$\begin{aligned} \sigma_H^{\,2} &= \sigma_R^{\,2} \sum HDOP_m^{\,2}/M \\ &= \sigma_R^{\,2} \bullet HDOP_{rms}^{\,2} \end{aligned} \qquad (1\text{-}4)$$

Therefore, the 2drms accuracy is given by:

$$\begin{aligned} 2drms &= 2\,\sigma_R\, HDOP_{rms} \\ &= 2\,\sigma_R \sqrt{\left[XDOP_{rms}^{\,2} + YDOP_{rms}^{\,2}\right]} \end{aligned} \qquad (1\text{-}5)$$

In this manner the 2drms accuracy measure can be applied to a region of coverage and over a period of time without the assumption of elliptical error distributions.

The 2drms accuracy measure tends to be unstable if there are occasional large errors. Such errors occur in the GPS system: for example, with the 21- satellite constellation there is an area around San Francisco where once a day the GDOP actually becomes momentarily infinite. If the GDOP analysis happened to use a sample of the dilution-of-precision near this moment, it could cause the 2drms value to be significantly higher than it would otherwise be. To see this, consider an artificial situation where 999 samples each had errors of $\pm$2 meters, and one sample had an error of 500 meters. The 2drms computation would give:

$$\begin{aligned} 2drms &= 2 \bullet \sqrt{\left[(999 \bullet 2^{\,2} + 500^2)/100\right]} \\ &= 2 \bullet \sqrt{\left[(3696 + 250{,}000)/1000\right]} \end{aligned}$$

It is obvious that the 2drms measure would be completely dominated by the one singular point.

It is necessary to modify the definition of 2drms to accommodate these large errors. One way to do this is to recognize that when the GDOP becomes too high, the navigation information should not be used anyhow, and the 2drms measure should only include usable measurements. (A receiver can estimate HDOP and thus be aware when the accuracy is poor.) It is proposed that only HDOPs less than 6 be considered as part of the SPS, and that the anomalous points be considered as a service outage. Such a definition removes the problem, and is not sensitive to the HDOP criterion used -- an HDOP of 10 would result in virtually identical 2drms accuracy estimates.

Selective Availability (SA) will complicate measurements aimed at establishing accuracies at a particular place. The SA errors are bias-like, with correlation times of many tens of minutes. An ensemble of receivers would give positional errors with a large bias error (up to 100 meters) and a small variance. To obtain enough measurements over a period of time to get a high confidence level may take a long time.

1.2.5 Circular Error Measure: CEP and R95

The CEP measure is useful in describing the accuracy of systems where the "likelihood" of a given measure being within the stated accuracy is at issue. In a bombing run, for example, one would be interested in the value of accuracy that gives a better than even chance that the target was destroyed; here a 50 percent measure is appropriate. In navigation the interest is usually in being "fairly certain" that the vehicle position is within the prescribed accuracy. In one-dimensional applications, this points to something like a 2-sigma value, which is associated with a 95 percent certainty. To obtain a measure more in line with the 95 percent figure and with the 2drms two-dimensional measure, the R95 measure is defined similarly to CEP, but using a 95th percentile.

Conceptually, finding a circular error measure from a set of measurement data involves drawing an ever-enlarging circle until 50 percent (CEP) or 95 percent (R95) of the data points are included; the radius thus obtained is the circular accuracy measure. Computationally, it would be found by sorting the radial errors and finding the magnitude of the radial error at the 50th percentile (CEP) or 95th percentile (R95). Unless the number of measured data points is very large, sorting the errors will not prove unduly time-consuming. As with the 2drms measure, the measures would be minimized if the coordinate origin were located at the center of gravity of the measurements. The same considerations of bias errors apply here as well.

Estimation of the circular measures using a Monte Carlo technique follows a similar line. A set of position errors is computed, a radial error assigned to each one; the errors are then sorted and the magnitude of the radial at the appropriate percentile determined.

Finding the circular error measures from GDOP analysis is difficult, and poses some problems in interpretation. At a particular location and time, the circular measure can be found from the standard deviation by taking advantage of the fact that the error distribution is elliptical. This is explored in detail in Section 4.1. There it is shown that CEP and R95 could be related to the standard deviation of pseudorange by:

$$CEP = \frac{K_{50}^{(c)}}{\sqrt{\left[1+c^2\right]}} HDOP \bullet \sigma_R \qquad (1\text{-}6)$$

$$R95 = \frac{K_{95}^{(c)}}{\sqrt{\left[1+c^2\right]}} HDOP \bullet \sigma_R \qquad (1\text{-}7)$$

where c is the ellipticity ratio, and K(c) can be found from Figure 1-2.

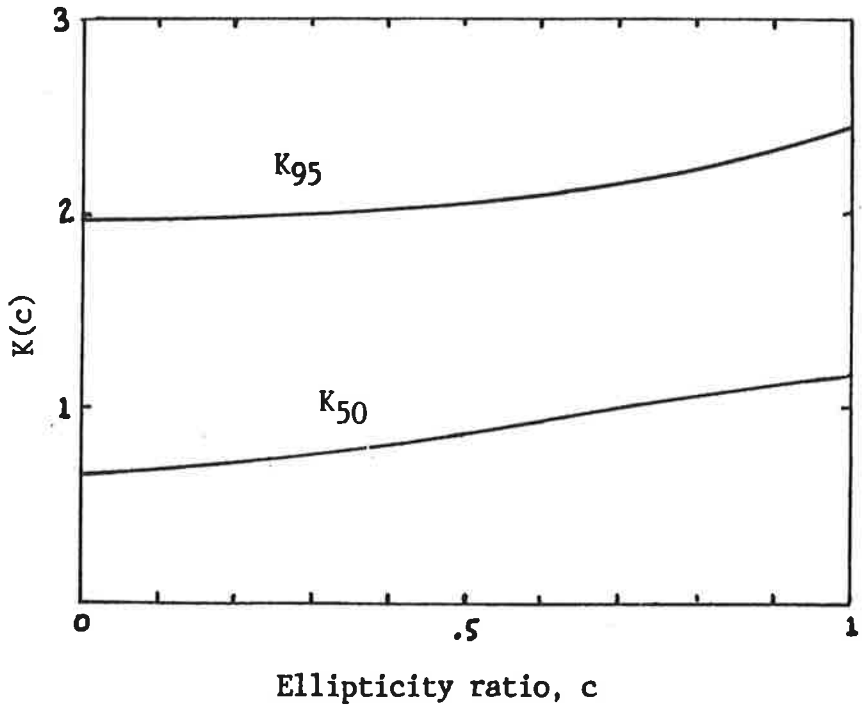


FIGURE 1-2. RELATION OF K(c) TO ELLIPTICITY RATIO

Generalizing this to a region of service coverage and a period of time is difficult because the ellipticity assumption breaks down. It is possible to define rms values for CEP and R95 as with HDOP above, but there is no readily interpretable meaning that could be associated with them. It would also be possible to determine the mean value of ellipticity over the region and period and to use that value of K(c), but such a measure would also have only an intuitive basis. As a result, the median value will be used to compare different accuracy measures.

1.2.6 Comparison of Accuracy Measures for Elliptical Distributions

A number of relationships have been worked out between the accuracy measures and the ellipticity parameters, assuming normally distributed variables. These are helpful in describing variations in the ratios between the different accuracy measures.

Figure 1-3 illustrates graphically the 2drms measure. An equal probability density ellipse is drawn having a semi-major axis of 2 σ_x and a semi- minor axis of 2 σ_y; it will be referred to as the 2-sigma ellipse. It contains 86.5 percent of the measurements for normally distributed variables. The 2drms circle (equation 1-1) is always larger, and contains from 95.4 percent to 98.2 percent of the measurements, depending on the ellipticity. This dependence is given in Figure 1-3 (based on a figure from Burt[3]).

Figure 1-4 similarly illustrates the R95 measure. The R95 circle is by definition the circle that contains 95 percent of the measurements. The figure also shows an equal-probability-density ellipse containing 95 percent of the measurements for comparison.

Figure 1-5 to 1-8 show progressively the relative sizes of R95 circles and 2drms circles, as well as a 2-sigma ellipse (86.5 percentile). Figure 1-5 shows the limiting case of a one-dimensional distribution, where none of the errors has a y-component. The 2-sigma ellipse becomes a line segmentwith a half- length of 2 σ_x. The 2drms circle contains 95.4 percent of the errors, so it is only slightly larger (2 percent) than the R95 circle.

Figure 1-6 shows a highly elliptical distribution. Note that the ellipse almost touches the 2drms circle, and that there are errors within the ellipse that lie outside the R95 circle. It can be seen from Figure 1-7 that for narrow elliptical distributions (low ellipticity ratios), the 2-sigma ellipse (86.5 percentile) contains errors that lie outside the R95 circle. At the limit of zero ellipticity, the distribution is circular, and the 2drms accuracy measure is 15.5 percent larger than the R95 measure. The ratio of 2drms to R95 and CEP is given in Figure 1-9 for different value of ellipticity ratio.

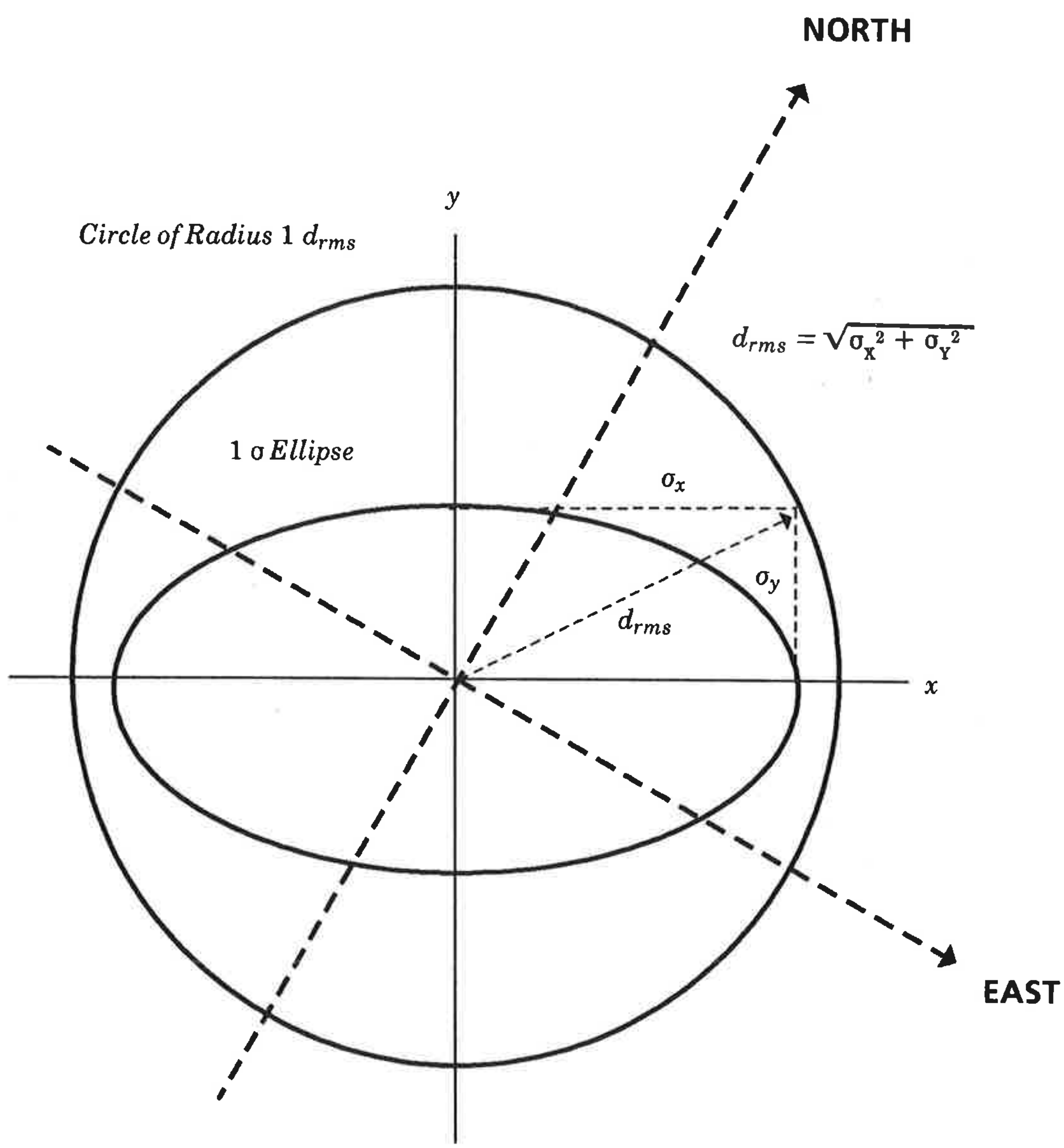


FIGURE 1-3. ILLUSTRATION OF THE 2DRMS ACCURACY MEASURE, ELLIPTICAL ERROR DISTRIBUTION

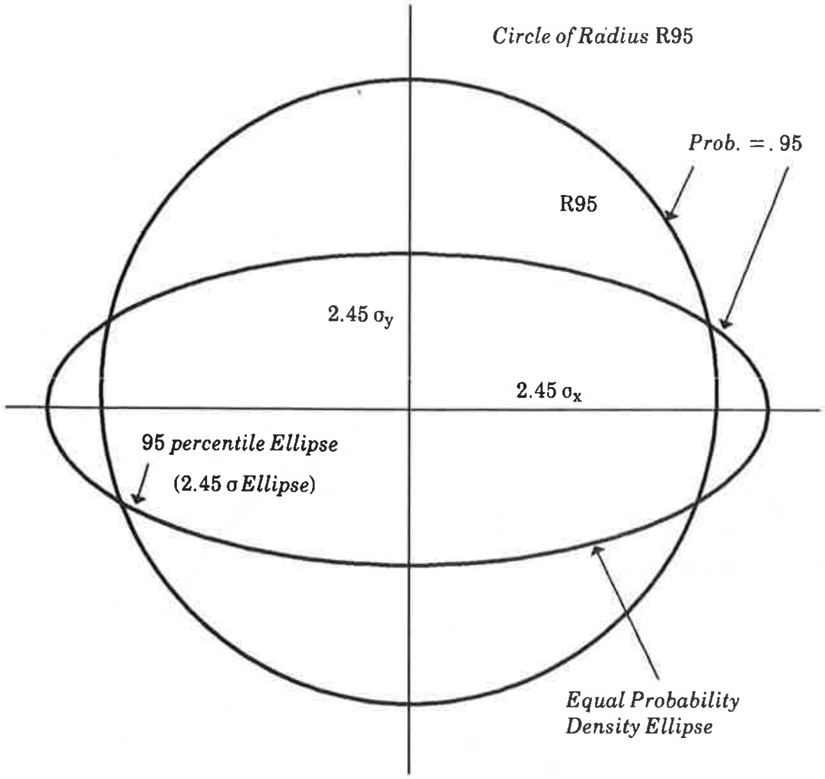


FIGURE 1-4. ILLUSTRATION OF THE R95 ACCURACY MEASURE, ELLIPTICAL ERROR DISTRIBUTION

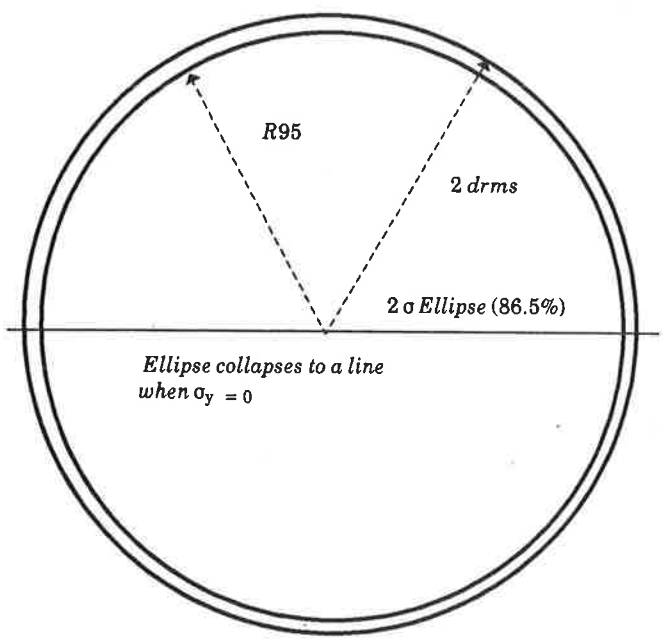


FIGURE 1-5. RELATIVE SIZES OF 2DRMS AND R95, SINGLE AXIS ERRORS

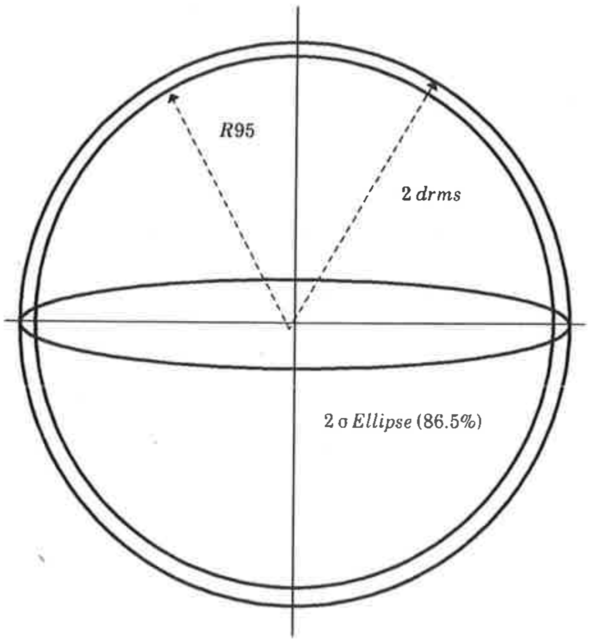


FIGURE 1-6. RELATIVE SIZES OF 2DRMS AND R95, HIGHLY SKEWED ERRORS

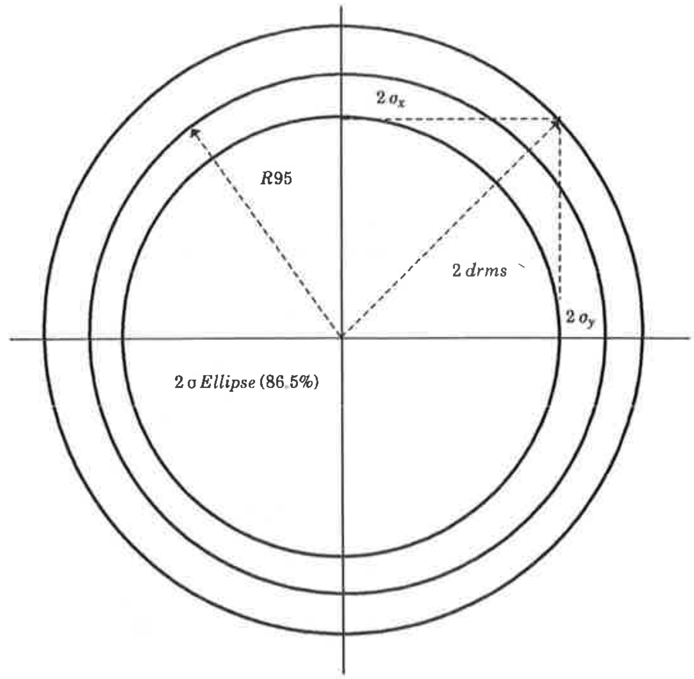


FIGURE 1-8. RELATIVE SIZES OF 2DRMS AND R95, CIRCULARLY DISTRIBUTED ERRORS

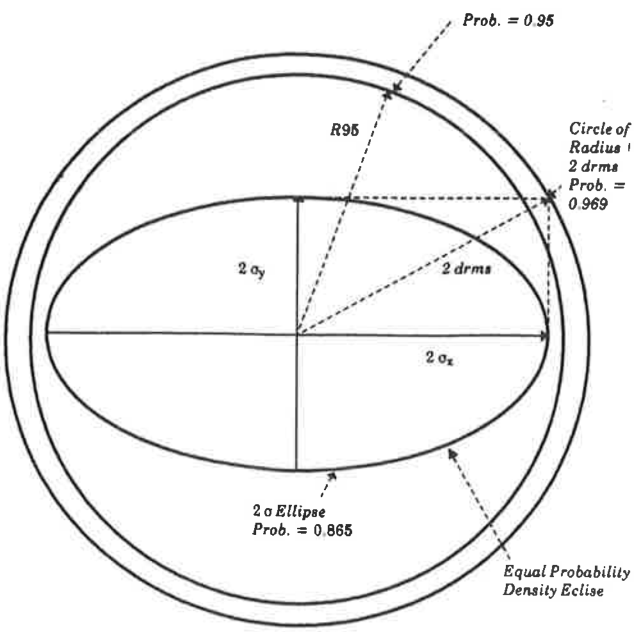


FIGURE 1-7. RELATIVE SIZES OF 2DRMS AND R95, SOMEWHAT SKEWED ERRORS

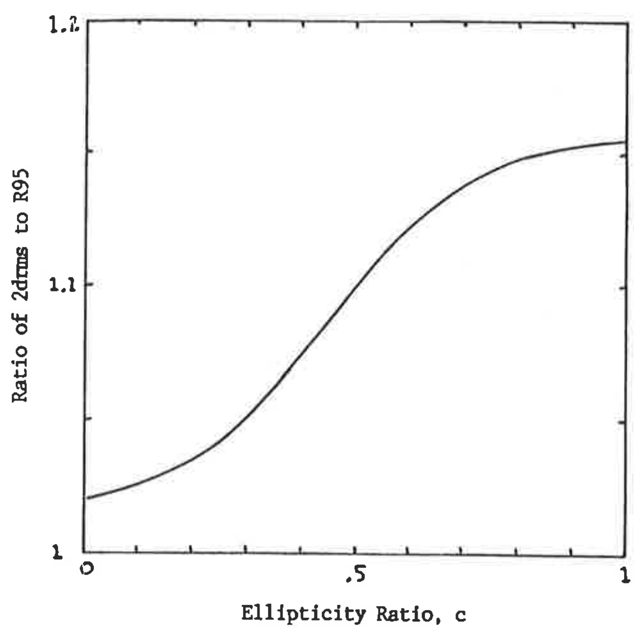


FIGURE 1-9. RATIO OF 2DRMS TO R95 ERRORS VERSUS ELLIPTICITY RATIO

The highly elliptical situation is often cited as a justification for the 2drms accuracy measure; the 2drms measure, being more conservative, would contain the errors within the ellipse but outside the R95 circle. While these errors would occur infrequently (5 percent of the time for R95), they would cluster, and if they happened to line up unfavorably with the navigation path, they would always be associated with lateral deviation errors. It might be concluded that the 2drms accuracy measure, which contains these errors and is thus more conservative, is a better accuracy measure. While this is true, the difference between the two-measures is only about 2 percent for highly skewed distributions. That is, if the R95 error measure were 100 meters, the 2 drms measure would be 103 meters, not a significant difference.

1.2.7 Accuracy Measures Applied to the Global Positioning System

Using a GDOP analysis, the various error measures were computed for the CONUS for a 24-hour period. A mask angle of 7.5 degrees was assumed, and two different satellite selection algorithms were employed: "all-in-view" and "best-set-of-four". The results cited below assume a standard deviation of pseudorange errors of 30 meters, which is approximately what the SPS will provide.

Table 1 shows the median and rms values of XDOP (east-west), YDOP (north- south), and HDOP, plus the 2drms, CEP and R95 median accuracies of GPS over the CONUS. The rms values of the circular measures are not given because of the difficulty of attaching any meaning to them. Also given are ellipticity, probability percentile of 2drms, and ratios between the accuracy measures.

It can be seen that the rms HDOP value of 2drms exceeds the median value by 6 percent, which indicates there are not a lot of large values of HDOP in the 21-satellite constellation coverage of the CONUS. This ratio would probably be larger for the 18-satellite coverage, where numerous outages exist which would raise this number.

The table shows that the east-west accuracy of GPS is better than the north-south accuracy by about 34 percent. This is because there is a region of low elevation angles to the north where no satellites ever appear, because of the 55^{o} inclination of the orbital planes. As a consequence, the distribution of the locations of the satellites will be skewed toward east-west, which improves the east-west accuracy. Intuitively, one would expect that the best-set-of-four satellite selection algorithm would reduce the difference between east-west and north-south accuracies, because it would tend to select satellites that (by resulting in lower HDOPs) would leave out satellites that would primarily improve the east-west accuracy. Indeed, the east-west accuracy is still better than the north-south accuracy, but only by 23 percent.

TABLE 1. DILUTION-OF-PRECISION AND ACCURACY MEASURES OF THE GLOBAL POSITIONING SYSTEM

(The results below assume a standard deviation of pseudorange errors, σ_R = 30 meters)

QUANTITY	ALL-IN-VIEW	BEST-SET OF-FOUR
MEDIAN XDOP	0.739	0.916
MEDIAN YDOP	0.948	1.110
MEDIAN HDOP	1.206	1.476
RMS XDOP	0.764	0.996
RMS YDOP	1.025	1.222
RMS HDOP	1.278	1.576
MEAN ELLIPTICITY RATIO	0.70	0.64
EAST-WEST ACCURACY, meters (2-SIGMA)	45.8	59.8
NORTH-SOUTH ACCURACY, meters (2-SIGMA)	61.5	73.3
2drms ACCURACY, meters	76.7	94.6
PROBABILITY PERCENTILE OF 2drms (Estimated from ellipticity)	97.7	97.5
R95 ACCURACY (mean), meters	66.5	83.4
CEP ACCURACY (mean), meters	30.3	37.4
RATIO OF 2drms TO MEAN R95	1.15	1.13
RATIO OF 2drms TO MEAN CEP	2.53	2.53

The ellipticity ratio was also computed over the CONUS, and a mean value of 0.70 was obtained for the all-in-view algorithm, 0.64 for the best-set-of-four. From the ellipticity the probability percentile of the 2drms accuracy measure can be estimated, but since the overall distribution is not exactly elliptical, the estimate is open to question.

