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FIBER COMPOSITE ANALYSIS AND DESIGN

VOLUME II STRUCTURES

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Fiber Composite Analysis and Design

VOLUME II STRUCTURES

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CT88/18
Vol. II

MATERIALS FOR TOMORROW

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Prepared For:
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PREFACE

Fiber composites have been developed to take advantage of unique properties found in materials in fibrous form. Many materials in the form of fine filaments exhibit strength and stiffness characteristics far greater than those of the same material in bulk form. The combination of such filaments with a suitable matrix can yield a composite with properties superior to the same material in bulk.

The concept is not new, but the emphasis has changed. Perhaps the earliest composites were the mud bricks reinforced with straw used in Biblical times. Here the mud (the matrix material) was the important element, and the straw was added to make the mud serve adequately as a material of construction. For today's composites, the matrix serves rather to make the fiber properties accessible. Hence, the title of this book, Fiber Composite Analysis and Design. The fiber comes first, and the technology with which this book is concerned is primarily directed toward the utilization of the potential provided by advanced fibers.

During the last twenty years, a wide range of new fibers has been introduced into the marketplace, creating both new design opportunities and new problems for the designer. The technology of using such fibers is not a simple one, but progress has been made toward its mastery. The current state of that art is described in the following chapters.

The material presented is adapted from a series of lectures on fiber composites given to FAA engineers in the fall of 1980 by senior staff members of the Materials Sciences Corporation. The object of this adaptation is to provide an adequate background of knowledge of the characteristics and mechanics of fiber composites so that an engineer experienced in the assessment of the performance of conventional materials can extend his competence to the assessment of composites. A broad spectrum of technologies is involved, ranging from detailed methods of analysis to more qualitative considerations of methods of design and evaluation. Throughout, the emphasis is directed toward the definition of the nature and magnitude of the differences associated with the use of composites rather than conventional materials, and especially toward the elucidation of the reasons for the differences and their implications for design evaluation.

FORWARD

This is the second volume of a two-volume book treating analysis and design problems of fiber composites, ranging from mechanics of composite materials to response of composite structures. The first volume treated the properties of fiber composite materials, including laminates. This second volume addresses structures made from these materials.

This volume represents the results of technical contributions by a number of authors, each of whom was the principal author of a particular section of the book. Dr. Sailendra N. Chatterjee was the principal author of the section on beams; Dr. James Whitney, plates and shells; Dr. John Hart-Smith, joints; Dr. John Kibler, structures; and Dr. David Taggart, fabrication methods and material property tests.

Mr. Lawrence M. Neri, Technical Specialist from the Federal Aviation Administration (FAA) Technical Center, contributed extensive editorial effort to this project. Mr. Joseph Soderquist, FAA National Resource Specialist-Advanced Nonmetallic Materials, also provided technical consultation.

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CHAPTER I. INTRODUCTION

The availability of a diversity of high performance composite materials represents, for the structural designer, a step increase in his capability to satisfy performance requirements. The field of structures and materials has entered the "Age of Discontinuity" (Drucker, P. F., Harper and Row, New York) wherein there is a "shift from starting with what nature provides to what man wants to accomplish." Material design and structural design can now be performed simultaneously.

Recent years have witnessed a significant increase in the understanding and utilization of fibrous composite materials. There has also been a much larger increase in the amount of published literature in this field. This book builds upon existing literature to present a review of the available capability for composite structural design and analysis. The aim is to provide guidance for one who seeks to become familiar with the tools required for designing with fibrous composites. Thus, the book identifies the key concepts associated with the use of these unique materials.

The required design methods may be discussed conveniently under three broad categories. The first of these is the selection of constituent materials and the evaluation of resulting properties of the basic unidirectional fibrous composite. This is the basic element in composite structural design. This unidirectional material may be used directly in a structural configuration (e.g. as a stiffener) or layers of this material may be laid up in prescribed patterns to form structural laminates. The design and analysis of such laminates is the second major subdivision of this discussion. The design and analysis of fiber composite materials and laminates were the subjects of Volume I of this two-volume set. This second volume addresses the design and analysis of structural configurations for the practical and efficient utilization of fiber composite materials.

The inclusion of the potential for material design into the structural design process has a significant impact upon the preliminary design phase. Specifically, it means that the number of materials which may be considered generally includes many for which material properties have not been obtained experimentally. Thus, preliminary materials selection may be based on analytically predicted properties. The important study of the relationship between effective properties of composites and the properties of their constituents is frequently described by the misnomer "micromechanics." The understanding of the relationships between the overall or average response of a composite and the properties of its constituents permits the representation of the inhomogeneous (and generally, anisotropic) material. The properties of this homogeneous material are the effective properties of the composite; that is, the properties which relate the average values of the state variables in the composite. Thus, the ratio of the volume average of a stress component to the volume average of a strain component defines an effective elastic stiffness of the composite. When the effective properties of a unidirectional composite have been determined, either analytically or experimentally, the material may be viewed as a homogeneous anisotropic material for many aspects of the design process. Thus, properties of laminates formed from layers of unidirectional composites may be determined by methods developed for layered media, such as plywood. When the overall structural properties of a laminate or any other configuration of a fibrous material can be assessed, the problem of structural

design becomes that of design with anisotropic materials. Reliable analyses of anisotropic beam, plate and shell structures considering static and dynamic stresses and structural stability are available. On the other hand, much remains to be done, particularly in such areas as failure criteria, reliability assessment, etc.

It is well to note that, although the development of the so-called "advanced" composites has taken place primarily within the aerospace industry, even the most expensive of these new composite materials have found usage in the commercial market place. Continued reduction in fiber material costs, coupled with continued attention to sound design practices will undoubtedly lead to a substantial multiplication of the number of applications of these composites.

1.1 APPROACH TO STRUCTURAL DESIGN.

A key viewpoint in the approach to design with composites is the recognition that there are many more steps involved in a composite structural design process than there are in the equivalent metallic structural design process. The important factor in this composite design cycle is the recognition that materials design must be performed simultaneously with structures design. Thus the structural design engineer starts with a selection of constituents and a choice of the volume fractions of these two constituents. These selections define a unidirectional composite, which is the basic element of a composite structure. This unidirectional composite is characterized by certain effective properties which relate the average values of the state variables in much the same way that physical properties of homogeneous materials relate the local values of these state variables.

When available, high-strength, high-stiffness fibers are utilized to reinforce plastics, they yield composites which are very strong and stiff in the fiber direction and generally have very poor properties in the transverse directions. Thus, for the practical structural design where a multiplicity of loading directions and conditions must be considered, the elementary structure of this material is obtained by forming a laminated composite wherein layers of the unidirectional material are oriented in various directions. At this point, effective properties of the laminate can be determined. These define the properties of the basic material with which to perform structural design. It is at this stage, that the composite design cycle first reaches the starting point for structural design with isotropic metals. From this stage, the designer proceeds to the determination of the configuration of each of the structural elements and to the overall structural design configuration. In the case of composites, the availability of these many different materials enhances the opportunity to close the design loop by performing a structural efficiency analysis of the resulting structure and determining the designed improvements or changes in constituent properties which can generate improved performance for the composite structure.

1.2 LAMINATE STRUCTURAL ANALYSIS.

When the effective physical properties of a unidirectional composite are known, the material may be viewed as a homogeneous anisotropic (transversely isotropic or orthotropic) material. This representation is valid strictly only for homogeneous states of average stress and average strain. However, it appears reasonable to use the effective properties whenever the average stresses and

average strains vary slowly over a dimension which is large compared to a fiber cross-section dimension. Thus, when a laminate is formed from layers of unidirectional carbon fiber reinforced plastics, it is reasonable to treat each layer as an anisotropic continuum and study the laminate using layered plate theory. On the other hand, a similar procedure for boron fiber reinforced plastics might be viewed with some uncertainty, inasmuch as the layer thickness for this case is only slightly greater than the fiber diameter. Nevertheless, this anisotropic layered plate theory is in widespread use for all types of composites and represents a suitable starting point for this discussion. The structural analysis of a laminated material seeks to define for the basic laminate element, the relationship between the stress resultants, on the one hand, and the middle surface strains and curvatures on the other hand. For the elastic portion of the design problem, a general linear relationship between these quantities can be postulated. The constants in this relationship can be evaluated by utilizing the stress-strain relations for the individual layers and the Kirchhoff-Love assumptions of plate theory. Basically, this involves a transformation, of the stress-strain relations of each layer, from the layer principal axes to the laminate principal axes. This is followed by integration through the thickness to define the force and moment resultants in terms of the displacements and curvatures. This theory was described in detail in Volume I.

The results of this laminate analysis can be represented by a 6 X 6 matrix array of coefficients in the generalized force displacement relations. In general, the laminate does not have the symmetries of a homogeneous material, and as a result of this, all of the elastic coefficients in the matrix may be non-zero indicating coupling effects in the laminate. These effects are the generalization of the phenomenon found in a bi-metallic strip which exhibits a change of curvature, as well as a strain, when subjected to an inplane force. The constitutive relations may be written in the form:

$$\begin{Bmatrix} N_i \\ M_i \end{Bmatrix} = \begin{bmatrix} a_{ij} & b_{ij} \\ b_{ij} & d_{ij} \end{bmatrix} \begin{Bmatrix} \epsilon_j^0 \\ \kappa_j^0 \end{Bmatrix}$$

where N_i are the three stress resultants

M_i are the three moment resultants

ϵ_j^0 are the three reference surface strain components

κ_j^0 are the three change of curvature components

and a_{ij} , b_{ij} , and d_{ij} are the laminate properties. In these relations, the b_{ij} vanish for a homogeneous isotropic material with principal elastic axes, x_L and x_T . Thus in these cases, there is no coupling between bending and extension. However some forms of coupling do exist for even simple laminates. For example, for a laminate composed of an even number of identical orthotropic layers whose principal axis are alternately oriented at $+\theta$ to the x_L axis, there is coupling between the twisting curvature and the extensional forces; between the bending curvatures and the shear force; between the shear strain and the bending moments; and between the extensional strains and the twisting moment. When the same layers are alternated symmetrically about the middle surface, there is coupling between twisting curvature and bending moments, and between bending curvatures and twisting moments only. This is generally the minimum type of coupling found for laminates having orthotropic laminae oriented at other than 0° or 90° , although even this can be eliminated by certain special laminate configurations.

In both of the last two cases, the coupling terms become relatively smaller as the number of layers increases. The coupling effects introduce significant complications into the analysis of laminated plates. However, methods of solution utilizing plate and shell theories modified to incorporate these effects are available.

1.3 STRUCTURAL ELEMENT.

The laminate analysis described previously, defines the effective elastic properties which relate the force and moment variables to the displacement and curvature variables. With these properties known, the composite laminate generally may be treated as a homogeneous anisotropic material and existing analyses for such materials may be utilized.

Although the approach outlined to this point defines a method for analysis of most composite structures, several problems do remain. First, much of the literature on analysis of homogeneous anisotropic structures applies to orthotropic materials. Thus the coupling effects discussed above are not considered. Second, the existence of layers creates a possible problem due to interlaminar stresses. Additionally, the low shear stiffness of unidirectional composites results in an increase in the effects of transverse shear upon plate and shell behavior. These problems are the subject of current research and are described briefly herein.

1.4 STRUCTURAL DESIGN.

With methods of structural analysis available, there then exists the problem of structural design. Design procedures which are in use generally involve the automated repetitive use of structural analysis methods similar to those described herein. Thus, to design a laminate at a given point in a structure at which load and moment resultants are defined, one may consider a variety of possible laminates in terms of constituent properties and geometry of the phases. A search is made, usually in a semi-automated fashion, to determine which of the various possible laminates which satisfy the load requirements yields a minimum weight design. One of the problems with this procedure is that a significant amount of computer time is required in the search process. Another problem is, as the optimization of the configuration is performed on a point by point basis, it may be that the resulting configurations will not provide overall compatibility or continuity of pattern which are, of course, required for reasonable fabrication techniques. It has also been shown that it is possible to integrate the requirements from point to point and yield an overall configuration which does the best job of satisfying requirements in various points on the structure by limiting the number of laminae orientations. The use of these computer procedures enables the designer to generate a composite laminate configuration which satisfies all the design constraints and is a reasonable representation of an optimized configuration.

1.5 JOINT ANALYSIS DESIGN.

Given the capability to analyze structural elements, the remaining gap is the need to analyze and design attachments between them. The complex response of composite structural elements in the vicinity of stress concentrations elevates the treatment of joints to a major sub-division of the structural analysis and design problem.

1.6 STRUCTURAL EFFICIENCY.

With the existence of a variety of possible structural designs for a given application, the designer reaches the stage where he requires the ability to assess the relative merits of different types of structural configurations. In the case of composites, the differences between various potential configurations include changes in material configuration; thus, variation of fiber orientation patterns is equivalent to a change in the material of construction. The choice of the material of construction for any structural application is, of course, a complex decision. One major factor contributing to this choice is structural weight. In the case of relatively expensive materials, (such as advanced fiber composite materials), the cost of the material used can have an important influence upon the cost effectiveness of the structural component. In the case of laminated composite materials, the reduction in amount of material used can also have a significant effect upon fabrication costs. Additionally for applications in the transportation field, reduction in material weight can provide performance improvement (e.g. greater payload, range, speed, etc.). For all these reasons, structural weight is an important criterion for judging the relative merits of various materials. For composites, the choice of material cannot be made on the basis of any simple parameter, such as modulus to density ratio. A more thorough investigation of material performance is required.

The concept of structural efficiency provides a useful approach to the evaluation of structural weight as a function of design conditions as well as of material properties. In this approach, a measure of structural weight is determined as a function of the structural index, (a quantity which is a function only of the design conditions - such as load and overall structural dimension). The general result of a structural efficiency analysis of a particular component is the definition (usually displayed graphically) of the minimum structural weight for each material considered as a function of structural loading intensity (structural index). Thus the best material for each range of design conditions can be defined readily. Examples are presented herein along with an overview of the methodologies.

1.7 CONCLUDING REMARKS.

Rational methods for analysis of composite materials and structures do exist. This book may serve to guide new workers in the field of composites. Many basic problems remain unsolved. However, there is a continually decreasing need to resort to the analyses based on the sometimes crude and approximate assumptions made in earlier years.

The understanding of the relationship between physical properties of laminated fibrous composites and those of their constituents rests on a sound base. On the other hand, much remains to be done; particularly in such areas as: failure criteria, reliability assessment, non-destructive evaluation, damage tolerance, fatigue creep, and joints and attachment problems.

It appears that composite materials are only beginning to have the anticipated impact on structural design which will lead to creative, new structural configurations. One need not be disappointed in this fact, for we are only at the beginning of the new materials engineering era. Time to come will undoubtedly see changes in structural design methods that will lead to widespread

utilization of composites. The improvements in properties; the existence of unique combinations of properties; and the fabrication simplifications which can result from the use of fibrous composites will lead to their utilization as the prominent materials of construction in all engineering disciplines.

CHAPTER II. STRUCTURAL ELEMENTS

2.0 INTRODUCTION.

The analytical methodology for treatment of composite structural elements is presented separately for each principal load carrying form. Thus, basic principles are described first for beams, and then for plates and for shells.

2.1 MATERIAL PROPERTIES.

An essential factor in a discussion of design with composites is the recognition that there are many more steps involved in a composite structural design than there are in the equivalent metallic structural design process. This design cycle for composite structures is illustrated in figure 2.1.1. The most important factor in this composite design cycle is the recognition that materials design must be performed simultaneously with structures design. Thus, the structural design engineer starts with a selection of constituents and a choice of the volume fractions of these two constituents. These selections define a unidirectional composite, which is the basic element of a composite structure. This unidirectional composite is characterized by certain effective properties which relate the average values of the state variables in much the same way that physical properties of homogeneous materials related the local values of these state variables.

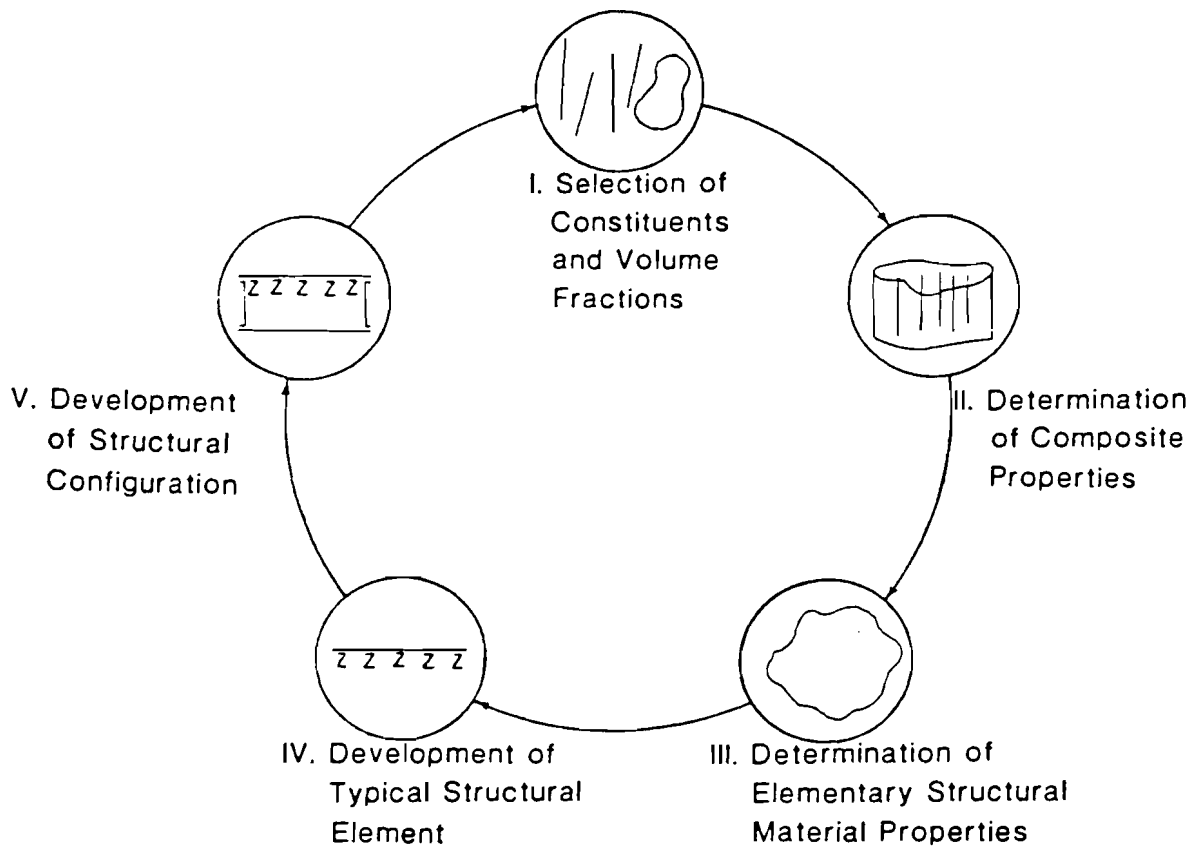


FIGURE 2.1.1 THE DESIGN CYCLE FOR COMPOSITE STRUCTURES INTRODUCES THE REQUIREMENT FOR DESIGNING MATERIAL AS WELL AS STRUCTURES

When contemporary high-strength, high-stiffness fibers are utilized to reinforce plastics, they yield composites which are very strong and stiff in the fiber direction and generally have very poor properties in the transverse directions. Thus, for the practical structural design where a multiplicity of loading directions and conditions must be considered, the elementary structure of this material is obtained by forming a laminated composite wherein layers of the unidirectional material are oriented in various directions. At this point effective properties of the laminate can be determined. These define the properties of the basic material with which to perform structural design. It is at this stage, that the composite design cycle first reaches the starting point for structural design with isotropic metals. From this stage, the designer proceeds to the determination of the configuration of each of the structural elements and to the overall structural design configuration. In the case of composites, the availability of these many different materials enhances the opportunity to close the design loop by performing a structural efficiency analysis of the resulting structure and determining the desired improvements or changes in constituent properties which can generate improved performance for the composite structure. The first steps in the design cycle have been treated in Volume I. Certain essential concepts are reviewed below. The design steps are treated in detail in the body of this text.

2.1.1 Effective Properties Concepts.

Certain definitions are of importance in the discussion of composites, particularly in the treatment of anisotropy and heterogeneity. Materials whose properties at a point vary in different directionals are anisotropic; those with properties which vary from point to point are heterogeneous. On a small enough scale even the commonly considered homogeneous materials are inhomogeneous. That is, the common structural metals, on a small scale, consists of crystals. Each of these crystals is anisotropic. Its properties in different directions are different and a group of polycrystals randomly oriented represents a heterogeneous material, since the properties in a given direction vary from point to point. Thus, there are two concepts, heterogeneity and anisotropy, which are pertinent to the study of composite materials. A composite can be one or both or neither of these, depending upon the constituents and the scale of interest. For example, consider a fiber composite material; that is, a mixture of fibers contained in a matrix material which binds the fibers together. The two phases may individually be isotropic or anisotropic materials. When the fibers are oriented within the matrix - for example, a set of filaments, all parallel to a given line, embedded in an otherwise isotropic and homogeneous matrix - the composite material is heterogeneous but isotropic. That is, the properties differ. This is on a small scale; however, for contemporary filaments whose cross-sectional dimensions are extremely small, practical interest focuses on the average of stresses and strains over a dimension which is large compared to this cross-sectional dimension. For that purpose it is possible to consider the materials response in an average way to be anisotropic but homogeneous; that is, one may consider a material which has the same average properties as a given fiber composite material. This new average or "effective," material will have properties in the fiber direction which differ substantially from those in a direction transverse to the filaments. This makes it an anisotropic material. Since one is not concerned with the local perturbations associated with the individual filaments, this may be considered to be a homogeneous material. This replacement of actual by effective material provides the transition from

micromechanics to macromechanics. On a microscopic scale, the composite is a heterogeneous isotropic material; whereas on a macroscopic scale, it is an anisotropic but homogeneous material. In general, the problem can be compounded by recognition that the individual phases by themselves may be anisotropic, and secondly, that the geometry of the phases may be such as to produce a great variety of macroscopic anisotropies.

The concept of designing a material to yield a desired set of properties has received impetus from the growing acceptance of composite materials. This utilization of the diversity of contemporary high strength and high stiffness fibers in various structural applications has motivated a new interest in the study of relationships between the mechanical and physical properties of composites and those of their constituents. The aims of such studies are, first, to provide the ability to analyze the performance of structures utilizing these heterogeneous materials, and second, to provide guidelines for the development of improved materials.

Inclusion in the structural design process of the material design phase has had a significant impact upon the entire design process; particularly upon the preliminary design phase. In this preliminary design, the number of materials which may be considered for a design generally will include many for which experimental materials properties data are not available. Thus, preliminary material selection may be based on analytically predicted properties. These analytical methods are the result of studies of the relationship between effective properties of composites and the properties of their constituents (studies which are generally denoted as "micromechanics"). The understanding of the relationships between the overall or average response of a composite and the properties of its constituents permits the representation of the inhomogeneous composite by an effective homogeneous (and generally, anisotropic) material. The properties of this homogeneous material are the effective properties of the composite; that is, they are the properties which relate the average values of the state variable in the composite. When the effective properties of a unidirectional composite have been determined, the material may be viewed as a homogeneous anisotropic material for many aspects of the design process.

2.1.2 Laminate Structural Analysis.

When the effective physical properties of a unidirectional composite are known, the material may be viewed as a homogeneous anisotropic (transversely isotropic or orthotropic material). This representation is valid strictly only for homogeneous states of average stress and average strain. However, it appears reasonable to use the effective properties whenever the average stresses and average strains vary slowly over a dimension which is large compared to a fiber cross-section dimension. Thus, when a laminate is formed from layers of unidirectional glass fiber reinforced plastics, it is reasonable to treat each layer as an anisotropic continuum and study the laminate using layered plate theory. On the other hand, a similar procedure for boron fiber reinforced plastics might be viewed with some uncertainty, inasmuch as the layer thickness for this case is only slightly greater than the fiber diameter. Nevertheless, this anisotropic layered plate theory is in widespread use for all types of composites and represents a suitable starting point for this discussion. The structural analysis of a laminated material seeks to define for the basic laminate element, the relationship between the stress resultants and the moment

resultants, on the one hand, and the middle surface strains and curvatures on the other hand. For the elastic portion of the design problem, a general linear relationship between these quantities can be postulated. The constants in this relationship can be evaluated by utilizing the stress-strain relations for the individual layers and the Kirchhoff-Love assumptions of plate theory. Basically, this involves a transformation, of the stress-strain relations of each layer, from the layer principal axes to the laminate principal axes. This is followed by integration through the thickness to define the force and moment resultants in terms of the displacements and curvatures. This theory was described in detail in Volume I.

The results of this laminate analysis can be represented by a 6 X 6 matrix array of coefficients in the generalized force displacement relations. In general, the laminate does not have the symmetries of a homogeneous material, and as a results of this, all of the elastic coefficients in the matrix may be non-zero indicating coupling effects in the laminate. These effects are the generalization of the phenomenon found in a bi-metallic strip which exhibits a change of curvature, as well as a strain, when subjected to an in-plane force.

$$\begin{Bmatrix} N_i \\ M_i \end{Bmatrix} = \begin{bmatrix} a_{ij} & b_{ij} \\ b_{ij} & d_{ij} \end{bmatrix} \begin{Bmatrix} \epsilon_j^0 \\ \kappa_j^0 \end{Bmatrix}$$

where N_i are the three stress resultants

M_i are the three moment resultants

ϵ_j^0 are the three reference surface strain components

κ_j^0 are the three change of curvature components

and a_{ij} , b_{ij} , and d_{ij} are the laminate properties. In these relations the b_{ij} vanish for a homogeneous material or an orthotropic material with principal elastic axes, x_L and x_T . Thus, in these cases, there is no coupling between bending and extension. However, some forms of coupling do exist for even simple laminates. For example, for a laminate composed of an even number of identical orthotropic layers whose principal axes are alternately oriented at $+\theta$ and $-\theta$ to the x_L axis, there is coupling between the twisting curvature and the extensional forces; between the bending curvatures and the shear force; between the shear strain and the bending moments; and between the extensional strains and the twisting moment.

When the same layers are alternated symmetrically about the middle surface, there is coupling between twisting curvature and bending moments, and between bending curvatures and twisting moments only. This is generally the minimum type of coupling found for laminates having orthotropic laminae oriented at other than 0° or 90° , although even this can be eliminated by certain special laminate configurations.

The laminate analysis defines the effective elastic properties which relate the force and moment variables to the displacement and curvature variables. With these properties known, the composite laminate generally may be treated as a homogeneous anisotropic material and existing analyses for such materials may be utilized.

2.2 BEAMS.

2.2.1 Introduction.

Fiber composite beams are utilized in many structures and they are also being considered for use in an increasing number of new applications. For design and analysis purposes many other fiber composite members may be treated as an assemblage of such beam elements. A propeller (or turbine) blade is an example. Figure 2.2.1 shows the design concept (a cross-section is shown) of a propeller blade using stacked cross-ply material and a honeycomb core material leading to a sandwich construction. Special design consideration is given to the leading edge. Similar blades may be manufactured by the process of filament winding over a core material. New production processes and improvements in adhesive bonding properties are making possible innovative design concepts of beams of constant and varying cross-sections.

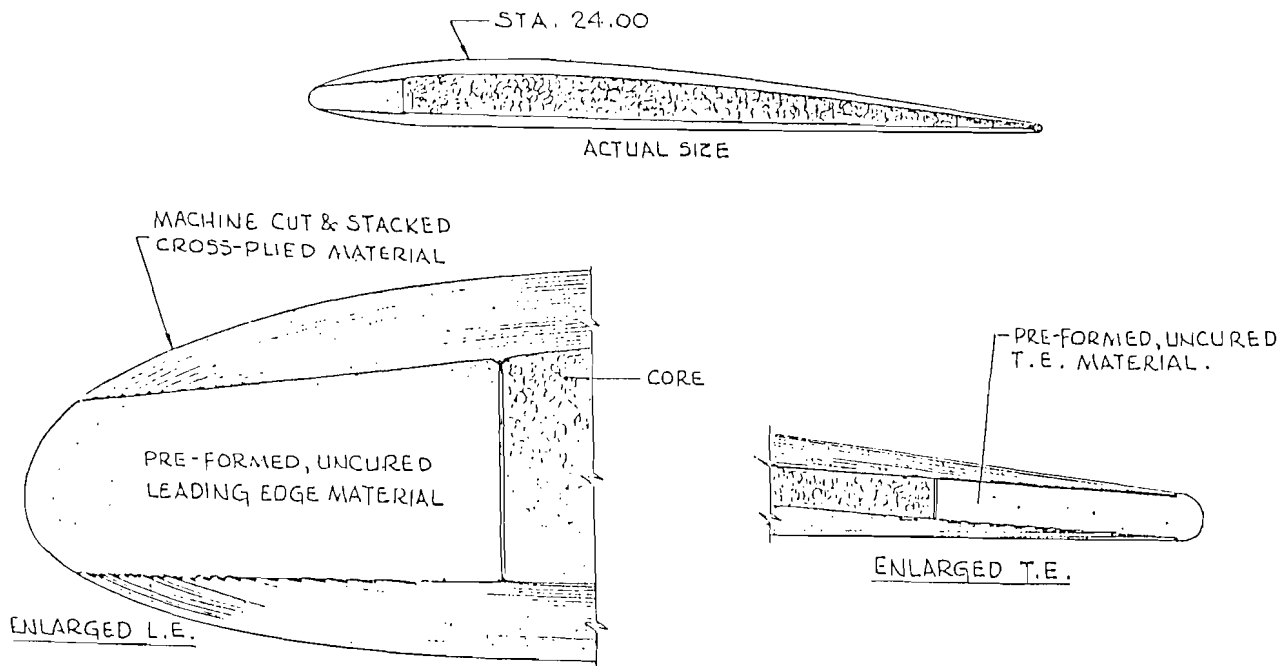


FIGURE 2.2.1 A DESIGN FOR COMPOSITE PROPELLER BLADE

Fiber composite beam theory is useful for many material characterization studies (reference 2.2.1). The tensile coupon is a beam carrying only an axial force. Torsion of a tube is often used for determining the shear modulus and strength of unidirectional and laminated composites. The short beam flexure test gives a measure of interlaminar shear properties. A double cantilever beam test is now being used widely for interlaminar fracture characterization studies. Some of these applications will be considered in later sections.

By definition beams are structural members subjected to forces perpendicular to their axes. In many applications they also carry axial forces as well as twisting moments or torques. Both straight and curved beams are used as structural components. However, we will mainly discuss straight beams and briefly touch upon curved members. The length or span of a beam is quite large as compared to the lateral dimensions of its cross-section and because of this fact, simple stress states exist over a large part of such beams except possibly near the supports. As discussed in many texts of strength of materials various types of idealized supporting conditions like clamped, free or simple supports are possible. In practice such idealized conditions are sometimes difficult to achieve and complex states of stress may exist there. However, for design of the beam cross-section such disturbances are usually not considered and special considerations are given to design details near supports. As in ordinary beam theory the dominant stress in a fiber composite beam is in the direction of beam axis unless shear forces or twisting moments are present. Shear stresses can have significant influence on failure and deflection as will be discussed later.

Following the standard strength of materials approach (see reference 2.2.2) we will first consider "planar" problems of fiber composite beams in which three internal force components, e.g., axial and shear forces and a bending moment are recognized at any cross-section. We will also consider the case of twisting later.

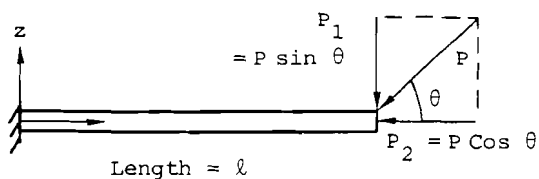
A beam can be statically determinate or indeterminate. In the first case the supporting forces or moments can be calculated by using the three equations of planar equilibrium, i.e.

$$\begin{aligned}\Sigma F_x &= 0 \\ \Sigma F_y &= 0 \\ \Sigma M &= 0\end{aligned}\tag{2.2.1}$$

For indeterminate beams consideration has to be given to displacements at specific locations. We will assume that the reader is familiar with such problems and to the application of method of sections for obtaining shear (or axial) force and bending moment diagrams. Some simple examples are given in figure 2.2.2. We have employed the sign convention shown in figure 2.2.2d which is similar to laminated composite plate theory. The following differential equations govern the equilibrium under shear forces (Q) and moments (M) when this convention is employed.

$$\frac{dM}{dx} = Q\tag{2.2.2}$$

$$\frac{dQ}{dx} + p = 0$$



Axial Force

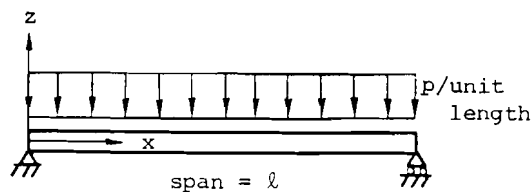


Shear Force

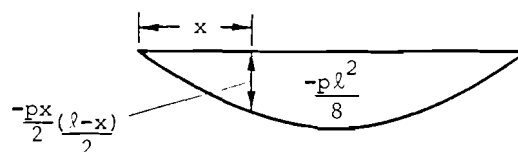
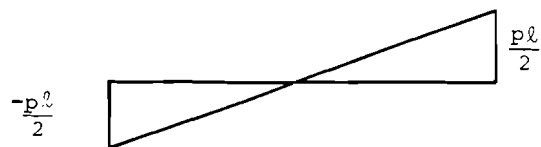


Bending Moment

a. TIPOLOADED CANTILEVER

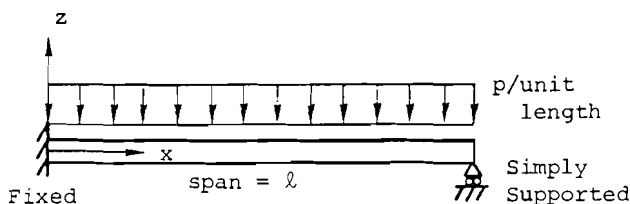


Axial Force = 0

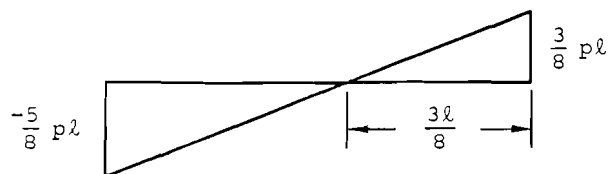


Bending Moment

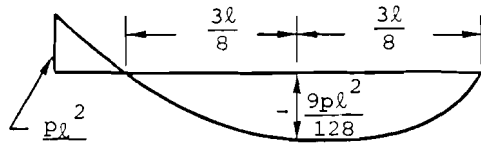
b. SIMPLY SUPPORTED BEAM, UNIFORMLY DISTRIBUTED LOAD



Axial Force = 0

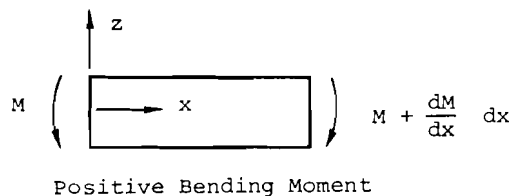
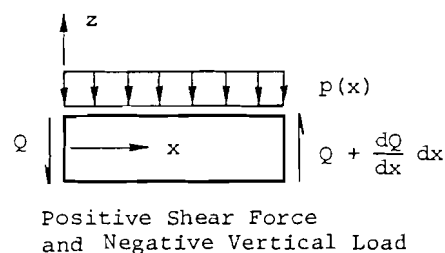
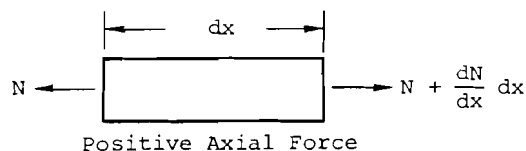


Shear Force



Bending Moment

c. TIP SUPPORTED CANTILEVER UNIFORMLY DISTRIBUTED LOAD



d. SIGN CONVENTION

FIGURE 2.2.2 EXAMPLES OF FORCE AND MOMENT DIAGRAM

p being the intensity of prescribed vertical load in positive z direction. The axial force is constant unless forces are applied on the beam element in that direction.