The 2drms accuracy measure is larger than the mean R95 measure by 13 - 15 percent, which from Figure 1-9 can be seen to be consistent with the mean ellipticity ratio. Thus for a 30-meter pseudorange standard deviation, the mean R95 accuracy over the CONUS for an all-in-view receiver would be 67 meters, while the 2drms accuracy would be about 77 meters. The corresponding values for the best-of-four receiver would be 83 (R95) and 95 meters (2drms).

1.2.8 Conclusions; 2drms Versus CEP or R95

The chief advantage of the 2drms measure is the ease with which it can be computed when using a GDOP analysis to estimate it. The definition of 2drms must be modified to ignore periods of high GDOP, but this is not a serious deficiency. While it is frequently preferred because it is more conservative than R95, this is due primarily to the fact that it is associated with a higher probability percentile for more circular distributions, not because it is more conservative for highly elliptical distributions. The percentile of probability cannot be directly computed, but must be estimated from the mean ellipticity. The fact that it is not tied to a specific probability thus makes it less desirable, since there are no redeeming features (other than computational convenience for GDOP analyses) that make it attractive.

The CEP is not desirable as a measure for navigation systems because the probability attached to it is too small. The R95 value discussed here is better since it used the 95 percent probability level more commonly associated with navigation requirements. The argument that could be levied against the R95 value is that it says nothing about how large the large errors can become. That is, large errors in the upper 5 percent of the distribution are not limited, and have no effect on the R95 measure. However, it is seen from this discussion that large errors cause problems with the 2drms measure as well, which require treating them as outages. Using the same criteria, the larger values of the upper 5 percent are of no consequence, so the argument is not valid.

1.3 FEDERAL RADIONAVIGATION PLAN: REQUIREMENTS IN STATISTICAL MEASURE OF ACCURACY

The Federal Radionavigation Plan[1] (FRP) states, "When specifying linear accuracy, or when it is necessary to specify requirements in terms of orthogonal axes, e.g., along-track or cross-track, the 95 percent confidence level will be used. ... When two dimensional accuracies are used, the two drms (distance root mean squared) uncertainty estimate will be used. Two drms is the radius of a circle that contains at least 95 percent of all possible fixes that can be obtained with a system at any one place. DOD specifies horizontal accuracy in terms of Circular Error Probable (CEP -- the radius of a circle containing 50 percent of all possible fixes). For the FRP, it is agreed that the conversion of CEP to 2 drms would be accomplished by using 2.5 as the multiplier."

But the exact definitions of 1 drms and 2 drms (cf. Section 3.1) are such that 1 drms varies from 0.683 to 0.632 and 2 drms varies from 0.954 to 0.982 depending on the ellipticity of the equal probability density ellipses associated with the positional error.

Both the drms radius and the circular measures such as R95 and CEP may be expressed in terms of the semi-major and the semi-minor principal axis of the equi-probability error ellipse. It is convenient, therefore, first to treat the Elliptical Error Probability (the probability that the measured position lies within a given equi-probability density ellipse) and the Circular Error Probability (the probability that the measured position lies within a given circle). Then we discuss the probability for the 1 drms and 2 drms circles.

The CEP (50 percent probability circle) is compared with the 1 drms circle and also the R95 (95 percent probability circle) is compared with the 2 drms probability circle for various values of the ellipticity parameter σ_y/σ_x. The advantages of the two measures for satellite and ground-based navigation systems, in particular LORAN-C and GPS, are discussed.

The reader who is primarily interested in practical applications and in the probability measures within a circle may skim over the earlier sections and proceed quickly to Sections 2, 3, 4, 5 and 6. Definitions and commentary on the terms and abbreviations are provided in the Glossary.

1.4 DEFINITION OF VARIANCE AND STANDARD DEVIATION IN A TWO DIMENSIONAL PROBABILITY DISTRIBUTION

1.4.1 Continuous Distribution

Suppose a large number of measurements were to be made to determine the position of a vehicle in a two-dimensional coordinate system (X,Y). In this section X and Y are two arbitrary directions (at right angles to one another); X and Y are not necessarily in the North or East direction. In the other sections X is the Eastern and Y is the Northern direction. Let the probability density distribution of the measurements be f (X,Y). We can consider the general product moments (about the origin of the distribution as [4]

$$E[X^i Y^j] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X^i Y^j f(X,Y)\, dX dY \tag{1-8}$$

where E [w] is the expectation value of w

When i=1 and j=0 we obtain from equation 1-8 the mean of the distribution in the X direction.

$$m_X = E[X] = \int\int X f(X,Y)\, dX dY \tag{1-9}$$

(The integrations for X and Y are from - ∞ to + ∞)

When i=0 and j=1 we obtain from equation 1-8 the mean of the distribution in the Y direction.

$$m_Y = E[Y] = \int\int Y f(X,Y)\, dX dY \tag{1-10}$$

We also consider the general product (central) moments about $X = m_X$ and $Y = m_Y$ as given by

$$E[(X-m_X)^i (Y-m_Y)^j] = \int\int (X-m_X)^i (Y-m_Y)^j f(X,Y)\, dX dY \tag{1-11}$$

When i=2 and j=0 we obtain from equation 1-11 σ_X^2, the variance of the distribution along the X direction.

$$\begin{aligned}\sigma_X^{\ 2} &= E[(X-m_X)^2] \\ &= \int\int (X-m_X)^2 f(X,Y)\, dX dY\end{aligned} \tag{1-12}$$

When i=0 and j=2 we obtain from equation 1-11 σ_Y^2, the variance of the distribution along the Y direction.

$$\begin{aligned}\sigma_Y^{\ 2} &= E[(Y-m_Y)^2] \\ &= \int\int (Y-m_Y)^2 f(X,Y)\, dX dY\end{aligned} \tag{1-13}$$

σ_X is the standard deviation with respect to X. σ_Y is the standard deviation with respect to Y. Since the Greek letter sigma is used, it is common to use the terms sigma X and sigma Y for the standard deviations.

When i = 1 and j = 1, we obtain from equation 1-11 σ_{XY}^2 the covariance of the joint distribution of X and Y.

$$\sigma_{XY}^2 = E[(X - m_X)(Y - m_Y)]$$
$$= \int\int (X - m_X)(Y - m_Y) f(X,Y)\, dX dY \qquad (1\text{-}14)$$

The variances give us the spread of the probability distribution around the mean along the X and Y directions. The covariance takes into account jointly the X and Y deviations.

1.4.2 Discrete Distribution of Measurements

Suppose that N_0 measurements were to be made on the vehicle postition. Let the n^{th} measurement be (X_n, Y_n). The average or mean value of the measurement distribution in the X direction is

$$m_X = E[X_n]$$
$$= (1/N_o) \sum_{n=1}^{N_o} X_n \qquad (1\text{-}15)$$

where X_n is the n^{th} position measurement. Likewise we obtain for the Y direction

$$m_Y = E[Y_n]$$
$$= (1/N_o) \sum_{n=1}^{N_o} Y_n \qquad (1\text{-}16)$$

We shall assume that m_X and m_Y are both equal to zero in the coordinate system of interest to us.

The variance of the distribution of measurements (in length squared units) in the X direction is

$$V_X = \sigma_X^2 = E[X_n^2]$$
$$= (1/N_o) \sum_{1}^{N_o} (X_n^2) \qquad (1\text{-}17)$$

The standard deviation (or root mean squared) is

$$\sigma_X = \sqrt{\left[V_X\right]}$$

$$= \sqrt{\left[(\sum_{1}^{N_o} X_n^2) / N_o\right]} \qquad (1\text{-}18)$$

σ_X is called rms or one sigma X. It is in length units.

Similarly, we have for the Y direction the Variance

$$V_Y = E[Y_n^2] = (1/N_o) \sum_{1}^{N_o} (Y_n^2)$$

$$= \sigma_Y^2 \qquad (1\text{-}19)$$

The standard deviation of the distribution in the Y direction is

$$\sigma_Y = \sqrt{\left[V_Y\right]}$$

$$= \sqrt{\left[(\sum_{1}^{N_o} Y_n^2)/N_o\right]} \qquad (1\text{-}20)$$

The covariance of the distribution is (in length squared units) given by

$$V_{XY} = \sigma_{XY}^2 = E[X_n Y_n]$$

$$= (1/N_o) \sum_{n=1}^{N_o} (X_n Y_n) \qquad (1\text{-}21)$$

In Appendix A are some examples which illustrate the contributions to the covariance. These examples are rather simple but they help us to understand the more complex elliptical distributions examined in Section 2.9.

1.5 DEFINITION OF COVARIANCE MATRIX OF THE POSITION ERROR VECTOR

$$Let\, X_n = \begin{bmatrix} X \\ Y \end{bmatrix}$$

be a vector.

Its transpose is the row vector

$$X_n^{\,T} = \begin{bmatrix} X, Y \end{bmatrix}$$

The <u>outer product</u> $X_n X_n^T$ can be expressed as a matrix

$$X_n X_n^{\,T} = \begin{bmatrix} X^2 & XY \\ YX & Y^2 \end{bmatrix}$$

$$E(X_n X_n^{\,T}) = \begin{bmatrix} (\Sigma XX)/N_o & (\Sigma XY)/N_o \\ (\Sigma YX)/N_o & (\Sigma YY)/N_o \end{bmatrix} = \begin{bmatrix} \sigma_X^{\,2} & \sigma_{XY}^{\,2} \\ \sigma_{YX}^{\,2} & \sigma_Y^{\,2} \end{bmatrix} \qquad (1\text{-}22)$$

The above expression is the <u>covariance matrix</u> of the position error vector.
$\sigma_X^{\,2}$ is often also written by some authors as $\sigma_{XX}^{\,2}$ and $\sigma_Y^{\,2}$ as $\sigma_{YY}^{\,2}$.

Let X and Y be jointly normal, or Gaussian, random variables with zero mean and with the symmetric <u>covariance matrix of the position error vector</u> given by(5)

$$M = \begin{bmatrix} \sigma_X^{\,2} & \sigma_{XY}^{\,2} \\ \sigma_{XY}^{\,2} & \sigma_Y^{\,2} \end{bmatrix} \qquad (1\text{-}23)$$

A detailed discussion of the covariance matrix in its expanded form and its relation to the dimensionless Q matrix and the G matrix is given in the Glossary.

In GPS application the positive X axis is in the Eastern direction while the positive Y axis is in the Northern direction.

From M the symmetric matrix, equation 1-23, the eigenvectors and eigenvalues can be computed. Then it is possible to construct an orthogonal matrix L_ϕ such that

$$L_\phi^T M L_\phi = N \tag{1-24}$$

where N is a diagonal matrix (i.e., a matrix having zero value of the non- diagonal elements).

Defining new variables x and y by

$$\begin{bmatrix} x \\ y \end{bmatrix} = L_\phi^T \begin{bmatrix} X \\ Y \end{bmatrix} \tag{1-25}$$

it follows that x and y are zero mean independent normal, or Gaussian, random variables. The subscript ϕ is an angle of rotation. An explanation will be given in Section 2.9.

$$N = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix} \tag{1-26}$$

Moreover the trace of M (i.e. the sum of the diagonal terms) is equal to the trace of N. That is to say

$$\sigma_X^2 + \sigma_Y^2 = \sigma_x^2 + \sigma_y^2 \tag{1-27}$$

The sum of the variances in the diagonal are then invariant with respect to an orthogonal coordinate transformation. A proof of this statement is given in Appendix B.

The quantity σ_X (or σ_x) in equation 1-4 is in length units (say meters). In matrix N, $\sigma_x^2 = \lambda_1$ and $\sigma_Y^2 = \lambda_2$, where λ_1 and λ_2 are the eigenvalues of the matrix M. By definition

$$1\ drms = \sqrt{\left[\sigma_x^2 + \sigma_y^2\right]} = \sqrt{\left[\sigma_X^2 + \sigma_Y^2\right]} \qquad (1\text{-}28a)$$

$$2\ drms = 2\sqrt{\left[\sigma_x^2 + \sigma_y^2\right]} = \sqrt{\left[(2\sigma_x)^2 + (2\sigma_y)^2\right]} \qquad (1\text{-}28b)$$

Care should be taken in the definition of the angle of rotation in the orthogonal coordinate transformation so that one can identify the angle between the X (Eastern) direction and x, the direction of the major principal axis. For details of the procedure see Section 2.9 and Appendix C.

1.6 PROPERTIES OF THE TWO DIMENSIONAL NORMAL (GAUSSIAN) PROBABILITY DENSITY DISTRIBUTION

1.6.1 Elliptical Error Probability

Assuming that the covariance matrix is nonsingular, it follows that

$$\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} = k^2 \qquad (1\text{-}29)$$

is the equation of an ellipse as shown in Figure 1-10. The major axis of the error ellipse is in the direction of the x axis, and the minor axis of the ellipse is in the direction of the y axis. The x and y axis are in general not in the North (+X) and East (+Y) directions. With $k \geq 0$, the probability that the measured position (X,Y) lies within the ellipse denoted by $P_E(k)$ is given by

$$P_E(k) = \iint_{\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} \leq k^2} \frac{exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)\right] dxdy}{2\pi\sigma_x\sigma_y} \qquad (1\text{-}30)$$

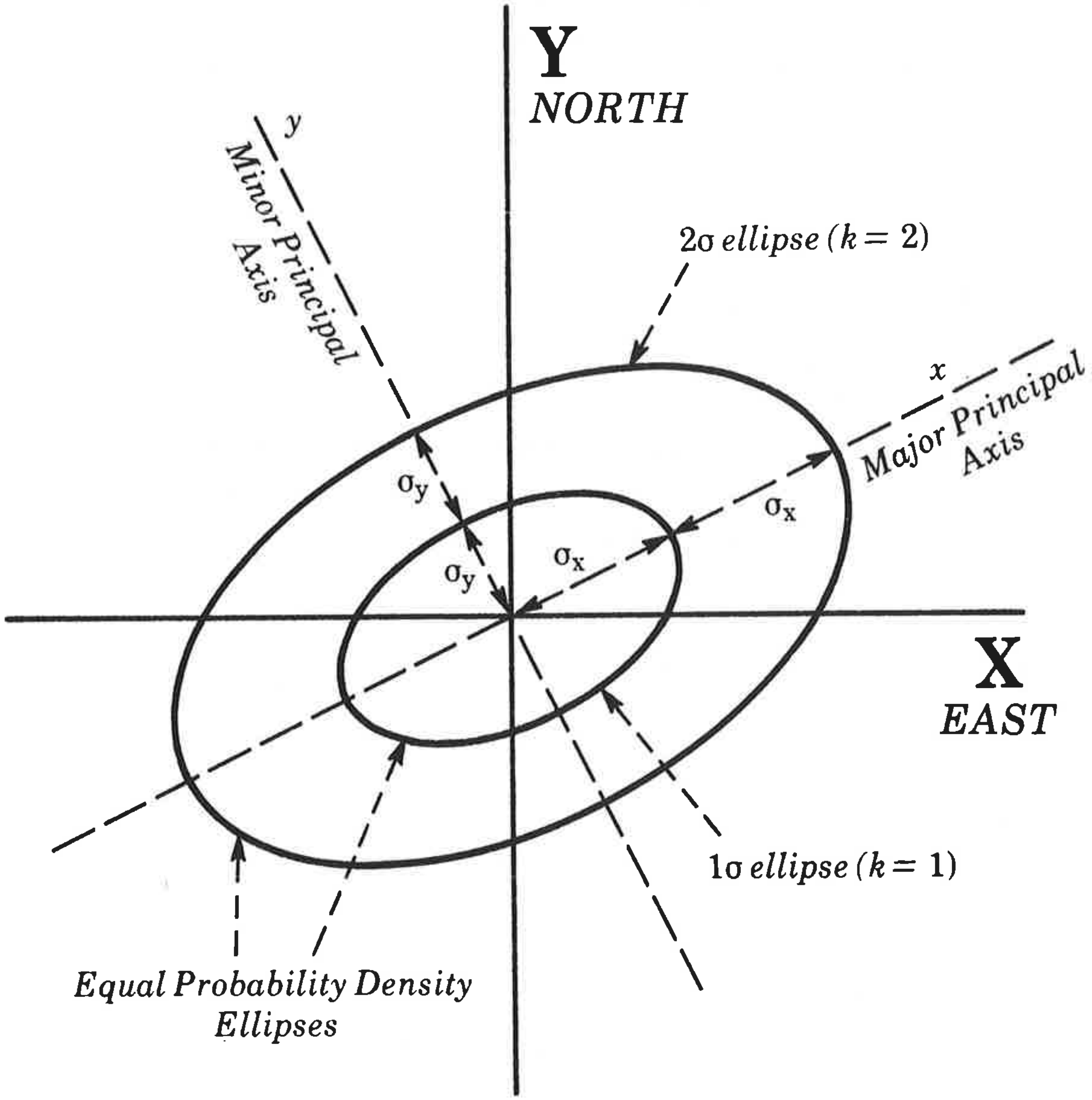


FIGURE 1-10. ELLIPSES OF EQUAL PROBABILITY DENSITY

Note: Probability of finding the measured position within the 1σ ellipse is 39.347 percent and within the 2 σ ellipse is 86.466 percent (cf. Table 2).

(Note that the lower case symbol k denotes a quantity that is different from the quantity denoted by the upper case symbol K, employed by H.L. Harter(6) in his circular integral tables. We deliberately use the symbol k (instead of the symbol K used by Brooks et al.)(5) in order to avoid confusion with Harter's K.)

$P_E(k)$ is the Elliptical Error Probability in 2 dimensions, and

$$P_E(k) = 1 - e^{-k^2/2} \qquad (1\text{-}31)$$

1.6.2 Probability Density Distribution and Equal Probability Ellipse

The probability density of the measured position is given by the two dimensional normal (Gaussian) probability density distribution

$$f(x,y) = \frac{exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^{\,2}} + \frac{y^2}{\sigma_y^{\,2}}\right)\right]}{2\,\pi\,\sigma_x\,\sigma_y} \qquad (1\text{-}32)$$

This is a distorted bell shape distribution when the bell shape is stretched out along the x direction and compressed along the y direction.

For a given value of k, the values of x and y which satisfy the equation

$$\frac{x^2}{\sigma_x^{\,2}} + \frac{y^2}{\sigma_y^{\,2}} = k^2 \qquad (1\text{-}33)$$

define an equal-probability density ellipse. On this ellipse f(x,y) has a constant value,

$$f(x,y) = \frac{exp\,[(-1/2)\bullet k^2]}{2\,\pi\,\sigma_x\,\sigma_y} \qquad (1\text{-}34)$$

The new variables are such that the x axis is along the major principal axis and y is along the minor principal axis of the constant probability density ellipses (see Figure 1-10).

1.6.3 Probability of a Position Measurement Within an Error Ellipse

From equation 1-31 the values of k for P_E percent probability ellipses can be calculated.[4]

TABLE 2. P_E PERCENT IS THE PROBABILITY OF FINDING THE MEASURED POSITION WITHIN THE ELLIPSE DEFINED BY k

P_E percent	k
25	0.7585
39.347	1.0 (1 σ ellipse)
50	1.17741
75	1.665
86.466	2.0 (2 σ ellipse)
90	2.146
95	2.44775
98.8	3.0 (3 σ ellipse)
99	3.035

The semi-major principal axis of a given 'k σ ellipse' (i.e. an equal probability density ellipse corresponding to a given value of k) is given by k σ_x. Thus a 2σ ellipse would have k = 2, and its semi-major principal axis would have the length 2 σ_x. Note that no matter what is the shape or ellipticity of equi-probability ellipses, the k σ ellipse contains a certain probability of finding a position measure. For example, within the '2 σ ellipse' the probability of finding a position measure is always 86.466 percent (no matter what is the ellipticity). This point was not made clear in the writings of Burt et al.[3] (cf. p. 45 and p. 46 in their paper).

In particular if the 'ellipse' is a circle then $\sigma = \sigma_x = \sigma_y$. For this special case of a circular normal probability density distribution

$$k^2 = (x^2 + y^2)/\sigma^2 \qquad (1\text{-}35)$$

$$= R^2/\sigma^2$$

$$k = R/\sigma \qquad (1\text{-}36)$$

W. Allan Burt et al. gave the formula for the circular normal distribution but they did not present the general elliptical case in equation 1-31, which was described by Brooks et al.(5) and by Burington and May.(4)

In the representation of the normal probability density distribution (equation 1-32) in the x and y coordinate system (x the major principal axis and y the minor principal axis of the equal density ellipse) the covariance of the joint distribution in x and y is equal to zero on account of the symmetry of the normal distribution function (see Appendix A for a simple illustration).

It is of interest in passing to refer to the mathematical form of the two-dimensional normal (Gaussian) distribution expressed in the coordinate system (say X East and North) where the angle between the major (or minor) principal axis of the equal density ellipse and the Eastern axis is not equal to zero. In this case the covariance of the joint distribution in the Eastern and Northern axis is generally non-zero. The mathematical representation of the probability density function is given by Burington and May on page 97 of their handbook in terms of the variances and also the covariance.(4) We shall not be using that representation in this study. However the above mentioned representation may be useful if one wishes to obtain the probabilities within a specified corridor parallel to a certain vehicle track (or crosstrack) direction. We give the general form for use in future research

$$f(X, Y) = \frac{exp\,(-W/2)}{2\,\pi\,\sigma_X\,\sigma_Y\sqrt{\left[1-r^2\right]}} \qquad (1\text{-}37)$$

(r is the correlation coefficient of X and Y)

$$-W/2 = \frac{-1}{2(1-r^2)}\left[\frac{X^2}{\sigma_X^2} - \frac{2rXY}{\sigma_X\sigma_Y} + \frac{Y^2}{\sigma_Y^2}\right], \tag{1-38}$$

with $$r = \sigma_{XY}^2/\sigma_X\sigma_Y \ , \tag{1-39}$$

(here x and y may be in any two orthogonal directions).

Some work has been completed at TSC on the probability corridors along and perpendicular to the vehicle on-track direction for GPS Navigation.(8)

2. CIRCULAR ERROR MEASURES

2.1 CIRCULAR ERROR PROBABILITY, CEP (CIRCULAR ERROR PROBABLE), AND THE ELLIPTICITY PARAMETER $c \equiv \sigma_y / \sigma_x$

When the elliptical probability density distribution is integrated over a circle of radius R, we obtain the probability that the position measurement (X,Y) lies within this circle. This probability is denoted by

$$P(R) = \iint_{x^2 + y^2 \le R^2} \frac{exp\left[-1/2\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)\right]}{2\pi\sigma_x\sigma_y} dxdy \tag{2-1}$$

and is called the Circular Error Probability.

The term CEP, the Circular Error Probable, which is widely used by the Department of Defense, denotes the value of a radius R or R50 which results in the probability being equal to 50 percent. Note that the Circular Error Probability P(R) is dimensionless while the Circular Error Probable (CEP) is in length units (say, meters); they are different quantities.

We should remind ourselves that in general the probability density distribution is elliptical so that σ_x is not equal to σ_y. Only in a special situation is $\sigma = \sigma_x = \sigma_y$. In this special case we have a circularly normal distribution.

Burt et al.[3] call the quantity $c = \sigma_y / \sigma_x$ the ellipticity parameter. This quantity is very useful and the symbol c, where $c = \sigma_y / \sigma_x$ is used by H. Leon Harter,[6] whose tables we shall be using extensively. This ellipticity parameter is different from the ellipticity definition used by astronomers (for example, Hubble's The Realm of the Nebulae, p. 41.); nor is it the same as the eccentricity of an ellipse as defined, for example, in G.B. Thomas' Calculus and Analytic Geometry (p. 243). Thomas' eccentricity is the distance from the center of the ellipse to one of the foci, divided by the semi-major principal axis of the ellipse.

When c (or σ_y / σ_x) is equal to 1 the ellipse becomes a circle, and when $c = 0$ the ellipse degenerates into a line (linear degeneracy). Since the ellipticity is most pronounced when $c = 0$, we could call c the inverse ellipticity or just the ellipticity parameter σ_y / σ_x.

Burt et al.[3] pointed out that "referring to the position accuracy of a system as '600 ft (2 σ)'" is ambiguous, that "Such a description leaves the reader wondering whether the measure is of circular error, in which case the numbers describe the 86 percent probability circle, or whether the numbers are to be interpreted as one-dimension sigmas along each axis, in which case the 95 percent probability circle is indicated (assuming the distribution to be circular, which actually it may not be)." Here Burt et al. were referring to the linear measure for a normal curve where the linear probability between $\pm$ 3 σ_x is 68.27 percent, between $\pm$2 σ_x is 95.45 percent and between $\pm$ 3σ_x is 99.73 percent as contrasted with the measure given in Table 2.

The linear measure as pointed out by Burt et al. is for a function of one dimension.

$$f(x) = \frac{exp\left[-\frac{1}{2}\frac{1}{\sigma^2}(x-a)^2\right]}{\sqrt{\left[2\pi\right]}\sigma}$$

The probability computed in Table 2, on the other hand, is for finding the position within the ellipse defined by the index k.

The probability of a two-dimensional normal distribution enclosed by the limits x= − L and x= + L are as follows:

$$P = \int_{-\infty}^{+\infty} dy \frac{exp\left[\frac{-1}{2}\left(\frac{y^2}{\sigma_y^2}\right)\right]}{\sqrt{\left[2\pi\right]}\sigma_y} \int_{-L}^{+L} dx \frac{exp\left[\frac{-1}{2}\left(\frac{x^2}{\sigma_x^2}\right)\right]}{\sqrt{\left[2\pi\right]}\sigma_x}$$

The integral in y is just 1. Therefore the enclosed probabilities for a multiple of σ_x are just these values, 68.27 percent, 95.45 percent, etc. mentioned above. The above probability 'corridors' are parallel to the x axis (or the y axis).

2.2 CIRCLE OF PROBABILITY 95 PERCENT WITH RADIUS R95

We can likewise define other radii whose circles contain a certain probability of error. R95 (or R_{95}) is, for example, a radius whose circle contains the position measure with a probability of 95 percent. The circular probability integral of equation 2-1 has been computed by various workers (cf. the references in Burt et al. p. 121 and p. 78).[3] R95 is in general not equal to 2 drms. Tables 3 and 4 are from H. Leon Harter.[6]

The probability (Circular Error Probability) that a measured position (X,Y) or (x,y) will lie within a circle with center at the origin and radius R, where R = K σ_x, can be obtained from Table 3.

Given σ_x σ_y and the Radius R , to Calculate the Probability P of Containing a Position Within the Circle

Suppose we are given the standard deviations σ_x, the semi-major principal axis, and σ_y, the semi-minor principal axis, of the 1 σ equal probability density ellipse and also we are given some specified radius R. Then by definition

$$K = R/\sigma_x \qquad (2\text{-}2)$$

(note K is different from k of the previous discussion in Section 1.5.).

Also $c = \sigma_y / \sigma_x$

c is the ellipticity parameter that is a characteristic of an equal probability density ellipse associated with the probability density function in the specific situation, $0 \leq c \leq 1$. Then we simply obtain the value of P(K,c) from Table 3 by interpolation.

Given σ_x,σ_y and the Probability of Containing a Position Measure, to Obtain the Radius of the Circle

Now suppose we want to obtain a radius of a certain probability of containing the measured position, given σ_x, and σ_y. Then we can specify the probability P, compute c (which is equal to σ_y / σ_x), and obtain K from Table 4. Then R can be calculated from the equation

$$R = K\sigma_x \qquad (2\text{-}3)$$

TABLE 3. CIRCULAR ERROR PROBABILITIES P (K,c) (FROM H.L. HARTER[6])

K \ c	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	.0796557	.0443987	.0242119	.0164170	.0123875	.0099377	.0082940	.0071157	.0062299	.0055400	.0049875
0.2	.1585194	.1330783	.0884533	.0628396	.0482413	.0390193	.0327123	.0281415	.0246824	.0219757	.0198013
0.3	.2358228	.2213804	.1739300	.1318281	.1039193	.0851535	.0719102	.0621386	.0546598	.0487639	.0440025
0.4	.3108435	.3010228	.2635181	.2139084	.1742045	.1451808	.1237982	.1076237	.0950495	.0850326	.0768837
0.5	.3829249	.3755884	.3481790	.3003001	.2532953	.2152886	.1857448	.1626829	.1443941	.1296286	.1175031
0.6	.4514938	.4457708	.4255605	.3846374	.3357384	.2914682	.2548177	.2251114	.2000707	.1811783	.1647298
0.7	.5160727	.5115048	.4960683	.4633258	.4170862	.3699305	.3280302	.2925654	.2629373	.2381583	.2172955
0.8	.5762892	.5725957	.5604457	.5349387	.4941882	.4474207	.4025628	.3627122	.3283453	.2989700	.2738510
0.9	.6318797	.6288721	.6191354	.5993140	.5651504	.5213998	.4759375	.4333628	.3953279	.3620135	.3330232
1.0	.6826895	.6802325	.6723586	.6568242	.6291249	.5900053	.5461319	.5025700	.4021421	.4257553	.3934693
1.1	.7286679	.7266597	.7202682	.7079681	.6859367	.6524489	.6116316	.5687467	.5272462	.4887873	.4539256
1.2	.7698607	.7682215	.7630305	.7532175	.7359558	.7070973	.6714269	.6306168	.5893494	.5498736	.5132477
1.3	.8063990	.8050648	.8008554	.7929968	.7793550	.7567265	.7249673	.6873122	.6474394	.6079822	.5704426
1.4	.8384867	.8374049	.8340018	.8277048	.8160851	.7989288	.7720889	.7383089	.7007900	.6623035	.6240889
1.5	.8663856	.8655127	.8627728	.8577302	.8493071	.8350816	.8129287	.7833062	.7489500	.7122546	.6753475
1.6	.8904014	.8897008	.8875000	.8834914	.8768044	.8657559	.8478303	.8226246	.7917194	.7574708	.7219627
1.7	.9108691	.9103102	.9085619	.9053766	.9001746	.8915536	.8773116	.8562471	.8291137	.7977882	.7462539
1.8	.9281394	.9276964	.9263125	.9237989	.9197275	.9130680	.9019110	.8846024	.8613238	.8332175	.8021013
1.9	.9425669	.9422182	.9411299	.9391586	.9359855	.9308615	.9222277	.9083600	.8880731	.8639149	.8355255
2.0	.9544997	.9542272	.9533775	.9518415	.9493815	.9454546	.9388418	.9278799	.9115762	.8901495	.8646647
2.1	.9642712	.9640598	.9634011	.9622127	.9603170	.9573205	.9522999	.9437668	.9305013	.9122714	.8897495
2.2	.9721931	.9720304	.9715237	.9706100	.9691507	.9668845	.9631017	.9565522	.9459380	.9306821	.9110784
2.3	.9785518	.9784275	.9780408	.9773450	.9762419	.9745239	.9716934	.9667306	.9583739	.9458085	.9289946
2.4	.9836049	.9835108	.9832180	.9826918	.9818594	.9805703	.9784661	.9747495	.9682098	.9580804	.9438652
2.5	.9875807	.9875100	.9872900	.9868953	.9862720	.9853112	.9837560	.9810035	.9760522	.9670136	.9560631
2.6	.9906776	.9906249	.9904612	.9901674	.9897045	.9889934	.9878527	.9858331	.9821023	.9756969	.9659525
2.7	.9930661	.9930271	.9929062	.9926894	.9923483	.9918260	.9909944	.9895268	.9867530	.9817837	.9738786
2.8	.9948897	.9948612	.9947727	.9946141	.9943649	.9939842	.9933821	.9923249	.9902888	.9864876	.9801589
2.9	.9962684	.9962477	.9961834	.9960684	.9958878	.9956126	.9951798	.9944246	.9929482	.9900803	.9850792
3.0	.9973002	.9972853	.9972391	.9971564	.9970206	.9968294	.9965205	.9959854	.9946274	.9927025	.9888010

$P(K, c)$ = the probability that a point falls inside a circle whose center is at the origin and whose radius is K times the larger standard deviation, c being the ratio of the smaller standard deviation to the larger standard deviation.

TABLE 3. CIRCULAR ERROR PROBABILITIES P (K,c) (Cont'd.) (FROM H.L. HARTER[6])

K \ c	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
3.1	.9980648	.9980542	.9980212	.9979622	.9978699	.9977296	.9975109	.9971348	.9963851	.9948168	.9918113
3.2	.9986257	.9986182	.9985949	.9985533	.9984880	.9983892	.9982356	.9979733	.9974478	.9963105	.9940240
3.3	.9990332	.9990279	.9990116	.9989824	.9989308	.9988677	.9987607	.9985792	.9982147	.9974004	.9956822
3.4	.9993261	.9993225	.9993112	.9992909	.9992503	.9992115	.9991370	.9990120	.9987620	.9981808	.9969113
3.5	.9995347	.9995323	.9995245	.9995105	.9994888	.9994550	.9994053	.9993204	.9991502	.9987480	.9978125
3.6	.9996818	.9996801	.9996748	.9996653	.9996505	.9996281	.9995938	.9995304	.9994218	.9991442	.9984662
3.7	.9997844	.9997832	.9997797	.9997733	.9997033	.9997482	.9997251	.9996807	.9996102	.9994208	.9989352
3.8	.9998553	.9998545	.9998522	.9998478	.9998412	.9998311	.9998157	.9997902	.9997300	.9996119	.9992682
3.9	.9999038	.9999033	.9999018	.9998989	.9998945	.9998878	.9998776	.9998008	.9998270	.9997426	.9995020
4.0	.9999367	.9999363	.9999353	.9999334	.9999305	.9999261	.9999105	.9999085	.9998870	.9998309	.9996645
4.1	.9999587	.9999585	.9999578	.9999566	.9999547	.9999519	.9999475	.9999404	.9999266	.9998900	.9997763
4.2	.9999733	.9999732	.9999727	.9999720	.9999707	.9999689	.9999661	.9999616	.9999527	.9999292	.9998523
4.3	.9999829	.9999828	.9999820	.9999821	.9999813	.9999801	.9999783	.9999754	.9999698	.9999548	.9999034
4.4	.9999892	.9999891	.9999889	.9999886	.9999881	.9999874	.9999863	.9999845	.9999809	.9999715	.9999375
4.5	.9999932	.9999932	.9999931	.9999929	.9999925	.9999921	.9999914	.9999902	.9999881	.9999822	.9999599
4.6	.9999958	.9999957	.9999957	.9999955	.9999954	.9999951	.9999947	.9999939	.9999926	.9999889	.9999746
4.7	.9999974	.9999974	.9999973	.9999973	.9999971	.9999970	.9999967	.9999963	.9999955	.9999932	.9999840
4.8	.9999984	.9999984	.9999984	.9999983	.9999983	.9999982	.9999980	.9999977	.9999972	.9999959	.9999901
4.9	.9999990	.9999990	.9999990	.9999990	.9999990	.9999989	.9999988	.9999986	.9999983	.9999975	.9999939
5.0	.9999994	.9999994	.9999994	.9999994	.9999994	.9999993	.9999993	.9999992	.9999990	.9999985	.9999963
5.1	.9999997	.9999997	.9999997	.9999996	.9999996	.9999996	.9999996	.9999995	.9999994	.9999991	.9999978
5.2	.9999998	.9999998	.9999998	.9999998	.9999998	.9999998	.9999998	.9999997	.9999997	.9999995	.9999987
5.3	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999998	.9999998	.9999997	.9999992
5.4	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999998	.9999995
5.5	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	.9999999	.9999999	.9999999	.9999997
5.6								1.0000000	1.0000000	.9999999	.9999998
5.7										1.0000000	.9999999
5.8											1.0000000
5.9											
6.0											

$P(K, c)$ = the probability that a point falls inside a circle whose center is at the origin and whose radius is K times the larger standard deviation, c being the ratio of the smaller ... deviati...

TABLE 4. VALUES OF K CORRESPONDING TO CUMULATIVE PROBABILITY P (FROM H.L. HARTER[6])

P \ c	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
.5000	0.67449	0.68109	0.70585	0.74093	0.80785	0.87042	0.93365	0.99621	1.05769	1.11807	1.17741
.7500	1.15035	1.15473	1.10825	1.10246	1.23100	1.28534	1.35143	1.42471	1.50231	1.58271	1.66511
.9000	1.64485	1.64701	1.65731	1.67383	1.69918	1.73708	1.79152	1.86253	1.94761	2.04236	2.14597
.9500	1.95996	1.96253	1.97041	1.98420	2.00514	2.03586	2.08130	2.14598	2.23029	2.33180	2.44775
.9750	2.24140	2.24365	2.25053	2.26255	2.28073	2.30707	2.34581	2.40356	2.48494	2.58999	2.71620
.9900	2.57583	2.57778	2.58377	2.59421	2.60995	2.63257	2.66533	2.71515	2.79069	2.89743	3.03435
.9950	2.80703	2.80883	2.81432	2.83289	2.83830	2.85894	2.88859	2.93347	3.00431	3.11073	3.25525
.9975	3.02334	2.02500	3.03010	3.03898	3.05234	3.07144	3.09871	3.13969	3.20586	3.31099	3.46164
.9990	3.29053	3.29206	3.29673	3.30489	3.31715	3.33464	3.35949	3.39647	3.45698	3.55939	3.71692

From Table 4 we see that the values of R50 (or CEP) and R95 are functions of the ellipticity parameter $c = \sigma_y / \sigma_x$.

2.3 TABULATION OF RATIOS CEP/σ_x, R95/ σ_x AND R95/2 σ_x VERSUS σ_y / σ_x

σ_x and σ_y are respectively the semi-major axis and the semi-minor axis of the 1 σ ellipse.

TABLE 5. RATIOS CEP/σ_x, R95/ σ_x, AND R95/2 σ_x VERSUS σ_y / σ_x

$c = \sigma_y / \sigma_x$	CEP/σ_x	R95/σ_x	R95/2σ_x
0.0	0.67	1.96	.980
0.1	0.68	1.96	.9813
0.2	0.71	1.97	.99
0.4	0.81	2.01	1.01
0.5	0.87	2.036	1.018
0.6	0.93	2.08	1.02
0.8	1.06	2.23	1.12
1.0	1.18	2.45	1.224

We see that the ratio of the probability radius to the length of σ_x becomes larger as c increases. R95/2σ_x ranges from .98 to 1.224.

Let R95 be expressed as 1 unit. Then for c = 0.1

$$\frac{2\sigma_x}{R95} = \frac{1}{0.9813}$$
$$= 1.019$$

and for c = 1.0

$$\frac{2\sigma_x}{R95} = \frac{1}{1.224}$$
$$= 0.8170$$

In Figure 2-1 are drawn an R95 circle (i.e., a radius whose circle contains 95 percent of the position measures) and two '2σ ellipses': (a) one ellipse with $\sigma_y/\sigma_x = c = .1$ (b) the second '2σ ellipse' is in the form of a circle with $\sigma_y / \sigma_x = c = 1$. The 2σ ellipses both have probabilities of 86.47 percent (cf. Table 2).

Although the radius of the 95 percent probability circle becomes smaller than the length of $2\sigma_x$ as c becomes smaller, the area ratio of the R95 circle to the 2σ ellipse increases as the following section demonstrates (see also Figure 2-1).

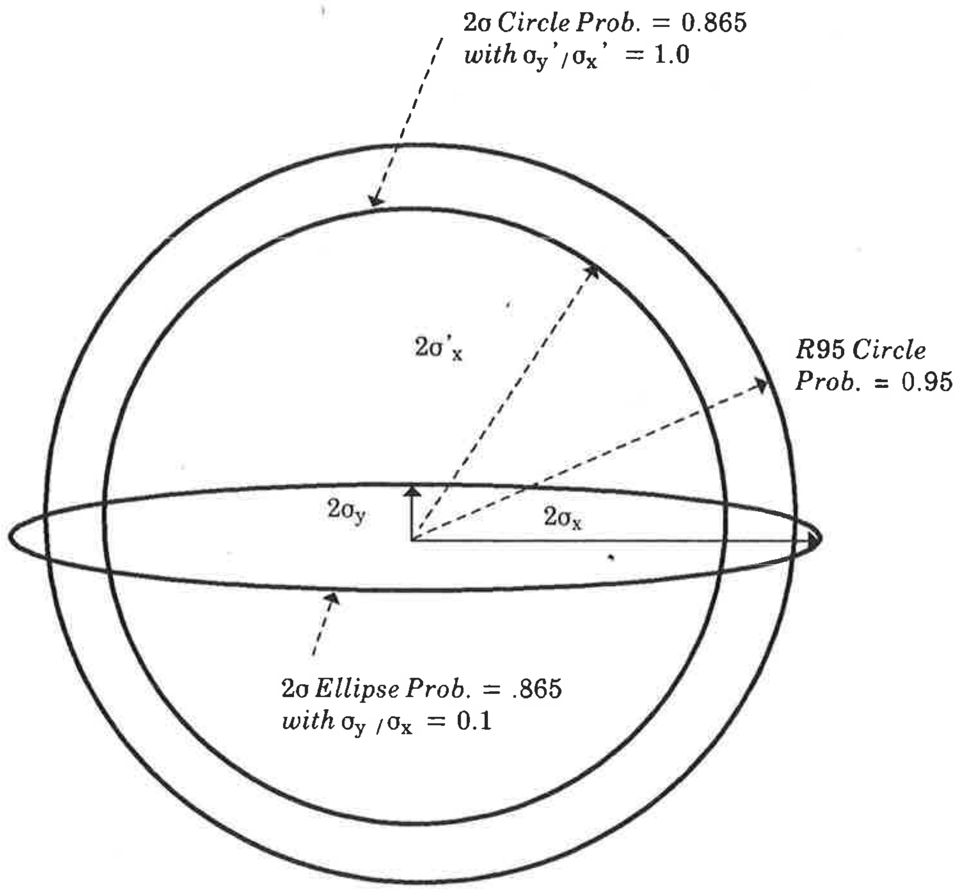
2.4 AREA RATIO OF R95 CIRCLE TO 2σ ELLIPSE

The ratio of the area within the circle of radius R95 (containing 95 percent of the position measures) to the area within the 2σ ellipse (containing 86.47 percent of the position measures) for a definite ratio of σ_y to σ_x can be computed. The area of the R95 circle is

$$\pi R^2 = \pi (K \sigma_x)^2$$
$$= \pi K^2 \sigma_x^2 \qquad (2\text{-}4)$$

The area of the 2σ ellipse is

$$\pi AB = \pi(2\sigma_x)(2\sigma_y)$$
$$= 4\pi\sigma_x^2 c \qquad (2\text{-}5)$$



Area Ratio of R95 Circle to 2σ Ellipse

$c = \sigma_y / \sigma_x = 0.1; 2\,\sigma_x = 1.019$ R95   c = 0.1, Ratio = 9.63

$c = \sigma_y' / \sigma_x' = 1.0; 2\,\sigma_x' = 0.8170$ R95   c = 1.0, Ratio = 2.498

FIGURE 2-1. R95 CIRCLE DRAWN ALONG WITH TWO '2σ ELLIPSES' (From Burt et al.(3))

Area ratio of R95 circle to 2σ Ellipse

$$= K^2/4C \tag{2-6}$$

Using Table 4 from Harter (P = .95) we obtained the following table.

TABLE 6. AREA RATIO OF R95 CIRCLE TO 2σ ELLIPSE VERSUS σ_y / σ_x

$c = \sigma_y / \sigma_x$	
0	∞
.1	9.629
.5	2.071
1.0	1.498

We see that the ratio increases with the decrease of the value of c. As $c \rightarrow 0$, $K \rightarrow 1.95996$ (cf. Table 4) and $K^2(4c) \rightarrow \infty$.

2.5 AREA RATIO OF CEP CIRCLE TO 1σ ELLIPSE

The area ratio of the circle of radius R50 (or CEP) which contains the measured position with a probability of 50 percent to the 1σ ellipse which contains the measured position with a probability of 39.3 percent can be computed as follows.

The area of the circle is

$$\begin{aligned} \pi R^2 &= \pi (K \sigma_x)^2 \\ &= \pi \sigma_x^2 K^2, \end{aligned} \tag{2-7}$$

where the values of K are obtained from the row, where P = .50, in Table 4 (from Harter).

The area of the 1σ ellipse (which contains a probability of 39.3 percent (cf. Table 2) is given by

$$\pi \sigma_x \sigma_y = \pi \sigma_x^2 c \tag{2-8}$$

Then the area ratio is

$$\frac{\pi \sigma_x^2 K^2}{\pi \sigma_x^2 c} = \frac{K^2}{c} \tag{2-9}$$

In Table 7 are computed several values of the area ratio versus σ_y/σ_x. When $c \to 0$, $K \to 0.6744$.

TABLE 7. AREA RATIO OF CEP (R50) TO 1σ ELLIPSE VERSUS σ_y/σ_x

$c = \sigma_y/\sigma_x$	
0	∞
.1	4.651
.5	1.632
1.0	1.386

2.6 AREA RATIO OF CEP CIRCLE TO 50 PERCENT ELLIPSE

In Tables 6 and 7 the probabilities of the circles are different from the probabilities of containing a position measure in the case of the corresponding ellipses. We ought to show a case where the probability of containment for the ellipse is the same value as that of the circle.

We consider the case of a 50 percent ellipse and the 50 percent equivalent circle (i.e. CEP (or R50)). For the ellipse, the index k is given by (1-24) for $P_E(k) = .50$. From Table 2 we see that k = 1.17741 (remember lower case k is that in Table 2 while upper case K is from Harter's tables).

The area of the circle is

$$\pi(R50)^2 = \pi\,(cep)^2 = \pi K^2 \sigma_x^2$$

The area of the 50 percent ellipse (or 1.177 σ ellipse) is

$$\pi AB = \pi k \sigma_y k \sigma_x = \pi \sigma_x^2 c k^2$$

The area ratio is

$$K^2/(c\,k^2) \qquad \text{(2-10)}$$

where K, which is a function of c, is given by the first row in Table 4 (P = .50) and k = 1.17741.

TABLE 8. AREA RATIO OF 50 PERCENT CIRCLE WITH RADIUS CEP TO 50 PERCENT ELLIPSE VERSUS σ_y/σ_x

$c = \sigma_y/\sigma_x$	
0.0	∞ (ellipse linear degenerate)
.1	3.355
.2	1.797
.5	1.093
.8	1.009
1.0	1.0 (ellipse becomes a circle)

Burt et al.[3] have in their Table A-1 compared the areas that we just considered. Our results are consistent with the numbers given in their table. As Burt et al. pointed out, the area of the CEP circle is always greater than the corresponding ellipse (except for the case where $\sigma_y/\sigma_x = 1$): "The divergence between the actual area of the ellipse of interest and the circle of equivalent probability increases as the ellipse becomes thinner and more elongated" (p. 47).

2.7 LENGTH RATIO OF CEP TO 1.177 σ_x, THE SEMI-MAJOR AXIS OF THE 50 PERCENT ELLIPSE

The length ratio is given by (with k = 1.17741)

$$\frac{CEP}{k\,\sigma_x} = \frac{K_{50}\,\sigma_x}{(k\sigma_x)} = \frac{K_{50}}{1.17741} \qquad \text{(2-11)}$$

Using Table 4 from Harter[6] with P = 50 we obtain the following table:

TABLE 9. LENGTH RATIO OF CEP TO THE SEMI-MAJOR AXIS OF THE 50 PERCENT ELLIPSE VERSUS σ_y/σ_x

$c = \sigma_y/\sigma_x$	
0.0	.573 (ellipse linear degenerate)
.1	.579
.2	.599
.5	.739
.8	.898
1.0	1.0 (ellipse becomes circle)

We see that the ratio varies from .573 to 1.0. We could call the 50 percent ellipse the 1.7741 σ ellipse.

2.8 RELATIONSHIP BETWEEN R(P), THE RADIUS OF A GIVEN PROBABILITY, AND σ_x

The relationship of the radius of a given probability R_P and σ_x is given in Table 4. In general

$$R_P = K_P \sigma_x \tag{2-12}$$

K_P is given in Table 4 for a given value of the probability P for the circle and for a given ellipticity $c = \sigma_y/\sigma_x$.

2.9 RELATIONSHIP BETWEEN R(P) AND CEP

For the Circular Error Probable

$$\begin{aligned} CEP &= R50 \\ &= K_{50}\sigma_x \end{aligned} \tag{2-13}$$

then

$$R_p/CEP = K_p/K_{50} \\ = Z \qquad (2\text{-}14)$$

where

$$Z = K_p/K_{50} \qquad (2\text{-}15)$$

Then the radius of a certain probability circle R(P), is given by

$$R\,(P) = Z\,(P,C) \bullet CEP \qquad (2\text{-}16)$$

Z(P,c) is a function of the probability and also of the ellipticity c, where $c \equiv \sigma_y/\sigma_x$.

Figure 2-2 from Burt et al.[3] illustrates the ratio of the radii of probability circles to CEP versus c, the ellipticity parameter. We have replaced Burt's R with Z to make matters clearer. Note that for a given probability (say 95 percent) there is a dependency of the ratio to the ellipticity. In Table 10 are computed the values of Z when the radius is R95 for a 95 percent probability circle. It is obtained by dividing the P = .95 row in Table 4 by the first row for P = .50.

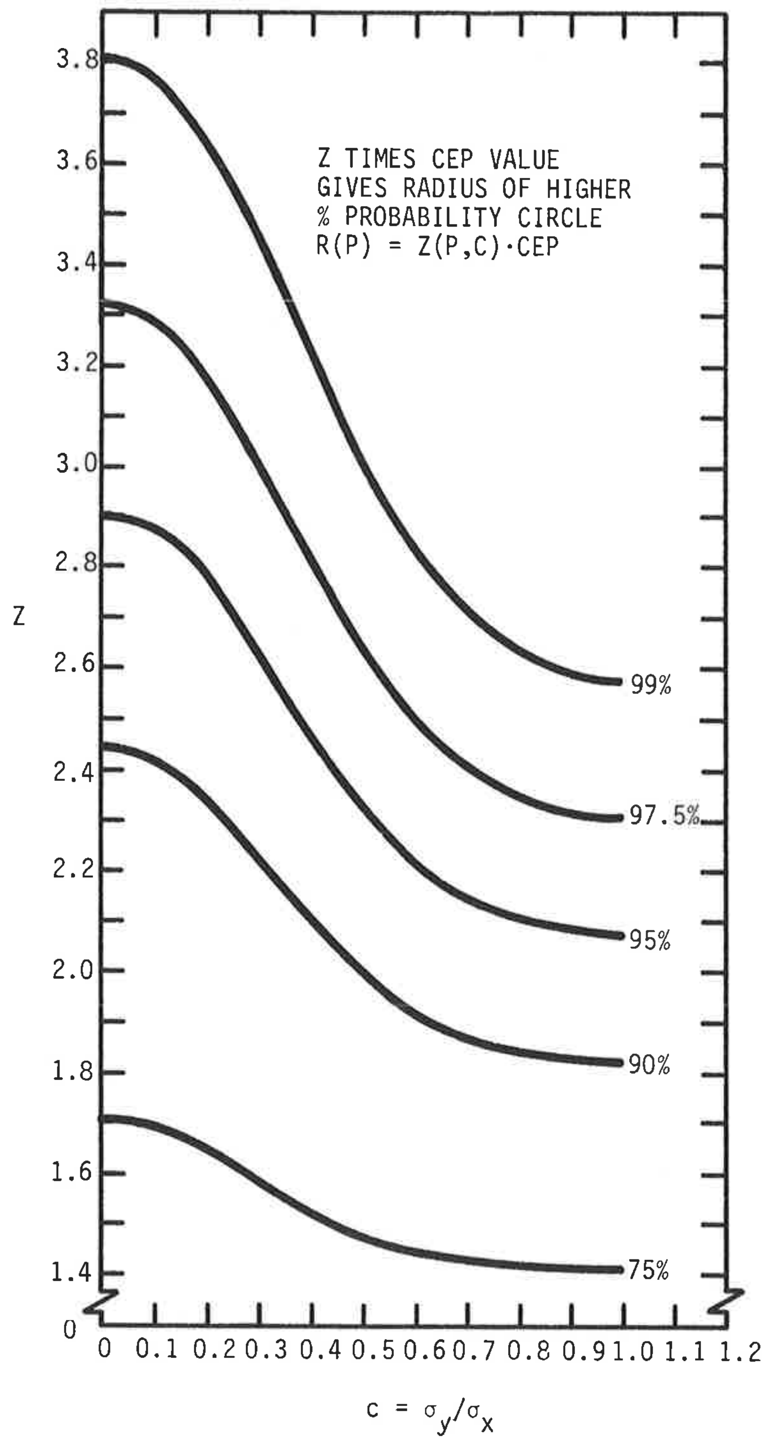


FIGURE 2-2. RATIO OF RADII OF PROBABILITY CIRCLES TO CEP VERSUS ELLIPTICITY σ_y/σ_x (FROM BURT, ET AL.[3])

TABLE 10. R(.95)/CEP VERSUS σ_y/σ_x

For a given σ_y/σ_x: R(.95) = Z CEP

$c = \sigma_y/\sigma_x$	Z(P = .95) (Computed from Harter Table 4)
.0	2.90584
.1	2.87765
.2	2.79154
.3	2.64585
.4	2.48207
.5	2.33894
.6	2.22920
.7	2.15414
.8	2.10864
.9	2.08556
1.0	2.07893

2.10 COMPARISON OF R95 CIRCLE AND THE 95 PERCENT ELLIPSE (OR 2.45 σ ELLIPSE)

The error probability containment of an ellipse with the k-index 2.44775 is just 95 percent (cf. Table 2). We could call this the 2.45 σ ellipse. Its semi-major principal axis is just 2.44775 times σ_x (the semi-major principal axis of the 1 σ ellipse).