Figures 2.2.2a and 2.2.2b show two common determinate problems whereas 2.2.2c shows the indeterminate problem of a tip supported cantilever of constant cross-section. Shear forces in the beam are not zero, but the effect of these forces on deflections are not considered here for determining force and moment distributions. Standard flexure theory has been applied to this problem. Effects of shear deformations on deflections will be considered later.

2.2.2 Pure Bending of Homogeneous Orthotropic Beams.

In "planar" problems of straight beams of constant cross-section, the plane is usually assumed to contain a vertical axis of symmetry. Applied loads on the beam and the bending moment at any cross-section lie in that plane which passes through the beam axis and the axis symmetry of the cross-section (figure 2.2.3). As in the standard beam bending problem a segment of a fiber composite beam may carry a constant bending moment only (problem of pure bending). In many cases, especially when the beam is made of an unidirectional fiber reinforced material with the fibers parallel to the axis of the beam the following assumptions used in standard theory are fairly accurate.

1. Bernoulli Euler Hypothesis which states that plane sections normal to the beam axis remain plane and normal to deflected axis (see deformed position C'D', E'F', in figure 2.2.3d of CD and EF in figure 2.2.3b).
2. Hooke's Law holds, i.e. axial stress at any point is proportional to axial strain. Other two extensional stresses are zero or can be neglected.

Because of assumption 1 the displacement u in x-direction and strain ϵ for a fixed value of x varies linearly with depth and there will be a neutral plane whose intersection with yz plane is called the neutral axis where displacements and strains are zero. Without any loss in generality we assume that the beam axis (x-axis) lies on this plane (see figure 2.2.3b).

If we consider the beam to be made of a homogeneous orthotropic material then it follows from assumption (2.2.2) that the distribution of stress σ_{xx} through the depth (variation in z-direction) is also linear and the flexure formula holds

$$\sigma_{xx}(z) = \frac{Mz}{I} \quad (2.2.3)$$

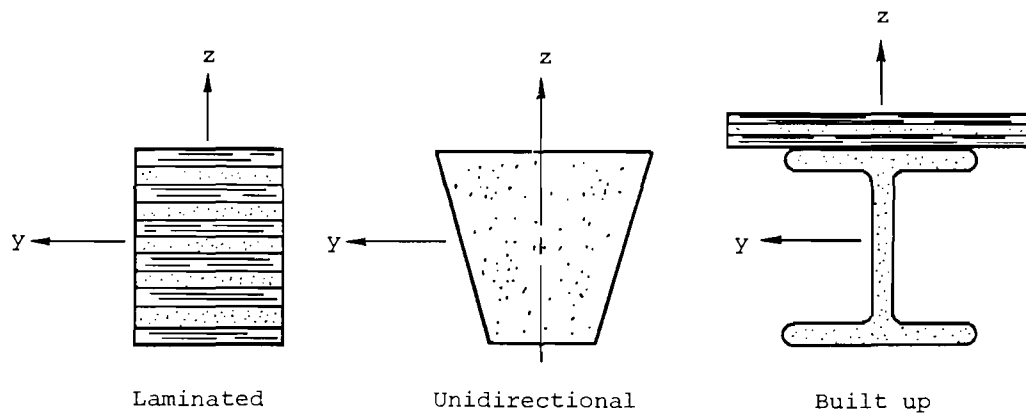
where

$$I = \int_C z^2 dC = \text{Moment of Inertia of the cross-section C about the neutral axis} \quad (2.2.4)$$

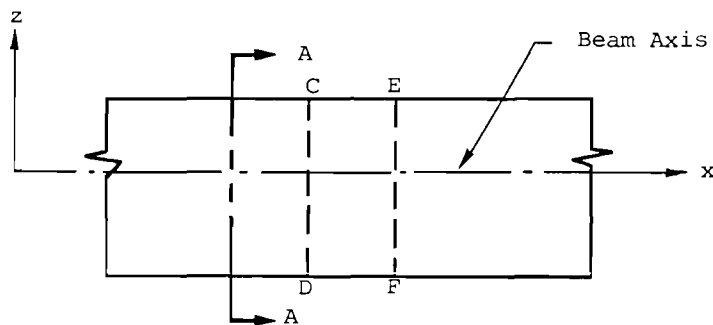
The location of the neutral axis has to be found from the condition

$$\int_C z dC = 0 \quad (2.2.5)$$

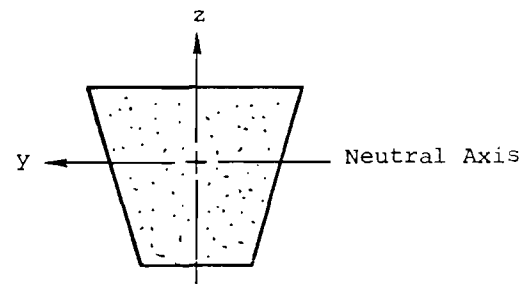
since there is no resultant force acting on the cross-section.



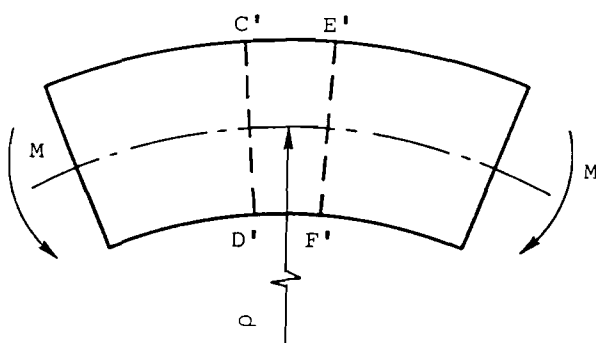
a. VERTICAL AXIS OF SYMMETRY OF BEAM CROSS-SECTION



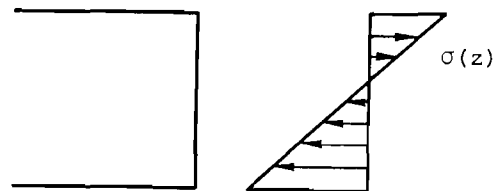
b. UNDEFORMED BEAM



c. VIEW OF SECTION AA



d. DEFORMATION - BERNOULLI-EULER THEORY



e. STRESS DISTRIBUTION IN A HOMOGENOUS BEAM

FIGURE 2.2.3 BENDING IN STRAIGHT BEAMS OF CONSTANT CROSS-SECTION

When the deformations are not too large, then the strain curvature and moment-curvature relation are

$$\frac{\epsilon_{xx}}{z} = \frac{1}{\rho} = \kappa = -\frac{d^2w}{dx^2} \quad (2.2.6)$$

where ρ and κ are the radius of curvature (figure 2.2.3d) and the curvature respectively. By Hooke's law (assumption 2.2.2) one has

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E_x} \quad (2.2.7)$$

E_x being the Young's modulus in x-direction. It should be noted that laminated composites containing many repeated $\pm\theta$ layups (θ being the inclination of fiber direction of a lamina to x-axis, plane of laminations being perpendicular to z-axis) or repeated (0/90/ $\pm\theta$) layups E_x should be taken equal to the effective modulus in x-direction. We will discuss this approximate method of homogeneization later.

From eqs. (2.2.6), (2.2.7) and (2.2.3) one obtains the differential equation governing the deflection curve which is

$$\frac{d^2w}{dx^2} = -\frac{M}{E_x I} \quad (2.2.8)$$

Although these equations are obtained on the assumption of pure bending (shear force = 0) they usually yield a good estimate of deflections in beams for large span to depth ratio. Thus, the vertical deflection at the tip loaded cantilever (figure 2.2.2a) can be obtained by substituting

$$M = P_1(L-x) \quad (2.2.9)$$

in eq. (2.2.8) and using the conditions

$$w = \frac{dw}{dx} = 0 \text{ at } x=0 \quad (2.2.10)$$

the result is

$$w = -\frac{P_1 x^2}{6E_x I} (3L-x) \quad (2.2.11)$$

We note, therefore, that many results from standard strength of materials are applicable to homogeneous orthotropic composites provided the span is large. It should be noted however that because of the nature of their construction unidirectional or laminated composites have low shear strength as compared to their flexural strength. Therefore shear stresses should be given due consideration in designing such beams.

2.2.3 Shear Stresses in Thin Homogeneous Beams.

A good estimate of shear stresses in homogeneous (or homogenized) composite beams of small width can be obtained from the concept of shear flow q (see figures 2.2.4a,b) and the assumption of constant shear stress τ through the width w for any value of z . It follows from the equilibrium of the shaded element (reference 2.2.1).

$$q = \frac{dM}{dx} \frac{C^* \bar{z}}{I} \quad (2.2.12)$$

and by using eq. (2.2.2)

$$\tau(z) = \frac{q}{b} = \frac{QC^* \bar{z}}{Ib} \quad (2.2.13)$$

where C^* is the area of the shaded cross-section (above z), $\bar{z}$ is its centroid and Q is the shear force at a cross-section.

A similar formula is applicable for calculation of shear flow in flanges and webs of beams of the type shown in figure 2.2.4c.

For a beam of a rectangular cross-section of height h and width b , eq. (2.2.13) reduces to the well known formula

$$\tau = \frac{3}{2} \frac{Q}{bh} \left[1 - \frac{4z^2}{h^2} \right] \quad (2.2.14)$$

which is very accurate for small values of b . Noting that the maximum bending moment and shear forces in a three-point loaded beam (figure 2.2.4d) are

$$M_{\max} = PL/4 \quad (2.2.15)$$

$$Q_{\max} = P/2$$

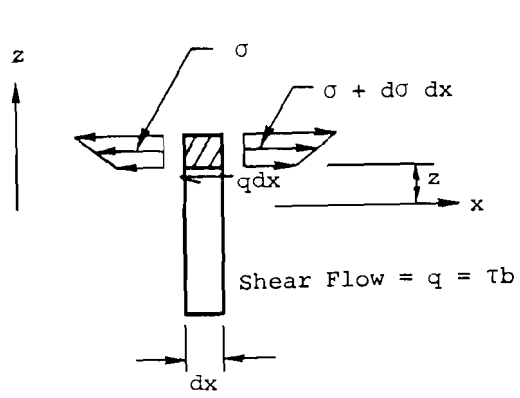
and using eq. (2.2.14), eq. (2.2.3) and $I = \frac{1}{12} bh^3$ one has

$$\sigma_{\max} = \frac{3}{2} \frac{PL}{bh^2} \quad (2.2.16)$$

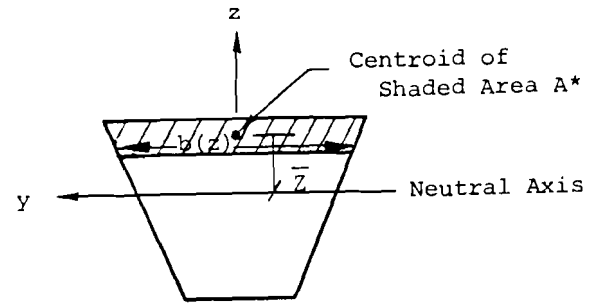
$$\tau_{\max} = \frac{3}{4} \frac{P}{bh}$$

and therefore,

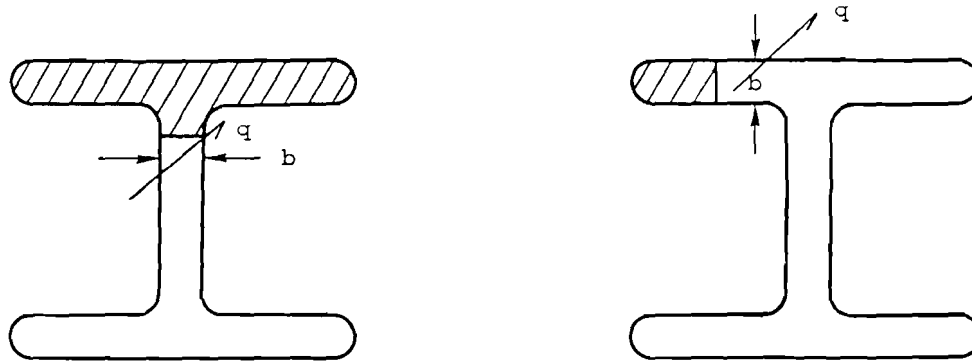
$$\tau_{\max} / \sigma_{\max} = h/2L$$



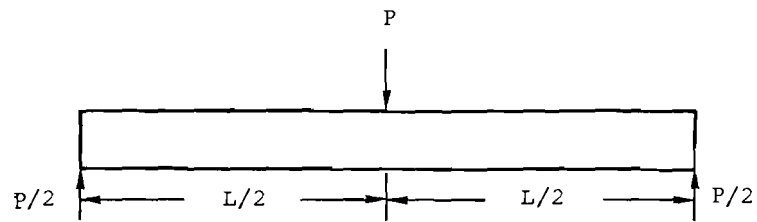
a. ELEMENT OF LENGTH dx



b. SECTION



c. SHEAR STRESSES IN BEAMS WITH THIN FLANGES AND WEBS



d. A BEAM UNDER 3-POINT BENDING

FIGURE 2.2.4 SHEAR STRESSES IN HOMOGENEOUS BEAMS

For many unidirectional composites, failure stress in shear is about 5% of the tensile strength in the fiber direction. Therefore for a $L/h = 40$, $\tau_{\max} = \sigma_{\max}/80$ and failure will invariably be in compression or tension. On the other hand for $L/h = 6$, $\tau_{\max} = \sigma_{\max}/12$ a high possibility of shear failure exists. Between these two configurations various combinations of failure modes under combined stress states are possible as illustrated in figure 2.2.5. Effect of concentrated forces at loading points has been ignored to permit clarity in this figure, but it is discussed in a later section.

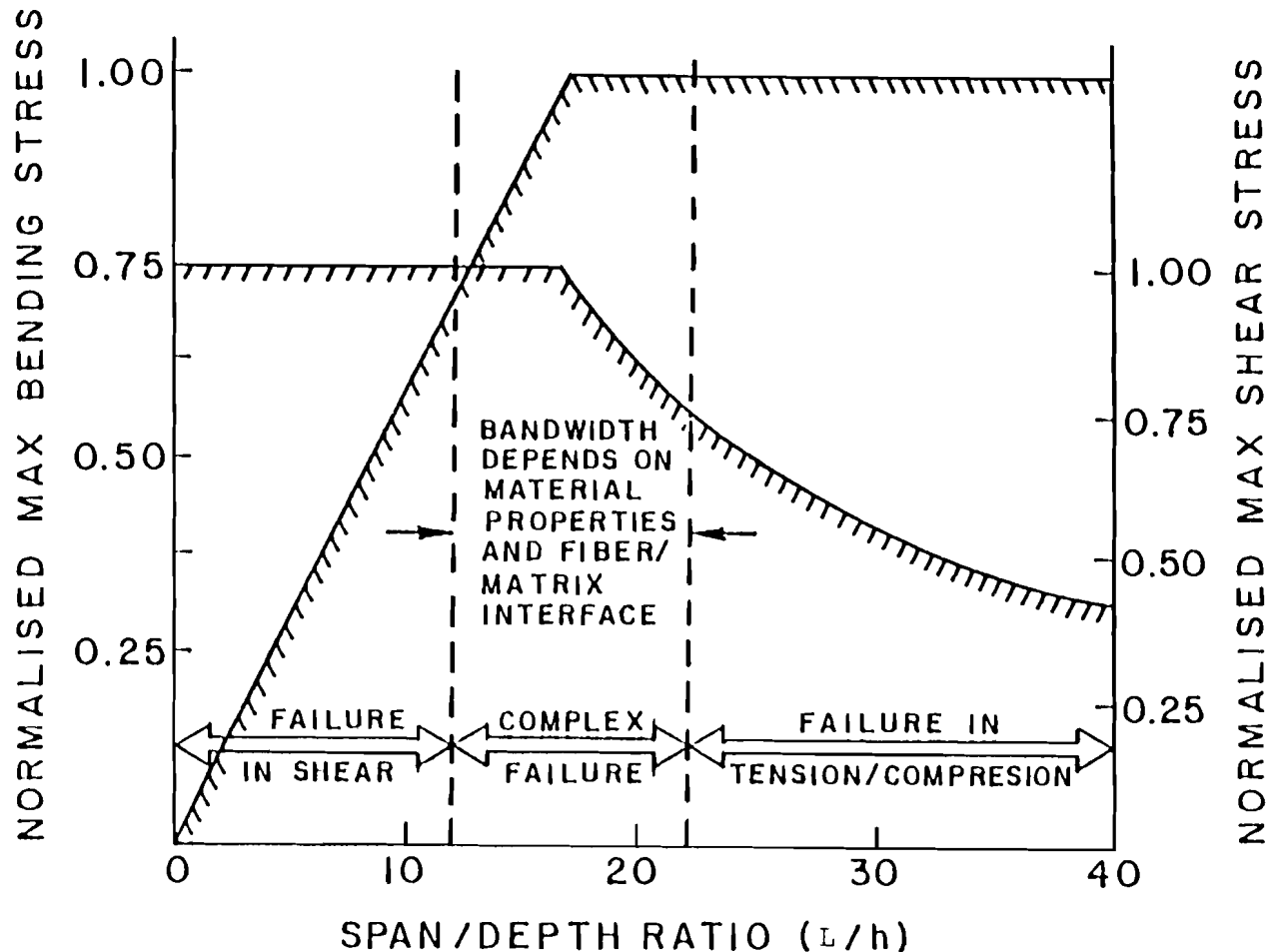


FIGURE 2.2.5 THE FAILURE MODE OF SIMPLE COMPOSITE BEAMS IN FLEXURE VS. SPAN-TO-DEPTH RATIO

2.2.4 Shear Deformations in Thin Homogeneous Beams.

Slope of the deflection curve of a beam not only depends on the rotation of the cross-section due to bending but also on shear. Due to shear stresses plane cross-sections perpendicular to neutral plane do not remain plane and perpendicular to that plane. Actual deformations in a beam element (figure 2.2.6a) due to flexure alone and including the effects of shear are shown in figures 2.2.6b,c, respectively. However, in a majority of cases a good simplifying assumption is that plane section remain plane, but not necessarily perpendicular to the neutral plane. Measuring z from the neutral plane one can write the displacements u_x and u_z in the following form

$$u_x = z\psi(x) \quad (2.2.17)$$

$$u_z = w(x)$$

where ψ is the rotation due to flexure.

With the following additional assumptions

$$\sigma_{yy} = \sigma_{zz} = 0 \quad (2.2.18)$$

it follows that the important strain and stress components are

$$\epsilon_{xx} = z \frac{\partial \psi}{\partial x} \quad (2.2.19a)$$

$$\sigma_{xx} = E_x z \frac{\partial \psi}{\partial x} \quad (2.2.19b)$$

$$\gamma_{xz} = \psi + \frac{\partial w}{\partial x} \quad (2.2.19c)$$

$$\tau_{xz} = G_{xz} \gamma_{xz} \quad (2.2.19d)$$

where γ_{xz} and τ_{xz} are the shear strain and stress at the neutral plane, G_{xz} is the xz shear modulus and E_x is the Young's Modulus in x -direction. Multiplying eq. (2.2.19d) by z , integrating over the cross-section and utilizing the shear stress distribution due to shear force Q one obtains

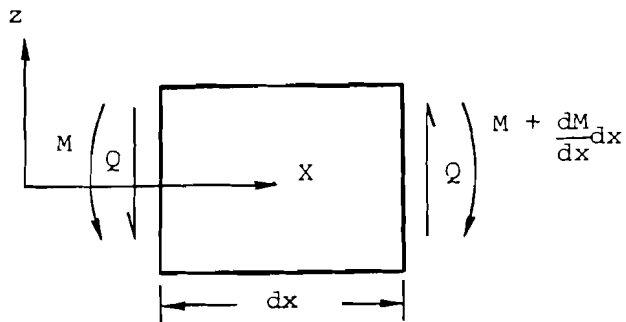
$$M = \int_c z \sigma_{xx} dC = D \frac{\partial \psi}{\partial x} \quad (2.2.20)$$

$$Q = k K \left(\psi + \frac{\partial w}{\partial x} \right)$$

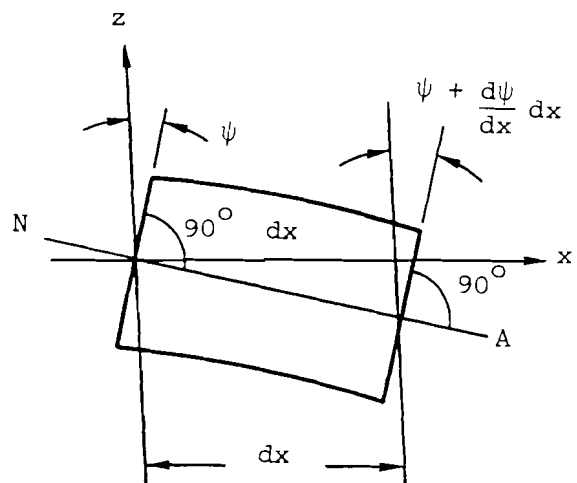
where

$$D = \int_c E_x z^2 dC = E_x I$$

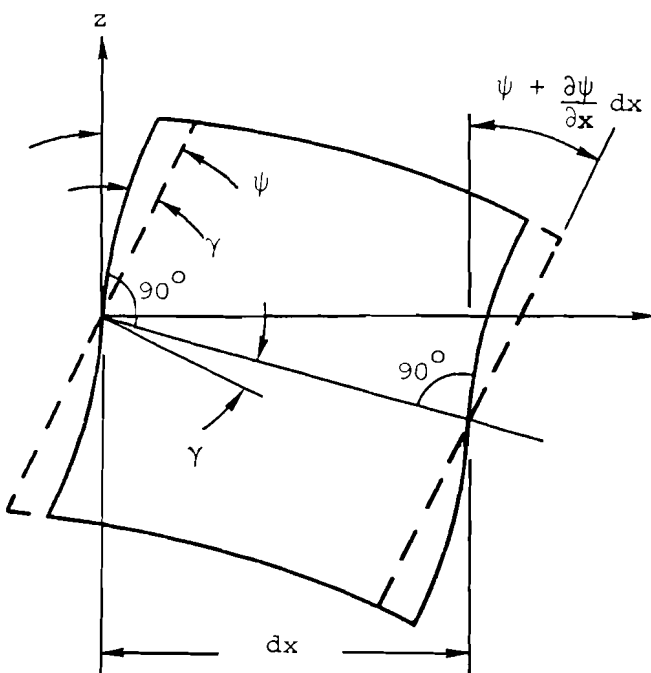
$$K = G_{xz} C$$



a. BEAM ELEMENT



b. DEFORMATION DUE TO FLEXURE



c. DEFORMATION INCLUDING EFFECTS OF SHEAR

Note: Dotted lines indicate Assumed Field for the Assumption for Plane Sections Remaining Plane

FIGURE 2.2.6 DEFORMATIONS IN BEAM ELEMENTS DUE TO FLEXURE ALONE AND FLEXURE INCLUDING EFFECTS OF SHEAR

C is the cross-sectional area and k is the shear coefficient to adjust for the nonuniform shear stress distribution. For a rectangular cross-section of width b and height h it follows that

$$D = E_x bh^3/12 \quad (2.2.21)$$

$$K = G_{xz} bh$$

Utilizing eq. (2.2.14) one can show that

$$k = 2/3 \quad (2.2.22a)$$

Instead of treating γ_{xz} as the shear strain at the midplane, one can use the Reissner's formulation (reference 2.2.3) in terms of complimentary energy (this is discussed in a later section) and obtain

$$k = 5/6 \quad (2.2.22b)$$

For the tip loaded cantilever (figure 2.2.2b) the shear force is $-P_1$ and the moment is given by eq. (2.2.9). Substituting in the first two equations of (2.2.20) one obtains

$$\frac{\partial \psi}{\partial x} = -\frac{P_1}{D} (L-x) \quad (2.2.23)$$

$$\text{and } \psi + \frac{\partial w}{\partial x} = -\frac{P_1}{kK}$$

Integrating eq. (2.2.23) and using the following conditions

$$\psi = w = 0 \text{ at } x = 0 \quad (2.2.24)$$

one obtains

$$\psi = -\frac{P_1}{D} \left[Lx - x^2/2 \right] \quad (2.2.25)$$

$$w = -\frac{P_1 x}{kK} - \frac{P_1 x^2}{6D} (3L-x)$$

Comparing eq. (2.2.25) with (2.2.11) one may note that the first term in second of eq. (2.2.25) is the additional deformation due to shear. At the loaded end the downward deflection is increased by a factor

$$f = 1 + \frac{3D}{kKL^2} \quad (2.2.26a)$$

as compared to that when shear deformation is neglected. For a rectangular beam with $k = 5/6$ [see eq. (2.2.22b)]

$$f = 1 + \frac{3}{10} \frac{E_x}{G_{xz}} \frac{h^2}{L^2} \quad (2.2.26b)$$

In case of a steel beam with Poisson's ratio $\nu = 0.3$, $E_x/G_{xz} = 2.6$

$$f = 1 + 0.78 \frac{h^2}{L^2} \quad (2.2.26c)$$

where as for a unidirectional graphite/epoxy composite beam $E_x/G_{xz} \approx 20$ and

$$f \approx 1 + 6 \frac{h^2}{L^2} \quad (2.2.26d)$$

Thus, for steel and composite beams with a span to depth ratio $L/h = 5$ the tip deflections are increased by about 3% and 24%, respectively, due to the effects of shear deformation. The large increase in shear deformation effect in the composite beam is due to the large ratio E_x/G_{xz} .

It should be noted that the approach is strictly applicable to beams with a arbitrary cross-section which is thin and with a depth that is also small compared to the span [see assumption (2.2.18)]. However, the results are often accurate for a wider class of cross-sections.

2.2.5 Orthotropic Beams of Other Cross-Sections.

Flexural stress σ_{xx} , bending rotation ψ and deflection w can be obtained by the use of eqs. (2.2.3 to 2.2.5) and eq. (2.2.20). However shear stress distribution and shear coefficients for beams of constant cross-section due to a constant shear force (as in a tip loaded cantilever) must be obtained by modifying the approach given in reference 2.2.4; which is based on two dimensional isotropic elasticity theory. We will not pursue the details of the methods of analyses (which can be found in references 2.2.5, 2.2.6, and 2.2.7) but present some results useful for practical applications.

The problem which has been considered by a number of investigators is that of a rectangular cross-section of height h and width b . For beams which are not thin the shear stress τ_{xz} in the cross-section varies not only in the thickness direction z , but also in the width direction y . Also the shear stress τ_{yz} is not zero. Figure 2.2.7 shows the variation of shear stress τ_{xz} at the neutral plane in unidirectional graphite epoxy and glass epoxy beams. The stress is lowest at the plane of symmetry ($y = 0$) and is highest at the sides ($y = \pm b/2$). The peak values increase with $b/2$ as well as lower anisotropy (E_{xx}/G_{xz}) in elastic properties. Thus glass/epoxy material shows higher stresses at $y = \pm b/2$ as compared to a graphite/epoxy system. This is in contrast to stress concentration near holes in an anisotropic plate which tends to increase with the degree of anisotropy. Variation of shear stress in a $\pm 45^\circ$ graphite/epoxy system is shown in figure 2.2.8 as compared to that in a unidirectional material. It is clear that stresses at $y = \pm b/2$ are much higher in the $\pm 45^\circ$ material. A simple way to calculate the stress τ_{xz} at $z = 0$ and $y = \pm b/2$ is to use the relation

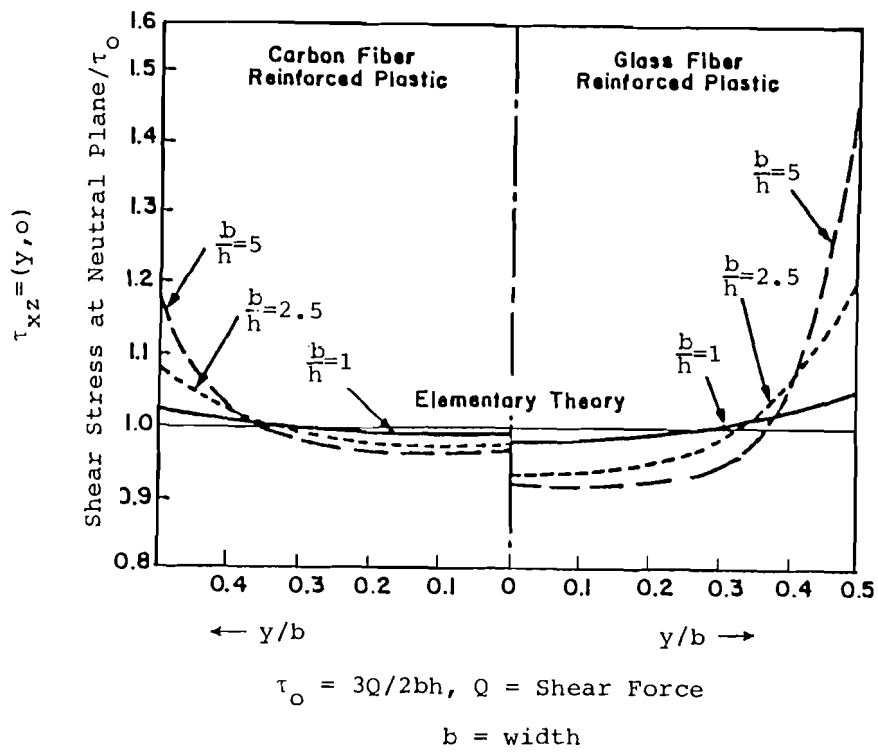


FIGURE 2.2.7 THE VARIATION OF SHEAR STRESS ACROSS THE WIDTH OF A RECTANGULAR UNIDIRECTIONALLY REINFORCED BEAM

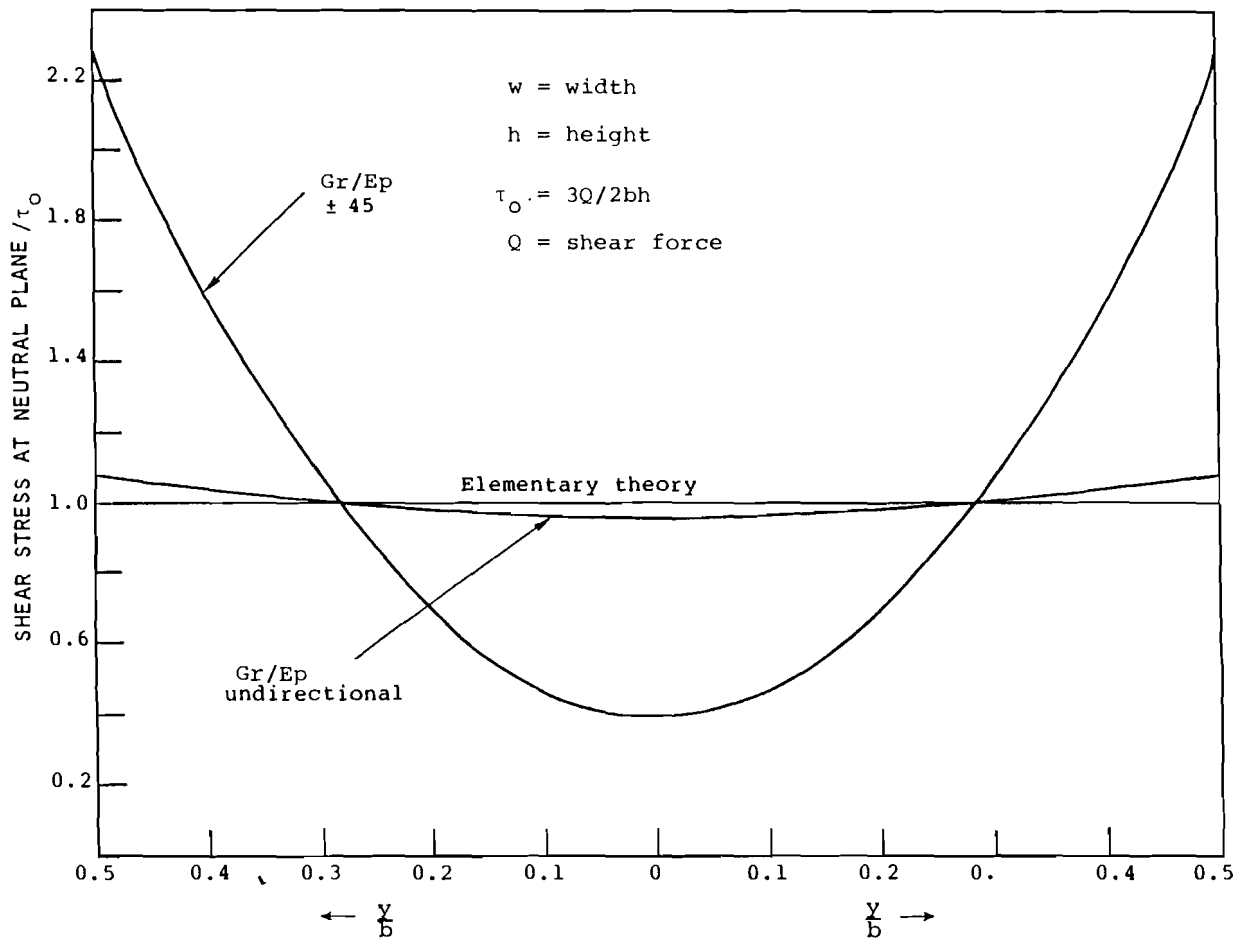


FIGURE 2.2.8 THE VARIATION IN SHEAR STRESS ACROSS BEAM WIDTH IN A ± 45 ANGLE-PLY LAMINATE $b/h = 2.5$

$$\tau_{xz}(\pm b, 0) = \frac{3Q}{2bh} \left[1 + \frac{2\nu_{xz}}{E_{xx}} G_{xy} \cdot \eta \right] \quad (2.2.27)$$

where η depends on the $\beta \sqrt{\frac{G_{xy}}{G_{xz}}} \cdot \frac{h}{b}$. The dependency is shown in figure 2.2.9.