It would be interesting to compare the R95 circle with the 95 percent ellipse for various values of $c = \sigma_y/\sigma_x$. The ratio of R95 to 2.45 σ_x is computed as follows.

$$R95 = K_{95}\,\sigma_x$$

(K_{95} from the row where P = .95 in Table 4 from Harter).

$$\frac{R95}{(2.45\,\sigma_x)} = \frac{K_{95}}{2.45}$$

The area ratio is calculated as follows. The semi-minor principal axis of the 95 percent ellipse is

$$2.45\,\sigma_y = 2.45\,c\,\sigma_x$$

The area of the ellipse is

$$\pi\,(2.45\,c\,\sigma_x)(2.45\,\sigma_x) = \pi < (2.45)^2\,\sigma_x^2$$

The area ratio is

$$\frac{\pi\,(R95)^2}{\pi\,c\,(2.45)^2\,\sigma_x^2} = \frac{K_{95}}{c\,(2.45)^2}$$

Table 11 gives the length ratio and area ratio of the R95 circle to the 95 percent ellipse vs. the ellipticity parameter $c = \sigma_y/\sigma_x$.

TABLE 11. LENGTH RATIO AND AREA RATIO OF R95 CIRCLE TO 95 PERCENT ELLIPSE VERSUS σ_y/σ_x

$c = \sigma_y/\sigma_x$	Length Ratio R95/2.45 σ_x	Area Ratio of R95 Circle to 95 percent Ellipse (2.45 σEllipse)
0	.801	∞
.1	.802	6.43
.5	.832	1.384
1.0	1.0	1.0

Figure 2-3 A, B and C compare the R95 Circle with the 95 percent ellipse for $\sigma_y/\sigma_x = 1.0, 0.5$ and 0.1.

The figures show that when the parameter $c = \sigma_y/\sigma_x$ becomes smaller the R95 Circle must have a large area in comparison to the area encompassed by the 95 percent ellipse.

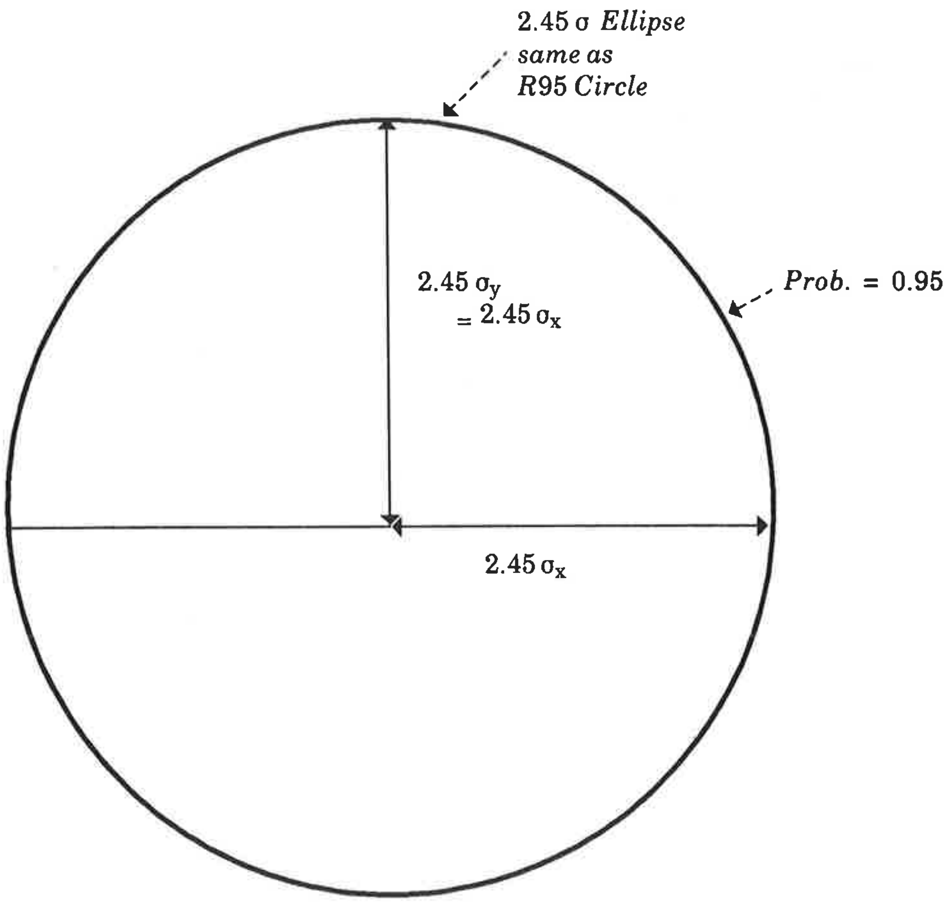


FIGURE 2-3A. R95 CIRCLE DRAWN WITH A 95 PERCENT ELLIPSE (WITH $\sigma_y / \sigma_x = 1.0$)

Note: A 2.45 σ ellipse is a 95 percent ellipse.
Area ratio of the R95 circle and the
2.45 σ ellipse is one.

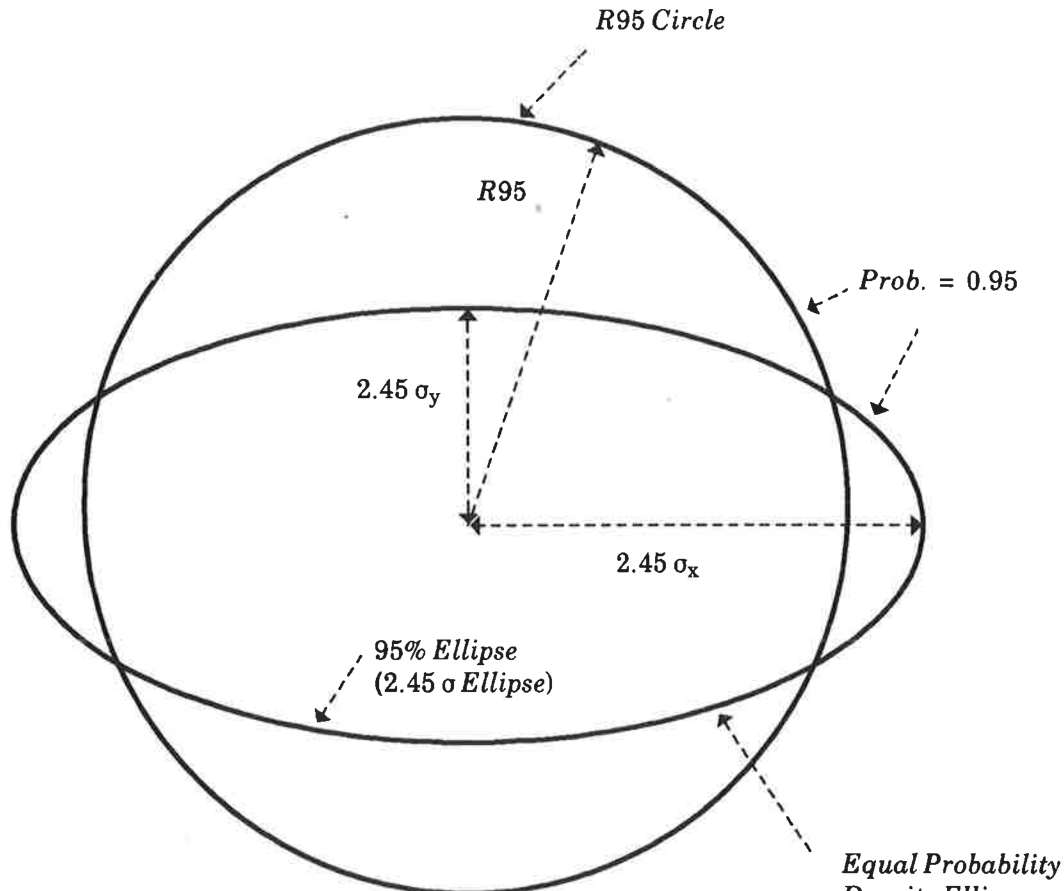


FIGURE 2-3B. R95 CIRCLE DRAWN WITH A 95 PERCENT ELLIPSE (WITH $\sigma_y / \sigma_x = 0.5$)

Note: $R95/(2.45\ \sigma_x) = 0.832$

Area ratio of R95 circle to 95 percent ellipse is 1.384

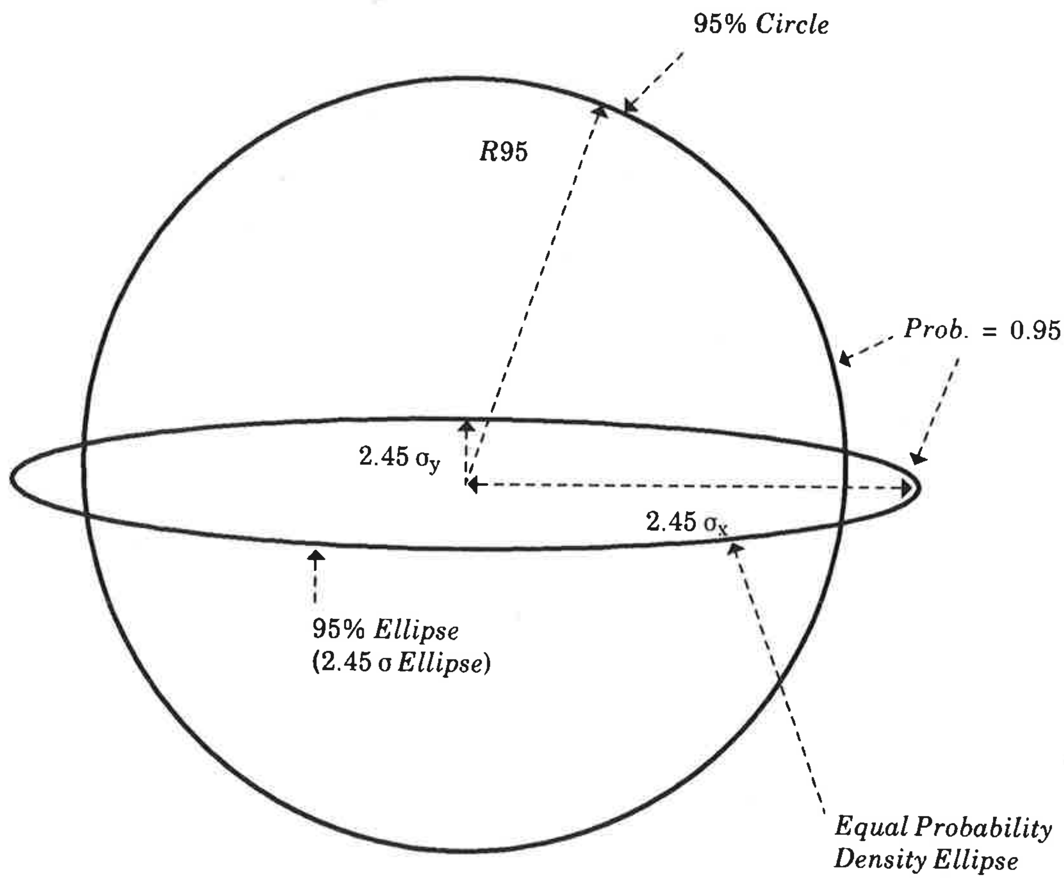


FIGURE 2-3C. R95 CIRCLE DRAWN WITH A 95 PERCENT ELLIPSE (WITH $\sigma_y / \sigma_x = 0.1$)

Note: $R95/(2.45\ \sigma_x) = 0.802$

Area ratio of R95 circle to 95 percent ellipse is 6.43

2.11 CALCULATION OF THE STANDARD DEVIATIONS ALONG THE PRINCIPAL AXES OF THE 1 σ ELLIPSE AND ITS ORIENTATION

The details of the derivation are given in Appendix C. Here we provide a summary of the results of the calculations. The standard deviations of a two-dimensional normal probability distribution along the principal axes of the equidensity ellipse are σ_x and σ_y (by definition $\sigma_x \geq \sigma_y$). We wish to obtain σ_x and σ_y in terms of the variance and covariance terms of the covariance matrix for the Eastern and Northern axes.

$$\sigma_x = \sqrt{\left[\lambda_1\right]} \tag{2-17a}$$

$$\sigma_y = \sqrt{\left[\lambda_2\right]} \tag{2-17b}$$

where λ1 and λ2 are the eigenvalues of M, the symmetric covariance matrix of the position error vector.

$$M = \begin{bmatrix} \sigma_X^2 & \sigma_{XY}^2 \\ \sigma_{XY}^2 & \sigma_Y^2 \end{bmatrix} \tag{2-17c}$$

The positive X axis is in the eastern direction and the positive Y axis is in the northern direction.

The eigenvalues are

$$\lambda_1 = 0.5\left[\sigma_X^2 + \sigma_Y^2 + \sqrt{\left[(\sigma_X^2 - \sigma_Y^2)^2 + 4\,\sigma_{XY}^4\right]}\right] \tag{2-18a}$$

$$\lambda_2 = 0.5\left[\sigma_X^2 + \sigma_Y^2 - \sqrt{\left[(\sigma_X^2 - \sigma_Y^2)^2 + 4\,\sigma_{XY}^4\right]}\right] \tag{2-18b}$$

$$(\sigma_x^2 = \lambda_1,\ \sigma_y^2 = \lambda_2)$$

The above equations that we obtained are the same as those given by D. J. Torrieri.(9) Our notation is different from that of Torrieri. Our σ_{XY}^2 are equivalent to his σ_{12}, so that our σ_{XY}^4 are the same as his σ_{12}^2 occurring in his equations 54 and 55.

Note that by equations 2-18a and b

$$\lambda_1 + \lambda_2 = \sigma_X^2 + \sigma_Y^2$$

$$= \sigma_x^2 + \sigma_y^2 \qquad \text{(by (2-17a and b))}$$

Orientation of the principal axes of the 1 σ ellipse are as follows:

Let θ be the angle between the nearest principal axis of the 1 σ ellipse and the vector $\hat{X}$ which points in the eastern direction. The rotation from $\hat{X}$ is counterclockwise when θ is positive and is clockwise when θ is negative. Then (see derivation in Appendix C)

$$\Theta = 0.5\, TAN^{-1}\left[\frac{2\sigma_{XY}^2}{\sigma_X^2 - \sigma_Y^2}\right] \qquad \text{(2-19)}$$

where

$$-\pi/4 < \Theta < \pi/4 \; (radians)$$

Our result is in agreement with that of Torrieri.(9)

The angle θ is by definition between the $\hat{X}$ (eastern) direction and the <u>nearest principal axis</u>, but this nearest principal axis may or may not be a major principal axis. The major principal axis of the equal probability density ellipse is by definition the direction of $\hat{x}$, and σ_x is along this direction. We therefore define an angle φ which is between $\hat{X}$ and the direction of $\hat{x}$, the nearest major principal axis (when φ is positive the rotation is in the counterclockwise direction). As a consequence of our definition of φ, we have

$$-\pi/4 < \phi < 3\pi/4 \; (radians)$$

(2-20)

while (2-21)

$$- \pi/4 < \theta < \pi/4 \; (radians)$$

The distribution of the orientation of the error ellipse is obtained by computer simulation. The distribution for a given value of ϕ is the same as that for $\phi + \pi$, so that one can plot the orientation distribution for ϕ from 0 to π.

Depending on the values of the elements of the covariance matrix there are several possible cases. These are illustrated as follows (cf. Figures 2-4 A, B and C).

Case 1. (Note $\sigma_{XY}{}^2$ may be positive or negative.)

When

$$\sigma_{XY}^2 \; (and \; \sigma_{YX}^2) \neq 0$$

and

$$\sigma_X > \sigma_Y$$

Then the major principal axis of the ellipse points in a direction between $\theta = \pi/4$ and $\theta = -\pi/4$. We may set $\phi = \theta$. (cf. Figure 2-4A, Top)

Case 2. (cf. Figure 2-4A, Bottom)

When

$$\sigma_{XY}^2 \; (and \; \sigma_{YX}^2) \neq 0$$

and

$$\sigma_Y > \sigma_X$$

Then the major principal axis of the ellipse points in a direction between $\pi/4$ and $3\,\pi/4$, and the minor principal axis points in a direction between $-\pi/4$ and $\pi/4$. We therefore set

$$\phi = \theta + \pi/2$$

Case 3a. (cf. Figure 2-4B, Top)

When

$$\sigma_{XY}^2 \; (and \; \sigma_{YX}^2) = 0$$

and

$$\sigma_X > \sigma_Y$$

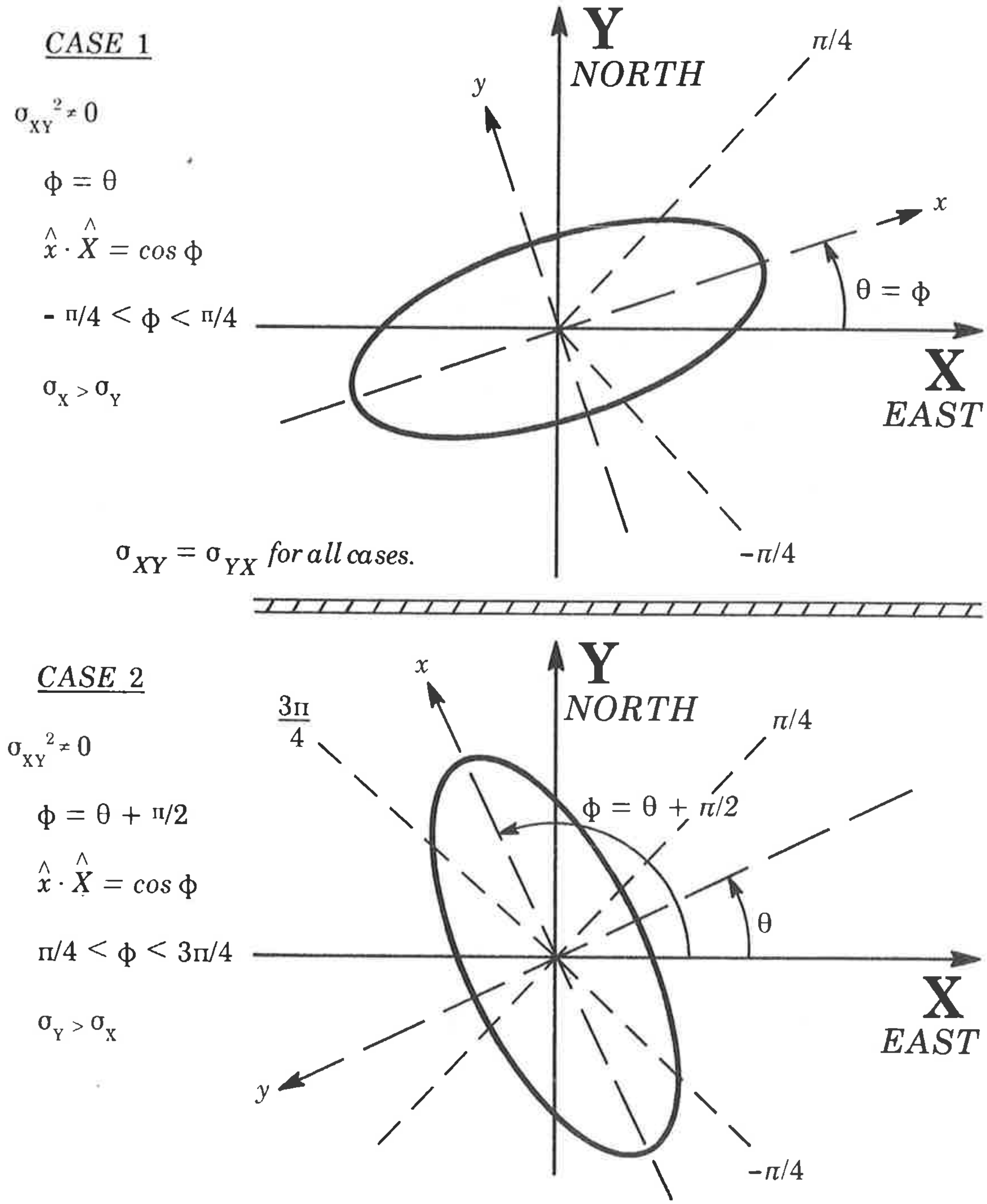


FIGURE 2-4A. ILLUSTRATIONS OF ORIENTATIONS OF ERROR ELLIPSE WHEN COVARIANCE $\sigma_{XY}^2 \neq 0$

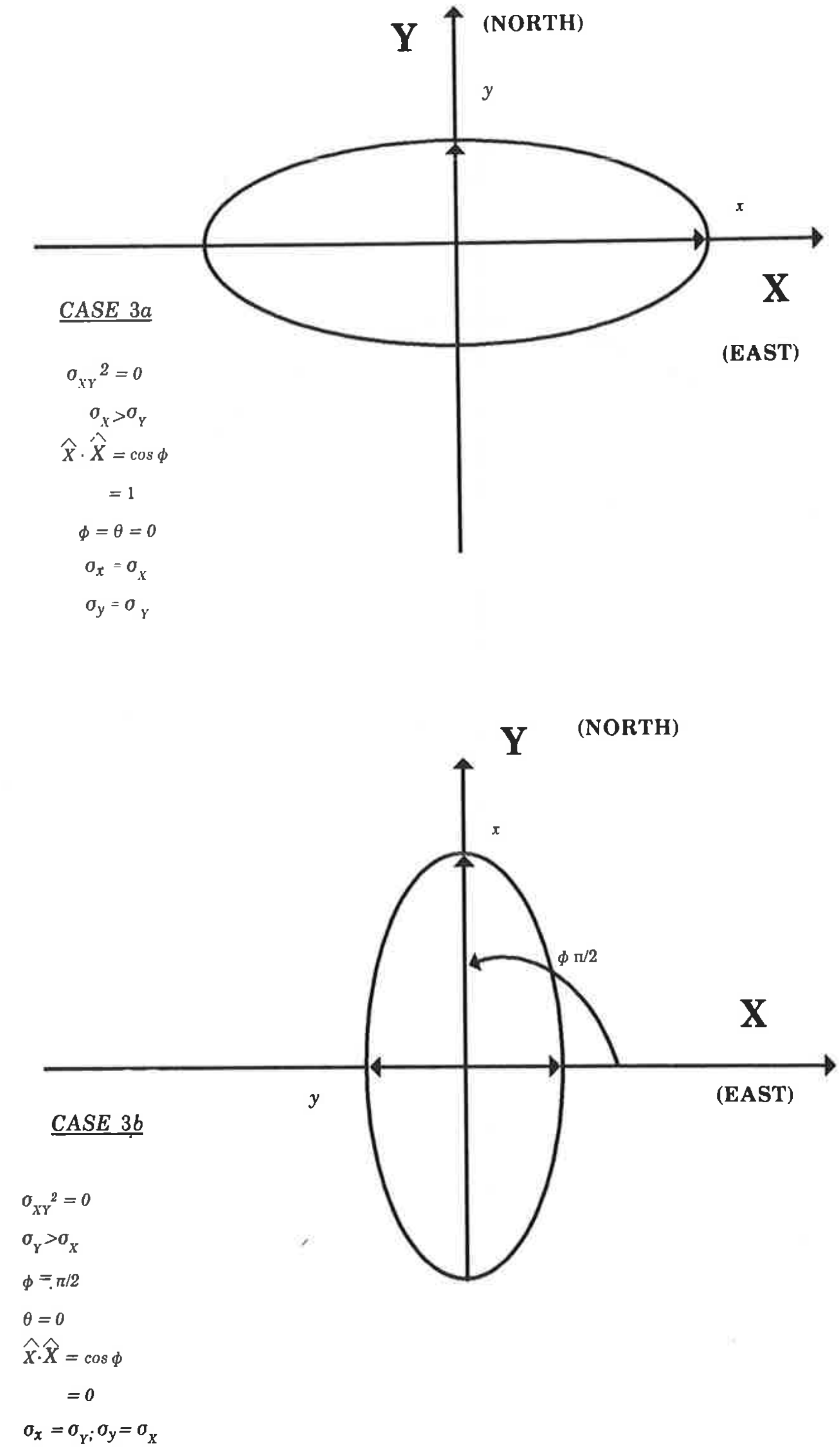


FIGURE 2-4B. ILLUSTRATIONS OF ORIENTATIONS OF ERROR ELLIPSE WHEN COVARIANCE $\sigma_{XY}{}^2 = 0$

3. DRMS, 2 DRMS ERROR MEASURES AND HDOP

3.1 THE TERMS DRMS AND 2 DRMS

The terms drms and 2 drms are used in the Federal Radionavigation Plan [1] (FRP). The FRP specifies that "when two dimensional accuracies are used, the two drms (distance root mean squared) uncertainty estimate will be used".

The term d_{rms} (or drms) is defined as the square root of the sum of the squares of the one sigma error components along the major and minor principal axes of an equal probability density ellipse (cf. Burt et al.[3] p. 54). The one sigma (1 σ) error is the same as a one standard deviation error (see Section 1.3 and Figure 1-1 for the formal definition). Burt et al. prefer to use the abbreviation d_{rms} with the subscript while the FRP used drms (or DRMS). Sometimes the term radial error and RMS (root mean square) error are used in place of drms when applied to two-dimensional errors. Figure 3-1 illustrates the definition of drms.

$$1\ drms = \sqrt{\left[\sigma_x^2 + \sigma_y^2\right]} \qquad (3\text{-}1)$$

A more general definition was given in Section 1.2.4. In Section 1 it was pointed out that when the covariance error matrix M is diagonalized and transformed into the diagonal matrix N, the trace (or the sum of the diagonal elements) of N equals the trace of M. That is to say

$$\sigma_x^2 + \sigma_y^2 = \sigma_X^2 + \sigma_Y^2 \qquad (3\text{-}2)$$

x is along the major principal axis and y is along the minor principal axis of the equi-probability density ellipse. X is along the East-West and Y is along the North-South axis.

Then

$$\begin{aligned} 1\ drms &= \sqrt{\left[\sigma_x^2 + \sigma_y^2\right]} \\ &= \sqrt{\left[\sigma_X^2 + \sigma_Y^2\right]} \end{aligned} \qquad (3\text{-}3)$$

Thus 1 drms (and therefore 2 drms) is invariant to rotation about the local vertical.

However, for the GPS the orthogonal system of σ_X and σ_Y are respectively in the East-West and North-South directions. So that drms can be calculated directly from these quantities.

3.2 AREA RATIO OF 2 DRMS CIRCLE TO 2 σ ELLIPSE

The area ratio of the circle of radius 2 drms to the 2 σ ellipse is given as follows:

$$\text{Area Ratio} = \pi (2\, drms)^2 / [\pi (2\sigma_x)(2\sigma_y)]$$
$$= \pi 4 \sigma_x^2 (1 + c^2) / \left[\pi 4 \sigma_x^2 c\right]$$
$$= (1 + c^2)/c, \tag{3-8}$$

where $c = \sigma_y / \sigma_x$

TABLE 12. LENGTH RATIO OF 2 DRMS/$2\sigma_x$ AND AREA RATIO OF 2 DRMS CIRCLE TO 2 σ ELLIPSE VERSUS σ_y / σ_x

$c = \sigma_y / \sigma_x$	(2 drms) / ($2\sigma_x$)	Area Ratio of 2 drms Circle to 2 σ Ellipse
0	1	∞
.10	1.005	10.10
.25	1.031	4.25
.50	1.118	2.50
.75	1.250	2.08
1.00	1.414	2.0

Now from equation 3-6 we have

$$1\ drms = \sigma_x \sqrt{\left[1 + c^2\right]}$$

$$and\ 2\ drms = 2\sigma_x \sqrt{\left|1 + c^2\right|}$$

For 1 drms $R = 1\ drms = \sigma_x \sqrt{\left[1 + c_2\right]}$

then

$$K = \sqrt{\left[1 + c^2\right]} \qquad (3\text{-}9)$$

We can then use Table 3 (from Harter[(6)]) to determine the probability of a circle of radius R = 1 drms containing a position measurement.

For example when c=o and K=1, then from Harter's Table 3, P = 0.683. And when C = 1, K = 1.414, P = 0.630.

3.3 PROBABILITIES OF CIRCLES WITH 1 DRMS RADIUS AND 2 DRMS RADIUS VERSUS ELLIPTICITY

Table 13 (from Burt et al.[(3)]) shows the probability of containing a vehicle position measurement within circles of radii 1 drms and 2 drms (centered at the actual position of the vehicle) versus the ellipticity parameter $c = \sigma_y/\sigma_x$. We have modified his headings to make the material clearer.

TABLE 13. PROBABILITY WITHIN 1 DRMS AND 2 DRMS CIRCLE VERSUS σ_y/σ_x (FROM BURT, ET AL.[(3)])

ELLIPTICITY RATIO PARMETER C	LENGTH	Probability of Containing A Position Measure Within A Circle Having A Radius	
σ_y/σ_x	1 drms/σ_x	of 1 drms	of 2 drms
0.0	1.000	0.683	0.954
0.1	1.005	0.682	0.955
0.2	1.020	0.682	0.957
0.3	1.042	0.676	0.961
0.4	1.077	0.671	0.966
0.5	1.118	0.662	0.969
0.6	1.166	0.650	0.973
0.7	1.220	0.641	0.977
0.8	1.280	0.635	0.980
0.9	1.345	0.632	0.981
1.0	1.414	0.632	0.982

Figures 3-2 and 3-3 from Burt et al. illustrate the variations graphically. We have corrected certain errors in the captions provided by Burt et al. Note that the probability of the circle of 1 drms decreases with the increase of σ_y/σ_x. However, the probability of the circle of 2 drms increases with the increase of the ratio σ_y/σ_x.

In the Federal Radionavigation Plan it is stated that "two drms is the radius of a circle that contains at least 95 percent of all possible fixes that can be obtained with a system at any one place." Inspection of Table 13 column 4 shows that this statement is valid. The table shows that the circle of radius 2 drms contains at least 95.4 percent of the vehicle position measurements. However, the probability of containing a position measure varies from 95.4 percent to 98.2 percent depending on the ellipticity parameter $c = \sigma_y/\sigma_x$. There is no single probability value associated with the circle of radius 2 drms.

3.4 RELATIONSHIP BETWEEN DRMS AND HDOP

The following notational equivalences are widely used:

$$\sigma_x \equiv \sigma_{xx}$$

$$\sigma_y \equiv \sigma_{yy}$$

$$\sigma_X \equiv \sigma_{XX}$$

$$\sigma_Y \equiv \sigma_{YY}$$

We have adopted the definitions of Brooks et al.(5) (cf. Brooks et al. p. 3.0-3). However, we use $\sigma_{XY}{}^2$ instead of σ_{XY} (Brooks et al.) to denote the off-diagonal covariance matrix element, since this is a length squared quantity. The sigma elements given in Milliken and Zoller(10) are normalized quantities and are dimensionless. One must multiply the sigma's of Milliken and Zoller by σ_R, the average pseudo-range error, to obtain the sigma's that we use, which are in length units.

The position error covariance matrix can be expressed as

$$COV\ \delta\ \overline{X}_u = (G_u{}^T G_u)^{-1} \bullet \sigma_R{}^2), \tag{3-10}$$

(cf. Milliken and Zoller(10)

Where σ_R is the RSS UERE (user-equivalent range error)

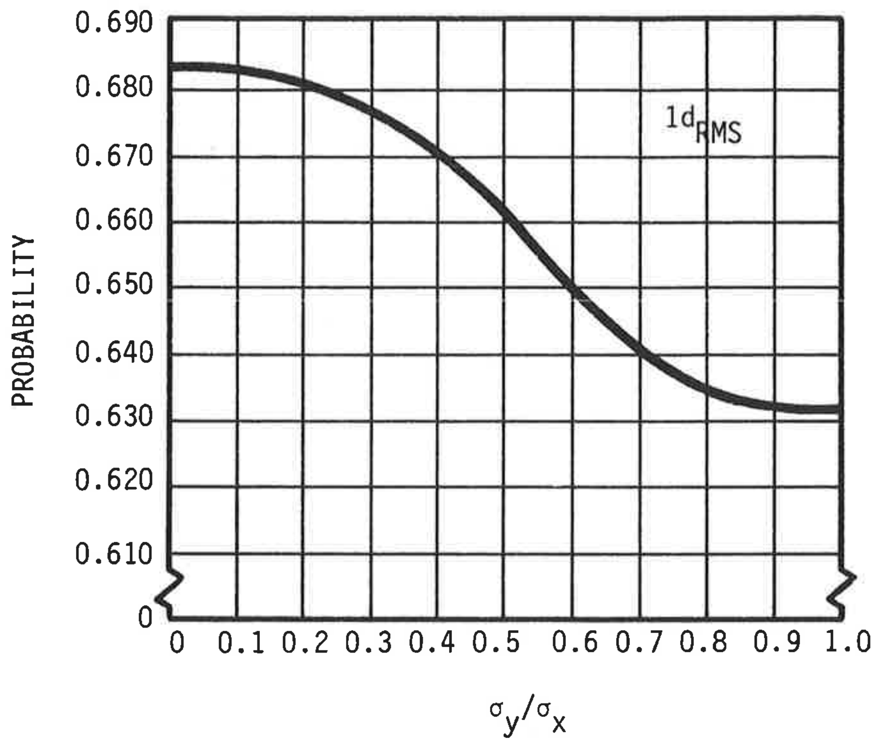


FIGURE 3-2. VARIATION IN PROBABILITIES OF CONTAINING THE POSITION MEASURE WITHIN A CIRCLE WITH RADIUS 1 DRMS

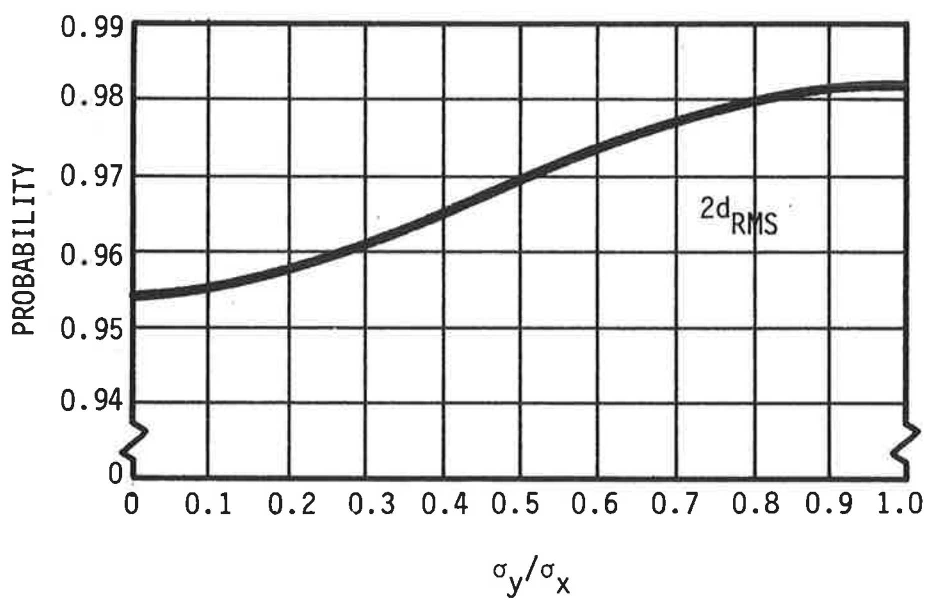


FIGURE 3-3. VARIATION IN PROBABILITIES OF CONTAINING THE POSITION MEASURE WITHIN A CIRCLE WITH RADIUS 2 DRMS

NOTE: Corrections of the original figure headings were made. (From Burt et al.[3])

The RSS Horizontal Position Solution Error is given by

$$\sigma_H = \sqrt{\left[\sigma_X^2 + \sigma_Y^2\right]} \qquad (3\text{-}11)$$

(σ_H is in length units)

The Horizontal Dilution Of Precision (HDOP) is defined as

$$HDOP = \sigma_H / \sigma_R$$
$$= (1/\sigma_R)\sqrt{\left[\sigma_X^2 + \sigma_Y^2\right]} \qquad (3\text{-}12)$$

(HDOP is dimensionless)

where σ_R is the RSS UERE (user-equivalent range error common to all the satellite measurements, and the UEREs are assumed to be statistically independent).

Then $\sigma_H = HDOP \bullet \sigma_R$

$$= \sqrt{\left[\sigma_X^2 + \sigma_Y^2\right]}$$
$$= \sqrt{\left[\sigma_x^2 + \sigma_y^2\right]} \qquad (3\text{-}13)$$

Since the trace of the matrix is independent in an orthogonal coordinate transformation, HDOP is then also independent of an orthogonal coordinate transformation.

Clearly

$$\sigma_H \equiv 1\ drms = HDOP \bullet \sigma_R \qquad (3\text{-}14)$$

Since 1 drms = HDOP σ_R, (3-15a)

2 drms = 2 HDOP σ_R. (3-15b)

$$XDOP = \sigma_X/\sigma_R \quad (3\text{-}16a)$$

and $$YDOP = \sigma_Y/\sigma_R \quad (3\text{-}16b)$$

(XDOP and YDOP are dimensionless; σ_X, σ_Y and σ_R are in length units)

The probability of finding a position measure within the circle of radius 2 drms depends on the ellipticity, and is good for LORAN-C or OMEGA where the ratio σ_y/σ_x is a constant value. This is not necessarily true for GPS, TRANSIT where the ellipticity may in some situations change rapidly. In some situations HDOP, which is a function of the geometric configuration of the satellites employed in the measurement, may change rapidly. Experimental determination of 2 drms makes sense only while the geometry hasn't changed much.

4. RELATIONSHIP BETWEEN CIRCULAR ERROR MEASURES AND DRMS MEASURES

4.1 THE RATIO 1 DRMS/CEP VERSUS σ_y/σ_x

The Federal Radionavigation Plan (Ref. 1, p. III-4) stated that "DOD specifies horizontal accuracy in terms of Circular Error Probable (CEP--the radius of a circle containing 50 percent of all possible fixes). For the FRP, it is agreed that the conversion of CEP to 2 drms would be accomplished by using 2.5 as the multiplier."

Now the ratio of 1 drms or 2 drms to CEP (or R50) can be calculated as follows:

$$1\ drms = \sigma_x \sqrt{\left[1 + c^2\right]}$$

$$\left[cf.\ (3-6)\ and\ 3-9)\right]$$

$$CEP = R50 \qquad (4\text{-}1)$$

$$= K_{50}\,\sigma_x$$

Then (4-2)

$$1\ drms/CEP = (1/K_{50})\sqrt{\left[1 + c^2\right]}$$

where K_{50} is the first row of entries in Table 4 (from Harter), being a function of the ellipticity parameter $c = \sigma_y/\sigma_x$. The ratio 1 drms / CEP versus c, the ellipticity parameter, is plotted in Figure 4-1 (from Burt et al). We see that this ratio is actually a function of σ_y/σ_x. The FRP's choice of 2.5 for 2 drms/CEP suggests an average ellipticity parameter $c = 0.6$.

It is worthwhile to determine the average value of c (or σ_y/σ_x) by conducting a computation of the values of c over representative space-time points of the CONUS . The results of the computation are shown in Section 6. We obtained a mean value of 0.698 for c. In Table 14 are tabulated the values of 1 drms/CEP versus σ_y/σ_x.

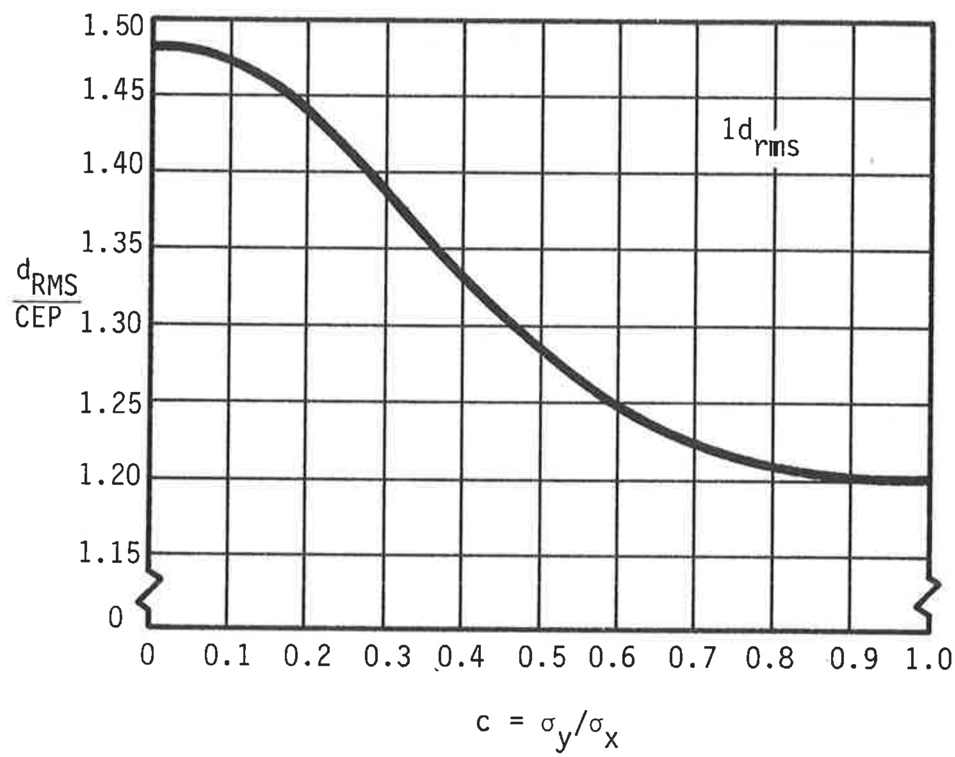


FIGURE 4-1. DRMS/CEP VERSUS σ_y/σ_x (FROM BURT, et al.[3])

TABLE 14. 1 DRMS/CEP VERSUS σ_y/σ_x

$c = \sigma_y/\sigma_x$	1 drms/CEP
0.0	1.481
.1	1.474
.2	1.445
.3	1.392
.4	1.333
.5	1.285
.6	1.249
.7	1.226
.8	1.210
.9	1.203
1.0	1.201

4.2 COMPARISON OF 2 DRMS AND R95 AS FUNCTIONS OF ELLIPTICITY PARAMETER σ_y/σ_x

Now

$$2\,drms = 2\sqrt{\left[\sigma_x^{\,2} + \sigma_y^{\,2}\right]} \qquad (4\text{-}3)$$

$$= 2\,\sigma_x\sqrt{\left[1 + c^2\right]} \qquad (4\text{-}4)$$

and

$$R95 = K_{95}\,\sigma_x$$

$$= (K_{95}/2)\,2\,\sigma_x \qquad (4\text{-}5)$$

$$= K_{95}/\sigma_x$$

R95 is the radius of a circle containing 95 percent of the position measures. The values of K_{95} can be obtained from the probability p = 95 percent in Table 4 (from Harter). In Table 15 length ratios are tabulated for several values of $c = \sigma_y/\sigma_x$.

TABLE 15. 2 DRMS/2σ_x, R95/2σ_x, AND 2 DRMS/R95 VERSUS σ_y/σ_x

$x = \sigma_y/\sigma_x$	2 drms/2σ_x	R95/2σ_x	2 drms/R95
0.0	1.0	.980	1.020
0.1	1.005	.981	1.024
0.3	1.044	.992	1.052
0.5	1.118	1.018	1.098
0.8	1.281	1.115	1.149
1.0	1.414	1.224	1.155

Figures 4-2A through 4-2D illustrate 4 cases of Table 15.

The 4th column in Table 15 shows the ratio of 2 drms to R95 as a function of c. The largest discrepancy occurs when c = 1 (i.e. a circle) where the ratio reaches a maximum of 1.155.

Figures 4-2A through 4-2D illustrate the relationships for several values of σ_y/σ_x. Notice that the 2 σ ellipse always contains a probability of 86.5 percent, the R95 circle contains a probability of 95 percent, but the 2 drms circle contains a probability that is a function of $c = \sigma_y/\sigma_x$ (cf. Table 13).

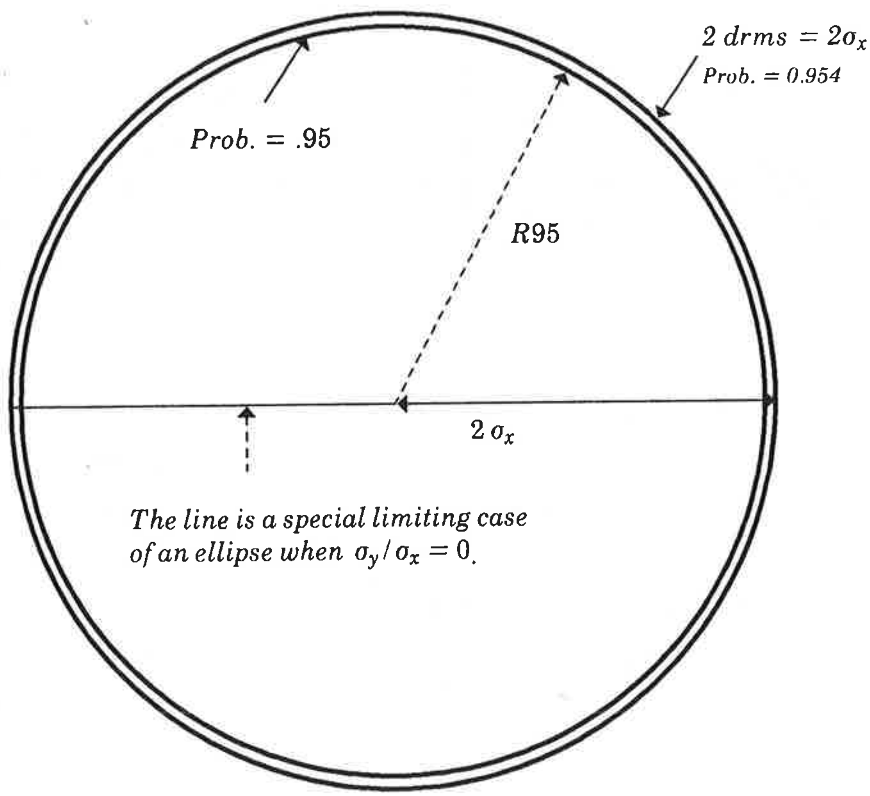


FIGURE 4-2A. 2 DRMS AND R95 CIRCLES WHEN $\sigma_y/\sigma_x = 0$

2 σ Ellipse degenerates into a line.

2 drms	= $2\,\sigma_x$ (This is true only for this special case)
R95	= .980 ($2\,\sigma_x$)
2 drms/R95	= 1.020

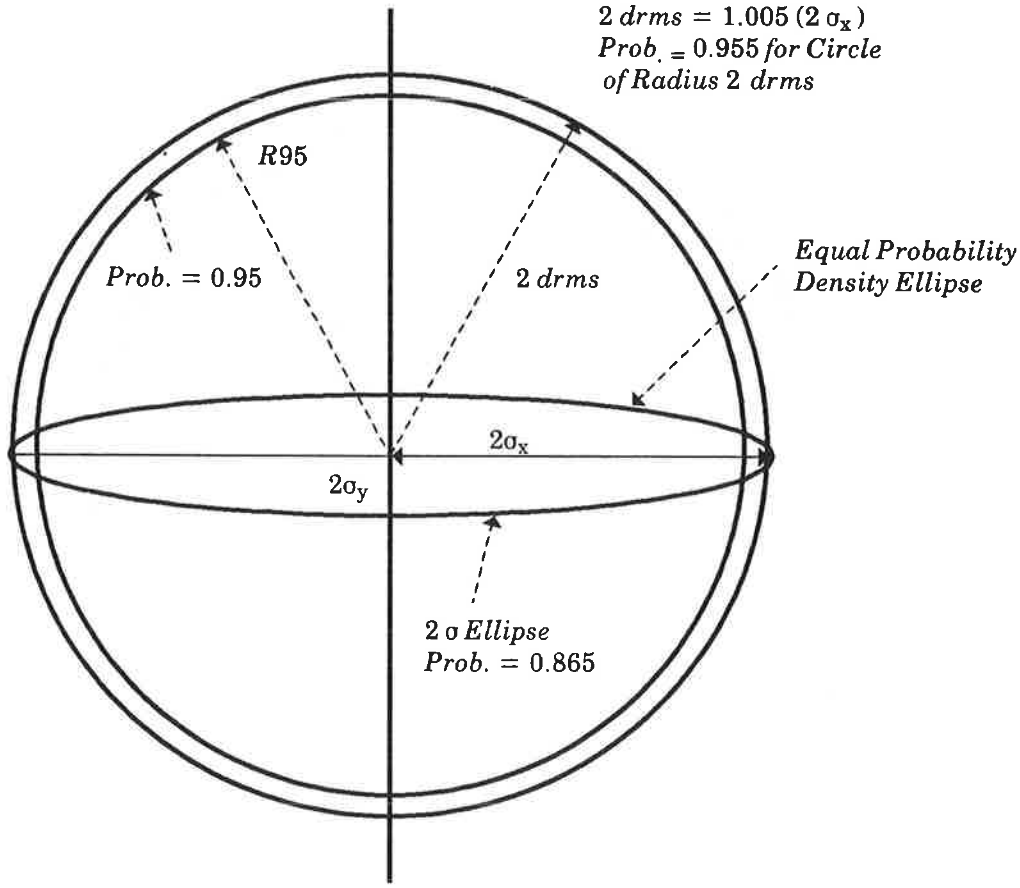


FIGURE 4-2B. 2 DRMS AND R95 CIRCLES WHEN $\sigma_y / \sigma_x = 0.1$

2 drms	$= 1.005\ (2\ \sigma_x)$ (cf. Table 12)
R95	$= 0.981\ (2\ \sigma_x)$ (cf. Table 4)
2drms/R95	$= 1.024$