It is worth noting at this point that the existence of a large Poisson's ratio in $(+\theta)_s$ and $(0^\circ/+\theta)_s$ materials results in large anti-elastic curvatures in wide rectangular beams thus creating a "plate-like" response as opposed to "beam-like" behavior. This topic will be discussed in a later section. The large anti-elastic curvatures in such highly anisotropic composites can sometimes cause deviations from the small deformation theory.

In addition to the stress field derived from the assumption of homogeneous material properties, there exist stresses associated with localized normal and tangential tractions on interfaces between the plies in a laminated beam. These interaction stresses arise from the tendency of individual layers to deform in shear in the plane of the fibers and the consequent inhibiting action of one ply on another. Within a laminated beam containing a large number of layers it is possible to show (reference 2.2.8) that greatest departure from stresses obtained from the assumption of homogeneity will occur near the top and bottom surface of the beam, whereas the macroscopic shear stresses will be distributed according to the parabolic law with maximum value at the center and zero value at the top and bottom surfaces. Both types of stresses will be a maximum at the sides ($y = \pm b/2$). Therefore, the probability of failure in angle ply beams at planes different from neutral plane exists, although failures observed in tests are rarely simple shear failures, because of the presence of flexural stresses at such planes. More commonly an element of transverse failure of the individual laminae usually exists.

Through the thickness shear response of many laminated composites shows low yield point and some strain hardening beyond yield which can have significant influence on the load deflection behavior of such beams. An interested reader may look at reference 2.2.9 where effects of such elastic plastic shear response on graphite composites which are elastic under extension are reported.

With all the cautionary notes given above, it should be mentioned that the simple beam theory discussed in this and previous sections is useful for many practical applications although some care may be needed in special circumstances. The problem of determining shear stress distribution and shear coefficient k [see eq.(2.2.20)] of orthotropic beams of other commonly used cross-sections is considered in reference 2.2.7. The shear coefficients of some cross-sections are given in table 2.2.1, which will be useful if deflections are to be determined considering shear deformations. The procedure is identical to that described in the last section on thin beams.

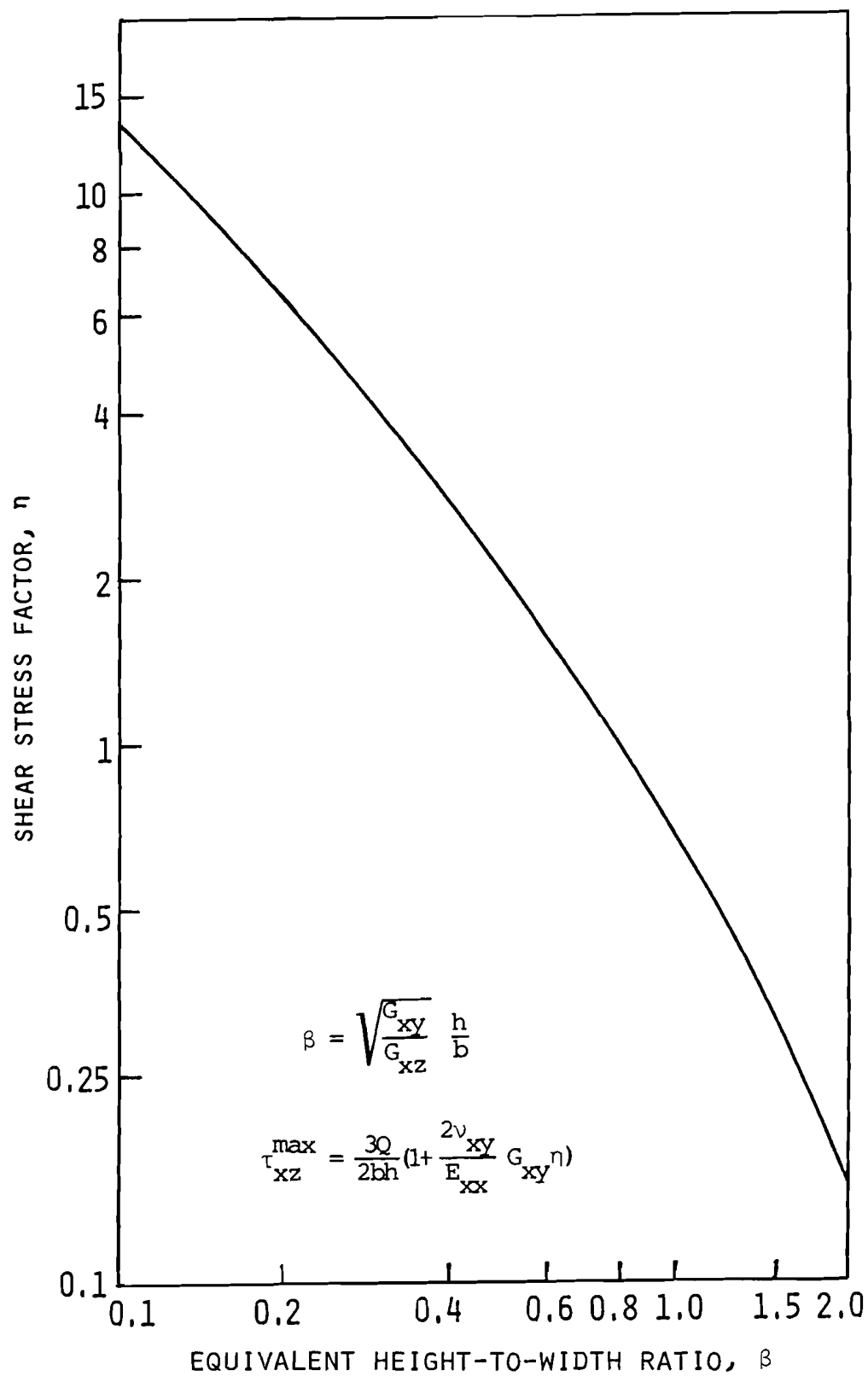


FIGURE 2.2.9 THE PEAK SHEAR STRESS ON CENTRAL PLANE OF BEAM IS OBTAINED FROM EQUIVALENT HEIGHT-TO-WIDTH RATIO

TABLE 2.2.1 SHEAR COEFFICIENTS "k" FOR COMMON BEAM CROSS-SECTION

	<u>ORTHOTROPIC</u>	<u>ISOTROPIC [$\nu=0.3$]</u>
RECTANGULAR SECTION	$\frac{5 E_{xx}}{6 E_{xx} - \nu_{xy} G_{xy}}$	0.850
CIRCULAR SECTION	$\frac{6 E_{xx}}{7 E_{xx} - 2 \nu_{xy} G_{xz}}$	0.886
THIN-TUBE SECTION	$\frac{E_{xx}}{2 E_{xx} - \nu_{xy} G_{xz}}$	0.531

2.2.6 Generally Laminated Beams.

We previously mentioned that for laminated beams containing many repeating patterns of 0° , 90° , and $+\theta$ layups it is possible to use the homogenization principle to obtain equivalent material properties E_{xx} and G_{xz} for use in the simple theory of thin beams. Poisson's ratio ν_{xy} and shear modulus G_{xy} are also required for beams of other cross-sections as described in the last section. To do this it is first necessary to compute the reduced stiffnesses of each layer (assuming $\sigma_{zz} = 0$) in the beam coordinate system shown in figure 2.2.3. This procedure is discussed in detail in Chapter III of Volume I (see eq. (3.15)) of this book for stiffnesses relating inplane stresses σ_{xx}^i , σ_{yy}^i , and τ_{xy}^i , to ϵ_{xx}^i , ϵ_{yy}^i and γ_{xy}^i for each layer i . Thus

$$\begin{vmatrix} \sigma_{xx}^i \\ \sigma_{yy}^i \\ \tau_{xy}^i \end{vmatrix} = \begin{vmatrix} \bar{Q}_{11}^i & \bar{Q}_{12}^i & \bar{Q}_{16}^i \\ \bar{Q}_{12}^i & \bar{Q}_{22}^i & \bar{Q}_{26}^i \\ \bar{Q}_{16}^i & \bar{Q}_{26}^i & \bar{Q}_{66}^i \end{vmatrix} \begin{vmatrix} \epsilon_{xx}^i \\ \epsilon_{yy}^i \\ \gamma_{xy}^i \end{vmatrix} \quad (2.2.28)$$

It is necessary to consider one sequence which is repeating in the beam. To obtain equivalent homogenous material properties we assume that the strains in each layer are the same and compute the average stresses over the sequence. Thus

$$\begin{bmatrix} \epsilon_x^i \end{bmatrix} = \begin{bmatrix} \epsilon^0 \end{bmatrix}$$

$$\begin{bmatrix} \sigma^0 \end{bmatrix} = \frac{1}{h^s} \sum_{i=1}^N \begin{bmatrix} \sigma^i \end{bmatrix} h^i = \text{Average Stresses}$$

$$h^i = \text{Thickness of layer of } i$$

(2.2.29)

$$h^s = \text{Thickness of the sequence} = \sum_{i=1}^N h^i$$

$$N = \text{Number of layers in the Sequence}$$

For a sequence with 0° , 90° and the same number of $+\theta$ and $-\theta$ layers one obtains

$$\begin{vmatrix} \sigma_{xx}^0 \\ \sigma_{yy}^i \\ \tau_{xy}^0 \end{vmatrix} = \begin{vmatrix} \bar{Q}_{11}^0 & \bar{Q}_{12}^0 & 0 \\ \bar{Q}_{12}^0 & \bar{Q}_{22}^0 & 0 \\ 0 & 0 & \bar{Q}_{66}^0 \end{vmatrix} \begin{vmatrix} \epsilon_{xx}^0 \\ \epsilon_{yy}^0 \\ \gamma_{xy}^0 \end{vmatrix} \quad (2.2.30)$$

where

$$\bar{Q}_{ij}^0 = \frac{1}{h^s} \sum_{i=1}^N \bar{Q}_{ij}^i h^i$$

which yields the following moduli of the homogenized material

$$G_{xy} = \bar{Q}_{66}^0$$

$$E_{xx} = \bar{Q}_{11}^0 - \frac{\bar{Q}_{12}^{0^2}}{\bar{Q}_{22}^0} \quad (2.2.31)$$

$$E_{yy} = \bar{Q}_{22}^0 - \frac{\bar{Q}_{12}^{0^2}}{\bar{Q}_{11}^0}$$

$$\nu_{xy} = \frac{\bar{Q}_{12}^0}{\bar{Q}_{22}^0}$$

$$\nu_{yx} = \frac{\bar{Q}_{12}^0}{\bar{Q}_{11}^0}$$

In the notation for Poisson's ratios the second subscript denotes the direction of strain and the first one for that of the applied stress.

Similarly to obtain the equivalent shear module G_{xz} and G_{yz} it is necessary to obtain the shear moduli in rotated co-ordinate system.

$$\begin{vmatrix} \tau_{yz}^i \\ \tau_{xz}^i \end{vmatrix} = \begin{vmatrix} \bar{Q}_{44}^i & \bar{Q}_{45}^i \\ \bar{Q}_{45}^i & \bar{Q}_{55}^i \end{vmatrix} \begin{vmatrix} \gamma_{yz}^i \\ \gamma_{xz}^i \end{vmatrix} \quad (2.2.32)$$

For a layer whose fiber direction makes an angle $+\theta$ with the x-axis,

$$\begin{aligned} \bar{Q}_{44} &= C_{44} \cos^2 \theta + C_{55} \sin^2 \theta \\ \bar{Q}_{45} &= \bar{Q}_{54} = \left[C_{55} - C_{44} \right] \sin \theta \cos \theta \\ \bar{Q}_{55} &= C_{44} \sin^2 \theta + C_{55} \cos^2 \theta \end{aligned} \quad (2.2.33)$$

where $C_{44} = G_T$, transverse shear modulus of 0° layer and $C_{55} = G_A$, axial shear modulus of 0° layer.

In contrast to the assumption that strains in the layers are the same (as was done for inplane properties), taking the shear stresses in the layers to be the same usually yields a better estimate of equivalence through the thickness shear moduli (reference 2.2.10). Thus writing [h^i, h^s being defined in eq. (2.2.29)]

$$\begin{aligned} \tau^i &= \tau^o \\ \gamma^o &= \frac{1}{h^s} \sum_{i=1}^N \gamma^i h^i, \text{ average shear strains} \end{aligned} \quad (2.2.34)$$

$$L_{44}^i = \bar{Q}_{55}^i / \Delta^i$$

$$L_{55}^i = \bar{Q}_{44}^i / \Delta^i$$

$$\Delta^i = \bar{Q}_{44}^i \bar{Q}_{55}^i - \bar{Q}_{45}^{i^2}$$

One obtains the following shear moduli of the homogenized material for a sequence with 0° , 90° and the same number $+\theta$ and $-\theta$ layers,

$$G_{yz} = h^s / \sum_{i=1}^N L_{44}^i h^i \quad (2.2.35)$$

$$G_{xz} = h^s / \sum_{i=1}^N L_{55}^i h^i$$

Therefore, for a high modulus graphite/epoxy system with the unidirectional material properties given in table 3.1 of Vol. I, the inplane properties of homogenized (0/90), (+45) and (0/90/+45) sequences (containing a moderately

large number of such arrangements) are the same as those given in table 3.2 of the same volume. On the other hand if transverse shear modulus (G_T) of the unidirectional material is 0.45×10^6 psi (from table 3.1 of Vol. I, $G_A = G_{12} = 0.65 \times 10^6$ psi) both of the equivalent shear moduli of (0/90), (+45) and (0/90/+45) sequences are equal to 0.532×10^6 psi. The reader may now determine the flexure and shear stresses as well as the deformation of cantilever beams made of these sequences of varying span to depth ratios by using methods described in previous sections.

Occasionally beams used in practice may be made of a small number of layers of different orientations. In such cases the homogenization process cannot be applied. The same is true if many layers are used but different orientations (like 0° , 90° , $\pm\theta$) are lumped at different locations through the thickness. In these circumstances the neutral axis may not lie in the mid-plane and an approach similar to that used in plate theory is suitable. Denoting u_0 to be the displacement in the x-direction at the mid-plane and measuring z from the mid-plane, one may assume the following displacements

$$u_x = u_0(x) + z\psi(x) \quad (2.2.36)$$

$$u_z = w(x)$$

which are adequate when the section has a plane of symmetry coinciding with that of loading and each layer behavior in an orthotropic manner, i.e. we assume that $\pm\theta$ layers are placed next to each other and can be replaced by a homogenized material.

$$\epsilon_{xx} = u_{0,x} + z\psi_{,x} \quad (2.2.37)$$

$$\sigma_{xx}^i = E_{xx}^i \left(u_{0,x} + z\psi_{,x} \right)$$

the superscript i denoting the layer i and the comma $(,)$ indicates differentiation $\left(f_{,x} = \frac{\partial f}{\partial x} \right)$. For wide beams use of reduced stiffness $\overline{Q}_{11}^i$ [see eq.

(2.2.28)] is more appropriate than E_{xx} . The axial force N and the moment M are given by

$$(N, M) = \int_{-h/2}^{h/2} \sigma_{xx}(1, z) b(z) dz \quad (2.2.38)$$

h being the total thickness of the cross-section and $b(z)$ is the variation of width with z . Therefore,

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} u_{0,x} \\ \psi, x \end{bmatrix} \quad (2.2.39)$$

$$(A, B, D) = \sum_{i=1}^N \int_{z_i}^{z_{i+1}} \begin{bmatrix} 1 & z & z^2 \end{bmatrix} b^i(z) E_{xx}^i dz$$

z_i, z_{i+1} being the z co-ordinate of the bottom and top of layer i , the layers being numbered in increasing order from the bottom surface of the beam towards the top. z_i will be negative if interface i is below the mid-plane. If $b^i(z) = b = \text{constant}$ or if the number of layers is not too small and variation of width is not too rapid then taking b^i to be the average width of layer i (figure 2.2.10).

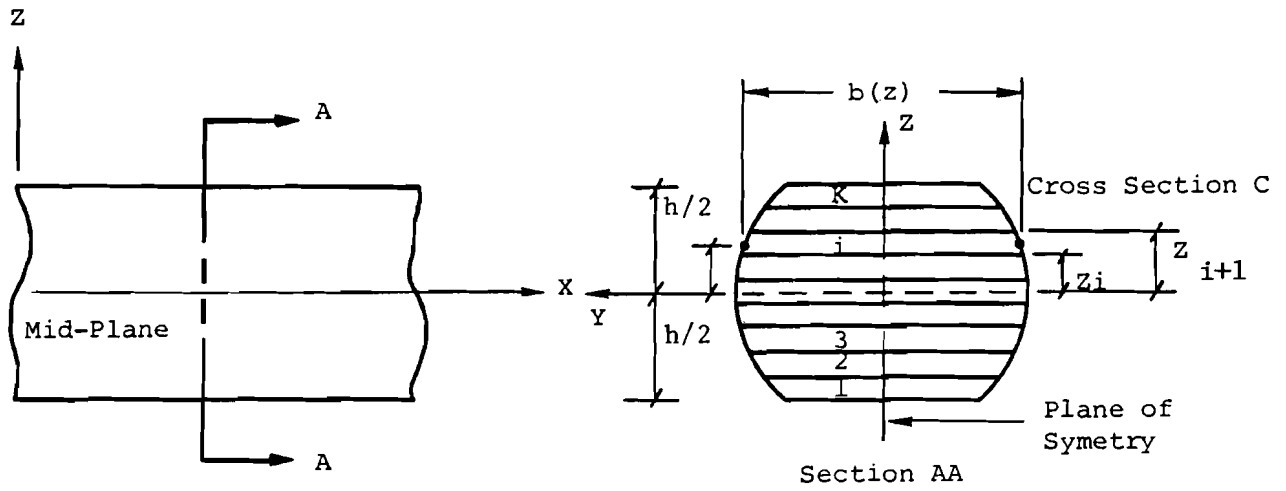


FIGURE 2.2.10. LAMINATED BEAM

$$A = \sum_{i=1}^N E_{xx}^i b^i (z_{i+1} - z_i) \quad (2.2.40)$$

$$B = \sum_{i=1}^N E_{xx}^i b^i (z_{i+1}^2 - z_i^2) / 2$$

$$D = \sum_{i=1}^N E_{xx}^i b^i (z_{i+1}^3 - z_i^3) / 3$$

Inverting first of eq. (2.2.39) one obtains

$$\begin{bmatrix} u_{0,x} \\ \psi, x \end{bmatrix} = \frac{1}{AD - B^2} \begin{bmatrix} D & -B \\ -B & A \end{bmatrix} \begin{bmatrix} N \\ M \end{bmatrix} \quad (2.2.41)$$

Substituting in the second equation of (2.2.37) and using the concept of shear flow shown in figures (2.2.4a,b) the shear stress at height z is given by

$$\tau_{xz}(z) = \frac{1}{b(z)\{AD-B^2\}} \left[\{DA'(z) - B B'(z)\} N_{,x} + \{AB'(z) - BA'(z)\} M_{,x} \right] \quad (2.2.42)$$

where

$$\{A'(z), B'(z)\} = \int_z^{h/2} \{1, z\} E_{xx} b(z) dz \quad (2.2.43)$$

For constant axial force N and shear force Q , $N_{,x} = 0, M_{,x} = Q$ and eq. (2.2.42) yields

$$\tau_{xz}(z) = Q\{AB'(z) - BA'(z)\}/b(z)\{AD-B^2\} \quad (2.2.44)$$

For computing the effects of shear deformations it is necessary to write the relationship between shear force and average shear strain [see eq. (2.2.20)]

$$Q = k K \bar{\gamma} = k K \left[\psi + w_{,x} \right] \quad (2.2.45)$$

k being the correction factor. For determining K with due consideration to lamination and width variation, we assume shear flow q to be constant for all z

$$q(z) = q_a = \text{assumed constant} = \frac{Q}{h} \quad (2.2.46a)$$

h being the height of the cross-section. It follows that

$$\tau_a(z) = \frac{q_a}{b(z)} = \text{assumed shear stress} \quad (2.2.46b)$$

$$\gamma_a^i = \frac{q_a}{G_{xz}^i b(z)} = \text{assumed strain in layer } i \quad (2.2.46c)$$

Taking the average of eq. (2.2.46c) over the cross-section and using eq. (2.2.45) without the correction factor k one obtains

$$K = Ch / \sum_{i=1}^N \frac{h^i}{G_{xz}^i} \quad (2.2.47)$$

for N layers, h^i being the thickness of the i th layer and C is the cross-sectional area. Note that this is equivalent to what has been suggested in eq. (2.2.20) with G_{xz} evaluated according to the homogenization principle. For a single material it reduces to (2.2.20).

To determine the factor K, the standard procedure (references 2.2.3, 2.2.11) is to equate the complimentary energy due to shear stresses to that obtained from the shear force average shear strain relation (2.2.45). Therefore

$$\frac{1}{2} \frac{Q^2}{kK} = \frac{1}{2} \int_{-h/2}^{h/2} \frac{\tau_{xz}^2(z) b(z)}{G_{xz}(z)} dz \quad (2.2.48)$$

which by eq. (2.2.44) reduces to

$$k = \left[AD - B^2 \right]^2 / K \int_{-h/2}^{h/2} \{ AB'(z) - BA'(z) \}^2 / \{ b(z) G_{xz}(z) \} dz \quad (2.2.49)$$

For a beam of rectangular cross-section eq. (2.2.49) yields $k = 5/6$ (see eq. [2.2.22b]).

As a simple example we consider a rectangular cross-ply (0/90) beam of height h and width b (figure 2.2.11a) where the properties of the 0° ply is given by

$$E_A = 25 \times 10^6 \text{ psi}, \quad E_T = 1 \times 10^6 \text{ psi} \quad (2.2.50a)$$

$$G_A = 0.5 \times 10^6 \text{ psi}, \quad G_T = 0.4 \times 10^6 \text{ psi}$$

Thus for the 0° layer

$$E_{xx} = E_A, \quad G_{xz} = G_A \quad (2.2.50b)$$

and for the 90° layer

$$E_{xx} = E_T, \quad G_{xz} = G_T \quad (2.2.50c)$$

which from eq. (2.2.39) yields

$$A = \frac{bh}{2} [E_A + E_T], \quad B = \frac{bh^2}{8} [E_A - E_T], \quad D = \frac{bh^3}{24} [E_A + E_T] \quad (2.2.50d,f)$$

using eq. (2.2.43) for the 0° layer

$$A'(z) = bE_A \left[\frac{h}{2} - z \right], \quad B'(z) = bE_A \left[\frac{h^2}{8} - \frac{z^2}{2} \right] \quad (2.2.50g,h)$$

and for the 90° layer

$$A'(z) = b \left[E_A \frac{h}{2} - E_T z \right], \quad B'(z) = b \left[E_A \frac{h^2}{8} - E_T \frac{z^2}{2} \right] \quad (2.2.50i,j)$$

Substituting the last three relations and numerical values of E_A , E_T one has

$$\tau_{xz}(z) = \frac{Q}{bh} \left\{ 7.9919 \left[1 - \frac{4z^2}{h^2} \right] + 7.3771 \left[1 - \frac{2z}{h} \right] \right\} \text{ in } 0^\circ \text{ layer} \quad (2.2.51a)$$

and

$$\tau_{xz}(z) = \frac{Q}{bh} \left\{ 7.9919 \left[1 - .04 \frac{4z^2}{h^2} \right] + 7.3771 \left[1 - .04 \frac{2z}{h} \right] \right\} \text{ in } 90^\circ \text{ layer} \quad (2.2.51b)$$

Which yields the shear stress distribution plotted in figure (2.2.11b), which is quite different from the well known parabolic distribution [eq. (2.2.14)].

Use of eq. (2.2.47) and (2.2.49) yields the following values of shear stiffness based on constant shear flow (or stress) and the corresponding shear correction factor.

$$\begin{aligned} \text{Shear Stiffness} \quad K &= 0.44 \times 10^6 \text{ bh lb.} \\ k &= 0.68 \\ \text{thus} \quad k K &= 0.302 \times 10^6 \text{ bh lb.} \end{aligned} \quad (2.2.52)$$

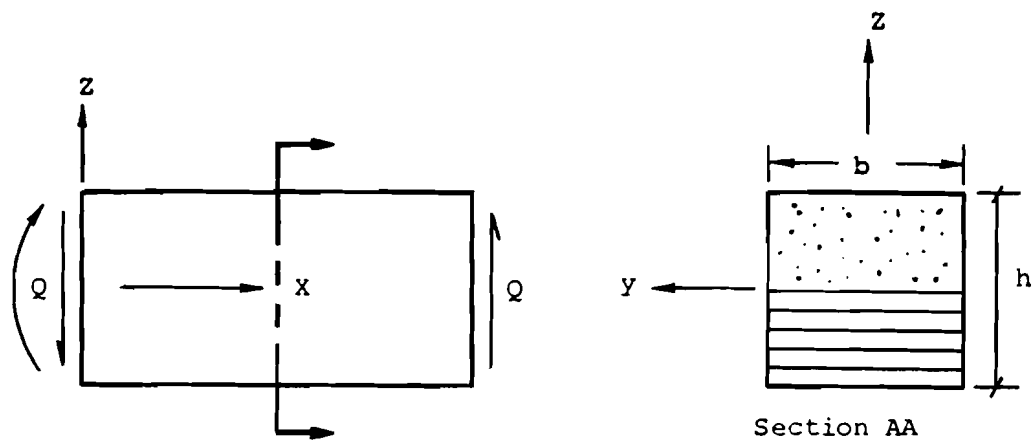
As an exercise the reader may perform similar calculations for a (0/90)₂ layup.

In reference 2.2.11, the stiffness is obtained on the assumption that the constant shear strain in each layer and the corresponding correction factor (calculated on the same principle) will be different, but the product $k K$ which is used to calculate shear deformation will be the same as obtained by the procedure given here. It should be noted, however, that if the number of layers in the (0/90) sequence or in any other repeating layup is gradually increased, the correction factor obtained by the use of the procedure in reference 2.2.11 will not approach 5/6 which is the value for a homogeneous material, but the factor obtained by the approach given here will be close to that value. It is an easy matter to determine the limiting value in the former case and the choice of using any of the two approaches may be left to the user. These remarks are made to show that for a repeating sequence the homogenization procedure described earlier along with a factor of 5/6 is indeed a good approximation for modeling effects of shear deformation. Before concluding this section it is worth mentioning that for problems involving dynamic response a better approach to evaluate correction factors is to match cut-off frequencies for thickness-shear waves with the ratio of wave length to thickness (height) approaching infinity (reference 2.2.12). For a laminate having many plies in a stacking sequence which consists of a large number of repetitions of given pattern (e.g. $[\phi_1, \phi_2, -\phi_2]_{sn}$), it was shown in reference 2.2.13 that the shear correction factor approaches the value $\pi^2/12$ which is the result for the homogeneous case.

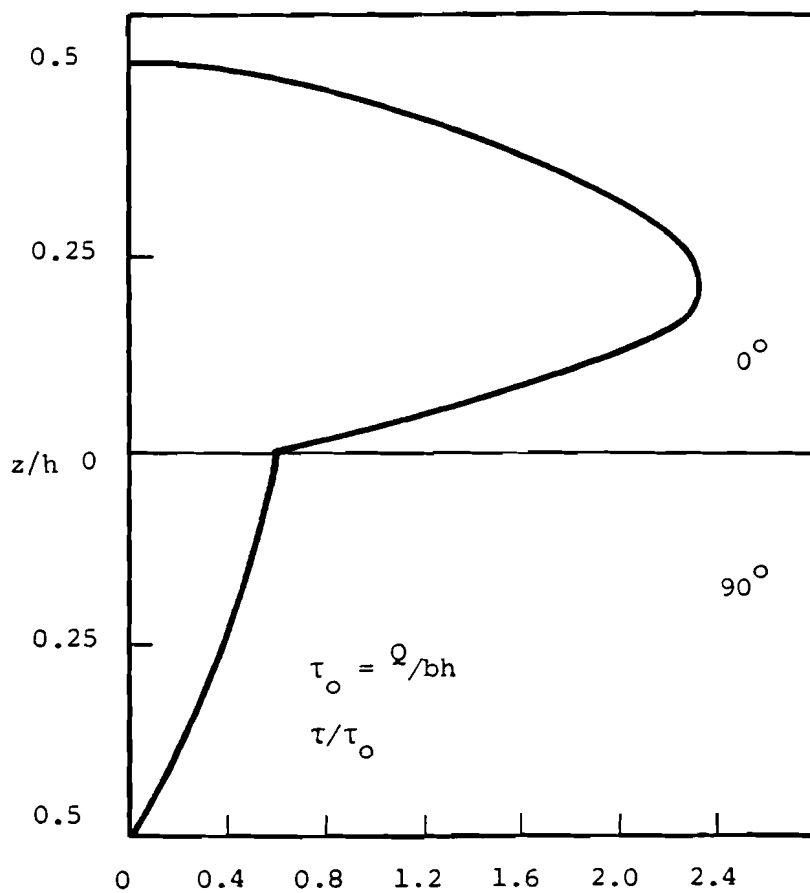
2.2.7 Short Beam Shear Test.

In view of the importance of interlaminar shear strength in composites' design, we consider the simple beam flexure problem with due consideration to the effects of concentrated loads. Justification for this attention is based on:

1. the importance of the beam as a basic structural element;
2. the use of the beam configuration to generate data and quality control information on composite systems, e.g. short beam shear test for interlaminar shear strength and the flexure strength test.



a. TWO LAYER CROSS-PLY



b. SHEAR STRESS VARIATION IN CROSS-PLY BEAM

FIGURE 2.2.11 SHEAR STRESS DISTRIBUTION FOR A RECTANGULAR CROSS-PLY BEAM

Figure 2.2.12 gives a typical configuration for the short beam shear test. We have previously considered the variation of shear stress across the width of the beam which has important implications in design. In this section we compare beam theory solutions for thin beams with more exact plane stress elasticity solutions. Because of its importance such comparisons have been performed by many investigators by the use of finite element methods as well as various types of series solutions (reference 2.2.14). Figure 2.2.13 shows the shear stress distribution in an unidirectionally reinforced beam at three locations for two L/h ratios, L, h being the span and depth of the beam, respectively. These results are obtained on the assumption that the overhangs are large (i.e. of infinite length). A Fourier transform solution is adequate for such a case. As can be seen from figure 2.2.14 the stresses die out rapidly as one moves outwards along the overhang and thus the results will be applicable when the overhang length is greater than two times the depth of the beam. Properties used for calculations are

$$E_A = 21 \times 10^6 \text{ psi}, \quad E_T = 1.5 \times 10^6 \text{ psi}$$

$$G_A = 0.8 \times 10^6 \text{ psi}, \quad \nu_{AT} = 0.3, \quad \nu_{TT} = 0.25$$

Locations A, B, C are at $x = h/4, L/4$ and $L/2 - h/4$ from the left hand support. The shear stress distribution at location B coincides with the beam theory result for $L/h = 16$, but the same is not true at $L/h = 4$. Near the loads (locations A and C) shear stresses are maximum near the surfaces where loads are applied for obvious reasons. This is true for $L/h = 4$ as well as for $L/h = 16$. Shear stresses at $z/h = \pm 0.375$ and 0 are plotted against x in figure 2.2.14 for $L/h = 4$. Because the length is short beam theory solution is not recovered anywhere on the beam. For these reasons use of beam theory for interpreting data from short beam shear tests with very small L/h ratios (which is commonly used to avoid flexural failure) is a questionable issue although such a test may possibly be used for quality control purposes. Obviously for large L/h ratios beam theory solution will hold over a large part of the beam although problems remain near load application points especially at the center where flexural stresses are also high. However, it has been observed that interlaminar shear failures became more confined to the central plane as L/h is increased.

2.2.8 Photoelastic Patterns in Composite Beams.

The patterns shown on the top of figure 2.2.15 are the experimentally derived isochromatic (principal strain difference) and isoclinics (inclination of principal strains) for a simple unidirectional graphite/epoxy beam under 3-point bending. The beam is sufficiently narrow to develop a plane stress state. The plane of polarization is vertical perpendicular to the plane of paper.

The patterns shown below are the corresponding isochromatics and isoclinics for an isotropic beam material.

It may be noted that transmission photoelasticity cannot be used for the composite beam. Consequently, the surface strain representation illustrated in the upper figure was obtained by bonding a birefringent coating onto the surface of the composite with a thin, reflecting, glue layer.

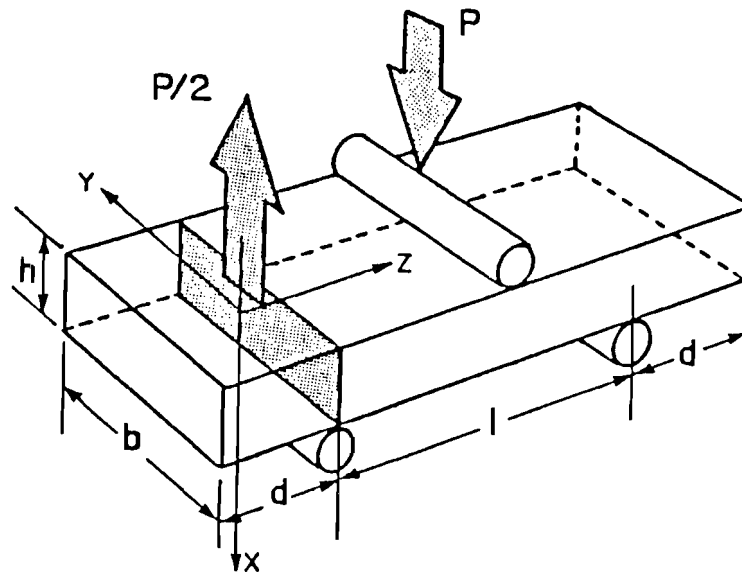


FIGURE 2.2.12 CONFIGURATION OF SHORT BEAM IN 3-POINT FLEXURE

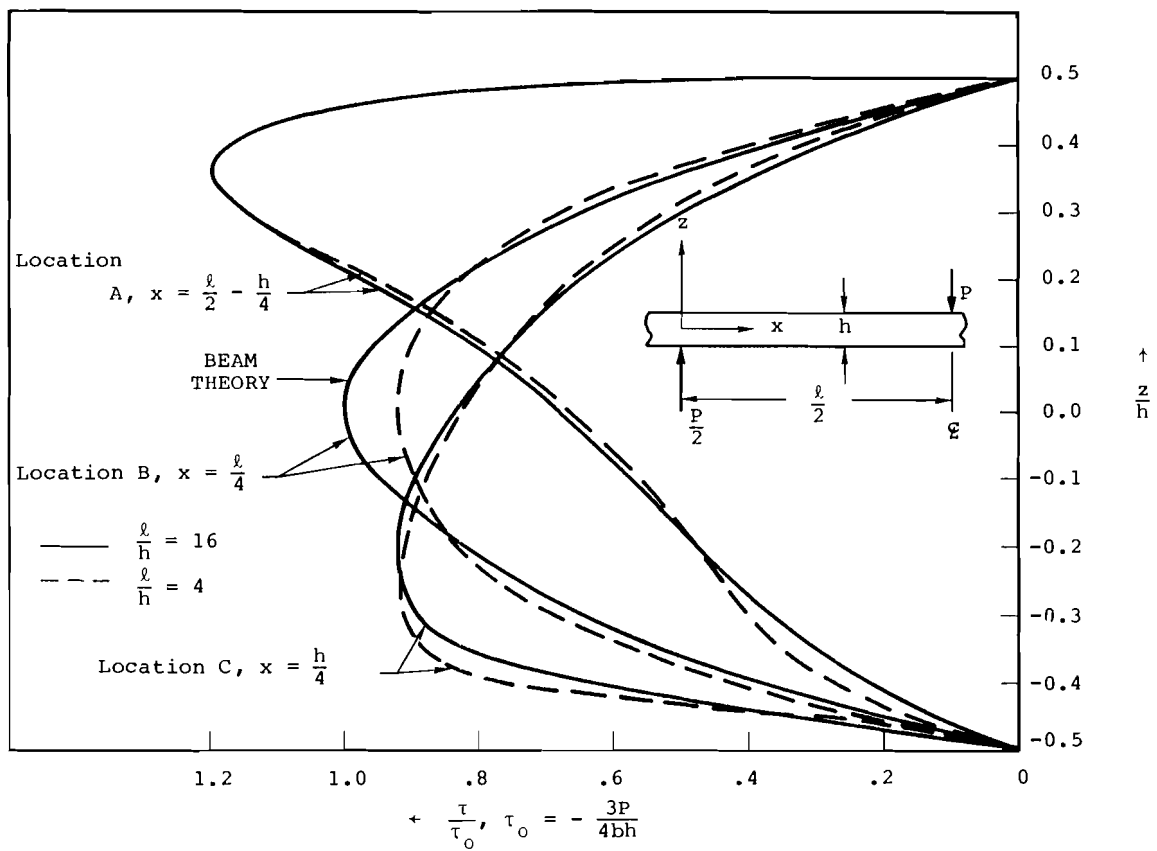


FIGURE 2.2.13. SHEAR STRESS DISTRIBUTIONS IN THREE-POINT LOADED BEAM

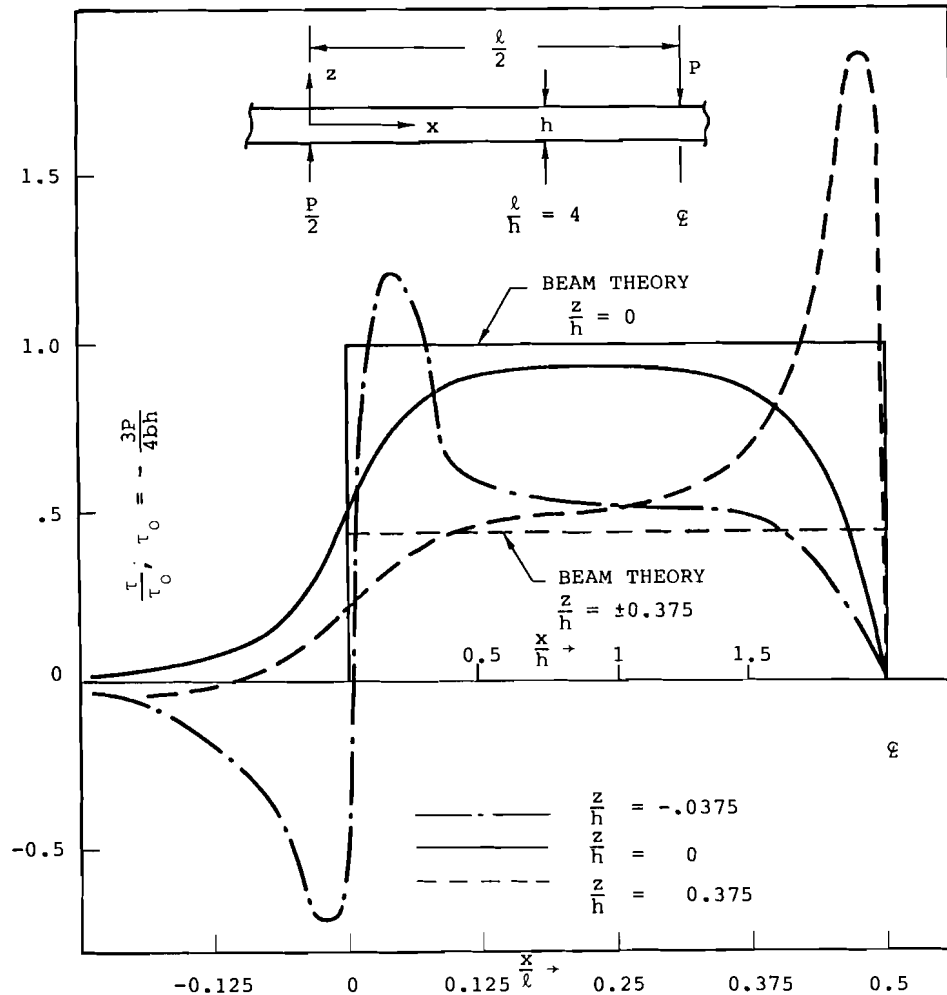


FIGURE 2.2.14 SHEAR STRESS VARIATION ALONG BEAM AXIS IN A SHORT BEAM $L/h = 4$

The reason for the dramatic differences illustrated on figure 2.2.15 is associated with the deviation between principal stress and principal strain directions in an orthotropic material. Figure 2.2.16 shows the nature of this deviation for two simple loading configurations. The lower diagram is representative of a general location in a simple beam subjected to 3-point loading.

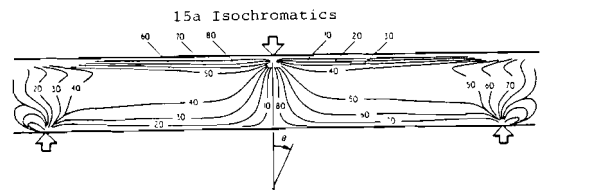
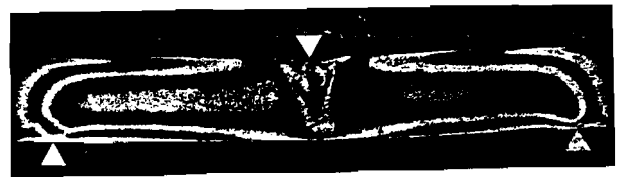
The isochromatic and isoclinic lines shown in figure 2.2.17 have been derived analytically from elasticity analysis of the 3-point loaded beam.

Figure 2.2.18 shows how the increase in orthotropy tends to expand the region which, in an isotropic beam, is very localized near the concentrated loading points. Effects near the left hand support are only shown. In these figures, λ is type of "orthotropic index" and is a measure of the magnitude of the difference between the directions of principal stress and principal strain.

The expression for λ is:

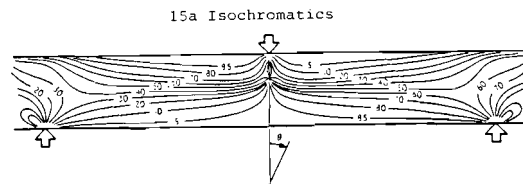
$$\lambda = \frac{E_x}{2G_{xy}(1+\nu_{xy})}$$

which of course is unity for the isotropic case. The results in figure 2.2.18 have been obtained by the use of simple engineering theory without any consideration to the detailed effects under concentrated loads. In this figure $\bar{x}$ and $\bar{y}$ are nondimensionalized co-ordinates obtained by dividing x and y by $h/2$, h being one depth of one beam.



15b Isoclinics

Unidirectional Graphite/Epoxy Composite Beam (3-Point Bend)



15b Isoclinics

Isotropic Beam (3-Point Bend)

FIGURE 2.2.15 PHOTOELASTIC PATTERNS IN COMPOSITES AND ISOTROPIC BEAMS

2.2.9 Torsion of Composite Bars.

Unidirectional, laminated or other types of composites are often used for carrying torques. In many cases such torsion members may be treated as homogeneous orthotropic (or isotropic) bars. Some examples are shown in figure 2.2.19. As in the case of isotropic bars undergoing small deformations two shear stresses τ_{xz} and τ_{yz} are the only non zero stress components (reference 2.2.4). Note that we have taken the z axis in the axial direction of the bar, which is commonly used in literature dealing with torsion. For a $\pm\theta$ layup two effective shear moduli of the homogenized material G_{xz} and G_{yz} are usually different. On the other hand for a unidirectional material with fibers parallel to the axis of the bar $G_{xz} = G_{yz} = G_A$, which is the axial shear modulus. If, however, the fibers are in planes parallel to the cross-sections $G_{yz} = G_A$ and $G_{xz} = G_T$, the transverse shear modulus. Torsional stiffnesses of the two bars shown on the

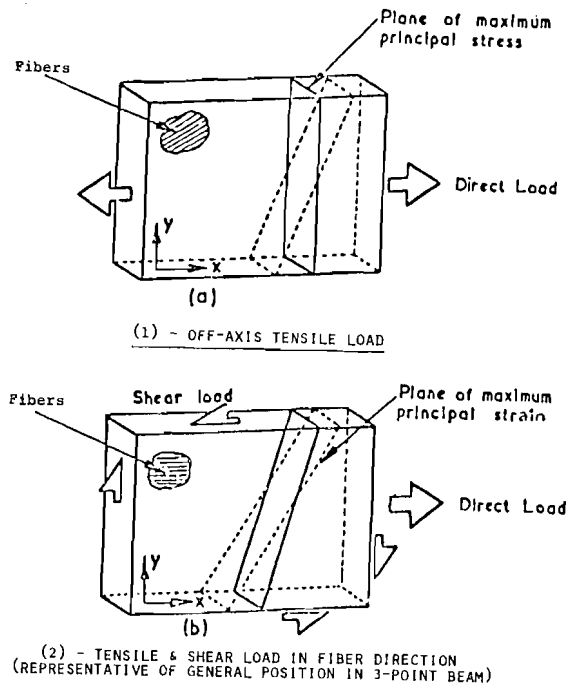


FIGURE 2.2.16 IN A COMPOSITE, THE PRINCIPAL STRESS AND PRINCIPAL STRAIN DIRECTIONS ARE NOT COINCIDENT

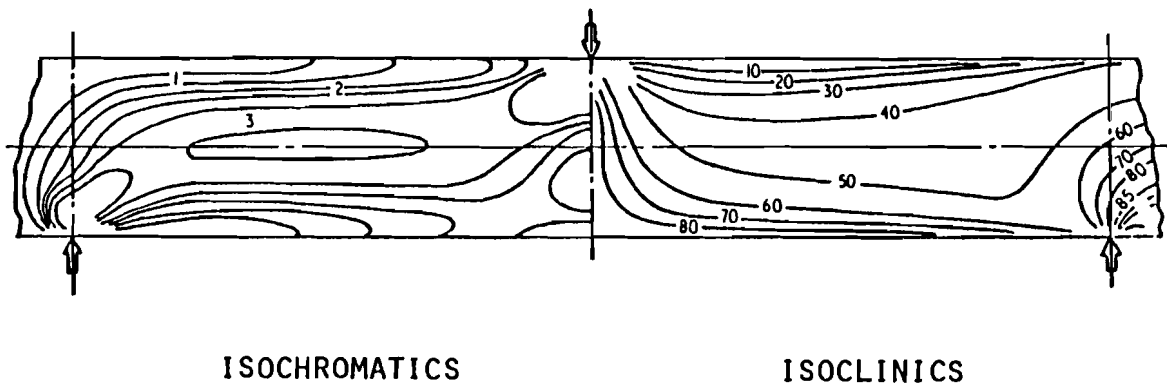


FIGURE 2.2.17 ISOCHROMATIONS AND ISOCLINCS PREDICTED FROM ELASTICITY ANALYSIS

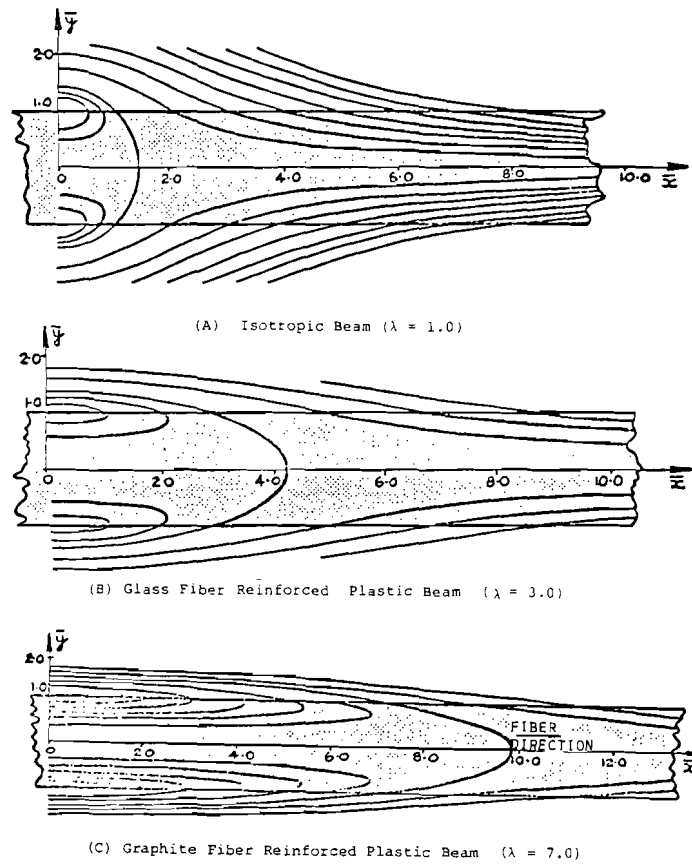


FIGURE 2.2.18 EFFECT OF ORTHOTROPY ON THE APPROXIMATE FORM OF ISOCHROMATIC LINES

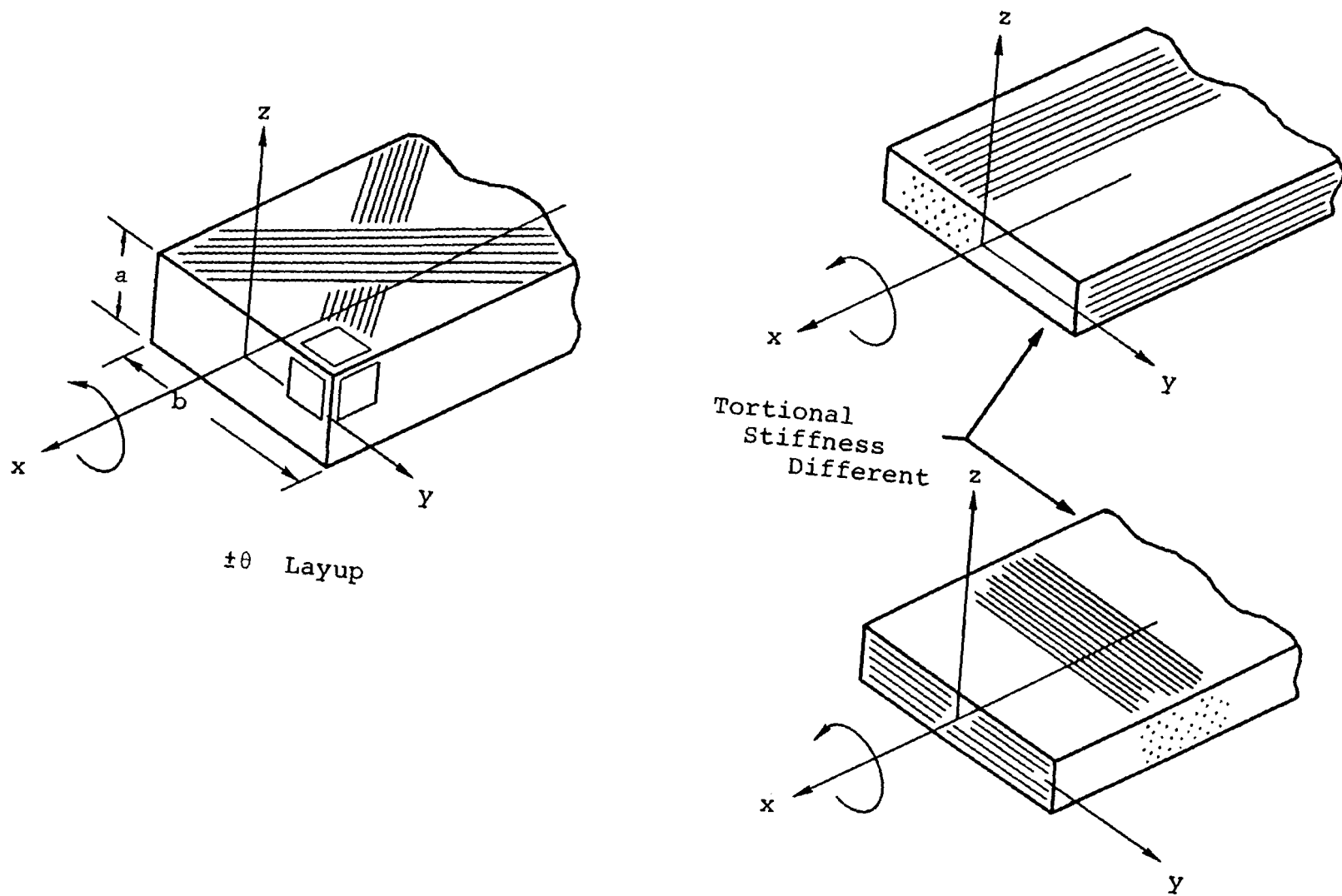


FIGURE 2.2.19. TORSION OF DIFFERENT LAYUPS

right will therefore differ. It should be pointed out that to compute the shear moduli of the homogenized material for a general repeating layup one may make use of eqs. (2.2.31) and (2.2.35) and other related formulae given in a previous section. However, the reader should be careful and note that in these formulae the x axis is taken in the direction of the beam axis. Thus G_{yz} for the $+\theta$ layup, shown in figure 2.2.19, must be taken as the inplane shear modulus [i.e. G_{xy} in eq. (2.2.31)].

In the notations described above the torsional stiffness C_t of rectangular bars relating the torque M_t to twist ϕ (rotation per unit length) in the equation

$$M_t = C_t \phi \quad (2.2.53a)$$

is given by

$$C_t = \frac{G_{yz} b a^3}{3} \left(1 - F \frac{a}{b} \right) \quad (2.2.53b)$$

$$\approx \frac{G_{yz} b a^3}{3} ; \frac{a}{b} \rightarrow 0 \quad (2.2.53c)$$

where a is the thickness and b is the other dimension of the cross-section (figure 2.2.19) and

$$F = \frac{192}{\pi^5} \sqrt{\frac{G_{yz}}{G_{xz}}} \sum_{n=0}^{\infty} (2n+1)^{-5} \tanh \left\{ \frac{2n+1}{2} \pi \frac{b}{a} \sqrt{\frac{G_{xz}}{G_{yz}}} \right\} \quad (2.2.53d)$$

To obtain these formulae use has been made of the deviation in Lekhnitski's treatise (reference 2.2.15, which also contains expressions for shear stresses) and some summation formulae (reference 2.2.4). It may be noted that for thin bars (small a/b) the argument of the hyperbolic tangent in eq. (2.2.53d) is large and the function approaches the value unity. In that case F can be taken as

$$F = 0.63 \sqrt{\frac{G_{yz}}{G_{xz}}} \quad (2.2.53e)$$

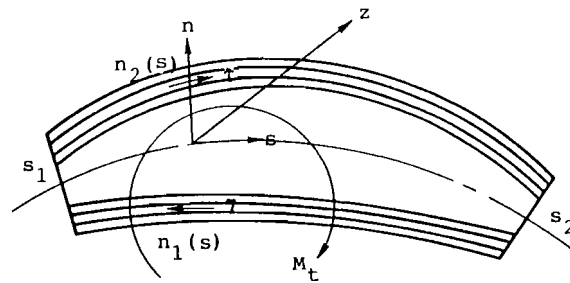
and therefore, for very small a/b the second term in the parenthesis in eq. (2.2.53b) may also be neglected thus yielding eq. (2.2.53c).

Lekhnitski's treatise, (reference 2.2.5), also deals with a wide variety of torsion, bending (uniaxial as well as biaxial) and the combined bending and torsion of homogeneous anisotropic bars of different types of cross-section which may be of interest to the reader. For the torsion of unidirectionally reinforced bars with fibers parallel to the axial direction, the results for different types of cross-sections given in reference 2.2.4 are also very useful in the many practical applications. In the case of laminated construction, simple strength

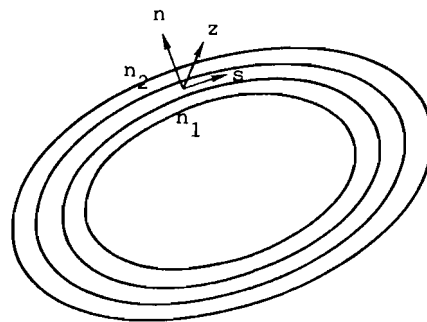
of material concepts are also very useful when one of the dimension of the cross-section is small (thin) compared to the other. For example, consider the case of the cross-section shown in figure 2.2.20a which is similar to the design concept of a propeller blade shown in figure 2.2.1. Referring the cross-section to a curvilinear co-ordinate system (s, n , with the z -axis perpendicular to the plane) one may assume that for any s

$$\gamma_{sz} \approx \phi \left\{ n - n_0(s) \right\} \quad (2.2.54a)$$

$$r_{sz}(s, n) = G_{sz}(s, n) \gamma_{sz} = G_{sz}(s, n) \phi \left\{ n - n_0(s) \right\} \quad (2.2.54b)$$



a. Thin Laminated Torsion Member



b. A Thin Tube

FIGURE 2.2.20 CROSS-SECTIONS WHERE STRENGTH OF MATERIALS TYPE SOLUTIONS ARE APPLICABLE

For a very thin cross-section (dimension in direction n is small) force and moment equilibriums are approximately satisfied by taking

$$\int_{n_1(s)}^{n_2(s)} \tau_{sz} \, dn = 0 \text{ for each } s \quad (2.2.54c)$$

$$\int_{s_1}^{s_2} \int_{n_1(s)}^{n_2(s)} \tau_{sz} \, n \, dn \, ds = M_t \quad (2.2.54d)$$

which yield

$$n_o(s) = B_{66}(s)/A_{66}(s) \quad (2.2.54e)$$

$$\begin{aligned} M_t &= \phi \int_{s_1}^{s_2} \left\{ D_{66}(s) - n_o(s) B_{66}(s) \right\} ds \\ &= \theta \int_{s_1}^{s_2} \left\{ D_{66}(s) - B_{66}^2(s)/A_{66}(s) \right\} ds \end{aligned} \quad (2.2.54f)$$

where

$$\left\{ A_{66}(s), B_{66}(s), D_{66}(s) \right\} = \int_{n_1(s)}^{n_2(s)} G_{sz}(s, n) \left\{ 1, n, n^2 \right\} dn \quad (2.2.54g)$$

Note that the term A_{66} , B_{66} , D_{66} can be obtained by using laminate theory (with z axis replacing n) except for the fact that they are functions of s . The integration in eq. (2.2.54f) can be performed numerically using trapezoidal or Simpson's rule. When the layup is symmetric about s , $B_{66}(s) = 0$ and one has

$$M_t = \phi \int_{s_1}^{s_2} D_{66}(s) \, ds \quad (2.2.54h)$$

which reduces to

$$M_t = G_{sz} \phi \int_{s_1}^{s_2} t^3(s) \, ds \quad (2.2.54i)$$

for a homogeneous or homogenized cross-section of varying thickness with equivalent shear modulus G_{sz} . This formula is widely used in practice for homogeneous materials and reduces to eq. (2.2.53c) for thin rectangular cross-sections.

On the other hand if the cross-section is a tube with a thin laminated wall of a constant thickness, it may be assumed that the shear strain is constant through the thickness and the laminated material may be replaced by a homogeneous material with shear modulus G

$$G \approx \frac{1}{t} \int G_{sz}(n) ds = A_{66}/t \quad (2.2.55)$$

where t is thickness of the wall and A_{66} is computed by laminate theory. The problem may now be solved by using the formulae for thin homogeneous tubes given in reference 2.2.4. With slight modification, results for walls with variable thickness may also be obtained.

In small deformation torsion problems it is usually assumed that the distance between any two cross-sections of the bar remains unchanged during torsion. This assumption however is not very accurate for thin beams, when the twist is large and becomes very conservative in case of unidirectional composites with fibers in the axial direction. It can be shown (reference 2.2.16) that the torque twist relationship for a rectangular bar of very narrow cross-section ($a/b \rightarrow 0$) for a moderate amount of twist is

$$M_t = \frac{G_{yz} b a^3 \phi}{3} \left[1 + \frac{E_{zz} b^4 \phi^2}{120 G_{yz} a^2} \right] \quad (2.2.56)$$

The second term comes from the contribution due to extensional stresses which have the direction of the longitudinal fibers (in composites as well as homogeneous materials) of the twisted strip. Results for a graphite/epoxy unidirectional bar are shown in figure 2.2.21 which also demonstrates the expected effect of stiffening on the first natural frequency of torsional vibration in graphite/epoxy and titanium bars. It can be seen that the effect is more pronounced in the case of graphite/epoxy materials. There is a common misconception in composites engineering that the addition of $+45^\circ$ plies near the surfaces will provide higher in-plane modulus and hence an increase twisting rigidity regardless of the pretwist configuration. The illustration demonstrates the importance of axial stiffness in thin pretwisted bars and in practice it has been found that an increase in the proportion of $+45^\circ$ plies reduces the torsional resonance frequency. This phenomenon may be of importance in turbine blade applications. Timoshenko (reference 2.2.16) also addressed the effect of longitudinal prestress on torsional stiffness. The reader may note that torsional resonance frequencies of helicopter rotor blades increase with their rotational speeds. This is because of the fact that the longitudinal tensile stresses increase with higher rotational speeds.

Equation (2.2.56) indicates that when a/b is not small the stiffening effect is less pronounced. It may be shown that for a circular cross-section of a homogeneous material with shear modulus G and axial Young's modulus E the result is

$$M_t = \frac{G \phi \pi d^4}{32} \left[1 + \frac{\phi^2 d^2 E}{48 G} \right] \quad (2.2.57)$$

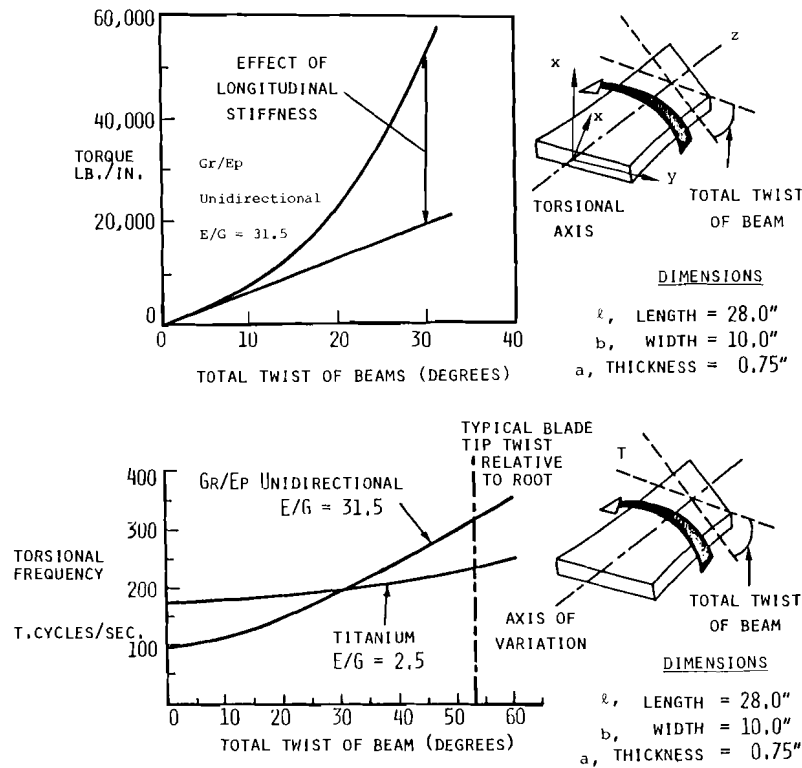


FIGURE 2.2.21. TORSIONAL STIFFNESS OF RECTANGULAR COMPOSITE BARS

d being the the diameter of the cross-section, which indicates that the stiffening effect will be small even for many composites unless d is comparable to the length of the beam. Thus for some cross-sections torsional rigidity can possibly be increased by increasing proportions of $\pm 45^\circ$ plies near the surfaces.

2.2.10 Lateral Instability in Composite Beams.

Lateral buckling of beams is of importance for practical design problems and the subject is treated in many text books (see reference 2.2.17). Many of the formulae given in such texts are applicable to composite beams with appropriate modifications for bending, torsional and warping rigidities. In this subsection we will only demonstrate how these modifications may be performed. Consider a beam whose cross-section can be approximated by a homogenized orthotropic cross-section as shown in figure 2.2.22 and which is carrying a constant bending moment M . During lateral buckling the beam deforms according to the pattern shown on the right. Solution of the governing equations are given in reference 2.2.17. If the ends of the beam cannot rotate about the x axis but are free to warp, the critical moment is given by

$$(M)_{cr} = \frac{\pi}{l} \sqrt{E_{xx} I_z C \left(1 + \frac{C_1}{C} \frac{\pi^2}{l^2} \right)} \quad (2.2.58)$$

where

- E_{xx} = Young's Modulus in x -direction
- I_z = Moment of Inertia about z axis
- C = Torsional Rigidity
- C_1 = Warping Rigidity.

Approximate expressions for C and C_1 of thin walled open sections can be found in reference 2.2.17 when $G_{xz} = G_{xy}$.

For a rectangular beam of height h and thickness b ($b \ll h$) warping rigidity C_1 can be taken equal to zero and using eq. (2.2.53b) and (2.2.53e)

$$C \approx \frac{G_{xz} h b^3}{3} \left[1 - 0.63 \sqrt{\frac{G_{xz} b}{G_{xy} h}} \right] \quad (2.2.59)$$

Similarly for a simply supported beam of span ℓ loaded by a concentrated force P at midspan, applied at the centroid of the cross-section, the critical load is

$$P_{cr} = \frac{16.94}{\ell^2} \sqrt{E_{xx} I_z C} \quad (2.2.60)$$

$$\text{where } I_z = \frac{1}{12} h^3 b$$

and C is given by eq. (2.2.59). For $G_{xz} \approx G_{xy}$ one has

$$P_{cr} = \frac{2.82 b^3 h}{\ell^2} \sqrt{\left[1 - 0.63 \frac{b}{h} \right] E_{xx} G_{xz}} \quad (2.2.61)$$

We will make use of eq. (2.2.61) in an example in the next section. It should be noted that if the load is shifted up or down (from the Centroid) the critical load will decrease or increase from that given by eq. (2.2.61). To obtain estimates of these changes as well as for results for various other end conditions and loadings the reader may refer to reference 2.2.17 or other similar texts.

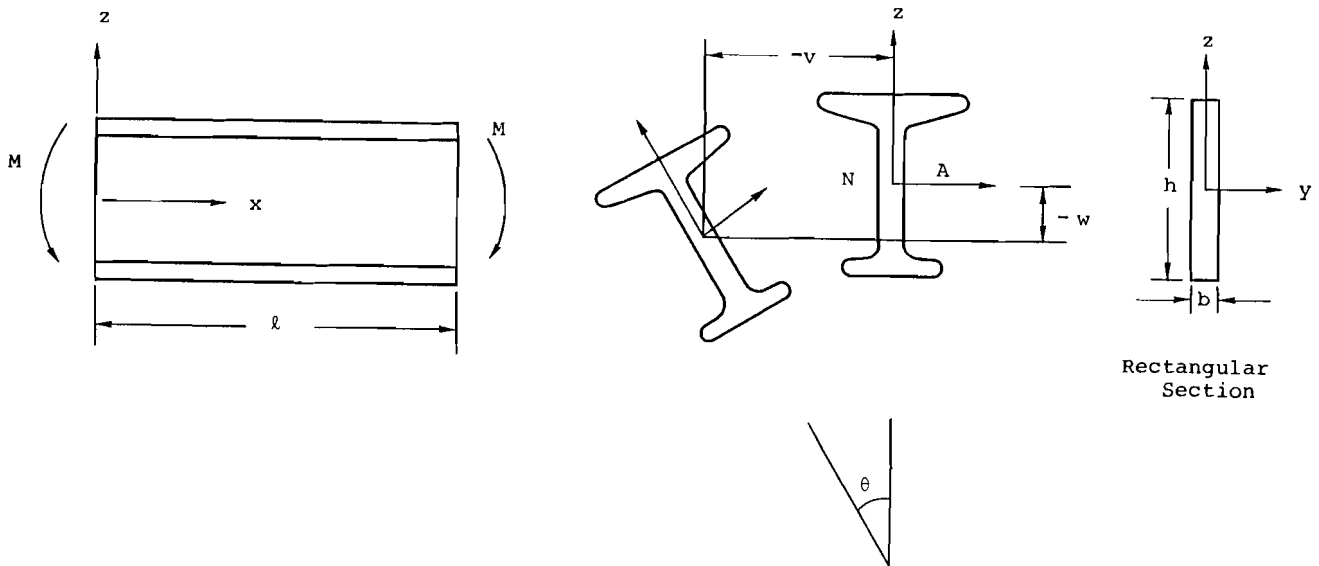


FIGURE 2.2.22. LATERAL BUCKLING OF A BEAM

2.2.11 A Simple Example.

To illustrate the usefulness of the simple beam theory for application to design and comparative studies with different types of composites we consider a simply supported rectangular beam shown in figure 2.2.23. Formulae which will be used in the example are also summarized in the figure. We consider the thickness and width of the beam to be 1 and 4 inches, respectively. Since the shear stress is maximum at the centroid the value of C^*z [see eq. (2.2.13)] is evaluated there. The numerical value of the quantities required for calculations and expressions for load carrying capacities in terms of maximum stresses are listed below:

$$\text{Bending} \quad I = \frac{1(4)^3}{12} = 5.33 \text{ in.}^4$$

$$\bar{h} = \frac{1(4)(2)}{1(4)} = 2 \text{ in.}$$

$$P = \frac{10.6667 \sigma_{\text{Max}}}{\ell}, \text{ lbs.}$$

$$\text{Shear} \quad C^*z = \frac{1(4)^2}{8} = 2 \text{ in.}^3$$

$$P = 5.33 \tau_{\text{Max}}, \text{ lbs.}$$

$$\text{Lateral Instability} \quad P_{\text{cr}} = 10.28 \sqrt{\frac{E_{xx} G_{xz}}{\ell^2}}$$

1. Moduli and maximum allowable stresses for different unidirectional materials are given in table 2.2.2.

2. Maximum loads and deflections which can be tolerated are plotted in figures 2.2.24 - 2.2.26 for various spans, these figures also illustrate which failure mode is likely for each span length.