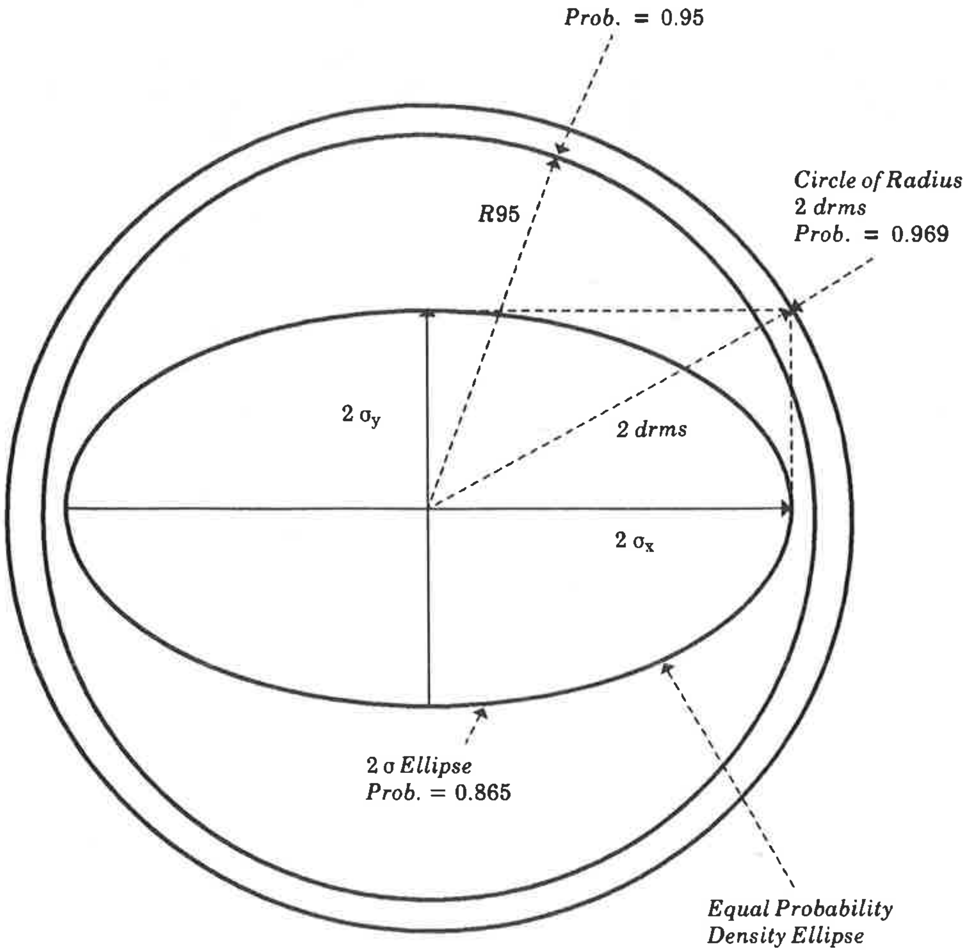


FIGURE 4-2C. 2 DRMS AND R95 CIRCLES WHEN $\sigma_y/\sigma_x = 0.5$

2 drms	$= 1.118\,(2\,\sigma_x)$
R95	$= 1.018\,(2\,\sigma_x)$
2 drms/R95	$= 1.098$

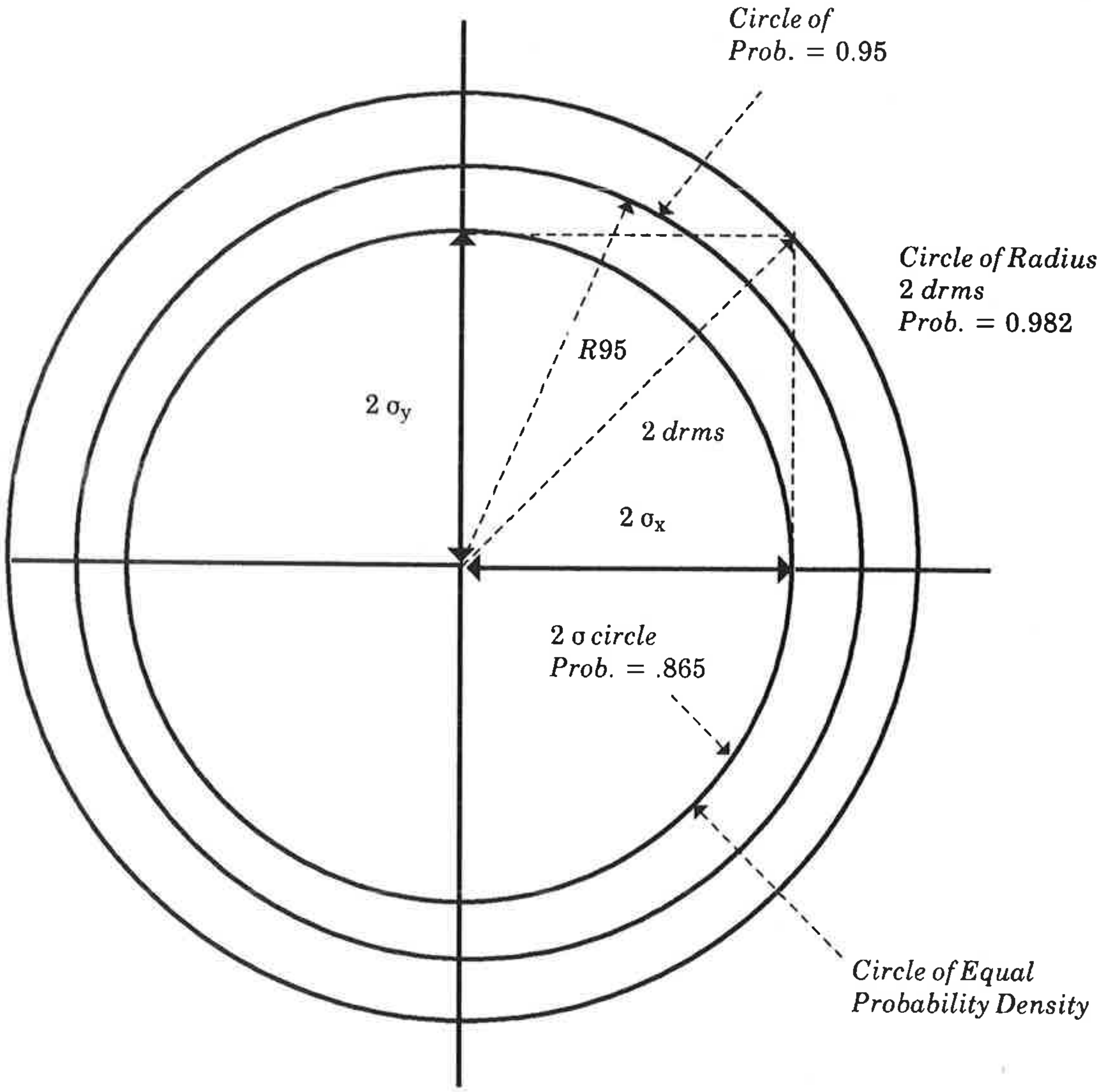


FIGURE 4-2D. 2 DRMS AND R95 CIRCLES WHEN $\sigma_y / \sigma_x = 1.0$

2 drms	$= 1.414\,(2\,\sigma_x)$
R95	$= 1.224\,(2\,\sigma_x)$
2 drms/R95	$= 1.155$

5. PRACTICAL CONSIDERATIONS

In the Global Positioning System (GPS) the satellite geometry is changing in time. Some of the most desirable satellites are setting at certain times while others are coming into view for the user. Now (from equation 3-15a)

$$1\ drms = HDOP \bullet \sigma_R$$

HDOP, which is a function of the satellite geometric configuration, is changing in time, so that the 2 drms measure can not be directly computed over a region and over a extended period of time (cf. Figures 5-3 through 5-6 on PDOP and HDOP temporal variation in Kalafus et al.[(11)]).

One could, however, define a "System 2 drms Accuracy" by summing errors over locations, time, and a fictitious set of instruments (ensemble average).

Consider a random space-time point on the CONUS. Let there be a possible selection at random of N instruments for each point. Let the index l indicate the location; m, the time; and n, the instrument. The position measure for the l,m,nth case is δX_{lmn}, δY_{lmn}.

The mean value of the variance in the X direction is

$$S_x^{\ 2} = (1/LMN) \sum^{L} \sum^{M} \sum^{N} \delta X_{lmn}^{\ \ 2}$$

$$= (1/LM) \sum^{L} \sum^{M} \delta \sigma_X^{\ 2} \qquad (5\text{-}1)$$

(the order does not matter).

We want to find the radius that encloses 95 percent of the sample.

If no blow-up occurs for HDOP,

$$\sigma_X^{\ 2} = (1/N) \sum_{n=1}^{N} \delta X_{lmn}^{\ \ 2} \qquad (5\text{-}2)$$

$$= (\sigma_R^{\ 2}/N) \sum^{N} (XDOP)^2 \qquad (5\text{-}3)$$

The above is just one of the elements of the covariance error matrix.

Then the "System 2 drms Accuracy" is defined as

$$\text{"2 drms error"} \triangleq 2\sqrt{\left[S_X^{\,2} + S_Y^{\,2}\right]}$$

$$= 2\sqrt{\left[(1/LM)\sum^{L}\sum^{M}(\sigma_X^{\,2} + \sigma_Y^{\,2})\right]} \qquad (5\text{-}4)$$

$$= 2\,\sigma_R\sqrt{\left[(1/LM)\right]\sum^{L}\sum^{M}(XDOP)^2} \qquad (5\text{-}5)$$

But if at some of the space-time points the σ_x becomes infinite, then this will distort the mean value (by definition $\sigma_x \geq \sigma_y$). This System 2 drms Accuracy measure would not be stable if applied during GDOP outages. Note that the circular measures do not have this problem.

It is possible to propose an Operational System 2 drms Accuracy measure that ignores data for which HDOP > 6. This is justified operationally because the system is not usable without some additional navigation inputs during these periods anyway; furthermore the user can determine when these outages are occuring. This measure is stable.

The System Accuracy measure is related to the covariance diagonal elements and to the individual HDOP's. The means of the components are zero for thermal noise errors (instruments average zero) and Selective Availability (time variations have zero mean). Still there exists a problem for ionospheric errors, where pseudorange errors have non-zero mean, and are partially correlated. Experimental determination of System Accuracy measure involves summations of errors for HDOP < 6. The results are independent of the order in which the experimental points are evaluated.

Unless error ellipse shape and orientation are provided along with an accuracy measure, no error measure provides a "better" indication of error variation. Since R95 (the radius of 95 percent probability containment) is fixed to a particular probability level of 95 percent, it is more meaningful than System 2 DRMS Accuracy.

6. EVALUATION OF ACCURACY MEASURES FOR NAVSTAR/GPS OVER THE CONUS

This section provides results in a GPS 24 hour interval with user's locations over the CONUS: HDOP distribution, the ellipticity parameter σ_y/σ_x distribution, error ellipse orientation distribution, N-S (north-south) versus E-W (east-west) errors, System 2 DRMS accuracy, CEP and R95 distributions.

6.1 HDOP, XDOP AND YDOP DISTRIBUTION FOR ALL-IN-VIEW STRATEGY

In this computation all the satellites that are visible (above the specified mask angle) to the user are utilized in the solution (All-In-View Strategy).

Representative user locations and GPS time points are selected as follows:

Longitude: 70.0 deg. to 120.0 deg. West (10 deg. increments)

Latitude: 30.0 deg. to 50.0 deg North (10 deg. increments)

GPS Time: 0.0 sec. to 86,400.0 sec. 720 sec. (or 12 min.) increments

Mask Angle: 7.5 deg. above the users mathematical horizon. The visibility region above the mask angle.

The total number of cases is 2178. There are 21 satellites: 18 regular and 3 spares. We assume that there is no altimeter aiding.

$$HDOP = \sigma_H/\sigma_R$$

$$= (1/\sigma_R)\sqrt{\left[\sigma_X^2 + \sigma_Y^2\right]} \qquad (6\text{-}1)$$

$$XDOP = \sigma_X/\sigma_R \qquad (6\text{-}2)$$

$$YDOP = \sigma_Y/\sigma_R \qquad (6\text{-}3)$$

Positive X is in the Eastern and positive Y is in the Northern direction.

Figure 6-1 shows the percentage of time that XDOP and YDOP is less than a given value.

XDOP: Median = 0.739	RMS XDOP = 0.764
YDOP: Median = 0.948	RMS YDOP = 1.025

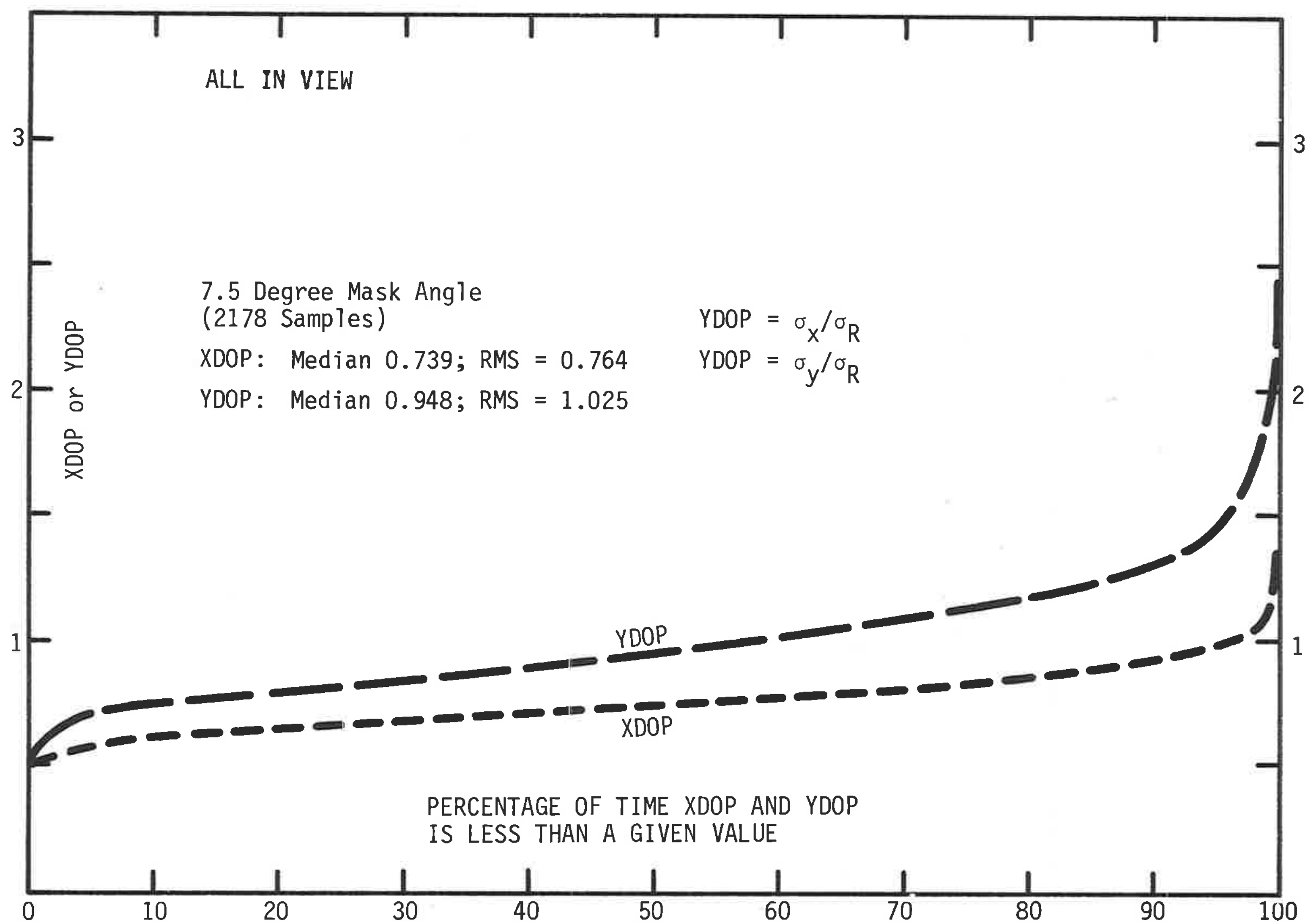


FIGURE 6-1. PERCENTAGE OF TIME XDOP AND YDOP IS LESS THAN A GIVEN VALUE VERSUS THE VALUE OF XDOP OR YDOP

Now σ_X is a measure of the accuracy in the E-W direction and σ_Y is a measure of the accuracy in the N-S direction. The results of the computation demonstrate that there is better overall accuracy in the E-W direction.

In the computer runs the results for each user's location and GPS time instant is placed in bins of width 0.1. Table 16 shows the results for HDOP.

Figure 6-2 shows the percentage of time that HDOP is less than a given value.

HDOP: Median = 1.206, RMS = 1.278

6.2 ELLIPTICITY PARAMETER σ_y / σ_x DISTRIBUTION

Figure 6-3 shows the distribution of the ellipticity parameter $c = \sigma_y / \sigma_x$ for 2178 samples. The mean value of σ_y / σ_x is 0.698. We noted in Section 4.1 that the Federal Radionavigation Plan (FRP) choice of 2.5 for 2 drms/CEP suggests an average ellipticity parameter $c = 0.6$ (cf. Figure 4-1). Our computed mean value of c, 0.7, suggests however a value of 1.226 for drms/CEP (cf. Table 13) or 2.452 for 2 drms/CEP.

6.3 ERROR ELLIPSE ORIENTATION DISTRIBUTION

Figure 6-4 shows the distribution of the orientation of the major principal axis of the equal probability density ellipse. The figure shows a noticeably greater number of samples in the North-South direction, the maximum being in the angular region 75 degrees to 85 degrees counterclockwise from the east. This suggests a greater inaccuracy in the N-S direction and is consistent with the result that the mean of YDOP is greater than the mean of XDOP (cf. Section 6.1 and Figure 6-1).

6.4 DISTRIBUTION OF R95 (RADIUS OF 95 PERCENT CONTAINMENT OF A POSITION MEASUREMENT) AND OF CEP (RADIUS OF 50 PERCENT)

Computations are made using the procedures outlined in Sections 4.1 and 4.2 (cf. Equations 4-1 and 4-5). The linear interpolation scheme using the K_{95} and K_{50} rows of Harter's Table 4 is outlined in Section 6.6 of this study (cf. Equations 6-5, 6-6 and 6-7).

TABLE 16. SAMPLED VALUES OF HDOP FOR 24 HOUR PERIOD OVER CONUS
(BIN WIDTH = 0.1, TOTAL SAMPLE SIZE = 2178)

BIN #	CENTER OF BIN WITH VALUE OF HDOP	FREQUENCY	PERCENT OF TOTAL SAMPLE	ACCUMULATIVE PERCENTAGE
8	0.75	0	0	0
9	0.85	46	2.11	2.11
10	0.95	215	9.87	11.98
11	1.05	445	20.43	32.42
12	1.15	364	16.71	49.13
13	1.25	333	15.29	64.42
14	1.35	263	12.08	76.49
15	1.45	228	10.47	86.96
16	1.55	111	5.10	92.06
17	1.65	53	2.43	94.49
18	1.75	45	2.07	96.56
19	1.85	24	1.10	97.66
20	1.95	19	0.87	95.53
21	2.05	13	0.60	99.13
22	2.15	8	0.37	99.49
23	2.25	1	0.05	99.54
24	2.35	0	0.00	99.54
25	2.45	2	0.09	99.63
26	2.55	4	0.18	99.82
27	2.65	1	0.05	99.86
28	2.75	1	0.05	99.91
29	2.85	1	0.05	99.95
30	2.95	0	0.00	99.95
31	3.05	1	0.05	100.00
32	3.15	0	0.00	100.00

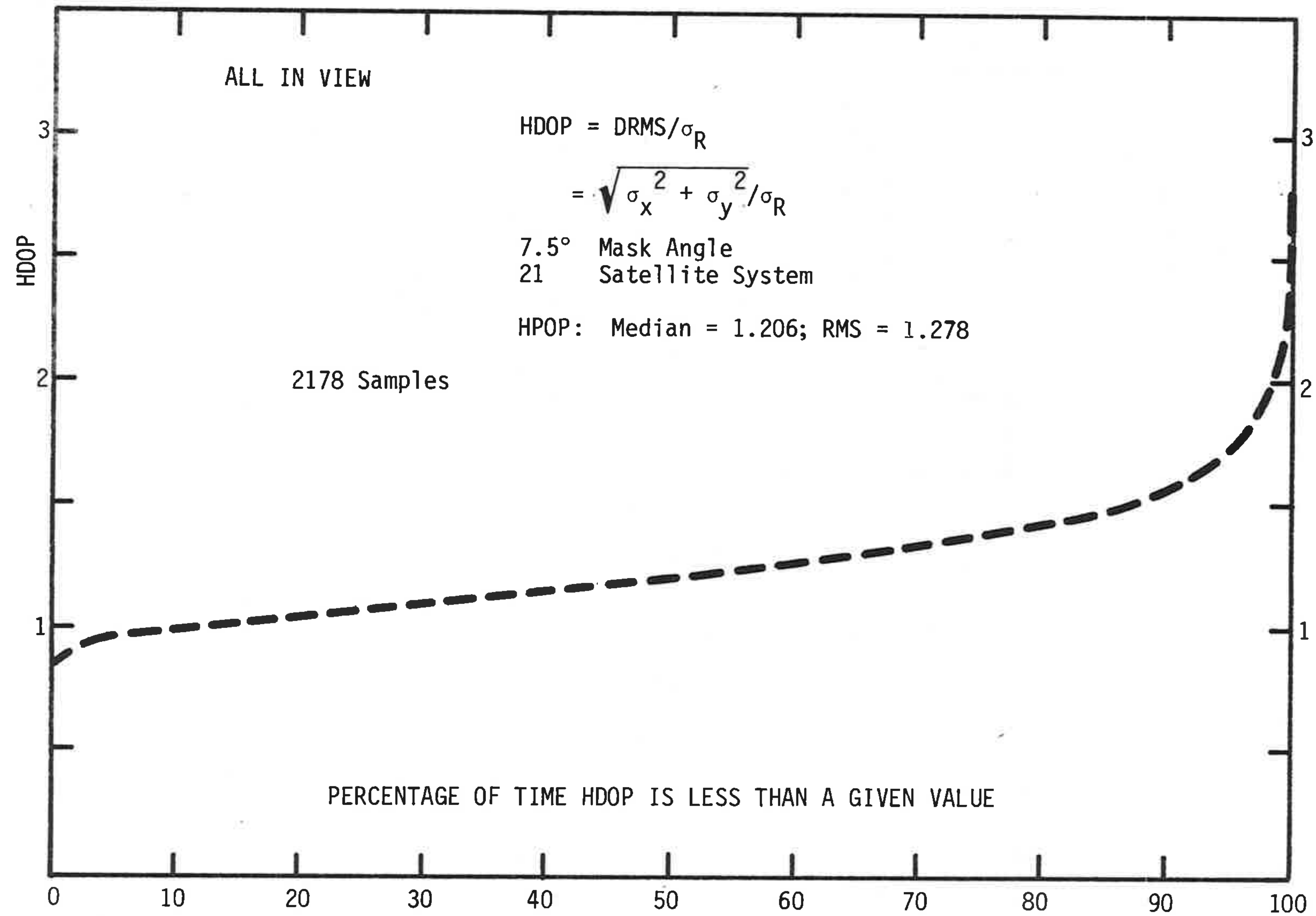


FIGURE 6-2. PERCENTAGE OF TIME HDOP IS LESS THAN A GIVEN VALUE VERSUS THE VALUE OF HDOP

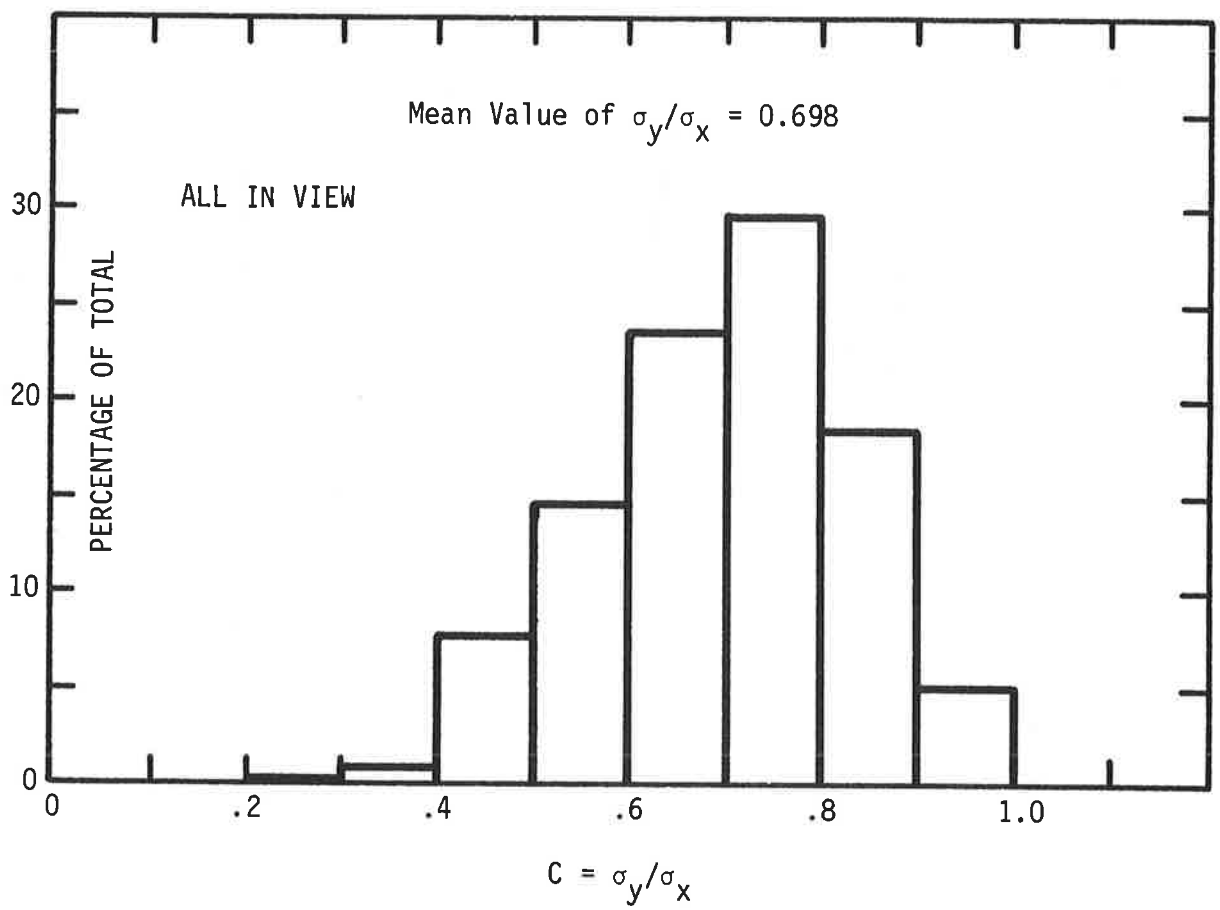


FIGURE 6-3. DISTRIBUTION OF ELLIPTICITY PARAMETER σ_y/σ_x OVER 24 HOUR PERIOD IN CONUS

2178 Samples

7.5 ° Mask Angle

21 Satellite System

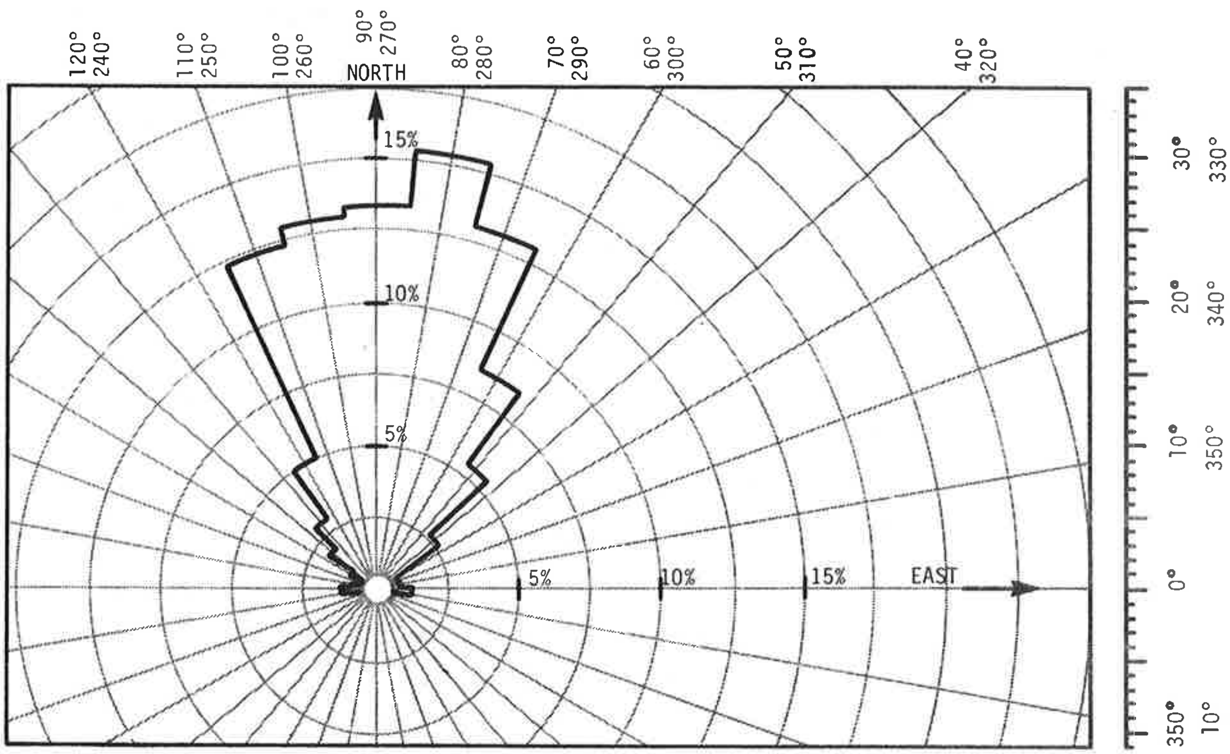


FIGURE 6-4. DISTRIBUTION OF ORIENTATION OF ERROR ELLIPSE (MAJOR AXES): 24 HOUR PERIOD OVER CONUS (ALL IN VIEW)

2718 Samples

7.5 ° Mask Angle

21 Satellite System

Table 17 shows the resulting values of $R95/\sigma_R$ placed in bins of width 0.1. σ_R is the pseudo-range error.

The Median of $R95/\sigma_R = 2.1209$

The Mean of $R95/\sigma_R = 2.2163$

Std. Dev. $\sigma = 0.48373$

$\sigma^2 = 0.234$

Table 18 shows the resulting values of CEP/σ_R (or $R50/\sigma_R$) placed in bins of width 0.1.

The Median of $CEP/\sigma_R = 0.985$

The Mean of $CEP/\sigma_R = 1.0103$

Std. Dev. $\sigma = 0.18439$

$\sigma^2 = 0.034$

Now from Section 6.1 we see that

(Mean of HDOP)/(Mean of CEP/σ_R) = Mean of DRMS/Mean of CEP

= 1.250/1.0103

= 1.2372

where HDOP = $DRMS/\sigma_R$

We also have from Section 6.2 the result that the mean value of $c = \sigma_y/\sigma_x$ is 0.698

We see that the above ratio of 1.2372 compares favorably with the result in Table 13, where for c = 0.7, 1 DRMS/CEP = 1.226.

Figure 6-5 shows the percentage of time that CEP/σ_R and $R95/\sigma_R$ is less than a certain value. The figure can be examined together with Tables 16 and 17 to obtain a clearer picture of the distribution of $R95/\sigma_R$ and CEP/σ_R.

TABLE 17. SAMPLED VALUES OF R95/σ_R (BIN WIDTH = 0.1, 2718 SAMPLES)

BIN #	CENTER OF BIN WITH VALUE OF R95/σ_R	FREQUENCY	ACCUMULATIVE PERCENTAGE
15	1.45	18	0.826
16	1.55	48	3.030
17	1.65	130	9.000
18	1.76	167	16.67
19	1.85	264	28.79
20	1.95	222	38.98
21	2.05	200	48.16
22	2.15	191	56.93
23	2.25	149	63.77
24	2.35	160	71.12
25	2.45	143	77.69
26	2.55	110	82.74
27	2.65	87	86.73
28	2.75	75	90.17
29	2.85	42	92.10
30	2.95	37	93.80
31	3.05	26	95.00
32	3.15	23	96.05
33	3.25	16	96.79
34	3.35	11	97.29
35	3.45	10	97.75
36	3.55	9	98.16
37	3.65	7	98.48
38	3.76	9	98.90
39	3.85	4	99.08
40	3.95	6	99.36
41	4.05	3	99.49
42	4.15	1	99.54

TABLE 17. SAMPLED VALUES OF R95/σ_R (BIN WIDTH = 0.1, 2718 SAMPLES) (CONTINUED)

BIN #	CENTER OF BIN WITH VALUE OF R95/σ_R	FREQUENCY	ACCUMULATIVE PERCENTAGE
43	4.25	0	99.54
44	4.35	0	99.54
45	4.45	0	99.54
46	4.55	1	99.59
47	4.65	1	99.63
48	4.75	2	99.72
49	4.85	2	99.82
50	4.95	0	99.82
51	5.05	1	99.86
52	5.15	1	99.90
53	5.25	0	99.90
54	5.35	0	99.90
55	5.45	1	99.95
56	5.55	0	99.95
57	5.65	0	99.95
58	5.75	0	99.95
59	5.85	1	100.00

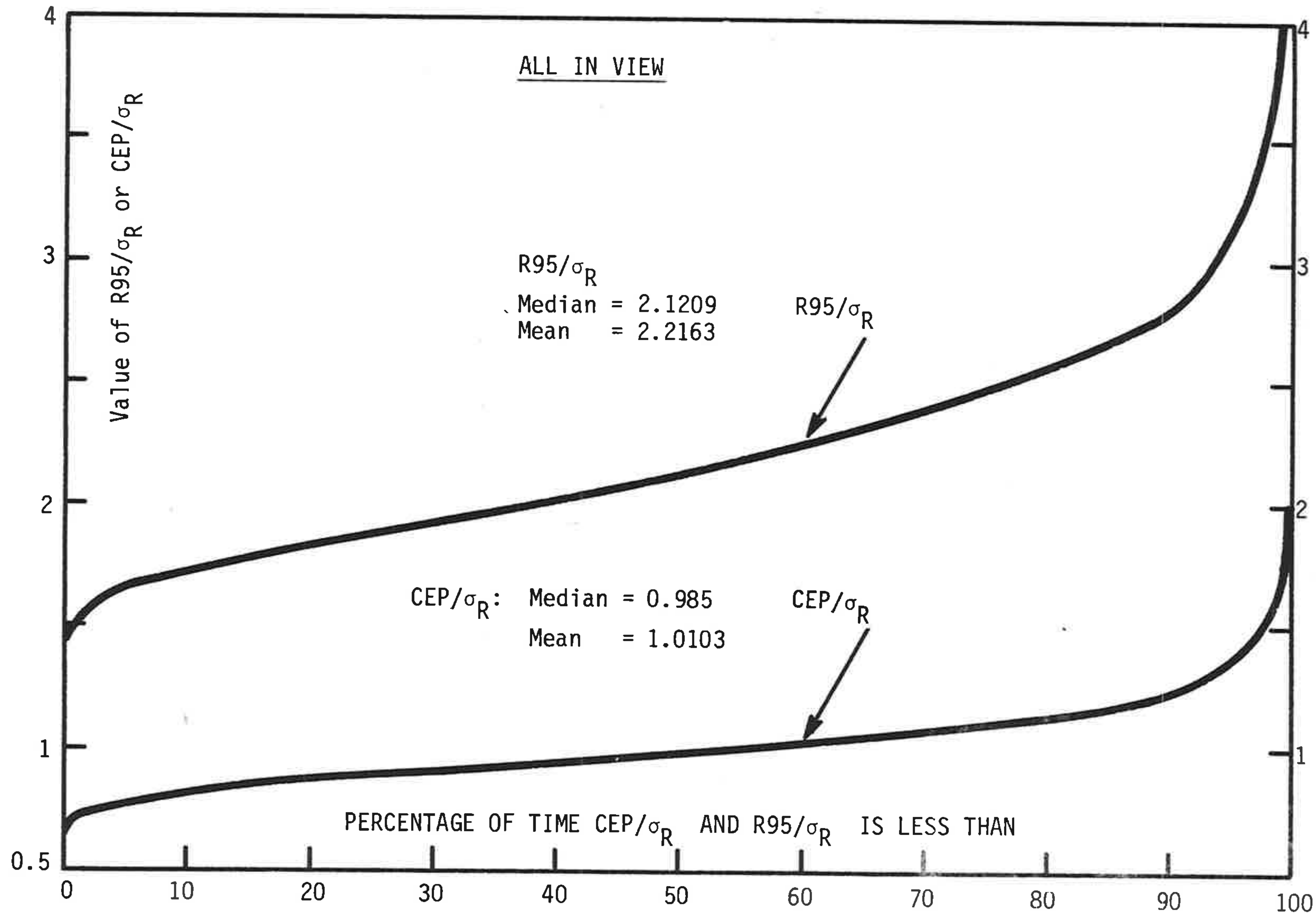


FIGURE 6-5. PERCENTAGE OF TIME R95 AND CEP IS LESS THAN A GIVEN VALUE VERSUS THE VALUE OF R95 OR CEP

TABLE 18. SAMPLED VALUES OF CEP/σ_R (BINS WIDTH = 0.1, 2178 SAMPLES)

BIN #	CENTER OF BIN WITH VALUE OF CEP/σ_R	FREQUENCY	ACCUMULATIVE PERCENTAGE
7	0.65	5	0.230
8	0.75	170	8.035
9	0.85	537	32.69
10	0.95	446	53.17
11	1.05	399	71.49
12	1.15	363	88.15
13	1.25	117	93.53
14	1.35	60	96.28
15	1.45	36	97.93
16	1.55	28	99.22
17	1.65	7	99.54
18	1.75	1	99.59
19	1.85	2	99.68
20	1.95	5	99.91
21	2.05	1	99.95
22	2.15	0	99.95
23	2.25	1	100.00

6.5 A SINGLE 95 PERCENT PROBABILITY CIRCLE FOR THE ENTIRE CONUS OVER A 24 HOUR PERIOD

It is possible to determine by computer program a close approximation of a single radius R_0 such that, if at any random space-time point on the CONUS one made a vehicle position measurement, this measurement would fall within a circle with the radius R_0 around the actual position location with an overall probability of 95 percent.

To assist our intuition for this problem, we first consider the following simple approach. Suppose at four space-time points we drew a circle of the same radius R_1 around the actual location of vehicles at each of the points and took 1000 measurements at each of the 4 points over a certain time period. Let the result be as follows:

Point	Measures falling within R_1	Measures Taken	Probability ratio at each point
#1	630	1000	.63
#2	960	1000	.96
#3	870	1000	.87
#4	920	1000	.92
Total	3380	4000	3.38

Now 3380/4000 = .845 which is the overall chance that any measure will fall within the radius R_1. We can obtain this probability by adding the individual probabilities in the fourth column and dividing by 4, the number of points (i.e. 3.38 ÷ 4 = .845). Note that 0.845 is the overall probability. For point #1, the probability of R_1 containing the measure is only .63.

Returning to our main problem, let the representative space points on the CONUS be labelled $i = 1$ to m (each point being specified by 2 coordinates) and the representative time instants over a 24 hour period be labelled $j = 1$ to n.

The overall probability of finding the measurement within a single radius R is given by

$$S = \frac{\left[\sum_{i=1}^{m} \sum_{j=1}^{n} P_{ij} (R, \sigma_x, \sigma_y)\right]}{mxn} \qquad (6\text{-}4)$$

We want to find the value of R such that S = .95. Using Table 3 (from Harter) and an interpolation method one can insert (with some educated guessing) tentative values of R and converge on R_0, at which radius S = 0.95.

Appendix D shows the details of the interpolation method for Table 3 (from Harter).

This method is such that, even when some of the σ_x corresponding to a given space-time point becomes infinite, there is a finite value for R_0.

It is possible however to define an operational value of S by throwing out all cases where σ_x is above some number. The remaining operational cases are used in equation 6-4.

6.6 THE RADIUS OF A CIRCLE SUCH THAT A MEASUREMENT WILL FALL WITHIN THIS CIRCLE WITH THE PROBABILITY 0.95 FOR A GIVEN USER LOCATION AND GPS TIME

This case is for a given user spatial location and, GPS time instant and is different from the case in Section 6.5. We first determine σ_x and σ_y for the given space-time point. We wish to obtain an estimate of the value of the radius R such that the probability of containing an error measurement is 0.95. From Table 4 (from Harter p. 728) we can obtain K(P,c) letting P = 0.9500.

$c = \sigma_y / \sigma_x$.

Having c, we find C_L and C_H, the nearest pair of values in Table 4, and also the corresponding values K_L and K_H.

$$C_L < C < C_H \quad (6\text{-}5)$$

Then

$$K = 10\,(C - C_L)(K_H - K_L) + K_L \quad (6\text{-}6)$$

Having K, we can calculate

$$R = K\sigma_x . \quad (6\text{-}7)$$

7. CONCLUSIONS

To signify the accuracy of a navigation system over a region and over a period of time, CEP or R95 is superior to 2 DRMS, because:

a. The geometry is allowed to change,

b. The probability level within the defined circle is the same when the configuration of satellites changes (NAVSTAR/GPS),

c. The definition applies without modification to changing geometry.

For an individual location and time, 2 DRMS is easier to compute, and relates directly to HDOP. But as we have shown, a computer program can be developed (using Harter's tables) to quickly compute the quantities $R50/\sigma_R$ and CEP/σ_R.

The probability of finding a position measure within the circle of radius 2 drms depends on the ellipticity of the error ellipse and is good for LORAN-C or OMEGA where the ratio σ_y/σ_x is a constant value. This is not necessarily true for GPS, TRANSIT where the ellipticity of the error ellipse may in some situation change rapidly. In some situations HDOP, which is a function of the geometric configuration of the satellite employed in the measurement, may change rapidly. Experimental determination of 2 drms makes sense only while the geometry hasn't changed much.

The 2 DRMS definition can be extended, except for outage periods, to provide a "System 2 DRMS Accuracy" measure. This overcomes some of the drawbacks, but is still not tied to a fixed probability level.

In the CONUS, NAVSTAR/GPS exhibits the following performance values, with Selective Availability imposed, <u>all-in-view strategy</u> (21 satellite system with 3 spares, without altimeter aiding, coarse acquisition code); we assume a pseudorange one-sigma error of 30 meters, and the mask angle is assumed to be 7.5°.

a. East-West. $\sigma_X = XDOP \bullet \sigma_R$.

Median $= 0.739\,\sigma_R$

$= 22.17$ meters

Median XDOP = 0.739

RMS XDOP = 0.764

East-West Accuracy, meters (2-Sigma) 45.8

b. North-South, $\sigma_Y = YDOP \bullet \sigma_R$.

Median $= 0.948\sigma_R$

$= 28.44$ meters

Median YDOP = 0.948

RMS YDOP = 1.025

North-South Accuracy, meters (2-Sigma) 61.5

c. CEP (50 Percent Probability Radius).

Mean		$= 1.0103\ \sigma_R$
		$= 30.3$ meters
Std. Dev. of CEP:	σ	$= 0.18439\ \sigma_R$
	σ	$= 5.53$ meters
Std. Dev. of CEP/σ_R:	σ	$= 0.184$
	σ^2	$= 0.034$

d. R95 (95 Percent Probability Radius).

Mean		$= 2.2163\ \sigma_R$
		$= 66.5$ meters
Std. Dev. of R95:	σ	$= 0.48373\ \sigma_R$
	σ	$= 14.51$ meters
Std. Dev. of R95/ σ_R:	σ	$= 0.484$
	σ^2	$= 0.234$

e. 1 drms $= \text{HDOP} \bullet \sigma_R$.

Median	$= 1.206\ \sigma_R$
	$= 36.18$ meters
RMS HDOP	$= 1.278$

2 drms ACCURACY, meters

$= 76.7$

f. Average Values of Ellipticity Parameter

$c = \sigma_y / \sigma_x$.

Mean $= 0.698$

We noted in Section 4.1 that the Federal Radionavigation Plan (FRP) choice of 2.5 for 2 drms/CEP (see Section 1.3) suggests an average ellipticity parameter $c = 0.60$ (cf. Figure 4-1). Our computed mean value for c is 0.698. This suggests a value of 1.226 for drms/CEP or 2.452 for 2 drms/CEP. This is slightly lower than the FRP suggested value.

The computed RMS of XDOP (E-W) is significantly smaller than that of YDOP (N-S). The RMS EAST-WEST ACCURACY meters (2 SIGMA) is 45.8, while the RMS NORTH- SOURTH ACCURACY meters (2-SIGMA) is 61.5. The overall greater accuracy in the East-West direction is consistent with our results for the orientation of the error ellipses. Figure 6-4 shows that a greater number of the major axes of the error ellipses fall in the North-South direction.

Calculations were made also for the "best-set-of-four" strategy. For a comparison of the results of the "all-in-view" and the "best-set-of-four" strategies, see Section 1.2 of the Summary: Table 1, page 1-15.

APPENDIX A

SAMPLE COVARIANCE CALCULATIONS ILLUSTRATING THE EFFECTS OF SYMMETRY

The following examples will illustrate the contributions to the covariance σ_{XY}^2. Four cases are considered. In each case the distribution consists of four points. We consider the pairs of points which are symmetric with respect to the origin. The variances σ_X^2 and σ_Y^2 of the 4 cases are all non-zero. The mean values are at the origin.

Case 1. (See Figure A-1a)

In this case the individual contributions to the resultant covariance are all zero. Hence the resultant covariance

$$\sigma_{XY}^2 = 0$$

Case 2. (See Figure A-1b)

In this case the individual contributions of the measurement points to the resultant covariance are non-zero. The contributions from A and B are positive while those from C and D are negative. The contributions cancelled so that the resultant covariance

$$\sigma_{XY}^2 = 0$$

Writing the contributions in detail we have

$$\sigma_{XY}^2 = (1/4)\,[3 \bullet 2 + (-3)(-2) + 3 \bullet (-2) + (-3) \bullet 2\,]$$
$$= 0$$

Case 3. (See Figure A-1c)

In this case the A + B contributions are positive

$24 + 24 = 48.$

The C + D contributions are negative

$-6 + (-6) = -12$

$$\sigma_{XY}^2 = \frac{48-12}{4}$$
$$= 9$$

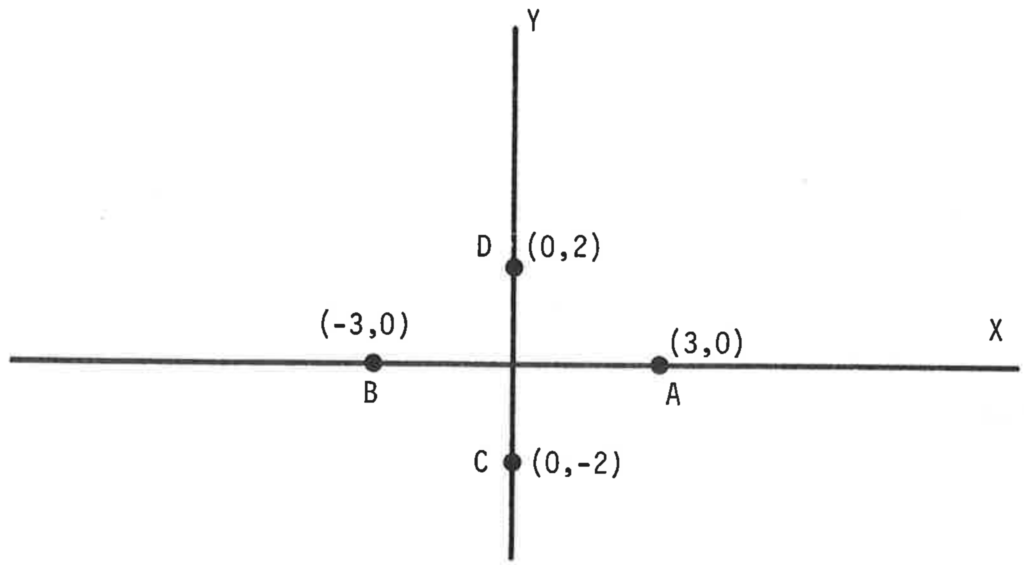


FIGURE A-1a. CASE 1. $\sigma_{XY}^2 = 0$

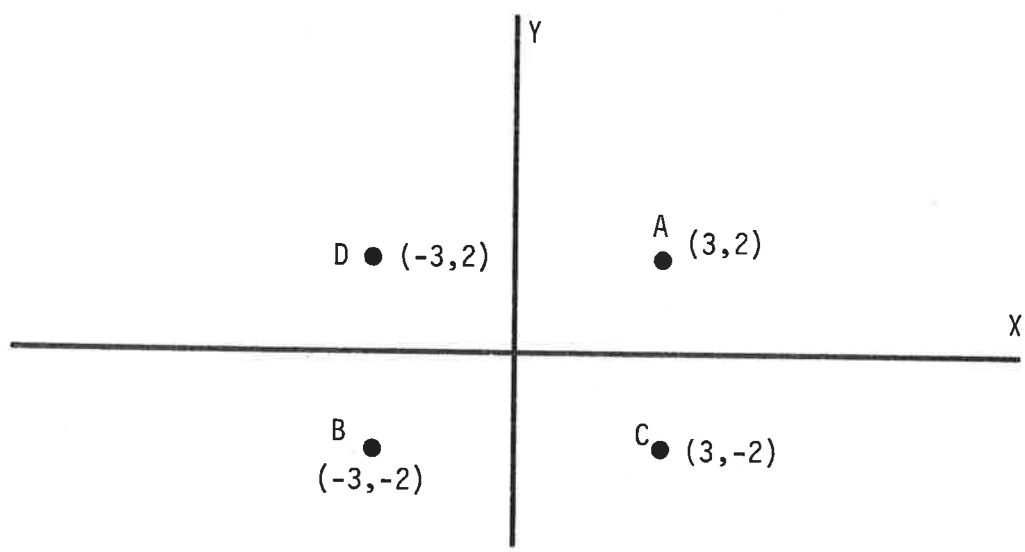


FIGURE A-1b. CASE 2. $\sigma_{XY}^2 = 0$

FIGURE A-1. ILLUSTRATIONS OF CONTRIBUTIONS TO COVARIANCE σ_{XY}^2

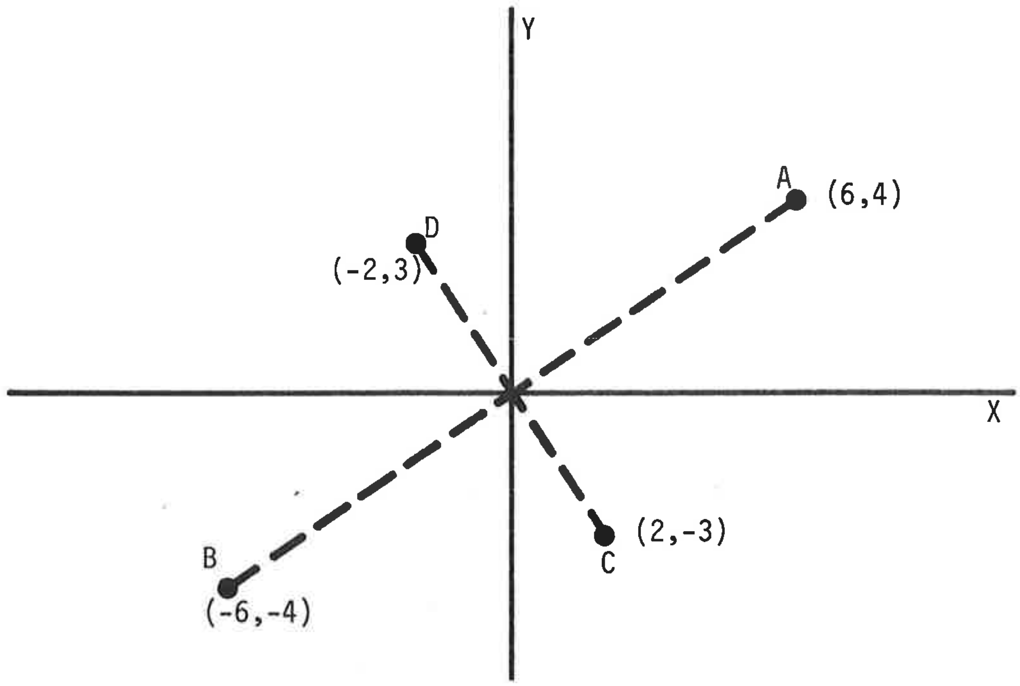


FIGURE A-1c. CASE 3. $\sigma_{XY}^2 = 9$

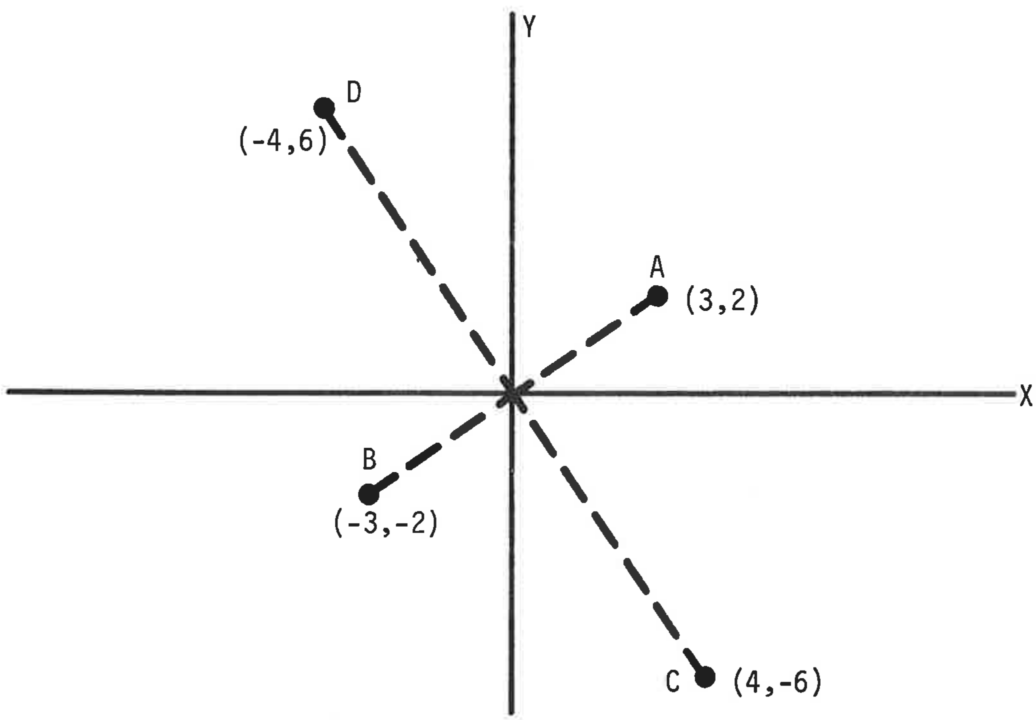


FIGURE A-1d. CASE 4. $\sigma_{XY}^2 = -9$

FIGURE A-1. ILLUSTRATIONS OF CONTRIBUTIONS TO COVARIANCE σ_{XY}^2 (CON'T)

Thus the A + B contributions predominate over the C + D contributions.