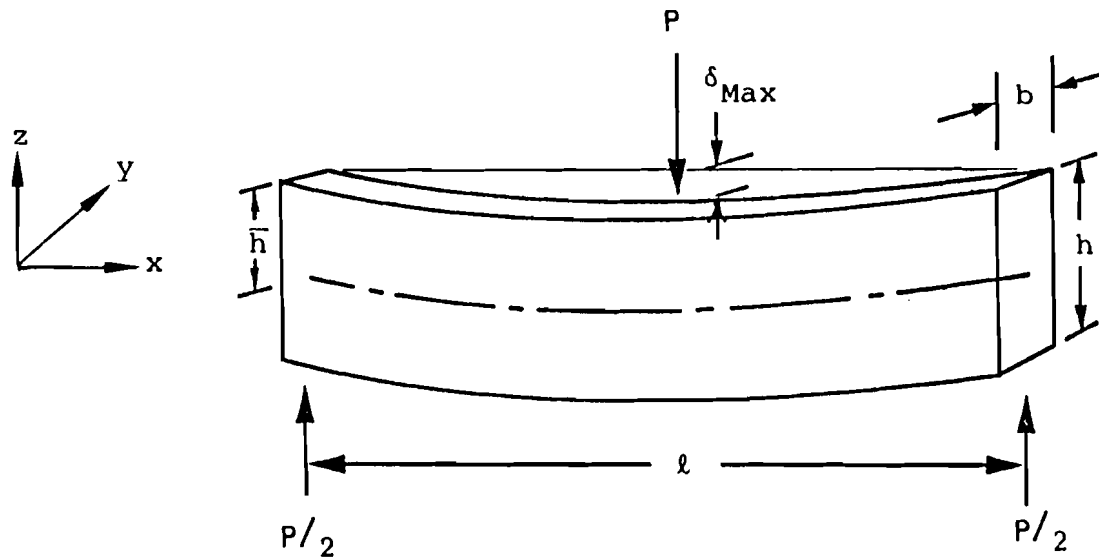
From these results the following general comments are appropriate:

o Differences in Strength Characteristics of Metal and Composite Simple Beams

Shear failures may be encountered in beams having ten times the span at which shear failures would be encountered in metal beams.

Lateral instability failures may be encountered in beams having less than one-half the span at which lateral instability would be encountered in metal beams

o Differences in Deflection Characteristics of Metal and Composite Simple Beams



$$I = \frac{bh^3}{12}, \quad M_{Max} = \frac{Pl}{4}, \quad \sigma_{Max} = \frac{M\bar{y}}{I}$$

$$C*\bar{z} = \frac{bh^2}{8}, \quad Q = \frac{P}{2}, \quad \tau = \frac{C*\bar{z}Q}{Ib}$$

$$\delta_{Max} = \frac{Pl^3}{48EI} + \frac{FPl}{4Gbh} \quad (F=1.2)$$

Lateral Instability:

$$P_{cr} = \frac{2.82b^3h}{l} \sqrt{C-0-63b/h) E_{xx} G_{xz}}$$

FIGURE 2.2.23 BENDING AND SHEAR - SIMPLE AND BEAM EXAMPLE

TABLE 2.2.2. PROPERTIES USED FOR SIMPLE BEAMS

MATERIAL	4130	2024-T3	E-GL/EP	KEV/EP	T-300/EP
σ_{Max} , ksi	95	50	80	40	100
τ_{Max} , ksi	55	40	3	3	3
E_{xx} , Msi	30	10.5	6.4	11.3	19.2
G_{xz} , Msi	12	4	0.6	0.7	0.6

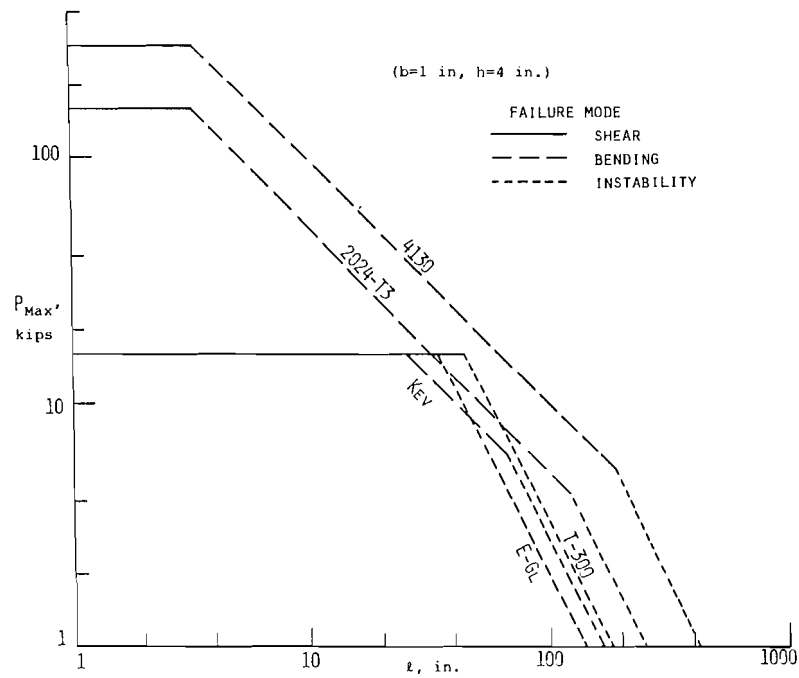


FIGURE 2.2.24 MAXIMUM LOADS ON SIMPLE BEAMS

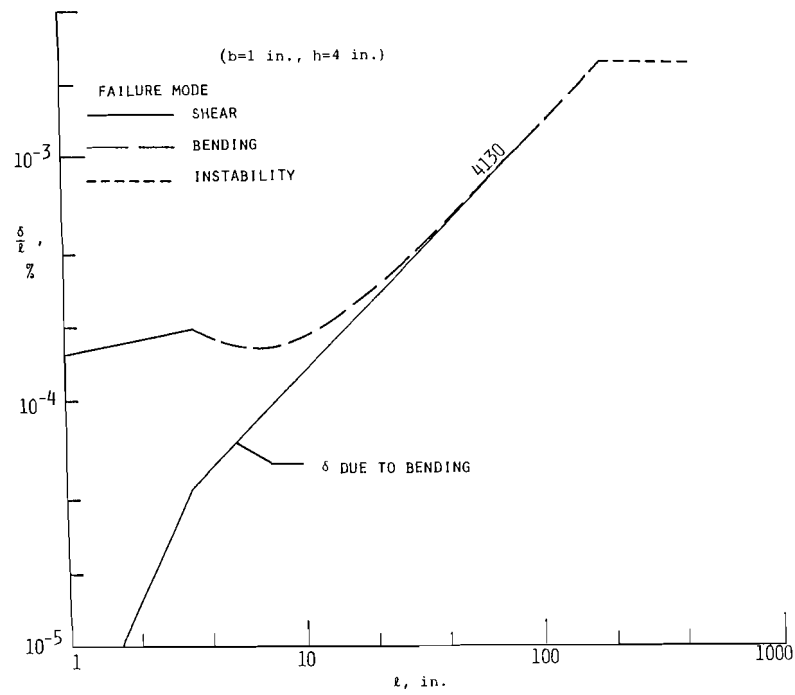


FIGURE 2.2.25 MAXIMUM DEFLECTIONS OF SIMPLE BEAMS

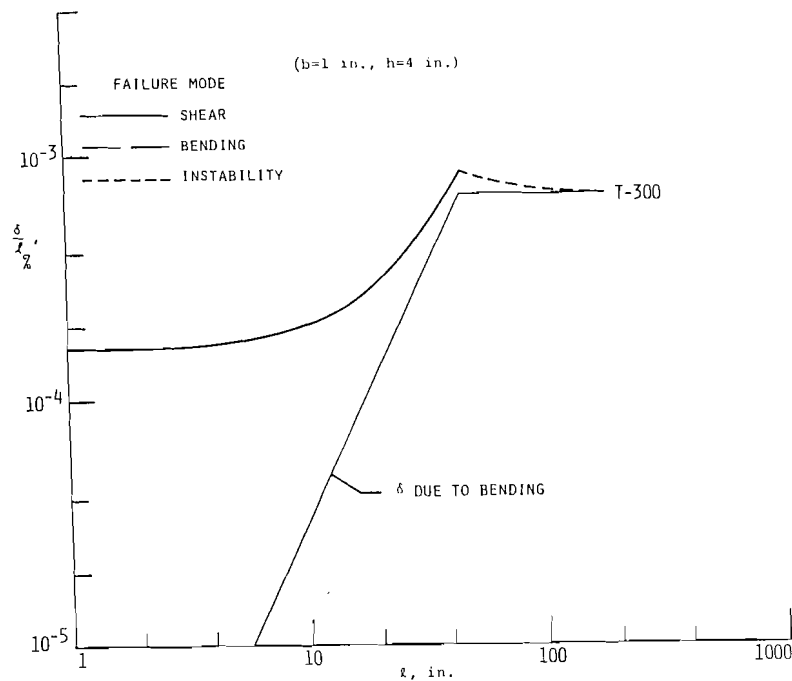


FIGURE 2.2.26 MAXIMUM DEFLECTIONS OF SIMPLE BEAMS

Shear deflections may not be negligible even for beams so long that they are laterally unstable.

The actual total deflection at failure of composite beams is not much different from that of metal beams.

o Possible Problems Encountered with Composite Beams not Encountered with Metal Beams

Low shear strength - short beams.

Lateral instability - long beams.

Low shear strengths and stiffnesses of unidirectionally reinforced composites are responsible for the differences discussed above.

2.2.12 Weight Optimization - Structural Efficiency.

Composite materials are usually chosen over conventional materials because of designs which yield low weights. Structural efficiency methodology is a systematic way (1) to select geometry and/or material, (2) to determine whether the limiting property is strength or stiffness, and (3) to provide quantitative evaluation of the effects of all changes including those in design condition themselves.

To perform such studies the following procedure is usually followed.

o Structural Efficiency Methodology

Define loads and governing dimensions

Derive indices incorporating design parameters

Define possible structural configurations

Define design limitations

Select candidate materials

Evaluate configurations/materials vs. indices.

To illustrate the procedure the example of a cantilever beam carrying a uniformly distributed load as shown in figure 2.2.27 has been selected. For simplicity the simple bending theory without any consideration to shear stresses or lateral instability will be used. The objective is to optimize the weight of the beam for different magnitudes of load and span. Required formulae are given below

$$\delta = \frac{q\ell^4}{8E_x I} \quad (2.2.62)$$

$$W = \rho A \ell \quad (2.2.63)$$

$$\sigma = \frac{q\ell^2 h}{4I} \quad (2.2.64)$$

where

- I - Moment of inertia of beam, in.⁴
- σ - Max, stress, psi
- W - Weight of beam, lbs.
- ρ - Density of beam material, pci
- A - Area of beam cross section, in.²

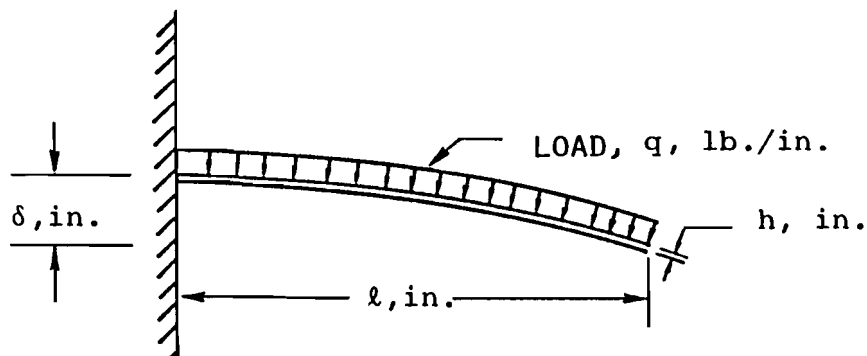


FIGURE 2.2.27 EVALUATION OF THE EFFICIENCY OF A UNIFORMLY LOADED CANTILEVER BEAM

For a given ratio of width (b) to depth (h) the cross-sectional dimensions (either b or h) can be eliminated from eq. (2.2.63) and (2.2.62) or (2.2.64). For a square cross-section results are given below

example: square cross-section beam, $\delta = 0.05\ell$

$$A = h^2$$

$$I = \frac{h^4}{12}$$

for strength

$$\frac{W}{\ell^3} = 2.08\rho^3 \sqrt[3]{\left(\frac{q/\ell}{\sigma}\right)^2} \quad (2.2.65)$$

for stiffness

$$\frac{W}{\ell^3} = 5.48\rho \sqrt[3]{\left(\frac{q/\ell}{E_x}\right)} \quad (2.2.66)$$

The efficiency plot is given in figure 2.2.28. Effects of configuration on weight efficiency are shown in figure 2.2.29. If different materials are considered, results can be plotted in the same figure (see figure 2.2.30 for example) which are very useful in many design processes.

Similar studies can be performed for cost optimization defining cost index with due consideration to weight savings (by defining premiums) and may be useful for selecting suitable material and cross-section before deciding on the fabrication of a component.

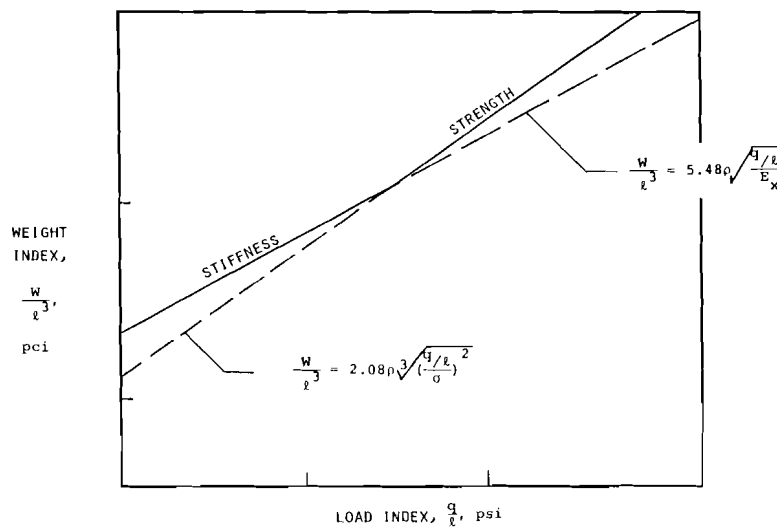


FIGURE 2.2.28 TYPICAL PLOT OF WEIGHT-EFFICIENCY OF SQUARE BEAMS

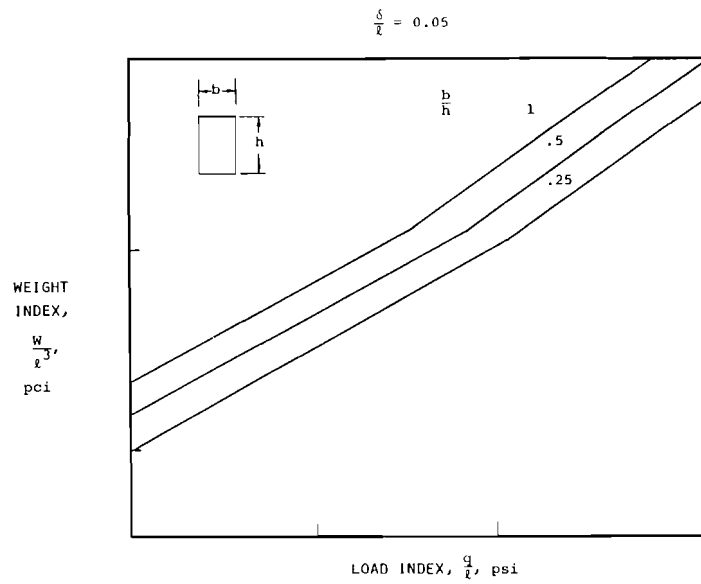


FIGURE 2.2.29 EFFECT OF CONFIGURATION ON WEIGHT-EFFICIENCY OF RECTANGULAR BEAMS

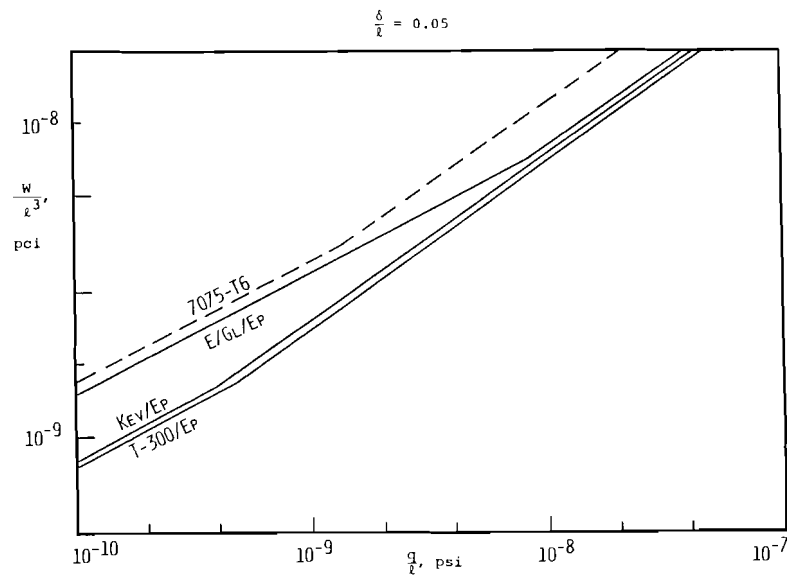


FIGURE 2.2.30 EFFECT OF MATERIAL ON WEIGHT-EFFICIENCY OF SQUARE BEAMS

2.2.13 Dynamics.

Timoshenko beam theory is directly applicable to planar dynamic problems. In terms of the displacement variable u_0 , ψ , w introduced in eq. (2.2.36) and stress resultants N , M , Q , the equations of motion (a dot over a quantity indicating time derivative) can be written in the following form:

$$\frac{\partial N}{\partial x} = P \ddot{u}_0 + R \dot{\psi} \quad (2.2.67)$$

$$\frac{\partial M}{\partial x} - Q = R \ddot{u}_0 + I \dot{\psi}$$

$$\frac{dQ}{dx} = P \ddot{w} + p(x, t)$$

where $p(x, t)$ is the time and space dependent applied pressure (it is assumed no other forces or moments are applied) and

$$(P, R, I) = \int_A \rho \left[1, z, z^2 \right] dA \quad (2.2.68)$$

ρ being the density at a point and A being one area of cross-section. For laminated beams

$$(P, R, I) = \int_{-h/2}^{h/2} \rho \left[1, z, z^2 \right] b(z) dz \quad (2.2.69)$$

$b(z)$ being the variation of width with respect to z (see figure 2.2.10)

Using the expressions for N , M , and Q in terms of $u_{0,x}$, ψ_x and $\psi + w_x$ [see eq. (2.2.39) and (2.2.45)] and writing $K' = kK$ where k is the shear correction factor and K is the shear stiffness one obtains.

$$\begin{aligned} A u_{0,xx} + B \psi_{,xx} &= P \ddot{u}_0 + R \dot{\psi} \\ B u_{0,xx} + D \psi_{,xx} - K' \left[\psi + w_{,x} \right] &= R \ddot{u}_0 + I \dot{\psi} \\ K \left[\psi_{,x} + w_{,xx} \right] &= P \ddot{w} + p(x, t) \end{aligned} \quad (2.2.70)$$

For a simply supported beam of span L the end conditions are

$$w = N = M = 0 \quad x = 0, L \quad (2.2.71)$$

For free vibration problems the solutions of (2.2.70) may be assumed in the form

$$w = -\sum_{n=1}^{\infty} \frac{a_n}{\alpha_n} \sin \alpha_n x e^{i\Omega t}$$

$$\psi + w, x = \sum_{n=1}^{\infty} b_n \cos \alpha_n x e^{i\Omega t} \quad (2.2.72)$$

$$U_0 = \sum_{n=1}^{\infty} C_n \cos \alpha_n x e^{i\Omega t}$$

$$\alpha_n = n\pi$$

which satisfy eq. (2.2.71). Substituting eq. (2.2.72) in (2.2.70) and using the orthogonality of sine and cosine functions of x , one obtains the following set of homogeneous equations

$$\left[\begin{array}{ccc} D & D & B \\ D & D+K'/\alpha_n^2 & B \\ B & B & A \end{array} \right] - \frac{\Omega^2}{\alpha_n^2} \left[\begin{array}{ccc} (P/\alpha_n^2 + I) & I & R \\ I & I & R \\ R & R & P \end{array} \right] \left[\begin{array}{c} a_n \\ b_n \\ c_n \end{array} \right] = 0$$

; $n = 1, 2, \dots$ (2.2.73)

from which three characteristic roots $\Omega = \Omega_n^i$ ($i=1,2,3$) for each n can be evaluated. These are natural frequencies of the beam. For the general case, coupling exists between the three displacement components u^0 , ψ and w for each n and i , the lowest value of Ω_n^1 is called the first natural frequency and the corresponding mode is called the first fundamental mode of vibration. For forced vibration problems with a given forcing frequency standard techniques of solution can be employed (see reference 2.2.18).

For the case when the beam is symmetric about the neutral axis $B = R = 0$, the two characteristic roots for each n are

$$\Omega_n^2 = \frac{1}{2} \left[\frac{D\alpha_n^2}{I} + K' \left(\frac{\alpha_n^2}{P} + \frac{1}{I} \right) \right] \pm \sqrt{\left\{ \frac{D\alpha_n^2}{I} + K' \left(\frac{\alpha_n^2}{P} + \frac{1}{I} \right) \right\}^2 - \frac{4DK'}{IP} \alpha_n^4} \quad (2.2.74)$$

and u_0 is no longer coupled with ψ and w .

When the effect of shear deformation and rotary inertia I are neglected one may also assume $b_n = 0$ and eq. (2.2.73) yields

$$\Omega_n^2 = D\alpha_n^4/P \quad (2.2.75)$$

which is the solution based on Euler Bernoulli theory.

For beams of other boundary conditions the procedure for calculation of natural frequencies is not as simple as the simply supported case. Even for Euler Bernoulli theory, evaluation of characteristic roots needs solution of a transcendental equation (reference 2.2.18). Transcendental equations which have to be solved for Timoshenko theory are discussed in reference 2.2.19. Results based on approximate formulations for clamped and cantilever beams, which are orthotropic, are given in table 2.2.3. Effects of E/G ratio and l/h ratio on the first natural frequency are illustrated in figures 2.2.31 and 2.2.32 respectively. These figures illustrate the influence of the low shear modulus of composites on beam vibration problems.

2.2.14 Limitation of Beam Theory for Large Width.

The classification of a long structural element under bending loads as a beam or a plate depends upon the nature of cross-sectional distortions.

o When anticlastic curvature is completely unrestrained (figure 2.2.33a) such that $R_y = -R_x/\nu_{xy}$, then the flexural rigidity is $E_x I$ and the element is considered a beam.

o When anticlastic curvature is completely restrained (figure 2.2.33b) such that $R_y = \infty$ then the flexural rigidity is $E_x I / (1 - \nu_{xy} \nu_{yx})$ and the element is considered a plate.

It has been shown that highly anisotropic composites arranged in certain laminate configurations, e.g. $[\pm\theta_n]_s$ or $[0, \pm\theta_n]_s$, may exhibit very large Poisson's ratios. Consequently the anticlastic distortion can play an important role in the classification of a structural element. Applications in which a knowledge of this phenomenon is important include static bending, vibration and column stability.

A general guideline for deciding whether the structure is plate or a beam is given below. In the interest of simplicity these formulae are given for an orthotropic configuration where the axes of orthotropism are parallel to plate sides.

For an orthotropic case:

$$\beta_o = \frac{b^2}{R_x h} \sqrt{3 \frac{E_x}{E_y} (1 - \nu_{xy} \nu_{yx})} \quad (2.2.76a)$$

For an isotropic case this reduces to:

$$\beta_i = \frac{b^2}{R_x h} \quad (2.2.76b)$$

- if $\beta < 1$ the structure will function as a beam
- if $\beta > 100$ the structure will function as a plate

TABLE 2.2.3 RESONANT FREQUENCIES OF SIMPLE ORTHOTROPIC BEAMS
WITH THE EFFECTS OF SHEAR AND ROTATOTRY INERTIA

SIMPLY SUPPORTED

$$\Omega_n^2 = \frac{E_{xx}}{2\rho} \left(\frac{n\pi}{L}\right)^2 + \frac{kG_{xz}}{2\rho} \left[\left(\frac{n\pi}{L}\right)^2 + \frac{A}{I}\right] \pm \sqrt{\left\{\frac{E_{xx}}{2\rho} \left(\frac{n\pi}{L}\right)^2 + \frac{kG_{xz}}{2\rho} \left[\left(\frac{n\pi}{L}\right)^2 + \frac{A}{I}\right]\right\}^2 - \frac{E_{xx}kG_{xz}}{\rho^2} \left(\frac{n\pi}{L}\right)^4}$$

CLAMPED

$$\Omega_n^2 \approx \frac{E_{xx}}{2\rho} \left(\frac{n\pi}{2L}\right)^2 + \frac{kG_{xz}}{2\rho} \left[\left(\frac{n\pi}{L}\right)^2 \frac{1}{4} + \frac{A}{I}\right] \pm \sqrt{\left\{\frac{E_{xx}}{2\rho} \left(\frac{n\pi}{L}\right)^2 + \frac{kG_{xz}}{2\rho} \left[\left(\frac{n\pi}{L}\right)^2 \frac{1}{4} + \frac{A}{I}\right]\right\}^2 - \frac{E_{xx}kG_{xz}}{4\rho^2} \left(\frac{n\pi}{L}\right)^4}$$

CANTILEVER

$$\Omega_n^2 \approx \frac{E_{xx}}{2\rho} \left(\frac{\pi}{2L}\right)^2 + \frac{kG_{xz}}{2\rho} \left[\left(\frac{\pi}{2L}\right)^2 \frac{1}{3} + \frac{A}{I}\right] \pm \sqrt{\left\{\frac{E_{xx}}{2\rho} \left(\frac{\pi}{2L}\right)^2 + \frac{kG_{xz}}{2\rho} \left[\left(\frac{\pi}{2L}\right)^2 \frac{1}{3} + \frac{A}{I}\right]\right\}^2 - \frac{4E_{xx}kG_{xz}}{3\rho^2} \left(\frac{\pi}{2L}\right)^4}$$

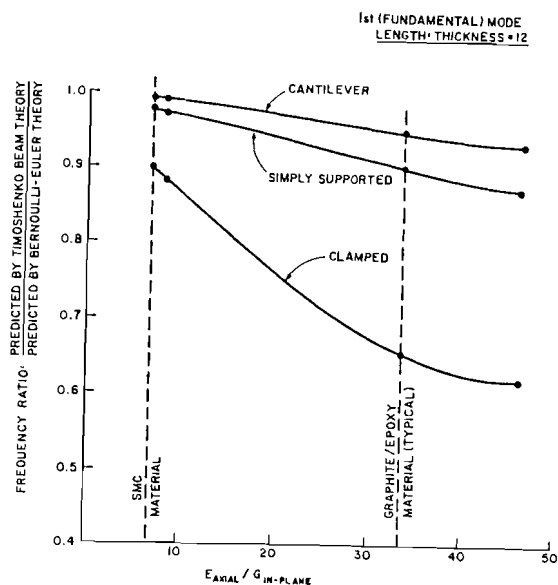


FIGURE 2.2.31 THE CORRECTION FOR SHEAR AND ROTATORY INTERIA OF SIMPLE COMPOSITE BEAMS

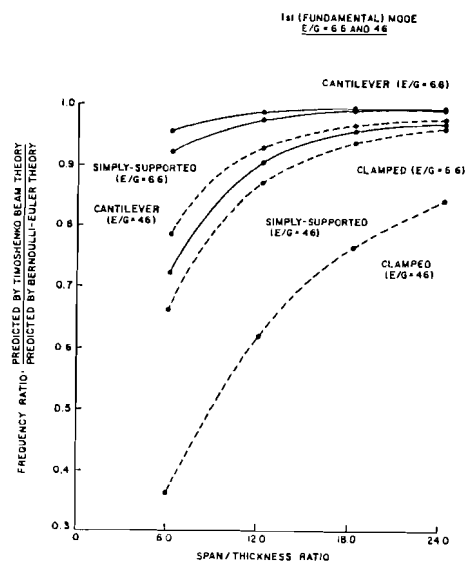
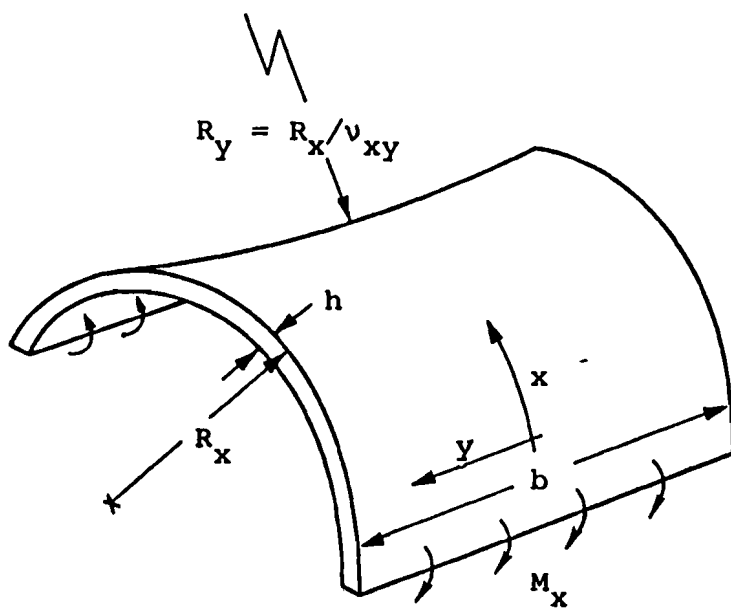


FIGURE 2.2.32 FREQUENCY REDUCTION VS. l/h IN COMPOSITE BEAMS

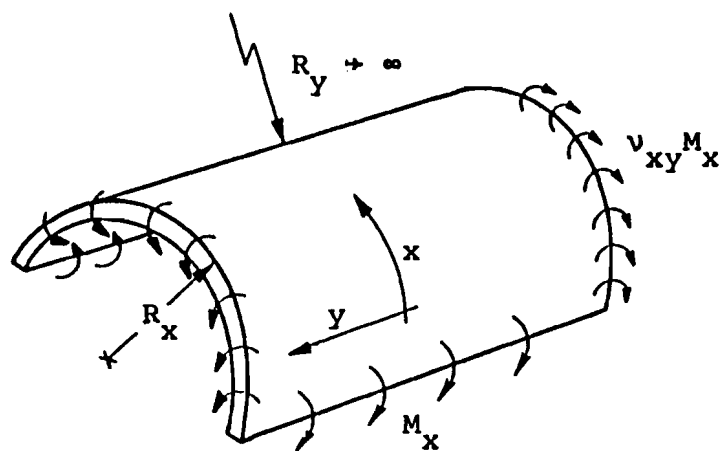


(A) Beam Element

ANTICLASTIC DISTORTION

UNRESTRAINED

FLEX. RIGIDITY = $E_x I$



(B) Plate Element

ANTICLASTIC DISTORTION

RESTRAINED

FLEX. RIGIDITY = $\frac{E_x I}{(1 - v_{xy} v_{yx})}$

FIGURE 2.2.33 BEAM AND PLATE ELEMENTS

- o if $1 < \beta < 100$ the structure will be in a transition region bounded by the former two conditions.

2.2.15 Curved Beams.

Low transverse properties of composites can have a significant influence in the case of curved beams of the configuration shown in figure 2.2.34.

A treatment of the curved orthotropic beam (figure 2.2.34) in bending has been presented in reference 2.2.20. A summary of the important expressions is given in the following equations.

Radial Stress

$$\sigma_r = \frac{-M}{b^2 h_g} \left\{ 1 - \left[\frac{1 - (a/b)^{k+1}}{1 - (a/b)^{2k}} \right] [r/b]^{k-1} - \left[\frac{1 - (a/b)^{k-1}}{1 - (a/b)^{2k}} \right] [a/r]^{k+1} \right\}$$

Tangential Stress

$$\sigma_\theta = - \frac{M}{b^2 b_g} \left\{ 1 - \left[\frac{1 - (a/b)^{k+1}}{1 - (a/b)^{2K}} \right] k [r/b]^{k-1} + \left[\frac{1 - (a/b)^{k-1}}{1 - (a/b)^{2k}} \right] k [a/r]^{k+1} \right\}$$

where $k = E_\theta / E_r$

$$\text{and } g = \frac{1}{2} \left[1 - (a/b)^2 \right] - \frac{k}{[1 - (a/b)^{2k}]} \left\{ \frac{[1 - (a/b)^{k+1}]^2}{k+1} + \frac{(a/b)^2 [1 - (a/b)^{k-1}]^2}{k-1} \right\}$$

The determination of transverse stress in curved composite beams or plates, is important to the composites engineer in that the curved configuration represents many features of detail in composite hardware. An awareness of the phenomenon is necessary for successful design with composites so that potential problems may be averted.

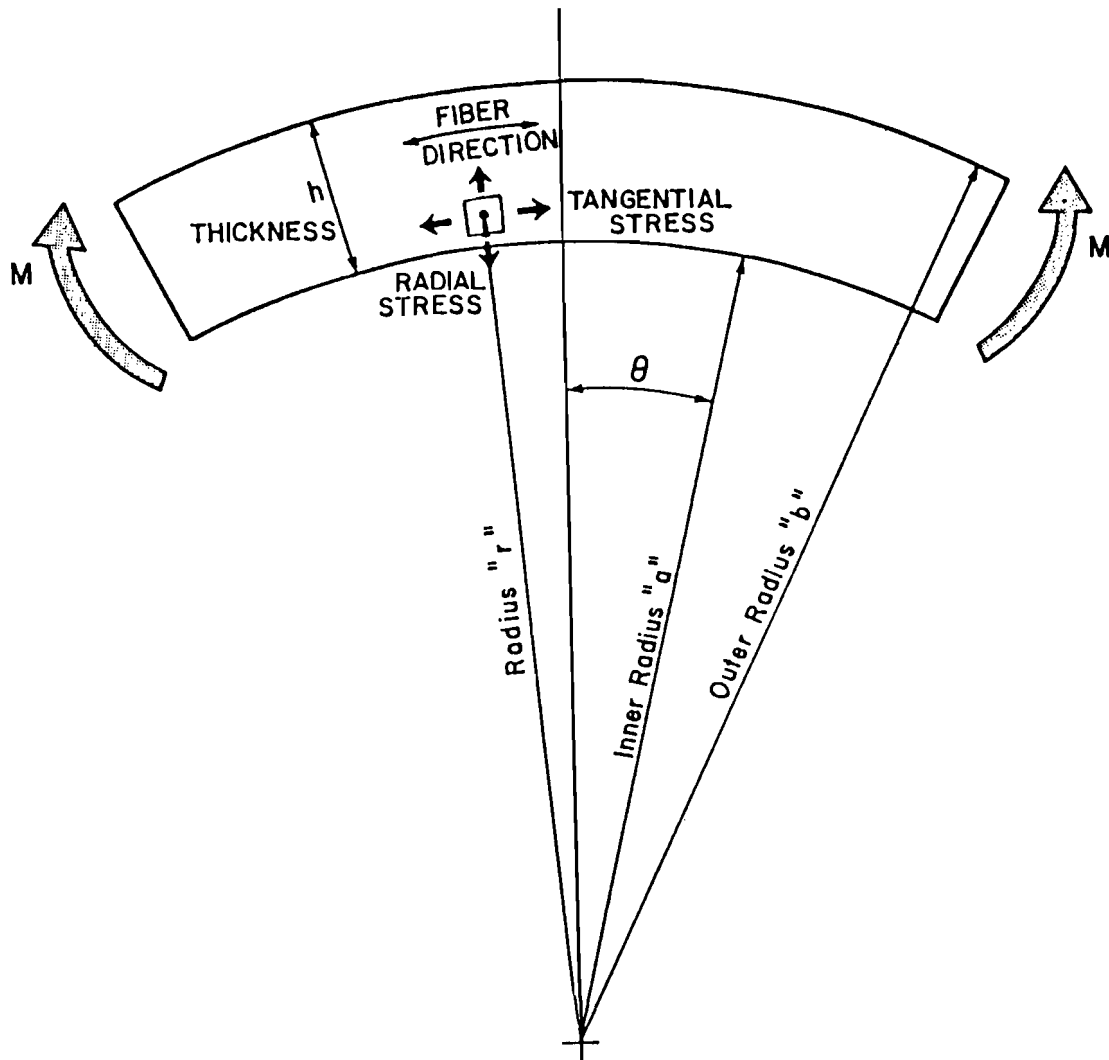


FIGURE 2.2.34 CURVED BEAM CONFIGURATION

2.2.16 Concluding Remarks.

In this chapter we have discussed in detail bending and torsion of composite beams with emphasis on shear stresses and shear deformations. We have also touched upon some important aspects like natural frequencies of vibration of straight beams and stresses in curved beams. There exist many other topics which are important for design considerations namely sandwich beams and beams of varying cross-section. Sandwich beams may be treated in the same fashion as laminated beams. However, some design formulae may be obtained by several simplifying assumptions. Such formulae and wrinkling effects are considered in reference 2.2.21. Varying cross-sections and assemblage of beam type members are best considered by the use of finite element methods using beam elements. Effects of bi-axial bending, combined bending and torsion in cross-sections without any vertical axis of symmetry are important and may be considered in the same fashion as 42 homogeneous beams with due consideration to low shear moduli of composites.

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2.3 PLATES.

2.3.1 Introduction.

Fiber composite materials may be efficiently used in the form of thin walled structures which are defined as surfaces of small thickness. When the surface is curved it is called a shell and when it is plane it is called a plate. In such cases the fiber composite is employed in the form of unidirectionally reinforced layers which are joined to form a laminate. The present chapter is concerned with the analysis of laminated plates. Such a plate may be subjected to loads in its plane, transversely to its plane and also to temperature and moisture changes. In all of these cases the loads produce internal membrane forces, bending and twisting moments and, of course, deformations in the plane and transversely to the plane.

In Volume I we have analyzed the internal stresses produced by membrane forces, bending and twisting moments in an element of a laminate. However, the determination of the forces and moments at any point is a major problem which will be addressed within this chapter. Unlike a beam where internal forces and moments are easily determined, it is necessary in the case of a plate to establish partial differential equations for the displacements and, to solve them for given support boundary conditions and to then determine internal forces and moments from the deflections. The internal stresses then follow from the results given in Chapter III of Volume I.

2.3.2 Static Elastic Behavior.

For purposes of continuity, the basic constitutive relations and kinematic relations from lamination theory as developed in Chapter III of Volume I are briefly reviewed here. In matrix notation the constitutive relations, including thermal effects, are of the form

$$\begin{Bmatrix} N \\ - \\ M \end{Bmatrix} = \begin{bmatrix} A & | & B \\ \hline B & | & D \end{bmatrix} \begin{Bmatrix} \epsilon^0 \\ \kappa \end{Bmatrix} - \begin{Bmatrix} N^T \\ - \\ M^T \end{Bmatrix} \quad (2.3.1)$$

where

$$\begin{aligned} \{N, M\} &= \int_{-h/2}^{h/2} \{\sigma\}(1, z) dz \\ \{A\} &= \int_{-h/2}^{h/2} \{\bar{Q}\} dz = \sum_{i=1}^N \{\bar{Q}\}_i (z_i - z_{i-1}) \end{aligned}$$

$$\{B\} = \int_{-h/2}^{h/2} \{\bar{Q}\} z \, dz = \frac{1}{2} \sum_{i=1}^N \{\bar{Q}\} (z_i^2 - z_{i-1}^2)$$

$$\{D\} = \int_{-h/2}^{h/2} \{\bar{Q}\} z^2 \, dz = \frac{1}{3} \sum_{i=1}^N \{\bar{Q}\} (z_i^3 - z_{i-1}^3)$$

$$\{N^T, M^T\} = \int_{-h/2}^{h/2} \{\Gamma\}_i \Delta T(1, z) \, dz$$

and the kinematic relations are of the form

$$\epsilon_{xx}^0 = \frac{\partial u^0}{\partial x}, \quad \epsilon_{yy}^0 = \frac{\partial v^0}{\partial y}, \quad 2\epsilon_{xy}^0 = \frac{\partial v^0}{\partial x} + \frac{\partial u^0}{\partial y} \quad (2.3.2)$$

$$\kappa_{xx} = -\frac{\partial^2 w}{\partial x^2}, \quad \kappa_{yy} = -\frac{\partial^2 w}{\partial y^2}, \quad \kappa_{xy} = -\frac{\partial^2 w}{\partial x \partial y}$$

In this context h is the total laminate thickness, and

$$u^0 = u_x^0, \quad v^0 = u_y^0, \quad w = u_z$$

Note that engineering strains are utilized, and that the last component of the curvature vector is twice the tensorial twisting stiffness. (See Section 3.3. of Volume I). The nomenclature for force and moment resultants are illustrated in figure 2.3.6 of Volume I.

2.3.2.1 Governing Equations.

The static equations of equilibrium, in terms of force and moment resultants, are those of classical plate theory (reference 2.3.1). Thus,

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = 0 \quad (2.3.3)$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yx}}{\partial y} = 0 \quad (2.3.4)$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \quad (2.3.5)$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = 0$$

$$\frac{\partial Q_{xx}}{\partial x} + \frac{\partial Q_{yy}}{\partial y} + q = 0 \quad (2.3.6)$$

where Q_x and Q_y are shear force resultants, i.e.

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz} dz, \quad Q_y = \int_{-h/2}^{h/2} \tau_{yz} dz \quad (2.3.7)$$

The surface normal stress term, q , is defined as

$$q = \sigma_{zz}(h/2) - \sigma_{zz}(-h/2)$$

Differentiating eqs. (2.3.4,5) and substituting the results into eq. (2.3.6) we obtain

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + q = 0 \quad (2.3.8)$$

Now eqs. (2.3.3,8) constitute the equilibrium equations in terms of force and moment resultants.

o Governing Equations in Terms of Displacements

Substituting the plate constitutive relations, eq. (2.3.1), into eqs. (2.3.3,8), we obtain the following equilibrium equations in terms of displacements:

$$\begin{aligned}
& A_{11} \frac{\partial^2 u^0}{\partial x^2} + 2A_{16} \frac{\partial^2 u^0}{\partial x \partial y} + A_{66} \frac{\partial^2 u^0}{\partial y^2} + A_{16} \frac{\partial^2 v^0}{\partial x^2} + (A_{12} \\
& + A_{66}) \frac{\partial^2 v^0}{\partial x \partial y} + A_{26} \frac{\partial^2 v^0}{\partial y^2} - B_{11} \frac{\partial^3 w}{\partial x^3} - 3B_{16} \frac{\partial^3 w}{\partial x^2 \partial y} - (B_{12} \\
& + 2B_{66}) \frac{\partial^3 w}{\partial x \partial y^2} - B_{26} \frac{\partial^3 w}{\partial y^3} - \frac{\partial N_{xx}^T}{\partial x} - \frac{\partial N_{xy}^T}{\partial y} = 0
\end{aligned} \tag{2.3.9}$$

$$\begin{aligned}
& A_{16} \frac{\partial^2 u^0}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 u^0}{\partial x \partial y} + A_{26} \frac{\partial^2 u^0}{\partial y^2} + A_{66} \frac{\partial^2 v^0}{\partial x^2} \\
& + 2A_{26} \frac{\partial^2 v^0}{\partial x \partial y} + A_{22} \frac{\partial^2 v^0}{\partial y^2} - B_{16} \frac{\partial^3 w}{\partial x^3} - (B_{12} + 2B_{66}) \frac{\partial^3 w}{\partial x^2 \partial y} \\
& - 3B_{26} \frac{\partial^3 w}{\partial x \partial y^2} - B_{22} \frac{\partial^3 w}{\partial y^3} - \frac{\partial N_{xy}^T}{\partial x} - \frac{\partial N_{yy}^T}{\partial y} = 0
\end{aligned} \tag{2.3.10}$$

$$\begin{aligned}
& D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3} \\
& + D_{22} \frac{\partial^4 w}{\partial y^4} - B_{11} \frac{\partial^3 u^0}{\partial x^3} - 3B_{16} \frac{\partial^3 u^0}{\partial x^2 \partial y} - (B_{12} + 2B_{66}) \frac{\partial^3 u^0}{\partial x \partial y^2} \\
& - B_{26} \frac{\partial^3 u^0}{\partial y^3} - B_{16} \frac{\partial^3 v^0}{\partial x^3} - (B_{12} + 2B_{66}) \frac{\partial^3 v^0}{\partial x^2 \partial y} - 3B_{26} \frac{\partial^3 v^0}{\partial x \partial y^2} \\
& - B_{22} \frac{\partial^3 v^0}{\partial y^3} - \frac{\partial^2 M_{xx}^T}{\partial x^2} + 2 \frac{\partial^2 M_{xy}^T}{\partial x \partial y} + \frac{\partial^2 M_{yy}^T}{\partial y^2} = q
\end{aligned} \tag{2.3.11}$$

For symmetric laminates ($B_{ij} = 0$) eqs. (2.3.9,10) will uncouple from eq. (2.3.11). Furthermore, for many practical applications, the temperature distribution will be independent of the inplane coordinates x, y , and derivatives of the thermal forces and moments will vanish. For such cases eqs. (2.3.9,10) become

$$A_{11} \frac{\partial^2 u^0}{\partial x^2} + 2A_{16} \frac{\partial^2 u^0}{\partial x \partial y} + A_{66} \frac{\partial^2 u^0}{\partial y^2} + (A_{12} + A_{66}) \frac{\partial^2 v^0}{\partial x^2} \quad (2.3.12)$$

$$+ A_{26} \frac{\partial^2 v^0}{\partial x \partial y} + A_{66} \frac{\partial^2 v^0}{\partial y^2} = 0$$

$$A_{16} \frac{\partial^2 u^0}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 u^0}{\partial x \partial y} + A_{26} \frac{\partial^2 u^0}{\partial y^2} + A_{66} \frac{\partial^2 v^0}{\partial x^2} \quad (2.3.13)$$

$$+ 2A_{26} \frac{\partial^2 v^0}{\partial x \partial y} + A_{22} \frac{\partial^2 v^0}{\partial y^2} = 0$$

Equations (2.3.12,13) are now analogous to the plane stress equations for homogeneous anisotropic elasticity. The difference is the units of the inplane stiffness terms (A_{ij}), which are in force per unit length rather than force per unit area.

Equation (2.3.11) reduces to the governing equation for the bending of a homogeneous, anisotropic plate (reference 2.3.2), i.e.

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3} \quad (2.3.14)$$

$$+ D_{22} \frac{\partial^4 w}{\partial y^4} = q$$

o Governing Equations in Terms of a Stress Function and Transverse Displacement

For certain static problems in which the thermal forces are independent of the x,y coordinates, a stress function formulation of the inplane problem is useful (reference 2.3.3). We now define a stress function such that

$$N_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \quad N_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \quad N_{xy} = - \frac{\partial^2 \phi}{\partial x \partial y} \quad (2.3.15)$$

Equations (2.3.3) are exactly satisfied by eq. (2.3.15)

Multiplying the first of eqs. (2.3.1) by A^{-1} (inverse of matrix A), we obtain

$$\epsilon^0 = A^{-1} N - A^{-1} B_K - A^{-1} N^T$$

Substituting this relationship into the second of eqs. (2.3.1) we obtain

$$M = BA^{-1}N + (D - BA^{-1}B)\kappa + BA^{-1}N^T - M^T$$

and eq. (2.3.1) can be written in the semi-inverted form

$$\begin{Bmatrix} \varepsilon^0 \\ M \end{Bmatrix} = \begin{bmatrix} A^* & B^* \\ (-B^*)^T & D^* \end{bmatrix} \begin{Bmatrix} N \\ \kappa \end{Bmatrix} + \begin{Bmatrix} N^{*T} \\ M^{*T} \end{Bmatrix} \quad (2.3.16)$$

where the subscript T denotes a transpose matrix, and

$$\begin{aligned} A^* &= A^{-1}, \quad B^* = -A^{-1}B, \quad D^* = D - BA^{-1}B \\ N^{*T} &= A^{-1}N^T, \quad M^{*T} = BA^{-1}N^T - M^T \end{aligned}$$

In the general case A^* and D^* are symmetric matrices while B^* is not. The D^* matrix is sometimes referred to as the reduced bending stiffness matrix (reference 2.2.3). More will be said later concerning the importance of the D^* matrix.

Substituting eq. (2.3.16) into eq. (2.3.8), and taking into account eq. (2.3.15), we obtain the bending equation

$$\begin{aligned} D_{11}^* \frac{\partial^4 w}{\partial x^4} + 4D_{16}^* \frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12}^* + 2D_{66}^*) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26}^* \frac{\partial^4 w}{\partial x \partial y^3} \\ + D_{22}^* \frac{\partial^4 w}{\partial y^4} + B_{21}^* \frac{\partial^4 \phi}{\partial x^4} + (2B_{26}^* - B_{61}^*) \frac{\partial^4 \phi}{\partial x^3 \partial y} + (B_{11}^* + B_{22}^* \\ - B_{66}^*) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + (2B_{16}^* - B_{62}^*) \frac{\partial^4 \phi}{\partial x \partial y^3} + B_{12}^* \frac{\partial^4 \phi}{\partial y^4} = q \end{aligned} \quad (2.3.17)$$

In order to obtain a second relationship, the inplane compatibility equation from classical theory of elasticity is considered, i.e.

$$\frac{\partial^2 \varepsilon_{xx}^0}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}^0}{\partial x^2} - \frac{\partial^2 \gamma_{xy}^0}{\partial x \partial y} = 0 \quad (2.3.18)$$

Combining eqs. (2.3.15,16,18), we obtain the compatibility equation

$$\begin{aligned}
& A_{22}^* \frac{\partial^4 \phi}{\partial x^4} - 2A_{26}^* \frac{\partial^4 \phi}{\partial x^3 \partial y} + (2A_{12}^* + A_{66}^*) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} - 2A_{16}^* \frac{\partial^4 \phi}{\partial x \partial y^3} \\
& + A_{11}^* \frac{\partial^4 \phi}{\partial y^4} - B_{21}^* \frac{\partial^4 w}{\partial x^4} + (B_{61}^* - 2B_{26}^*) \frac{\partial^4 w}{\partial x^3 \partial y} + (2B_{66}^* - B_{11}^* \\
& - B_{22}^*) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + (B_{62}^* - B_{16}^*) \frac{\partial^4 w}{\partial x \partial y^3} - B_{12}^* \frac{\partial^4 w}{\partial y^4} = 0 \quad (2.3.19)
\end{aligned}$$

For symmetric layups $B_{ij}^* = 0$, $D_{ij}^* = D_{ij}$, and eq. (2.3.17) reduces to eq. (2.3.14), while eq. (2.3.19) becomes

$$\begin{aligned}
& A_{22}^* \frac{\partial^4 \phi}{\partial x^4} - 2A_{26}^* \frac{\partial^4 \phi}{\partial x^3 \partial y} + (2A_{12}^* + A_{66}^*) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} \\
& - 2A_{16}^* \frac{\partial^4 \phi}{\partial x \partial y^3} + A_{11}^* \frac{\partial^4 \phi}{\partial y^4} = 0 \quad (2.3.20)
\end{aligned}$$

which is essentially the stress function equation for an anisotropic plane stress elasticity problem (reference 2.3.2). For homogeneous isotropic materials

$$\begin{aligned}
A_{11}^* &= A_{22}^* = \frac{1}{Eh}, \quad A_{12}^* = -\frac{\nu}{Eh}, \quad A_{66}^* = \frac{2(1+\nu)}{Eh} \\
A_{16}^* &= A_{26}^* = 0
\end{aligned}$$

and eq. (2.3.20) reduces to the well-known bi-harmonic equation

$$\nabla^4 \phi = 0 \quad (2.3.21)$$

The inplane displacements can be determined by integrating the strain-displacement relations, eq. (2.3.2), in conjunction with the constitutive relations (2.3.16). Thus,

$$\begin{aligned}
u^0 &= A_{12}^* \frac{\partial \phi}{\partial x} + A_{11}^* \int \frac{\partial^2 \phi}{\partial y^2} dx - A_{16}^* \frac{\partial \phi}{\partial y} - B_{11}^* \frac{\partial w}{\partial x} \\
&- B_{12}^* \int \frac{\partial^2 w}{\partial y^2} dx - 2B_{16}^* \frac{\partial w}{\partial y} + f(y) \quad (2.3.22)
\end{aligned}$$

$$\begin{aligned}
v^0 = & -A_{26}^* \frac{\partial \phi}{\partial x} + A_{22}^* \int \frac{\partial^2 \phi}{\partial x^2} dy + A_{12}^* \frac{\partial \phi}{\partial y} - 2B_{26}^* \frac{\partial w}{\partial x} \\
& - B_{21}^* \int \frac{\partial^2 w}{\partial x^2} dy - B_{22}^* \frac{\partial w}{\partial y} + g(x)
\end{aligned} \tag{2.3.23}$$

where $f(y)$ and $g(x)$ are functions of integration and represent rigid body displacements.

o Ply Stresses

The ply stresses are related to laminate strains through the reduced stiffness matrix $[\bar{Q}]$, as illustrated in eq. (6.12) of Volume I. In particular,

$$\{\sigma_x^i\} = [\bar{Q}^i] \{\epsilon^i\} - \{\Gamma^i\} \Delta T$$

or in terms of laminate strains

$$\{\sigma_x^i\} = [\bar{Q}^i] \{\epsilon^0\} - z^i [\bar{Q}^i] \{\kappa\} - \{\Gamma^i\} \Delta T \tag{2.3.24}$$

This relationship can be used in conjunction with the displacement formulation, eqs. (2.2.9-11), and the strain-displacement relations, eqs. (2.3.2), to obtain ply stresses.

For the case where the inplane problem is formulated in terms of a stress function, the constitutive relations as given by eq. (2.3.16) can be utilized, i.e.

$$\begin{aligned}
\{\sigma^i\} = & [\bar{Q}^i] [A^*] \{N\} + [\bar{Q}^i] [B^*] \{\kappa\} + [A^*] \{N^T\} \\
& - z^i [\bar{Q}^i] \{\kappa\} - \{\bar{\Gamma}^i\} \Delta T
\end{aligned} \tag{2.3.25}$$

Thus, the ply stresses can be obtained from the stress function and the transverse displacement.

o Interlaminar Stresses

It has been noted in Section 3.6.4 of Volume I, that stresses in a zone of a laminate near free edges are three-dimensional in nature. Thus, in such cases, the assumption of plane stress breaks down.

For the general case of laminates undergoing transverse bending, transverse shear stresses are induced throughout the laminate. These stresses are usually small compared to the maximum inplane bending stresses. Thus, the assumption of plane stress is justified in formulating gross laminate response.

Interlaminar failure, however, represents a potentially serious mode of failure in a laminated composite. Thus, the transverse shear stresses on interlaminar planes are of interest to the structural designer. If the inplane stresses are accurately represented by classical laminated plate theory, then the interlaminar stresses can be reasonably estimated by integrating the equilibrium equations from the classical theory of elasticity. Comparison of interlaminar stresses determined by this approach with the same stresses calculated from solutions obtained from elasticity theory applied to laminated plates (references 3.5-7) shows excellent agreement for thin plates ($a/h, b/h > 20$).

Now consider the following equilibrium equations from classical theory of elasticity as applied to the i th ply:

$$\frac{\partial \sigma_{xx}^i}{\partial x} + \frac{\partial \tau_{xy}^i}{\partial y} + \frac{\partial \tau_{xz}^i}{\partial z} = 0 \quad (2.3.26)$$

$$\frac{\partial \tau_{xy}^i}{\partial x} + \frac{\partial \sigma_{yy}^i}{\partial y} + \frac{\partial \tau_{yz}^i}{\partial z} = 0$$

$$\frac{\partial \tau_{xz}^i}{\partial x} + \frac{\partial \tau_{yz}^i}{\partial y} + \frac{\partial \sigma_{zz}^i}{\partial z} = 0 \quad (2.3.27)$$

The interlaminar shear stresses can be obtained by integrating eqs. (2.3.26), with the result

$$\tau_{xz}^i = - \int_{-h/2}^{z^i} \left(\frac{\partial \sigma_{xx}^i}{\partial x} + \frac{\partial \tau_{xy}^i}{\partial y} \right) dz \quad (2.3.28)$$

$$\tau_{yz}^i = - \int_{-h/2}^{z^i} \left(\frac{\partial \tau_{xy}^i}{\partial x} + \frac{\partial \sigma_{yy}^i}{\partial y} \right) dz$$

Solutions to boundary value problems in conjunction with eqs. (2.3.24,25) can be utilized to determine the inplane stresses. Results can then be substituted into eq. (2.3.28), and integrated to obtain the interlaminar shear stresses. Since the inplane stresses are linear in z , the resulting interlaminar shear stresses will be a second order polynomial within each ply with a discontinuity in slope at the ply interfaces.

Results from eqs. (2.3.28) can be substituted into the third equilibrium equation of elasticity, eq. (2.3.27), and integrated to obtain the interlaminar normal stress. Thus,

$$\sigma_{zz}^i = \int_{-h/2}^{z^i} \left(\frac{\partial \tau_{xz}^i}{\partial x} + \frac{\partial \tau_{yz}^i}{\partial y} \right) dz \quad (2.3.29)$$

Numerical examples will be presented in the next section which illustrate the use of eqs. (2.3.28,29).

o Boundary Conditions

Laminated plate theory is not complete without considering the proper boundary conditions. Such conditions guarantee unique solutions to the governing equations. Proper boundary conditions can be obtained from an energy formulation of the governing equations in a manner similar to Mindlin (reference 2.3.8). It can be shown that the necessary boundary conditions are those of classical homogeneous plate theory plus those of an inplane elasticity problem. Thus, one member of each pair of the following four quantities must be prescribed along the boundary:

$$u_n^0, N_n; \quad u_s^0, N_{ns}; \quad \frac{\partial w}{\partial n}, M_n; \quad w, \frac{\partial M_{ns}}{\partial s} + Q_n \quad (2.3.30)$$

where n and s denote coordinates normal and tangential to the plate edge, respectively. The quantity $\partial M_{ns} / \partial s + Q_n$ is the well-known Kirchhoff condition. Thus, for the general case of a laminated plate, four boundary conditions along each edge must be prescribed.

2.3.2.2 Stresses and Deflections.

The behavior of laminated plates under transverse loading will now be investigated by obtaining solutions to a number of meaningful boundary value problems. Closed form solutions are not available for the general case of laminated rectangular plates. Thus, in examining the response of various laminates to transverse loading, examples of increasing complexity of behavior will be discussed.

The one-dimensional problem of cylindrical bending does provide a source for obtaining solutions to laminated plates of arbitrary stacking geometry. Stress analysis can then be performed with a minimum of complexity. Thus, cylindrical bending will be the first example presented. Rectangular plates will then be discussed with emphasis being placed on understanding some of the unique behavior displayed by laminated composite materials. This will allow us to put various categories of laminate complexities in the proper perspective.

o Cylindrical Bending

Consider a laminated plate composed of an arbitrary number of layers having infinite length in the y -direction, uniformly supported along the edges $x=0$, a ,

and subjected to the transverse load $q = q(x)$ (see figure 2.3.1). Under these conditions the deflected surface is cylindrical in nature, i.e.

$$u^0 = u^0(x), \quad v^0 = v^0(x), \quad w = w(x) \quad (2.3.31)$$

Many practical problems involve a plate whose length is large compared to its width, a . In such problems, the major central region of the plate may be treated by the following methods, regardless of the nature of the boundary conditions on the short edges. Non-cylindrical bending in such plates will generally be confined to the regions close to the short edges.

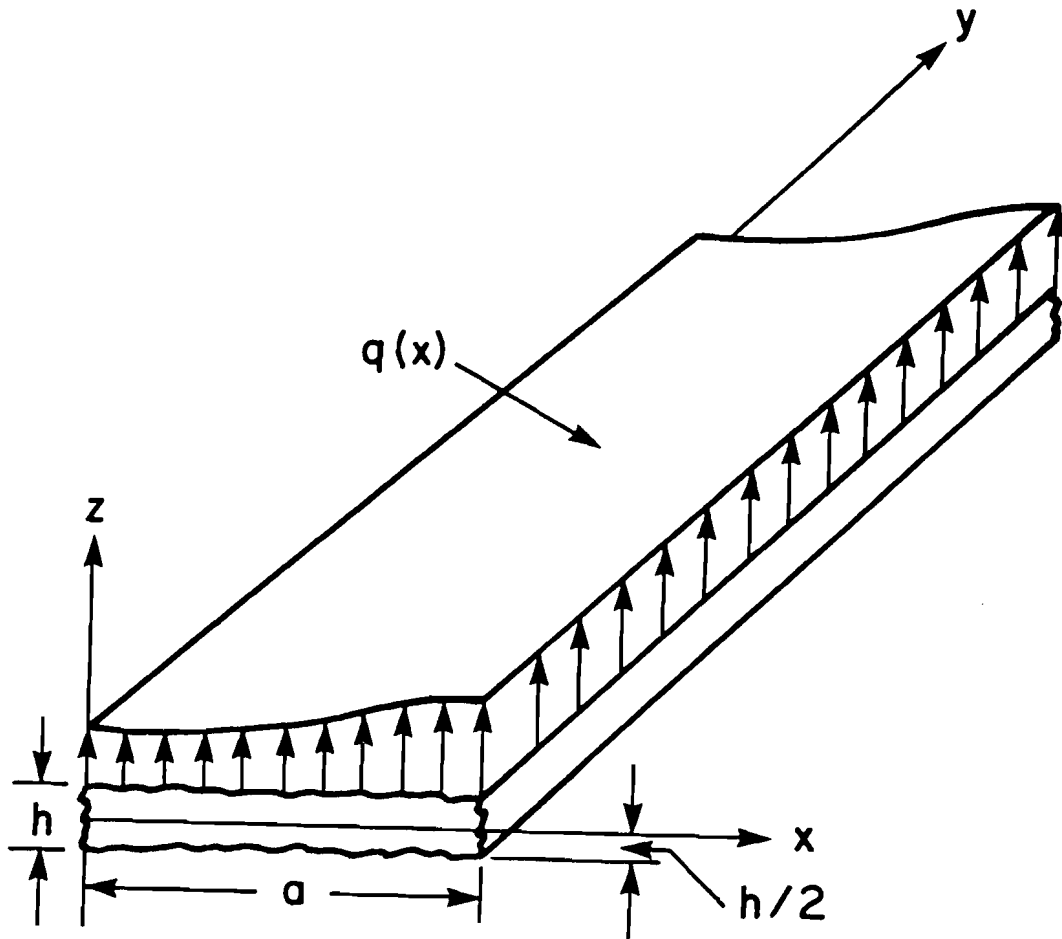


FIGURE 2.3.1 CYLINDRICAL BENDING

Using eq.(2.3.31) in conjunction with the displacement formulation of plate bending, eqs. (2.3.9-11), in the absence of thermal effects, we obtain the following relations for cylindrical bending:

$$A_{11} \frac{d^2 u^0}{dx^2} + A_{16} \frac{d^2 v^0}{dx^2} - B_{11} \frac{d^3 w}{dx^3} = 0 \quad (2.3.32)$$

$$A_{16} \frac{d^2 u^0}{dx^2} + A_{66} \frac{d^2 v^0}{dx^2} - B_{16} \frac{d^3 w}{dx^3} = 0 \quad (2.3.33)$$

$$D_{11} \frac{d^4 w}{dx^4} - B_{11} \frac{d^3 u^0}{dx^3} - B_{16} \frac{d^3 v^0}{dx^3} = q \quad (2.3.34)$$

These equations can be uncoupled by solving eqs. (2.3.32,33) for the following relationships:

$$\frac{d^2 u^0}{dx^2} = \frac{B}{A} \frac{d^3 w}{dx^3} \quad (2.3.35)$$

$$\frac{d^2 v^0}{dx^2} = \frac{C}{A} \frac{d^3 w}{dx^3} \quad (2.3.36)$$

where

$$A = A_{11}A_{66} - A_{16}^2, \quad B = A_{66}B_{11} - A_{16}B_{16},$$

$$C = A_{11}B_{16} - A_{16}B_{11}$$

Differentiating these expressions and then substituting them into eq. (2.3.34), we obtain

$$\frac{d^4 w}{dx^4} = \frac{A}{D} q \quad (2.3.37)$$

where

$$D = D_{11}A - B_{11}B - B_{16}C$$

Equation (2.3.37) can now be integrated and the result substituted into eqs. (2.3.35,36) with the result

$$\frac{d^2 u^0}{dx^2} = \frac{B}{D} \int q(x) dx \quad (2.3.38)$$

$$\frac{d^2 v^0}{dx^2} = \frac{C}{D} \int q(x) dx \quad (2.3.39)$$

From the laminated plate constitutive relations, eqs. (2.3.1), in the absence of thermal loads we obtain the following constitutive relations for cylindrical bending:

$$\begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{16} & B_{11} \\ A_{12} & A_{26} & B_{12} \\ A_{16} & A_{66} & B_{16} \\ B_{11} & B_{16} & D_{11} \\ B_{12} & B_{26} & D_{12} \\ B_{16} & B_{66} & D_{16} \end{bmatrix} \begin{bmatrix} \frac{du^0}{dx} \\ \frac{dv^0}{dx} \\ -\frac{d^2 w}{dx^2} \end{bmatrix} \quad (2.3.40)$$

and from the equilibrium eqs. (2.3.4,5),

$$Q_x = \frac{dM_{xx}}{dx}, \quad Q_y = \frac{dM_{xy}}{dx} \quad (2.3.41)$$

For the case $q = q_0 = \text{constant}$, direct integration of eqs. (2.3.37-39) leads to the results

$$u^0 = \frac{B}{D} (q_0 \frac{x^3}{6} + c_1 \frac{x^2}{2} + c_2 x + c_3) \quad (2.3.42)$$

$$v^0 = \frac{C}{D} (q_0 \frac{x^3}{6} + c_1 \frac{x^2}{2} + c_4 x + c_5) \quad (2.3.43)$$

$$w = \frac{A}{D} (q_0 \frac{x^4}{24} + c_1 \frac{x^3}{6} + c_6 \frac{x^2}{2} + c_7 x + c_8) \quad (2.3.44)$$

Now we consider two sets of boundary conditions

Case 1: Simple-Supports

at $x = 0, a$

$$N_x = N_{xy} = w = M_x = 0 \quad (2.3.45)$$

This is equivalent to classical simple-support boundary conditions (see figure 2.3.2a). In order to prevent rigid body displacements, the supports are assumed to be fixed to the plate at the origin, i.e.

$$u^0(0) = v^0(0) = 0 \quad (2.3.46)$$

Combining eq. (2.3.40) and eqs. (2.3.42-46), we obtain

$$u^0 = \frac{Bq_0 x^2}{12D} (2x - 3a) \quad (2.3.47)$$

$$v^0 = \frac{Cq_0 x^2}{12D} (2x - 3a) \quad (2.3.48)$$

$$w = \frac{Aq_0 x}{24D} (x^3 - 2ax^2 + a^3) \quad (2.3.49)$$

Force and moment resultants can easily be calculated from eq. (2.3.40).

Case 2: Clamped-Supports

at $x = 0, a$

$$u^0 = v^0 = w = \frac{dw}{dx} = 0 \quad (2.3.50)$$

This boundary condition is illustrated in fig. (2.3.2b). We derive the following displacement equations by combining eqs. (2.3.42-44) with eqs. (2.3.50,51)

$$u^0 = \frac{Bq_0 x}{12D} (2x^2 - 3ax + a^2) \quad (2.3.51)$$

$$v^0 = \frac{Cq_0 x}{12D} (2x^2 - 3ax + a^2) \quad (2.3.52)$$

$$w = \frac{Aq_0 x^2}{24D} (x^2 - 2ax + a^2) \quad (2.3.53)$$

In each of the above cases, the maximum plate deflection can be written in the form

$$w_{\max} = w' \left[1 + \frac{AD_{11}K}{D} \right] \quad (2.3.54)$$

where

$$K = \frac{B_{11}B + B_{16}C}{AD_{11}}$$

and w' denotes the maximum deflection obtained when the coupling coefficients are neglected in the governing equations. For simple-supports

$$w' = \frac{5q_0 a^4}{384D_{11}} \quad (2.3.55)$$

and for clamped-supports

$$w' = \frac{q_0 a^4}{384D_{11}} \quad (2.3.56)$$

It can be shown that A , D , and K in eqs. (2.3.55,56) are always positive. Thus, bending-extensional coupling tends to increase the maximum plate deflection. It will be shown in later sections that the magnitude of this increase depends on individual ply properties and the number of plies in the laminate.