Case 4. See (Figure A-1d)

Here the A + B contributions are positive

6+6 = 12

The C + D contributions are negative

-24 + (-24) = - 48.

$$\sigma_{XY}^2 = \frac{48 - 12}{4}$$

$$= - 9$$

Note: σ_{XY} has no physical meaning. Only σ_{XY}^2 is meaningful.

Thus the negative contribution predominates.

These examples are of course rather simple but they give us insight into the more complex elliptical distributions which we examined in Section 2.11. The distribution of the points in the above 4 cases may appear to be rather arbitrary. But in the cases of the elliptical distributions with which we shall be concerned, each element of the distribution is balanced by a corresponding element on the other side of the origin, so that the above 4 cases are of some interest in examining the more general elliptical distribution.

APPENDIX B

PROOF THAT THE TRACE OF A MATRIX IS INVARIANT WITH RESPECT TO AN ORTHOGONAL COORDINATE TRANSFORMATION

We wish to show that when the matrix A is transformed into the diagonal matrix A' that the trace of A is equal to the trace of A'.

$A \equiv a_{jk}$, is the covariance matrix having the elements σ_x^2, σ_y^2, σ_{xy}^2 etc., where the error ellipse is not aligned with the old x,y,z axis. There exists an orthogonal matrix U, (with the property $U^T = U^{-1}$) such that A' is a diagonal matrix.

$$A' = U^T A U$$
$$\bar{x} = U \, \bar{x'}$$

The elements of the normal eigenvectors of A are the elements of the successive columns of U.

$$U^{-1} A U \equiv A'$$
$$A' \equiv a'_{il}$$
$$A \equiv a_{jk}$$

$$U^{-1} \equiv (U^{-1})_{ij}$$
$$U \equiv u_{kl}$$

Then $$\sum_k \sum_j (u_{ij}^{-1} \; a_{jk} \; u_{kl}) = a'_{il}.$$

$$(only\, a_{ii}' \neq 0, a_{ij}' = 0 \;\; when\, i \neq j)$$

Let l = i (diagonal elements are considered only).

Then $$\sum_k \sum_j \left[u_{ij}^{-1} a_{jk} u_{ki} \right] = a'_{ii}$$
$$= \lambda_i$$

(where the λ_i's are the eigenvalues of A).

We rearrange the terms.

$$\sum_k \sum_j (u_{ki}\, u_{ij}^{-1}\, a_{jk}) = a'_{ii}$$

$$= \lambda_i$$

Now sum over the diagonal terms i = 1 to n.

$$\sum_i \sum_k (\sum_j (U_{ki}\, U_{ij}^{-1}\, a_{jk}) = \sum_i a'_{ii}$$

$$= \sum_i \lambda_i$$

$\sum_i a_{ii}'$ is the Trace of A′ and is the sum of the eigenvalues of A.

Now $\sum_k \sum_j (\sum_i U_{ki}\, u_{ij}^{-1}\, a_{jk}) = T_r (A')$

$$\sum_k \left[\sum_j \delta_{kj}\, a_{jk} \right] = T_r (A')$$

($\delta_{kj} = 0$ when $j \neq k$

= 1 when j = k)

δ_{ki} = I Identity matrix

$$\sum_{k=1}^{n} a_{kk} = \sum_{i=1}^{n} a'_{ii}$$

Trace (A) = Trace (A′)

= Sum of eigenvalues of A

$$= \sum_{i=1}^{n} \lambda'_i$$

<u>which is to be proven.</u>

In the well known text of F.B. Hildebrand, Methods of Applied Mathematics (p. 40), there is an example:

$$a = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 34 & -12 \\ 0 & -12 & 41 \end{bmatrix}$$

Trace (a) = 100 = 25 + 34 + 41

The eigenvalues of a are: $\lambda_1 = 25, \lambda_2 = 25, \lambda_3 = 50$.

The diagonal matrix a' is

$$a' = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 50 \end{bmatrix}$$

Trace (a') = 100 = Trace (a)

But the general theorem that we give here is not to be found in Hildebrand.

APPENDIX C

DERIVATION OF THE STANDARD DEVIATIONS OF THE 1 σ ELLIPSE ALONG THE PRINCIPAL AXES

In the derivation, it should at first be left undetermined what direction the major principal axis is in for the new coordinate axes., We only say that the a and b vectors are along the principle axis of the 1 σ ellipse. There exists an orthogonal matrix L (having by definition the property $L^T = L^{-1}$) such that

$$D = L^T M L$$

where D is a diagonal matrix and M is given in equation 1-7. Also the trace of M equals the trace of D.

$$D = \begin{bmatrix} \sigma_a^2 & 0 \\ 0 & \sigma_b^2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where λ_1 and λ_2 are the eigenvalues of M. By definition $\lambda_1 \geq \lambda_2$.

A vector $\underline{v}$ with components (v_X, v_Y) will have components (v_a, v_b) in the new coordinate system.

$$\begin{bmatrix} v_a \\ v_b \end{bmatrix} = L^T \begin{bmatrix} v_X \\ v_Y \end{bmatrix}$$

The matrix L is such that the elements of the normalized eigenvectors $\hat{a}$ and $\hat{b}$ of M are the elements of the successive columns of L:

$$\hat{a} = \begin{bmatrix} a_X \\ a_Y \end{bmatrix}$$

$$\hat{b} = \begin{bmatrix} b_X \\ b_Y \end{bmatrix}$$

$$L = \begin{bmatrix} a_X & b_X \\ a_Y & b_Y \end{bmatrix}$$

Eigenvalue Calculation

The eigenvector equations are

$$(M - \lambda I)\, b = 0$$
$$(M - \lambda I)\, a = 0$$

Solutions are possible only if the determinant of M-λI equal 0.

$$(M - \lambda I) = 0$$

Or

$$(\sigma_X^2 - \lambda)(\sigma_Y^2 - \lambda) - \sigma_{XY}^2\, \sigma_{XY}^2 = 0$$

Solution of the above equation for λ leads to the eigenvalues λ_1 and λ_2 in (2- 18a and b).

·Orientation of Equal Probability Density Ellipse

The new axis a and b can be obtained by a rotation of angle θ (counterclockwise when θ is positive) from the X direction. Then

$$v_a = v_X \cos\theta + v_Y \sin\theta$$
$$v_b = -v_X \sin\theta + v_Y \cos\theta$$

and

$$\begin{bmatrix} v_a \\ v_b \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_X \\ v_Y \end{bmatrix}$$

Hence

$$L^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

and

$$L = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Now the columns of L are the components of the eigenvectors of M. Then the eigenvectors of M are

$$\hat{a} = [\cos\theta, \sin\theta]^T$$

$$\hat{b} = [-\sin\theta, \cos\theta]^T$$

One of the eigenvector equations of M is

$$Ma = \lambda_1 \, a,$$

or

$$\begin{bmatrix} \sigma_x^2 & \sigma_{XY}^2 \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \lambda_1 \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

or

$$\sigma_X^2 \cos\theta + \sigma_{XY}^2 \sin\theta = \lambda_1 \cos\theta$$

$$\sigma_{XY}^2 \cos\theta + \sigma_Y^2 \sin\theta = \lambda_1 \sin\theta$$

From the above two equations we obtain (eliminating λ_1)

$$\frac{(\sigma_{XY}^2 \cos\theta + \sigma_Y^2 \sin\theta)}{(\sigma_X^2 \cos\theta + \sigma_{XY}^2 \sin\theta)} = Tan\,\theta$$

with some algebra we obtain equation 2-19.

APPENDIX D

ALGORITHM FOR DETERMINING A SINGLE 95 PERCENT PROBABILITY RADIUS FOR A 24 HOUR PERIOD OVER THE CONUS

First we insert some tentative value for R. Also for a given space-time point we have σ_x and σ_y. We calculate for (i, j)

$$c = \sigma_y / \sigma_x$$

$$K = R / \sigma_-$$

(i, j) are the space-time indices of a point on the CONUS.

Using Table 3 (from Harter[6]) we must determine the value P_{Rij}.

From the value of K and c, we first determine c_L and c_H, K_L and K_H where c_L and c_H are the nearest higher and lower values for the input value of c; and similarly, K_L and K_H for the input value of K.

Then from Table 3 we obtain

$$P_{LL}(K_L, c_L),$$

$$P_{LH}(K_L, c_H),$$

$$P_{HL}(K_H, c_L),$$

$$P_{HH}(K_H, c_H).$$

Then calculate

$$P_{mL} = \left[(K - K_L)/0.1 \right] (P_{HL} - P_{LL}) + P_{LL},$$

$$P_{mH} = \left[(K - K_L)/0.1 \right] (P_{HH} - P_{LH}) + P_{LH}.$$

Then $$P_{mc} = \left[(c - c_L)/0.1 \right] (P_{mH} - P_{mL}) + P_{mL}$$

Let $P_{Rij} = P_{mc}(i, j)$

Then $$S = \left[\sum_{i=1}^{m} \sum_{j=1}^{n} P_{Rij}(R, \sigma_x, \sigma_y) \right] / \left[m \, X \, n \right]$$

We then try several values of R, calculate S and approach R_0 by successive trials, so that S approaches .95.

APPENDIX E

BEST-SET-OF-FOUR SATELLITES IN VIEW:
COMPUTED RESULTS

RMS CALCULATION OF DOP VALUES
BEST-SET-OF-FOUR SATELLITE SELECTION ALGORITHM

BIN #	FREQUENCY	CENTER OF BIN WITH VALUE OF			
6	0	0.55	0.30	0	0
7	1	0.65	0.42	0.42	0.423
8	281	0.75	0.56	158.06	158.485
9	558	0.85	0.72	403.16	561.640
10	378	0.95	0.90	341.15	902.785
11	444	1.05	1.10	489.51	1392.295
12	274	1.15	1.32	362.37	1754.660
13	154	1.25	1.56	240.63	1995.285
14	77	1.35	1.82	140.33	2135.618
15	10	1.45	2.10	21.03	2156.643
16	0	1.55	2.40	0	2156.643
17	1	1.65	2.72	2.72	2159.365
18		1.75	3.06	0	2159.365
19		1.85	3.42	0	2159.365
20		1.95	3.80	0	2159.365
21		2.05	4.20	0	2159.365
22		2.15	4.62	0	2159.365
23		2.25	5.06	0	2159.365
24		2.35	5.52	0	2159.365
25		2.45	6.00	0	2159.365
26		2.55	6.50	0	2159.365
27		2.65	7.02	0	2159.365
28		2.75	7.56	0	2159.365
29		2.85	8.12	0	2159.365
TOTAL	2173		XDOP(rms)		0.996

RMS CALCULATION OF DOP VALUES
BEST-SET-OF-FOUR SATELLITE SELECTION ALGORITHM (CONTINUED)

BIN #	FREQUENCY	CENTER OF BIN WITH VALUE OF			
6	0	0.55	0.30	0	0
7	0	0.65	0.42	0	0
8	89	0.75	0.56	50.06	50.063
9	177	0.85	0.72	127.88	177.945
10	293	0.95	0.90	264.43	442.378
11	355	1.05	1.10	391.39	833.765
12	287	1.15	1.32	379.56	1213.323
13	347	1.25	1.56	542.19	1755.510
14	267	1.35	1.82	486.61	2242.118
15	160	1.45	2.10	336.40	2578.518
16	56	1.55	2.40	134.54	2713.058
17	33	1.65	2.72	89.84	2802.900
18	39	1.75	3.06	119.44	2922.338
19	27	1.85	3.42	92.41	3014.745
20	17	1.95	3.80	64.64	3079.388
21	6	2.05	4.20	25.22	3104.603
22	4	2.15	4.62	18.49	3123.093
23	4	2.25	5.06	20.25	3143.343
24	4	2.35	5.52	22.09	3165.433
25	4	2.45	6.00	24.01	3189.443
26	4	2.55	6.50	26.01	3215.453
27	5	2.65	7.02	35.11	3250.565
28	0	2.75	7.56	0	3250.565
29	0	2.85	8.12	0	3250.565
TOTAL	2178			YDOP(rms)	1.222

HDOP(rms) 1.576

ELLIPTICITY VALUES			
MM 1	0	0.05	0
MM 2	0	0.15	0
MM 3	0	0.25	0
MM 4	30	0.35	10.500
MM 5	261	0.45	117.450
MM 6	596	0.55	327.800
MM 7	514	0.65	334.100
MM 8	518	0.75	388.500
MM 9	213	0.85	181.050
MM 10	46	0.95	43.700
TOTAL	2178	MEAN VALUE, ELLIPTICITY 0.644	

BEST-SET-OF-FOUR

COMPUTED CEP (OR R50)

BIN #	FREQ.	ACCUM. FREQ.	PERCENT	Value of CEP/σ_R
5 =	0			
6 =	0			
7 =	0			
8 =	0			
9 =	0			
10 =	40	40	1.84	0.95
11 =	368	408	18.73	1.05
12 =	553	961	44.12	1.15
13 =	549	1510	69.33	1.25
14 =	361	1871	85.90	1.35
15 =	159	2030	93.20	1.45
16 =	65	2095	96.19	1.55
17 =	38	2133	92.93	1.65
18 =	21	2154	98.90	1.75
19 =	5	2159	99.13	1.85
20 =	11	2170	99.63	1.95
21 =	4	2174	99.82	2.05
22 =	4	2178	100.00	2.25

BEST-SET-OF-FOUR

R95

BIN #	FREQ.	ACCUM. FREQ.	PERCENT	Value of R95/σ_R
21 =	40	40	1.837	2.05
22 =	95	135	6.198	2.15
23 =	155	290	13.31	2.25
24 =	164	454	20.84	2.35
25 =	176	630	28.93	2.45
26 =	223	853	39.16	2.55
27 =	208	1061	48.71	2.65
28 =	157	1218	55.92	2.75
29 =	194	1412	64.83	2.85
30 =	225	1637	75.16	2.95
31 =	152	1789	82.14	3.05
32 =	118	1907	87.56	3.15
33 =	57	1964	90.17	3.25
34 =	36	2000	91.83	3.35
35 =	32	2032	93.30	3.45
36 =	22	2054	94.31	3.55
37 =	24	2078	95.41	3.65
38 =	20	2098	96.33	3.75
39 =	27	2125	97.57	3.85
40 =	7	2132	97.89	3.95
41 =	8	2140	98.35	4.05
42 =	6	2146	98.62	4.15
43 =	6	2152		4.25
44 =	4	2156		4.35
45 =	0	2156		4.45
46 =	1	2157		4.55
47 =	3	2160		4.65
48 =	2	2162		4.75
49 =	2	2164		4.85
50 =	4	2168		4.95
51 =	1	2169		5.05
52 =	1	2170		5.15
53 =	5	2175		5.25
54 =	1	2176		5.35
55 =	1	2177		5.45
56 =	1	2178		5.55

GLOSSARY OF TERMS AND SYMBOLS

1 σ Ellipse. See One σ Ellipse

All-In-View-Strategy. A procedure which uses all the satellites, visible from the user's location, in the calculation of the covariance matrix of the position error vector.

Best-Set-of-Four. A procedure which uses the best-set-of-four satelites out of all the combinations of four of the satellites visible from the user's location.

$c = \sigma_y/\sigma_x$, Ellipticity Parameter. The ratio of the semi-minor principal axis to the semi-major principal axis of a 1 σ error ellipse. Also known as Ellipticity Ratio.

CEP, Circular Error Probable. The radius of a circle containing 50 percent of the vehicle position measures. The same as R50. The probability density function for this measure is in general elliptical for GPS.

Circular Error Probability. The probability that a vehicle position measure lies within a circle of radius R.

CONUS. Conterminous United States, the main land mass of the United States not including Alaska and Hawaii.

Covariance Matrix of the Position Error Vector. This is a symmetric matrix, M (i.e.,$\sigma_{ij}^2 = \sigma_{ji}^2$),

$$M = [G^T G]^{-1} \bullet \sigma_R^2 = Q \bullet \sigma_R^2$$

$$M = \begin{bmatrix} \sigma_X^2 & \sigma_{XY}^2 & \sigma_{XZ}^2 & \sigma_{XT}^2 \\ \sigma_{YX}^2 & \sigma_Y^2 & \sigma_{YZ}^2 & \sigma_{YT}^2 \\ \sigma_{ZX}^2 & \sigma_{ZY}^2 & \sigma_Z^2 & \sigma_{ZT}^2 \\ \sigma_{TX}^2 & \sigma_{TY}^2 & \sigma_{TZ}^2 & \sigma_T^2 \end{bmatrix} = \sigma_{ij}^2$$

The spatial elements of the M matrix have a dimension of length squared (say, meter squared). The elements of the Q Matrix,

$$Q = [G^T G]^{-1} = $$
$$= M/\sigma_R^{\ 2} \qquad = \sigma_{ij}^2 / \sigma_R^2$$

are dimensionless.

The G matrix can be expressed as:

$$G(n \times 4) = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \\ \bullet \\ \bullet \\ \bullet \\ \vec{r}_n \end{bmatrix}$$

with $\vec{r}_i = (e_{i1}, e_{i2}, e_{i3}, 1)$.

The unit 3-vector e_i points from the user position to the i^{th} satellite. X (East), Y (North), Z (Up) (cf. Reference 10).

drms, DRMS, or d_{rms}. Distance Root Mean Squared. The square root of the sum of the squares of the one sigma error components along the semi-major and semi- minor principal axes of an equal probability density ellipse,

$$1\ drms = \sqrt{\left[\sigma_x^{\ 2} + \sigma_y^{\ 2}\right]}$$
$$= \sqrt{\left[\sigma_X^{\ 2} + \sigma_Y^{\ 2}\right]} \qquad \left(\begin{matrix} X\,(E-W), \\ Y\,(N-S). \end{matrix} \right)$$
$$Also,\ 1\ drms = HDOP \bullet \sigma_R$$
$$= \sigma_H$$

The magnitude of 1 drms is that of the hypotenuse of a right triangle with the legs being the semi-major principal axis and the semi-minor principal axis of a 1 σ error ellipse.

$$Also,\ 1\ drms = \sigma_x \sqrt{\left[1 + c^2\right]}$$
$$where\ c \equiv \sigma_y / \sigma_x$$

See also the general definition of drms in Section 1.2.4.

2 drms. This is twice the magnitude of 1 drms.

$$2\ drms = \sqrt{\left[(2\,\sigma_x)^2 + (2\,\sigma_y)^2\right]}$$
$$= 2\ HDOP \bullet \sigma_R$$
$$= \sqrt{\left[1 + c^2\right]} \bullet \sigma_x$$

Elliptical Error Probability (2 dimension). The probability that a vehicle position measure falls within an equal probability density ellipse defined by the index k (lower case),

$$P_E = 1 - exp\,(k^2/2).$$

Equal Probability Density Contours. The locus of equal values of the two-dimensional probability density distribution function.

Equal Probability Density Ellipses. The equal probability density contours of the normal (Gaussian) distribution function (in 2-dimension) are in the form of ellipses having a common center and ellipticity,

The 1 σ ellipse has a semi-major principal axis of magnitude σ_x and a semi- minor principal axis of σ_y.

The k σ ellipse has a semi-major principal axis of magnitude k σ_x.

Error Ellipse. Abbreviated term for an Equal Probability Density Ellipse.

FRP Federal Radionavigation Plan. A plan jointly developed by the U.S. Departments of Defense and Transportation to ensure efficient use of resources and full protection of national interests.

GPS. NAVSTAR/GLOBAL POSITIONING SYSTEM. A navigation system providing extremely accurate position and velocity information to users, based on the measurement of the transit time of RF signals from a system of satellites.

HDOP. Horizontal Dilutions Of Precision.

This is given by

$$HDOP = \sqrt{\left[\sigma_x^{\,2} + \sigma_y^{\,2}\right]}/\sigma_R$$

where σ_R is the user-equivalent range error (UERE).

k. Parameter occurring in the Elliptical Error Probability equation. This parameter defines the probability contained in a corresponding equi-probability density ellipse (i.e., the k σ ellipse).

σ_x. The standard deviation along the major principal axis of an equal probability density ellipse.

σ_y. The standard deviation along the minor principal axis of an equal probability density ellipse.

SIGMA X. Same as σ_X.

σ_X. Standard Deviation in the X (Eastern) direction (Sigma X) also called the RMS errors,

$$\sigma_X = \sqrt{\left[\sigma_X^2\right]},$$

where the Variance σ_X^2 is defined as follows:

$$\begin{aligned}\sigma_X^2 &= E\left[(X - m_X)^2\right] \\ &= \int\int (X - m_X)^2 f(X,Y)\, dX dY,\end{aligned}$$

m_X is the mean value of f(X,Y), the probability density distribution in the X direction.

σ_Y Standard Deviation in the Y (Northern) direction. (Sigma Y) also called the RMS error in the Y direction.

$$\sigma_Y = \sqrt{\left[\sigma_Y^2\right]}$$

σ_{XX}^2. (Same meaning as σ_X^2.) Some authors use two X's in the subscript. We use only one X in the subscript.

σ_{XY}^2. Covariance (of the joint distribution of X and Y).

$$\sigma_{XY}^2 = E\left[(X - m_X)(Y - m_Y)\right] = \int\int (X - m_X)(Y - m_Y) f(X,Y)\, dX dY$$

σ_H. RSS Horizontal Position Solution Error.

$$\begin{aligned}\sigma_H &= \sqrt{\left[\sigma_X^2 + \sigma_Y^2\right]} \\ &= 1\, DRMS.\end{aligned}$$

σ_R. (See UERE, User Equivalent Range Error).

Selective Availability (SA). Capability for the military to deny full system accuracy to unauthorized users. Under SA conditions civil users of SPS (Standard Positioning Service) signal will have accuracy not worse than 100 meters (2 DRMS) at all times.

(PPS, Precise Positioning Service, which will be available to military users, is not explicitly considered in this study.)

Trace of Matrix. The sum of the diagonal terms of the matrix,

cf. equation 1-20

UERE, User Equivalent Range Error. σ_R, Error in the measurement of range from the user's location to the satellites due to the imprecise clock of the user, propagation delays and other errors.

Variance. σ_X^2 (or σ_Y^2) Defined in Section 1.4 (cf. equation 1-12); see also, σ_x Standard Deviation, where its definition is given.

x. The direction of increasing or decreasing values of x is along the major principal axis of an equal probability density ellipse.

X. The direction of increasing or decreasing value of X is along the East-West direction.

XDOP. The square root of the X component of the diagnonal elements of the matrix Q,

$Q = (G^TG)^{-1}$, (see covariance matrix)

$= M/\sigma_R^2$ (Q is dimensionless)

$XDOP = \sigma_x/\sigma_R$ (σ_R is the Range Error)

$\sigma_x = XDOP \bullet \sigma_R$

(XDOP and YDOP are dimensionless quantities)

YDOP. The square root of the Y component of the diagonal elements of the matrix Q,

$Q = (G^TG)^{-1}$, (see covariance matrix)

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