It should also be noted that we are not completely free to pick boundary conditions in conjunction with eq. (2.3.30). In particular, consider eq. (2.3.3) for the case of cylindrical bending. These equations become

$$\frac{dN_{xx}}{dx} = 0, \quad \frac{dN_{xy}}{dx} = 0 \quad (2.3.57)$$

Thus, both N_{xx} and N_{xy} must be constant. This means that if we choose either of these quantities on one boundary, they are automatically fixed at the same value on the opposite boundary.

Cylindrical bending provides a convenient solution to generalized laminates which can be used to gain insight into plate stresses generated under transverse loading. As a simple example let us consider the $[0^\circ/90^\circ]$ class of laminates subjected to cylindrical bending under uniform transverse load. We will use the simple-support boundary conditions. Thus, the displacements will be given by eqs. (2.3.47-49).

Ply properties as given in table 3.1 of Volume I will be utilized for the numerical examples. In particular

$$E_1 = 25.0 \times 10^6 \text{ psi}, \quad E_2 = 1.7 \times 10^6 \text{ psi},$$

$$G_{12} = 0.65 \times 10^6 \text{ psi}, \quad \nu_{12} = 0.30$$

This yields the following ply stiffness matrix

$$[\bar{Q}] = \begin{bmatrix} 25.15 \times 10^6 & 5.131 \times 10^5 & 0 \\ & 1.71 \times 10^6 & 0 \\ \text{sym} & & 0.65 \times 10^6 \end{bmatrix}$$

As a first example, consider the symmetric laminate $[0^\circ/90^\circ]_s$. For this case $u^0 = v^0 = 0$, and eq. (2.3.49) becomes

$$w = \frac{q_0 x}{24D_{11}} (x^3 - 2ax^2 + a^3) \quad (2.3.58)$$

and

$$\kappa_{xx} = -\frac{d^2 w}{dx^2} = \frac{q_0 x}{2D_{11}} (a - x) \quad (2.3.59)$$

From the ply constitutive relations, eq. (2.2.13) of Vol. I and eq. (2.3.59)

$$\sigma_x^i = \bar{Q}_{11}^i \frac{q_0 x}{2D_{11}} (a - x) z^i \quad (2.3.60)$$

$$\sigma_y^i = \bar{Q}_{12}^i \frac{q_0 x}{2D_{11}} (a - x) z^i \quad (2.3.61)$$

$$\tau_{xy}^i = 0 \quad (2.3.62)$$

Substituting these relationships into eq. (2.3.28), we obtain the following interlaminar shear stresses

$$\tau_{xz}^i = \frac{q_0}{2D_{11}} \int_{-h/2}^{z^i} Q_{11}^i (2x - a) z \, dz \quad (2.3.63)$$

$$\tau_{yz}^i = 0$$

For the laminate under consideration, it can be shown that

$$\begin{aligned} D_{11} &= [7\bar{Q}_{11}(0^\circ) + Q_{11}(90^\circ)] \frac{h^3}{96} \\ &= [7(25.15) + 1.71] \frac{h^3 \times 10^6}{96} = 1.85 \times 10^6 h^3 \end{aligned}$$

Using this in conjunction with eqs. (2.3.60,61), we obtain the ply stresses.

Since $Q_{12}(0^\circ) = Q_{12}(90^\circ)$, the transverse stress, σ_{yy} , will be continuous through-the-thickness of the laminate. Thus, in normalized form

$$\begin{aligned}\bar{\sigma}_{yy} &= \frac{\bar{Q}_{12} \bar{x}(1 - \bar{x}) \bar{z}}{(2)(1.85)(10^6)} = \frac{(0.5131)}{(2)(1.85)} \bar{x}(1 - \bar{x}) \bar{z} \\ &= (0.139) \bar{x}(1 - \bar{x}) \bar{z}\end{aligned}\quad (2.3.64)$$

where

$$\bar{x} = \frac{x}{a}, \quad \bar{z} = \frac{z}{h}, \quad \bar{\sigma}_{yy} = \frac{\sigma_{yy} h^2}{q_0 a^2}$$

For the normal stress, σ_{xx} , the following results are obtained:

0° plies

$$\begin{aligned}\bar{\sigma}_{xx} &= \frac{Q_{11}(0^\circ)}{(2)(1.85)(10^6)} \bar{x}(1 - \bar{x}) \bar{z} = \frac{(25.15)}{(2)(1.85)} \bar{x}(1 - \bar{x}) \bar{z} \\ &= (6.80) \bar{x}(1 - \bar{x}) \bar{z} \quad \begin{cases} 0.25 \leq \bar{z} \leq 0.5 \\ -0.5 \leq \bar{z} \leq -0.25 \end{cases}\end{aligned}\quad (2.3.65)$$

where

$$\bar{\sigma}_{xx} = \frac{\sigma_{xx} h^2}{q_0 a^2}$$

90° plies

$$\begin{aligned}\bar{\sigma}_{xx} &= Q_{11}(90^\circ) \bar{x}(1 - \bar{x}) \bar{z} = \frac{(1.71)}{(2)(1.85)} \bar{x}(1 - \bar{x}) \bar{z} \\ &= (0.462) \bar{x}(1 - \bar{x}) \bar{z}, \quad -0.25 \leq \bar{z} \leq 0.25\end{aligned}\quad (2.3.66)$$

Performing the integration in eq. (2.3.63), we obtain the interlaminar shear stress distribution. This distribution is symmetric through-the-thickness. Thus, the integration only need be performed up to the center line.

0° plies

$$\begin{aligned}
 \bar{\tau}_{xz} &= \frac{\bar{Q}_{11}(0^\circ)(2\bar{x} - 1)}{(2)(1.85)(10^6)} \int_{-1/2}^{\bar{z}} \bar{z} \, d\bar{z} \\
 &= \frac{(25.15)}{(2)(1.85)} (2\bar{x} - 1) \left(\frac{\bar{z}^2}{2} - \frac{1}{8} \right) \quad (2.3.67) \\
 &= 0.850(1 - 2\bar{x})(1 - 4\bar{z}^2) \quad \begin{cases} -0.5 \leq \bar{z} \leq -0.25 \\ 0.25 \leq \bar{z} \leq 0.5 \end{cases}
 \end{aligned}$$

where

$$\bar{\tau}_{xz} = \frac{\tau_{xz} h}{q_0 a}$$

90° plies

$$\begin{aligned}
 \bar{\tau}_{yz} &= \frac{(2\bar{x} - 1)}{(2)(1.85)(10^6)} \left[\int_{-1/2}^{-1/4} \bar{Q}_{11}(0^\circ) \bar{z} \, d\bar{z} \right. \\
 &\quad \left. + \int_{-1/4}^{\bar{z}} \bar{Q}_{11}(90^\circ) \bar{z} \, d\bar{z} \right]
 \end{aligned}$$

The first integral is simply the integral we evaluated for the shear stresses in the 0° plies. Thus,

$$\begin{aligned}
 \bar{\tau}_{xz} &= (2\bar{x} - 1) \left[- (0.850)(0.75) + \frac{(1.71)}{(2)(1.85)} \int_{-1/4}^{\bar{z}} \bar{z} \, d\bar{z} \right] \quad (2.3.68) \\
 &= (2\bar{x} - 1) \left[- 0.638 + 0.462 \left(\bar{z}^2 - \frac{1}{32} \right) \right] \\
 &= 0.231(1 - 2\bar{x})(2.82 - \bar{z}^2), \quad -0.25 \leq \bar{z} \leq 0.25
 \end{aligned}$$

The interlaminar normal stress can now be calculated from eq. (2.3.29) in conjunction with eqs. (2.3.67,68).

Lower 0° ply

$$\begin{aligned}\bar{\sigma}_{zz} &= (0.850)(2) \int_{-1/2}^{\bar{z}} (1 - 4\bar{z}^2) d\bar{z} \\ &= 1.70 \left(\bar{z} - \frac{4\bar{z}^3}{3} + \frac{1}{2} - \frac{1}{6} \right) \\ &= 0.567 (1 + 3\bar{z} - 4\bar{z}^3), \quad -0.5 \leq \bar{z} \leq -0.25\end{aligned}\tag{2.3.69}$$

where

$$\bar{\sigma}_{zz} = \frac{\sigma_{zz}}{q_0}$$

90° plies

$$\begin{aligned}\bar{\sigma}_{zz} &= 1.7 \int_{-1/2}^{-1/4} (1 - 4\bar{z}^2) d\bar{z} \\ &\quad + (0.231)(2) \int_{-1/4}^{\bar{z}} (2.82 - \bar{z}^2) d\bar{z} \\ &= 0.177 + 0.462 \left(2.82\bar{z} - \frac{\bar{z}^3}{3} + \frac{2.82}{4} - \frac{1}{192} \right) \\ &= .50 (1 + 2.61\bar{z} - 0.308\bar{z}^3), \quad -0.25 \leq \bar{z} \leq 0.25\end{aligned}\tag{2.3.70}$$

Outer 0° ply

$$\begin{aligned}
 \bar{\sigma}_{zz} &= 1.7 \int_{-1/2}^{-1/4} (1 - 4\bar{z}^2) d\bar{z} \\
 &\quad + 0.462 \int_{-1/4}^{-1/4} (2.82 - \bar{z}^2) d\bar{z} \\
 &\quad + (0.850)(2) \int_{1/4}^{\bar{z}} (1 - 4\bar{z}^2) d\bar{z} \\
 &= 0.177 + 0.647 + 1.7 \left(\bar{z} - \frac{4\bar{z}^3}{3} - \frac{1}{4} + \frac{1}{48} \right) \\
 &= 0.434(1 + 3.92\bar{z} - 5.22\bar{z}^3), \quad 0.25 \leq \bar{z} \leq 0.5
 \end{aligned} \tag{2.3.71}$$

The normal stress σ_{xx} , interlaminar shear stress τ_{xz} , and the interlaminar normal stress σ_{zz} are illustrated in figures 2.3.3, 4, and 5, respectively. Values of x are chosen at cross-sections where σ_{xx} and τ_{xz} are maximum. It should be noted that σ_{zz} is independent of x .

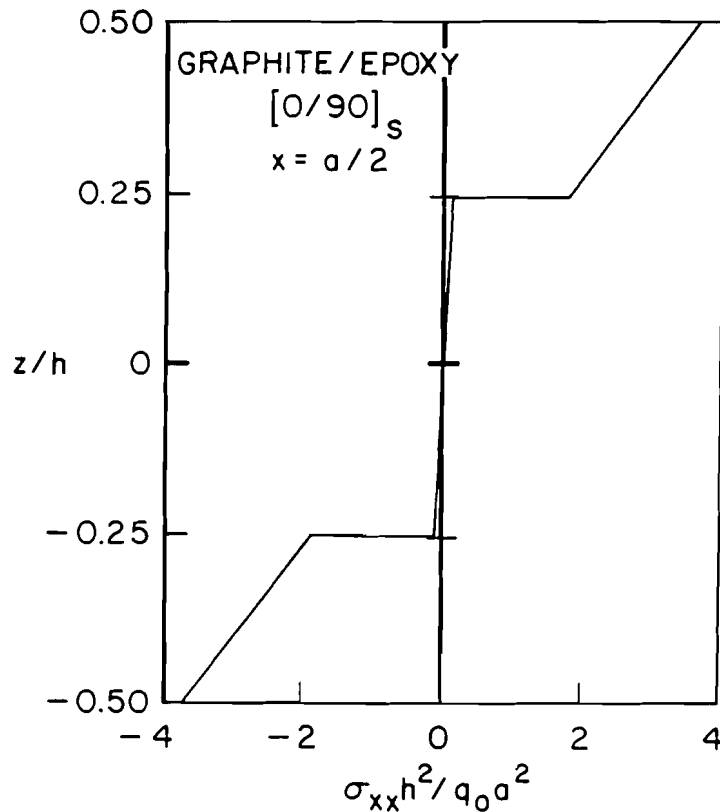


FIGURE 2.3.3. INPLANE NORMAL STRESS AT $x=a/2$ FOR [0/90]_s LAMINATE

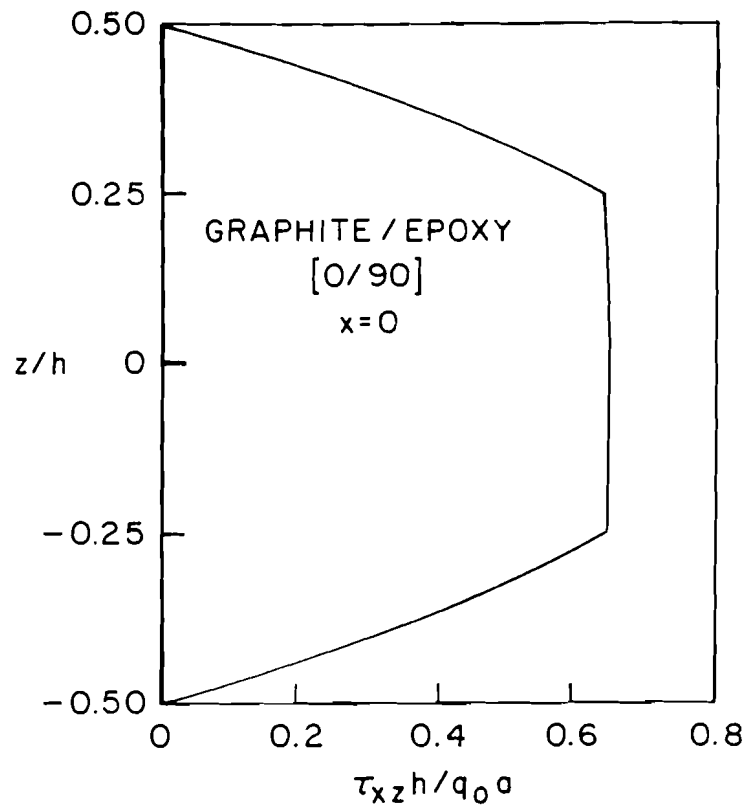


FIGURE 2.3.4. INTERLAMINAR SHEAR STRESS AT $x=0$ FOR $[0/90]_s$ LAMINATE

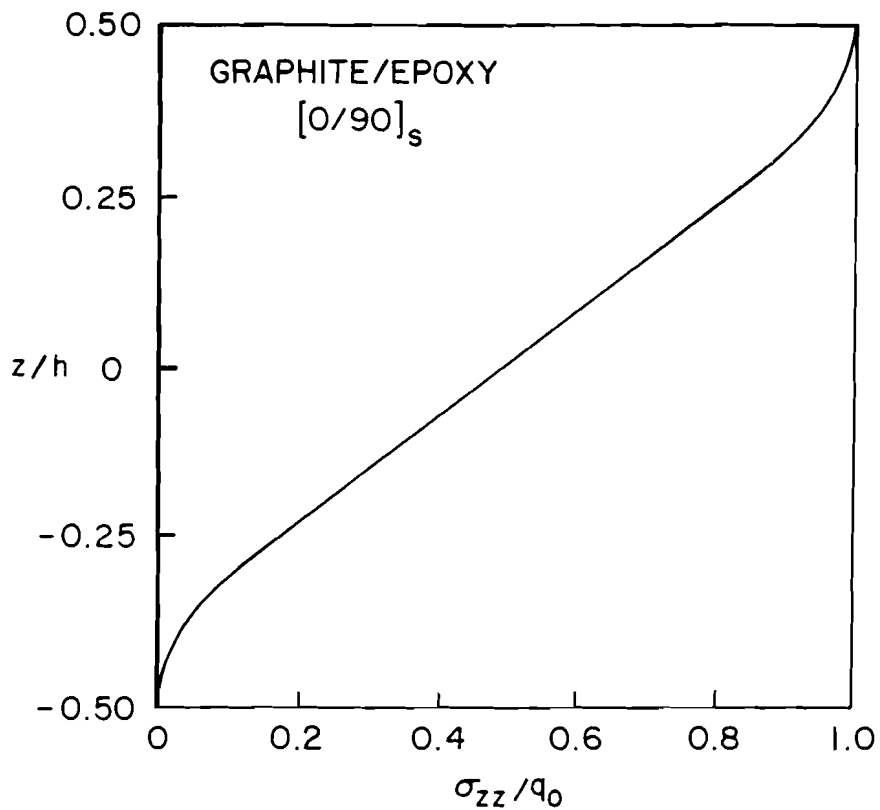


FIGURE 2.3.5. INTERLAMINAR NORMAL STRESS FOR $[0/90]_s$ LAMINATE

To illustrate the effect of stacking sequence on ply stresses, we now consider the symmetric laminate $[90/0]_s$. For this case

$$\begin{aligned} D_{11} &= [7\bar{Q}_{11}(90^\circ) + \bar{Q}_{11}0^\circ] \frac{h^3}{96} \\ &= [(7)(1.71) + 25.15] \frac{h^3 \times 10^6}{96} = 0.387 \times 10^6 h^3 \end{aligned}$$

Since $\bar{Q}_{12}(0^\circ) = \bar{Q}_{12}(90^\circ)$, σ_{yy} is still described by eq. (2.3.64).

Following the same procedure as with the original stacking sequence, we obtain the following stress distributions:

$$\bar{\sigma}_{xx} = (2.21)\bar{x}(1 - \bar{x})\bar{z} \quad \begin{cases} 0.25 \leq \bar{z} \leq 0.5 \\ -0.5 \leq \bar{z} \leq -0.25 \end{cases}$$

$$\bar{\sigma}_{xx} = (32.5)\bar{x}(1 - \bar{x})\bar{z}, \quad -0.25 \leq \bar{z} \leq 0.25$$

$$\bar{\tau}_{xz} = (0.276)(1 - 2\bar{x})(1 - 4\bar{z}^2) \quad \begin{cases} 0.25 \leq \bar{z} \leq 0.5 \\ -0.5 \leq \bar{z} \leq -0.25 \end{cases}$$

$$\bar{\tau}_{xz} = 16.3(1 - 2\bar{x})(0.0752 - \bar{z}^2), \quad -0.25 \leq \bar{z} \leq 0.25$$

Little new information is obtained by calculating the interlaminar normal stress σ_{zz} for this case. The normal stress σ_{xx} is illustrated in fig. 2.3.6, while the interlaminar shear stress τ_{xz} is shown in figure 2.3.7.

Examination of the results of these stress calculations illustrates certain unique aspects of the stress distribution in composite plates. Thus, figure 2.3.6 demonstrates that the maximum stresses associated with a bending moment do not necessarily occur at the outer surfaces. Comparison of figures 2.3.6 and 2.3.3 illustrates the substantial differences associated with stacking sequence for laminates having identical plies, with the identical orientations. Similarly, transverse shear stresses vary substantially with changes in stacking sequence.

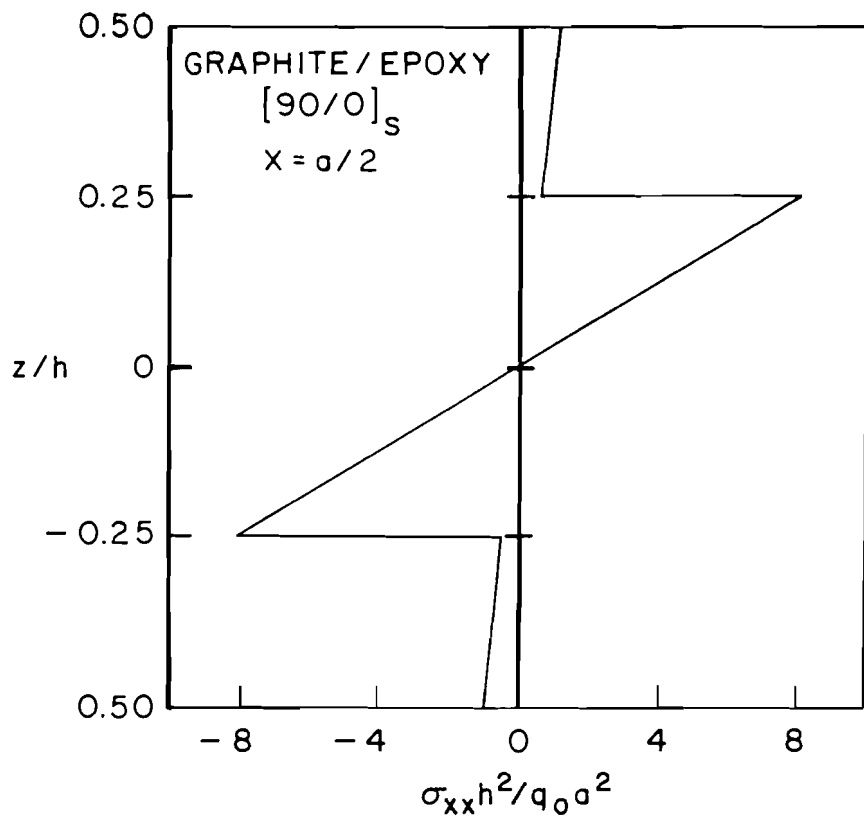


FIGURE 2.3.6. INPLANE STRESS AT $x=a/2$ FOR [0/90]_s LAMINATE

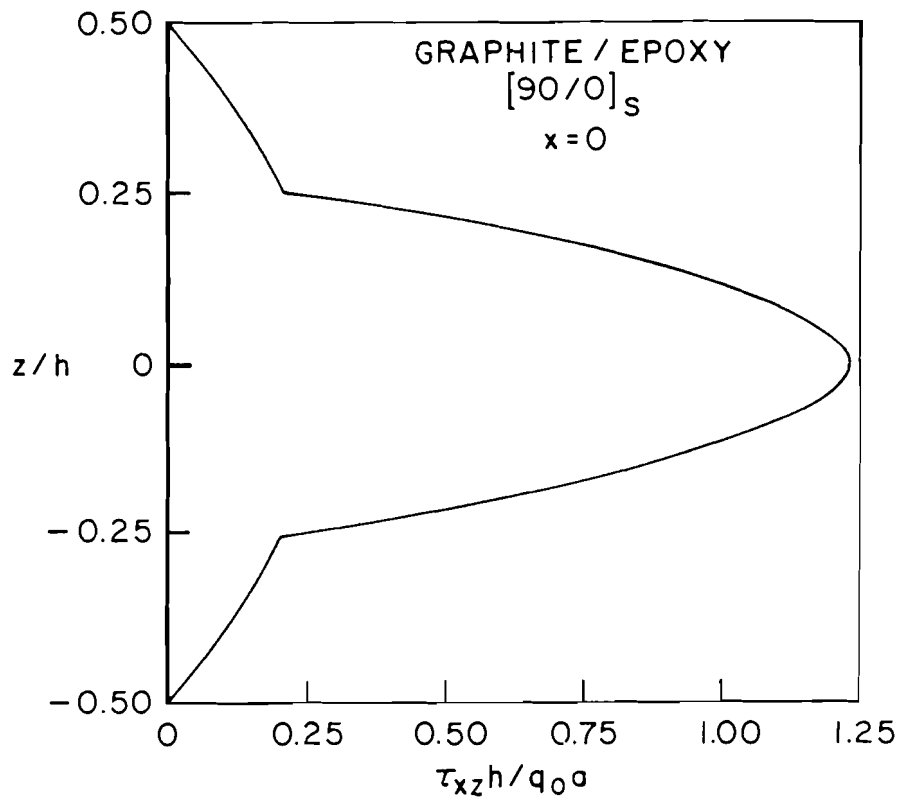


FIGURE 2.3.7. INTERLAMINAR SHEAR STRESS AT $x=0$ FOR [90/0]_s LAMINATE

Now let us consider an additional complexity by determining the stresses in an unsymmetric laminate with the stacking geometry [0/90]. Since the plies are orthotropic ($\bar{Q}_{16} = \bar{Q}_{26} = 0$), we know that $D_{16} = D_{26} = B_{16} = B_{26} = A_{16} = A_{26} = 0$. Furthermore, $\bar{Q}_{12}(0^\circ) = \bar{Q}_{12}(90^\circ)$ and $\bar{Q}_{66}(0^\circ) = \bar{Q}_{66}(90^\circ)$ which leads to the result $B_{12} = B_{66} = 0$. Thus, B_{11} and B_{22} are the only nonvanishing coupling stiffness terms. A cursory examination of eqs. (2.3.51-53) leads us to the result

$$u^0 = \frac{B_{11}q_0 x^2}{12(D_{11}A_{11} - B_{11}^2)} (2x - a) \quad (2.3.72)$$

$$v^0 = 0 \quad (2.3.73)$$

$$w = \frac{A_{11}q_0 x}{24(D_{11}A_{11} - B_{11}^2)} (x^3 - 2ax^2 + a^3) \quad (2.3.74)$$

and

$$\begin{aligned} \epsilon_{xx}^0 &= \frac{du^0}{dx} = \frac{B_{11}q_0 x}{2(D_{11}A_{11} - B_{11}^2)} (x - a) \\ \kappa_{xx} &= -\frac{d^2 w}{dx^2} = \frac{-A_{11}q_0 x}{2(D_{11}A_{11} - B_{11}^2)} (x - a) \end{aligned}$$

Again utilizing the ply constitutive relations from eq. (2.2.13) of Vol. I

$$\begin{aligned} \sigma_{xx}^i &= \bar{Q}_{11}^i \epsilon_{xx}^0 + z \bar{Q}_{11}^i \kappa_{xx} \\ &= \frac{\bar{Q}_{11}^i q_0 x}{2(D_{11}A_{11} - B_{11}^2)} (x - a) (B_{11} - z^i A_{11}) \end{aligned} \quad (2.3.75)$$

$$\begin{aligned} \sigma_{yy}^i &= \bar{Q}_{12}^i \epsilon_{xx}^0 + z \bar{Q}_{12}^i \kappa_{xx} \\ &= \frac{\bar{Q}_{12}^i q_0 x}{2(D_{11}A_{11} - B_{11}^2)} (x - a) (B_{11} - z^i A_{11}) \end{aligned} \quad (2.3.76)$$

$$\tau_{xy}^i = 0 \quad (2.3.77)$$

Using eq. (2.3.75) in conjunction with eqs. (2.3.28), we obtain the interlaminar stress distribution.

$$\tau_{xz}^i = \frac{q_0(a - 2x)}{2(D_{11}A_{11} - B_{11}^2)} \int_{-h/2}^{z^i} \bar{Q}_{11}^i (B_{11} - z^i A_{11}) dz$$

$$\tau_{yz}^i = 0 \quad (2.3.78)$$

For the laminate under consideration

$$\begin{aligned} A_{11} &= [\bar{Q}_{11}(0^\circ) + \bar{Q}_{11}(90^\circ)] \frac{h}{2} = (25.15 + 1.71) \frac{h \times 10^6}{2} \\ &= 13.4 \times 10^6 h \end{aligned}$$

$$\begin{aligned} B_{11} &= [\bar{Q}_{11}(0^\circ) - \bar{Q}_{11}(90^\circ)] \frac{h^2}{8} = (25.15 - 1.71) \frac{h \times 10^6}{8} \\ &= 2.93 \times 10^6 h^2 \end{aligned}$$

The bending stiffness D_{11} for the present unsymmetric laminate is the same as for a homogeneous plate, i.e.

$$D_{11} = \frac{A_{11} h^2}{12} = \frac{13.4}{12} \times 10^6 h^3 = 1.12 \times 10^6 h^3$$

and

$$\begin{aligned} (D_{11}A_{11} - B_{11}^2) &= [(1.12)(13.4) - (2.93)^2] (10^6)(10^6) h^4 \\ &= 6.42 \times 10^{12} h^4 \end{aligned}$$

Again, since $\bar{Q}_{12}(0^\circ) = \bar{Q}_{12}(90^\circ)$, the transverse normal stress σ_{yy} is continuous through the laminate thickness. From eq. (2.3.76)

$$\begin{aligned} \bar{\sigma}_{yy} &= \frac{\bar{Q}_{12} \bar{x} (\bar{x} - 1)}{2(D_{11}A_{11} - B_{11}^2)} (B_{11} - \bar{z} A_{11}) \\ &= \frac{(0.5131)}{(2)(6.42)} \bar{x} (\bar{x} - 1) (2.93 - 13.4 \bar{z}) \end{aligned}$$

$$= 0.117\bar{x}(\bar{x} - 1)(1 - 4.57\bar{z}) \quad (2.3.79)$$

The following results are obtained for σ_{xx} :

90° ply

$$\begin{aligned} \bar{\sigma}_{xx} &= \frac{\bar{Q}_{11}(90^\circ)\bar{x}}{(2)(6.42)(10^6)} (\bar{x} - 1)(2.93 - 13.4\bar{z}) \\ &= \frac{(1.71)\bar{x}}{(2)(6.42)} (\bar{x} - 1)(2.93 - 13.4\bar{z}) \\ &= 0.390(\bar{x} - 1)(1 - 4.57\bar{z}), \quad -0.5 \leq \bar{z} \leq 0 \end{aligned} \quad (2.3.80)$$

0° ply

$$\begin{aligned} \bar{\sigma}_{xx} &= \frac{\bar{Q}_{11}(0^\circ)\bar{x}}{(2)(6.42)10^6} (\bar{x} - 1)(2.93 - 13.4\bar{z}) \\ &= \frac{(25.15)\bar{x}}{(2)(6.42)} (\bar{x} - 1)(2.93 - 13.4\bar{z}) \\ &= 5.74\bar{x}(\bar{x} - 1)(1 - 4.57\bar{z}), \quad 0 \leq \bar{z} \leq 0.5 \end{aligned} \quad (2.3.81)$$

The interlaminar shear distribution will not be symmetric through-the-thickness in the case of unsymmetric laminates. Integration of eqs. (2.3.78) leads to the following interlaminar shear stress distribution

90° ply

$$\begin{aligned} \bar{\tau}_{xz} &= \frac{\bar{Q}_{11}(90^\circ)(1 - 2\bar{x})}{(2)(6.42)10^6} \int_{-1/2}^{\bar{z}} (2.93 - 13.4\bar{z}) d\bar{z} \\ &= \frac{(1.71)}{(2)(6.42)} (1 - 2\bar{x}) \left(2.93\bar{z} - \frac{13.4}{2} \bar{z}^2 + \frac{2.93}{2} + \frac{13.4}{8} \right) \\ &= 0.418(1 - 2\bar{x})(1 + 0.933\bar{z} - 2.13\bar{z}^2), \quad -0.5 \leq \bar{z} \leq 0 \end{aligned} \quad (2.3.82)$$

0° ply

$$\bar{\tau}_{xz} = \frac{(1 - 2\bar{x})}{(2)(6.42)10^6} \left[\int_{-1/2}^0 Q_{11}(90^\circ) (B_{11} - \bar{z}A_{11}) d\bar{z} + \int_0^{\bar{z}} Q_{11}(0^\circ) (B_{11} - \bar{z}A_{11}) d\bar{z} \right]$$

Again, the first integral is the value of $\bar{\tau}_{xz}(0)$. Thus

$$\begin{aligned} \bar{\tau}_{xz} &= (1 - 2\bar{x}) \left[0.418 + \frac{25.15}{(2)(6.42)} \int_0^{\bar{z}} (2.93 - 13.4\bar{z}) d\bar{z} \right] \\ &= (1 - 2\bar{x}) \left[0.418 + 1.96 \left(2.93\bar{z} - \frac{13.4}{2} \bar{z}^2 \right) \right] \\ &= 0.418(1 - 2\bar{x})(1 + 13.7\bar{z} - 31.4\bar{z}^2), \quad 0 \leq \bar{z} \leq 0.5 \quad (2.3.83) \end{aligned}$$

The interlaminar normal stress σ_{zz} will now be determined by combining eq. (2.3.82,83) with eq. (2.3.29)

90° ply

$$\begin{aligned} \bar{\sigma}_{zz} &= (0.418)(2) \int_{-1/2}^{\bar{z}} (1 + 0.933\bar{z} - 2.13\bar{z}^2) \\ &= -0.836 \left(\bar{z} + \frac{0.933}{2} \bar{z}^2 - \frac{2.13}{3} \bar{z}^3 + \frac{1}{2} - \frac{0.933}{8} - \frac{2.13}{24} \right) \\ &= 0.247(1 + 3.38\bar{z} + 1.58\bar{z}^2 - 2.40\bar{z}^3), \quad -0.5 \leq \bar{z} \leq 0 \quad (2.3.84) \end{aligned}$$

0° ply

$$\begin{aligned} \bar{\sigma}_{zz} &= (0.418)(2) \left[\int_{-1/2}^0 (1 + 0.933\bar{z} - 2.13\bar{z}^2) d\bar{z} + \int_0^{\bar{z}} (1 + 13.7\bar{z} - 31.4\bar{z}^2) d\bar{z} \right] \\ &= 0.247(0.418)(2) \left(\bar{z} + \frac{13.7}{2} \bar{z}^2 - \frac{31.4}{3} \bar{z}^3 \right) \\ &= 0.247(1 + 3.38\bar{z} + 23.2\bar{z}^2 - 35.4\bar{z}^3), \quad 0 \leq \bar{z} \leq 0.5 \quad (2.3.85) \end{aligned}$$

The stresses for this laminat are plotted in figures 2.3.8-11. Note that bending-extebsional coupling shifts the inplane normal stress distribution such that neither σ_{xx} or σ_{yy} are zero at the laminate midplane. In addition, the maximum value of the interlaminar shear stress is not at the laminate midplane as in the case of symmetric laminates or homogeneous plates.

o Specially Orthotropic Plates

There are varying degrees of complexity encountered in laminated plate analysis. The first degree of simplification is for the plate to be symmetric, thus uncoupling inplane and bending response. Uncoupled laminates can be broken into two general categories, specially orthotropic plates and midplane symmetric laminates (anisotropic plates). We now consider the first category.

Laminates which are symmetric, $B_{ij} = 0$, and for which the bending-twisting coupling terms vanish, $D_{16} = D_{26} = 0$, are referred to as specially orthotropic. Since solutions are more easily attained for such plates, composite laminates are often assumed to behave as specially orthotropic plates. Most engineering laminates, however, do not qualify as specially orthotropic because of the existence of D_{16} and D_{26} bending-twisting coupling terms. In many practical cases it is a reasonable approximation to assume specially orthotropic response. This will be discussed in more detail in the next section.

Consider the rectangular laminated plate shown in figure 2.3.12 to be subjected to the transverse load $q = q(x,y)$. For this loading in conjunction with specially orthotropic plates, eq. (2.3.14) becomes

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} = q \quad (2.3.86)$$

For a simply-supported plate the following boundary conditions are applicable:

$$\begin{aligned} \text{at } x = 0, a \\ W &= 0 \\ M_x &= -\left(D_{11} \frac{\partial^2 w}{\partial x^2} + D_{12} \frac{\partial^2 w}{\partial y^2}\right) = 0 \end{aligned} \quad (2.3.87)$$

$$\begin{aligned} \text{at } y = 0, b \\ W &= 0 \\ M_y &= -\left(D_{12} \frac{\partial^2 w}{\partial x^2} + D_{22} \frac{\partial^2 w}{\partial y^2}\right) = 0 \end{aligned} \quad (2.3.88)$$

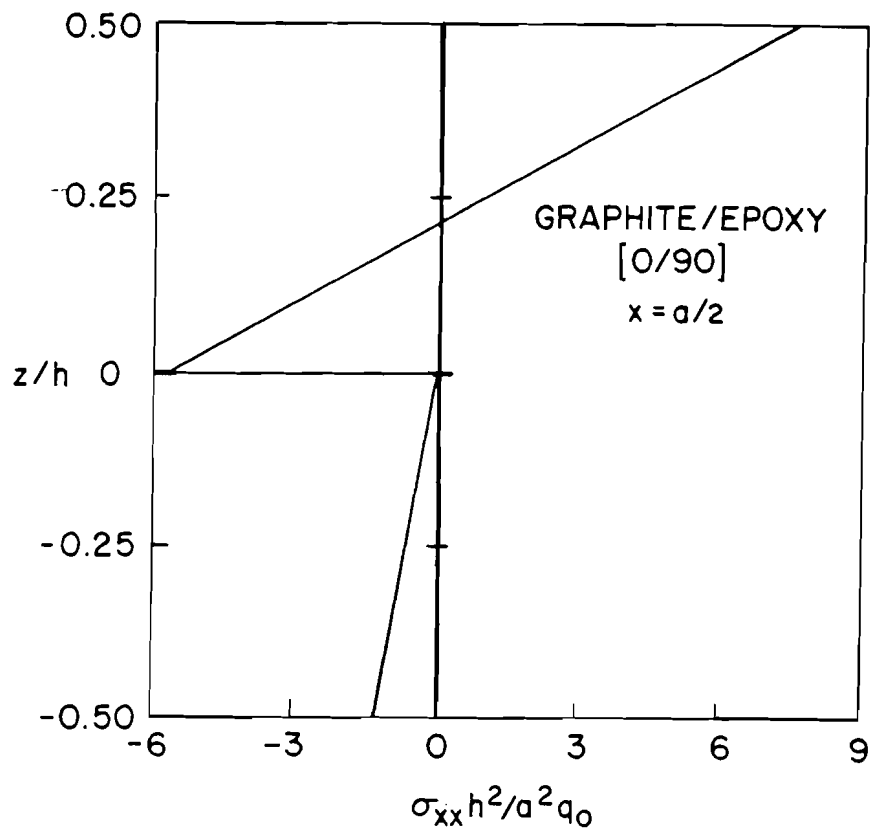


FIGURE 2.3.8. INPLANE NORMAL STRESS AT $x=a/2$ FOR [0/90] UNSYMMETRIC LAMINATE

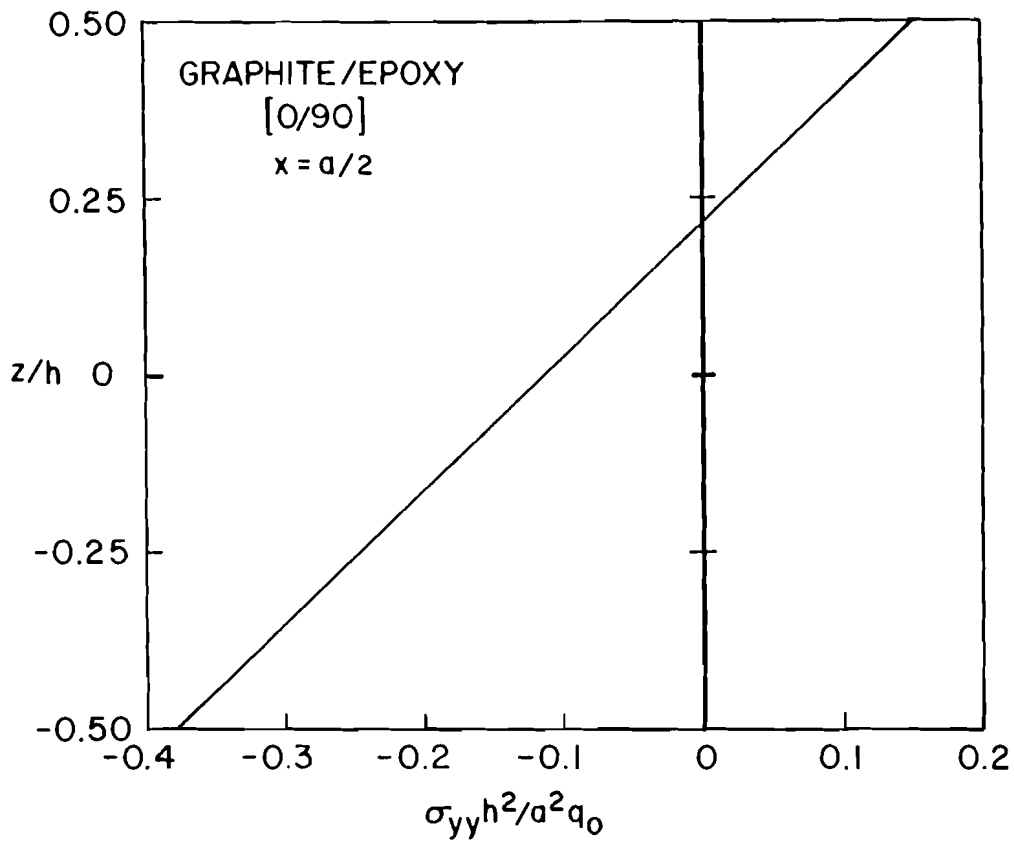


FIGURE 2.3.9. TRANSVERSE INPLANE NORMAL STRESS AT $x=a/2$ FOR [0/90] UNSYMMETRIC LAMINATE

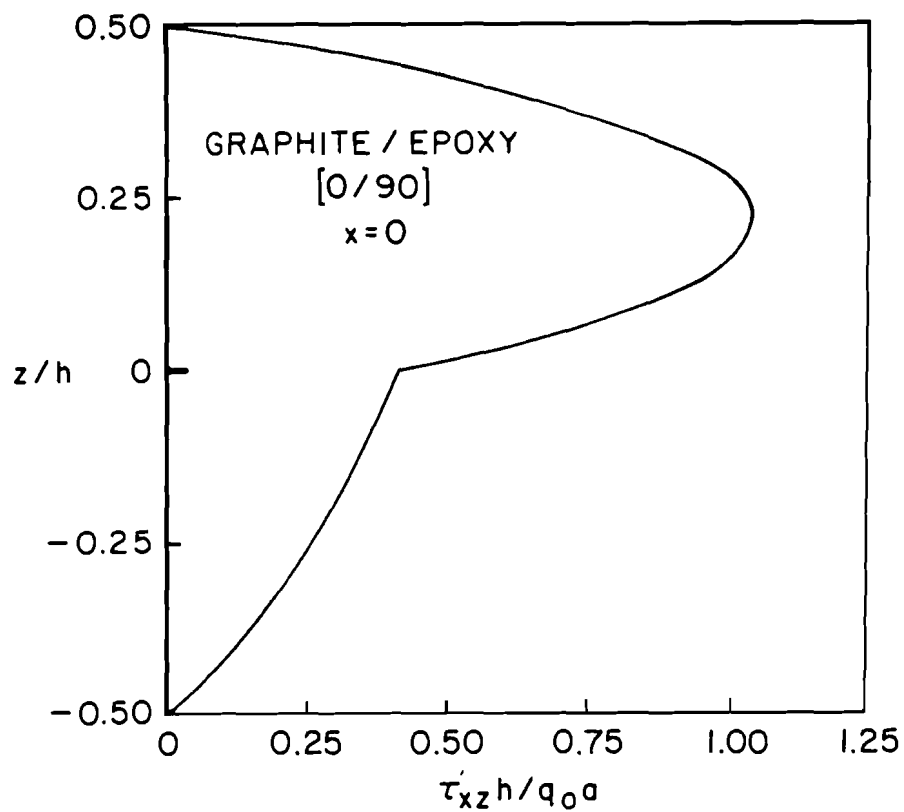


FIGURE 2.3.10. INTERLAMINAR SHEAR STRESS AT $x=0$ FOR [0/90] UNSYMMETRIC LAMINATE

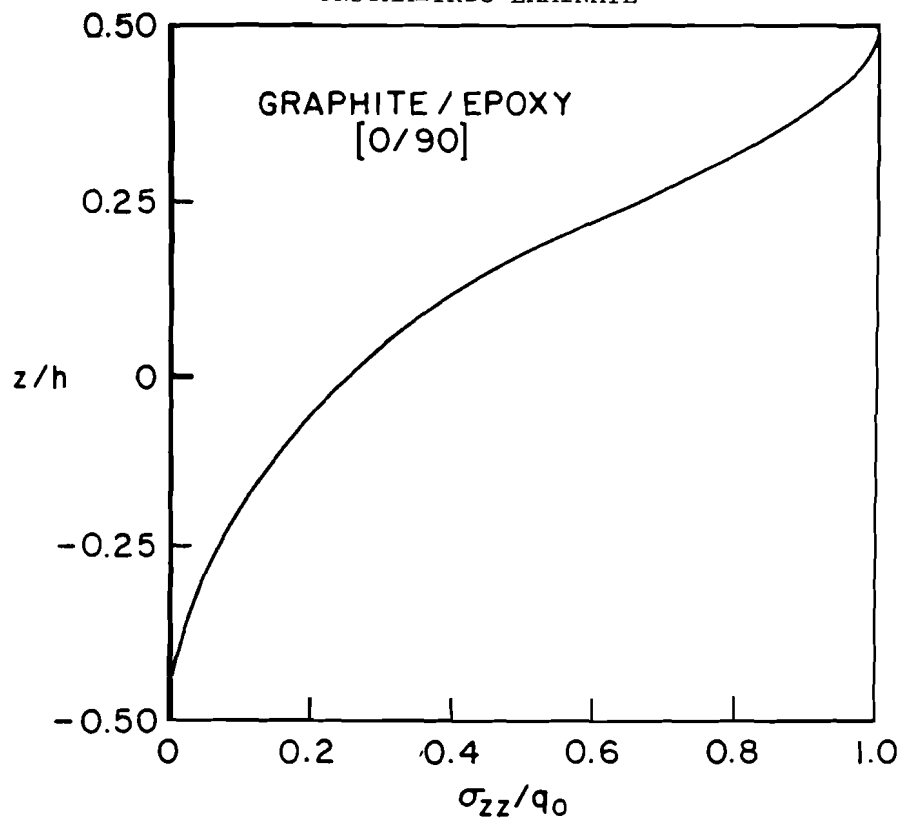


FIGURE 2.3.11. INTERLAMINAR NORMAL STRESS FOR [0/90] UNSYMMETRIC LAMINATE

The moment deflection relations are obtained from the laminate constitutive relations, eq. (2.3.1)

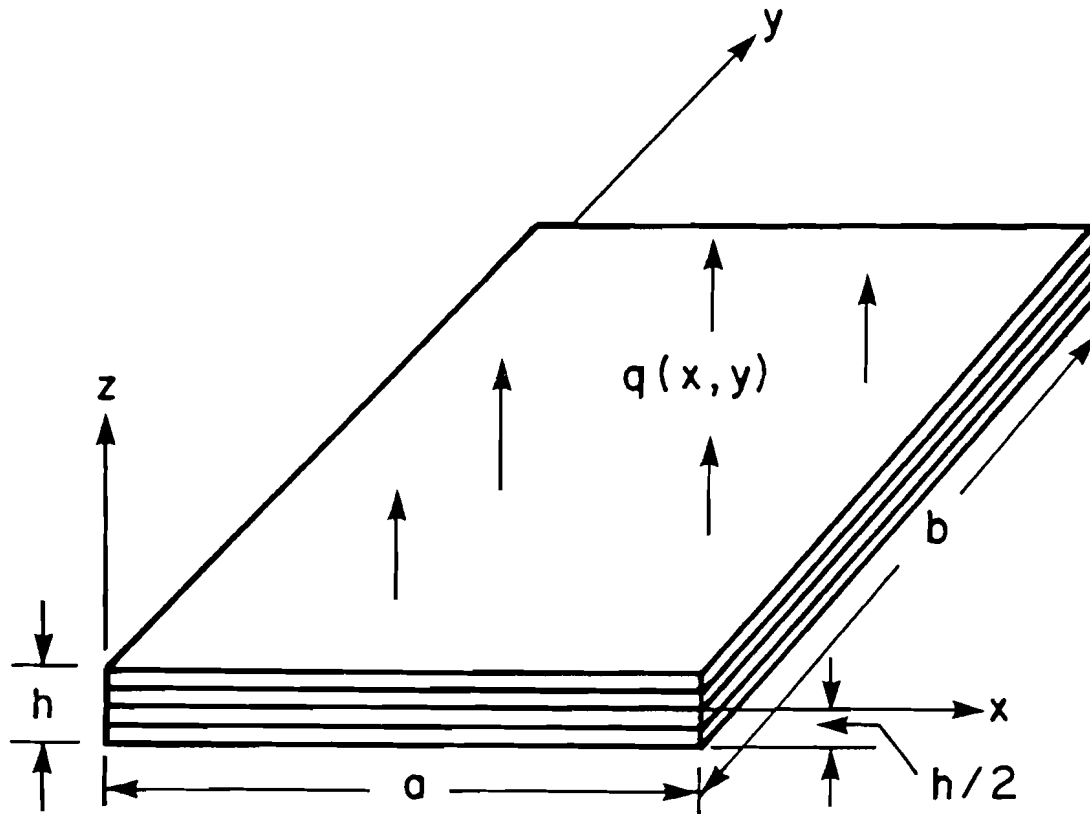


FIGURE 2.3.12. RECTANGULAR LAMINATED PLATE UNDER TRANSVERSE LOAD

A solution to the problem defined by eqs. (2.3.86-88) can be obtained in simple fashion in terms of double Fourier series. Let the load distribution be expanded in such a series

$$q(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (2.3.89)$$

where the Fourier coefficients are determined from the relationship

$$q_{mn} = \frac{4}{ab} \int_0^b \int_0^a q(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy \quad (2.3.90)$$

For example for uniform load q_0

$$\begin{aligned} q_{mn} &= \frac{16 q_0}{\pi^2 mn} \quad (m, n \text{ odd}) \\ q_{mn} &= 0 \quad (m, n \text{ even}) \end{aligned} \quad (2.3.91)$$

A solution can be constructed by representing the deflection by a similar double sine series. Thus

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (2.3.92)$$

Note that this solution does satisfy the boundary conditions eqs. (2.3.87,88) since the second derivative of a sine will reproduce the sine and all of the sines vanish on the plate boundary. It remains to satisfy the differential equation (2.3.6). Inserting eq. (2.3.89) and eq. (2.3.92) into eq. (2.3.86) and equating terms of the resulting Fourier series on both sides we have

$$w = \frac{a^4}{\pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{q_{mn}}{D_{mn}} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (2.3.93)$$

where

$$D_{mn} = D_{11} m^4 + 2(D_{12} + 2D_{66})(mnR)^2 + D_{22}(nR)^4 \quad (2.3.94)$$

Differentiating eq. (2.3.93) we obtain the plate curvatures

$$\begin{aligned} \kappa_{xx} &= - \frac{\partial^2 w}{\partial x^2} = \frac{a^2}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 q_{mn}}{D_{mn}} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ \kappa_{yy} &= - \frac{\partial^2 w}{\partial y^2} = \frac{a^2 R^2}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n^2 q_{mn}}{D_{mn}} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ 2\kappa_{xy} &= - 2 \frac{\partial^2 w}{\partial x \partial y} = - 2 \frac{a^2 R}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{mn q_{mn}}{D_{mn}} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \end{aligned} \quad (2.3.95)$$

From the ply constitutive relations, eq. (2.2.13) of Vol. I, and eq. (2.3.95) of this Volume we can derive the ply inplane stresses.

$$\sigma_{xx}^i = \frac{a^2}{\pi^2} z \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{q_{mn}}{D_{mn}} (\bar{Q}_{11}^i m^2 + \bar{Q}_{12}^i n^2 R^2) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (2.3.96)$$

$$\sigma_{yy}^i = \frac{a^2}{\pi^2} z \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{q_{mn}}{D_{mn}} (\bar{Q}_{12}^i m^2 + \bar{Q}_{22}^i n^2 R^2) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (2.3.97)$$

$$\tau_{xy}^i = -2 \frac{a^2 R}{\pi^2} \bar{Q}_{66}^i z \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{mn q_{mn}}{D_{mn}} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \quad (2.3.98)$$

The interlaminar shear stresses can be determined by substituting eqs. (2.3.93-98) into eq. (2.3.28), and performing the integration.

$$\begin{aligned} \tau_{xz}^i = & -\frac{a}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{-h/2}^{z^i} \frac{m q_{mn}}{D_{mn}} [\bar{Q}_{11}^i m^2 + (\bar{Q}_{12}^i \\ & + 2\bar{Q}_{66}^i) n^2 R^2] \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} z \, dz \end{aligned} \quad (2.3.99)$$

$$\begin{aligned} \tau_{yz}^i = & -\frac{a}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{-h/2}^{z^i} \frac{n q_{mn}}{D_{mn}} [(\bar{Q}_{12}^i + 2\bar{Q}_{66}^i) m^2 \\ & + \bar{Q}_{22}^i n^2 R^2] \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} z \, dz \end{aligned} \quad (2.3.100)$$

For a uniform load the deflection at the center of the plate is given by

$$w\left(\frac{a}{2}, \frac{b}{2}\right) = \frac{16q_0}{\pi^2} \sum_{m=1,3,\dots} \sum_{n=1,3,\dots} \frac{(-1)^{\frac{m+n+2}{2}}}{mn D_{mn}} \quad (2.3.101)$$

and for the curvatures

$$\begin{aligned} \kappa_{xx}\left(\frac{a}{2}, \frac{b}{2}\right) &= \frac{16q_0}{a^2} \sum_{m=1,3,\dots} \sum_{n=1,3,\dots} \frac{m(-1)^{\frac{m+n+2}{2}}}{m D_{mn}} \\ \kappa_{yy}\left(\frac{a}{2}, \frac{b}{2}\right) &= \frac{16q_0}{b^2} \sum_{m=1,3,\dots} \sum_{n=1,3,\dots} \frac{n(-1)^{\frac{m+n+2}{2}}}{n D_{mn}} \end{aligned} \quad (2.3.102)$$

The series for the deflection converges rapidly since the denominator is of the sixth order in m, n . The curvature convergence is less rapid but still of a practical nature.

Another load of particular interest is a single load q_0 uniformly distributed over a rectangle as shown in figure 2.3.12a. For this case the integral in eq. (2.3.94) yields

$$q_{mn} = \frac{16q_0}{\pi^2 mn cd} \sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b} \sin \frac{m\pi c}{2a} \sin \frac{n\pi d}{2b} \quad (2.3.102a)$$

where c and d are dimensions of the rectangle parallel to the x and y directions, respectively, and ξ and η are the x and y coordinates, respectively, of the center of the rectangle.

By allowing c and d to approach zero, we obtain an expression for a concentration load P located at the coordinates $x = \xi$, $y = \eta$.

$$q_{mn} = \frac{4P}{ab} \sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b} \quad (2.3.102b)$$

As a numerical example, let us again consider the symmetric laminate $[0/90]_s$ with the ply properties as given in the previous section. To simplify the calculation we assume the loading is a half sine wave ($m=n=1$).

For this laminate and ply properties, the following stiffness properties were determined in the section on cylindrical bending:

$$D_{11} = 1.85 \times 10^6 h^3$$

Recognizing that D_{22} of a $[0/90]_s$ laminate is identical to D_{11} of a $[90/0]_s$ laminate, then

$$D_{22} = 0.387 \times 10^6 h^3$$

Since $\bar{Q}_{12}(0^\circ) = \bar{Q}_{12}(90^\circ)$ and $\bar{Q}_{66}(0^\circ) = \bar{Q}_{66}(90^\circ)$, the stiffness terms D_{12} and D_{66} are the same as for a homogeneous plate, i.e.

$$D_{12} = \frac{\bar{Q}_{12}(0^\circ) h^3}{12} = \frac{0.5131}{12} \times 10^6 h^3 = 0.0428 \times 10^6 h^3$$

$$D_{66} = \frac{\bar{Q}_{66}(0^\circ) h^3}{12} = \frac{0.65}{12} \times 10^6 h^3 = 0.0542 \times 10^6 h^3$$

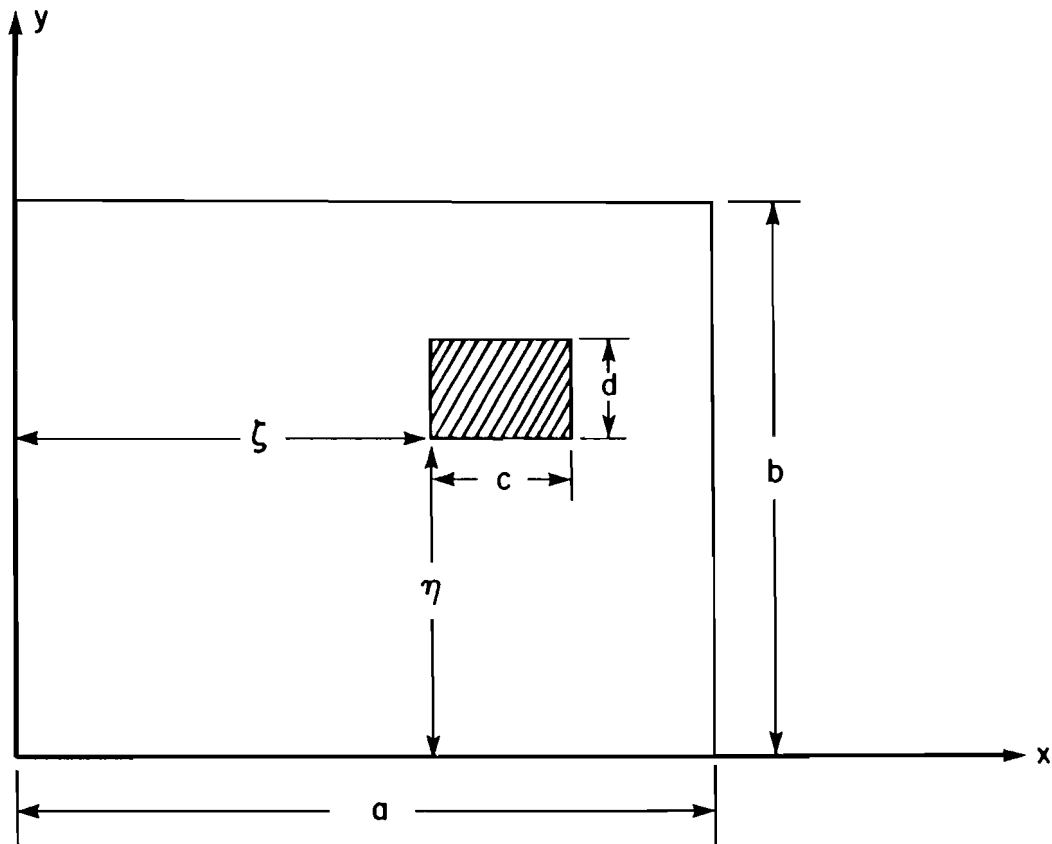


FIGURE 2.3.12A. COORDINATE LOCATION FOR UNIFORM LOAD APPLIED OVER A RECTANGLE

Now for a square plate ($R = 1$)

$$[D_{11} + 2(D_{12} + 2D_{66})R^2 + D_{22}R^4] =$$

$$[1.85 + 2(0.0423)(1)^2 + 2(2)(0.0542)(1)^2 + (0.387)(1)^4]$$

$$\times 10^6 h^3 = 2.54 \times 10^6 h^3$$

Now eq. (2.3.91) becomes

$$w = \frac{a^4 q_{11} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}}{(3.142)^4 (2.54) 10^6 h^3}$$

$$= (4.04 \times 10^{-9}) \frac{a^4 q_{11}}{h^3} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

The maximum deflection is at the center of the plate ($x = a/2, y = b/2$).

$$w_{\max} = (4.04 \times 10^{-9}) \frac{a^4 q_{11}}{h^3}$$

As just discussed, the bending stiffness terms D_{11} and D_{22} are the only nonhomogeneous stiffness components for the $[0/90]_s$ class of laminates. For an increasing number of layers, these stiffness terms approach a homogeneous relation. In particular, consider the class of laminates $[0/90]_{Ns}$ where N denotes the number of repeating 0/90 units above the centerline. For such laminates it can be shown that

$$D_{11} = [(1 + 3/4N)\bar{Q}_{11}(0^\circ) + (1 - 3/4N)\bar{Q}_{11}(90^\circ)] \frac{h^3}{24}$$

$$D_{22} = [(1 + 3/4N)\bar{Q}_{11}(90^\circ) + (1 - 3/4N)\bar{Q}_{11}(0^\circ)] \frac{h^3}{24}$$

For increasing N

$$D_{11} = D_{22} = [\bar{Q}_{11}(0^\circ) + \bar{Q}_{11}(90^\circ)] \frac{h^3}{12} = A_{11} \frac{h^2}{12} \quad (\text{Increasing } N)$$

The number of layers required to reach homogeneous behavior is strongly dependent on the ply elastic constant ratio E_1/E_2 . This ratio is often referred to as the anisotropy ratio and will be used in future numerical results. Let us consider another numerical example using the class of laminates $[0/90]_{Ns}$ with the following ply elastic properties

$$\frac{E_1}{E_2} = \text{variable}, \quad \nu_{12} = 0.3, \quad \frac{G_{12}}{E_2} = \frac{0.65 \times 10^6}{1.7 \times 10^6} = 0.382$$

Using these ply properties, the maximum deflection is calculated from eq. (2.3.91) for various anisotropy ratios and increasing values of N . Results are shown in table 2.3.1 for a uniform load $q = q_0 = \text{constant}$.

TABLE 2.3.1. EFFECT OF STACKING SEQUENCE ON BENDING OF $[0/90]_{Ns}$ LAMINATES

N	$w_{\max} q_0 h^3 \times 10^{-2} / a^4 E_2$		
	$E_1/E_2 = 5$	$E_1/E_2 = 10$	$E_1/E_2 = 15$
1	2.366	1.472	1.067
2	2.375	1.482	1.077
3	2.377	1.483	1.078
∞	2.378	1.485	1.079

Using eq. (2.3.95) in conjunction with eq. (2.3.102), the maximum deflection becomes

$$w_{\max} = \frac{16q_0 a^4}{\pi^4} \sum_{m=1,3,\dots} \sum_{n=1,3,\dots} \frac{(-1)^{\left(\frac{m+n+2}{2}\right)}}{mn D_{mn}}$$

Convergence of this series is quite rapid. Results are for $M = 13$, which yields 49 terms in the series. For this laminate geometry, heterogeneity is seen to have little effect on the maximum deflection. The anisotropy ratio does affect the plate stiffness, however, with reduced deflection with increasing E_1/E_2 .

The bending moment resultant M_{xx} along the centerline of the plate, $y = b/2$, is illustrated in figure 2.3.12b for $E_1/E_2 = 15$.

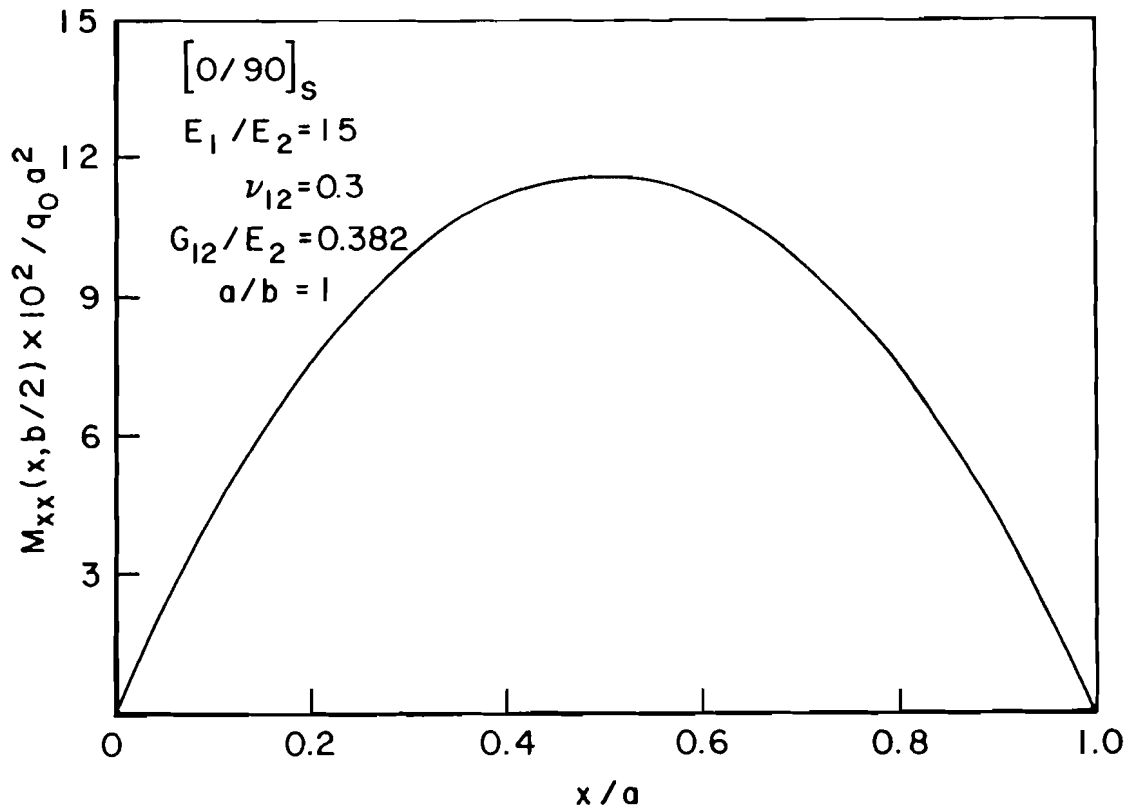


FIGURE 2.3.12b. BENDING MOMENT DISTRIBUTION AT CENTER OF A BIDIRECTIONAL LAMINATED PLATE

Now consider a laminate which is simply-supported along the edges $y = 0, b$ and subjected to an arbitrary transverse load q . If the surface load is represented by a double Fourier series as given by eq. (2.3.92), then the deflection can be assumed to be of the form

$$\begin{aligned}
w = & \sum_{n=1}^{\infty} \phi_n(x) \sin \frac{n\pi y}{b} \\
& + \frac{a^4}{\pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{q_{mn}}{D_{mn}} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\end{aligned} \tag{2.3.102c}$$

where D_{mn} is defined in eq. (2.3.95). This relationship satisfies the simply-supported boundary conditions, eq. (2.3.88). Substituting eqs. (2.3.92) and (2.3.102c) into eq. (2.3.86), we obtain the result

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left[D_{11} \frac{d^4 \phi_n}{dx^4} - 2(D_{12} + 2D_{66}) \frac{n^2 \pi^2}{b^2} \frac{d^2 \phi_n}{dx^2} \right. \\
& \left. + D_{22} \frac{n^4 \pi^4}{b^4} \phi_n \right] \sin \frac{n\pi y}{b} = 0
\end{aligned} \tag{2.3.102d}$$

It is easily seen that $\phi_n(x)$ must satisfy the differential equation

$$D_{11} \frac{d^4 \phi_n}{dx^4} - 2(D_{12} + 2D_{66}) \frac{n^2 \pi^2}{b^2} \frac{d^2 \phi_n}{dx^2} + D_{22} \frac{n^4 \pi^4}{b^4} \phi_n = 0 \tag{2.3.102e}$$

Solutions to this differential equation are of the form

$$\phi_n(x) = \exp \frac{n\pi \lambda}{b}$$

Substituting this function into eq. (2.3.102e), we obtain the algebraic equation

$$D_{11} \lambda^4 - 2(D_{12} + 2D_{66}) \lambda^2 + D_{22} = 0$$

which results in the solution

$$\lambda^2 = \frac{1}{D_{11}} \left[D_{12} + 2D_{66} \pm \sqrt{(D_{12} + 2D_{66})^2 - D_{11} D_{22}} \right] \tag{2.3.102f}$$

For many engineering laminates constructed from high modulus unidirectional layers, λ^2 will be complex and the roots of eq. (2.3.102f) can be written in the form $s \pm it$, $-s \pm it$, ($s, t > 0$). The solution of eq. (2.3.88) will then take the form

$$\begin{aligned}
w = & \sum_{n=1}^{\infty} \left[(A_n \cos \frac{n\pi tx}{b} + B_n \sin \frac{n\pi tx}{b}) \cosh \frac{n\pi sx}{b} \right. \\
& + (C_n \cos \frac{n\pi tx}{b} + D_n \sin \frac{n\pi tx}{b}) \sinh \frac{n\pi sx}{b} \\
& \left. + \frac{a^4}{\pi^4} \sum_{m=1}^{\infty} \frac{q_{mn}}{D_{mn}} \sin \frac{m\pi x}{a} \right] \sin \frac{n\pi y}{b}
\end{aligned} \tag{2.3.102g}$$

The constants in this solution are found from two boundary conditions each on the edges $x = 0, a$. Curvatures and bending moments can be determined in the same manner as for the case of all edges simply-supported.

Some simplification of eq. (2.3.102c) can be obtained for the case where $q = q_0 = \text{constant}$. In particular, it is only necessary to expand the transverse load in a single Fourier series, i.e.

$$q(x, y) = \frac{4q_0}{\pi} \sum_{n=1,3,\dots}^{\infty} \frac{1}{n} \sin \frac{n\pi y}{b}$$

and the second part of eq. (2.3.102c) is replaced by the particular solution

$$w_p = \frac{4b^4 q_0}{D_{22} \pi^5} \sum_{n=1,3,\dots}^{\infty} \frac{1}{n^5} \sin \frac{n\pi y}{b}$$

For this case eq. (2.3.102g) becomes

$$\begin{aligned}
w = & \sum_{n=1}^{\infty} \left[(A_n \cos \frac{n\pi tx}{b} + D_n \sin \frac{n\pi tx}{b}) \sinh \frac{n\pi sx}{b} \right. \\
& + (C_n \cos \frac{n\pi tx}{b} + D_n \sin \frac{n\pi tx}{b}) \sinh \frac{n\pi sx}{b} \\
& \left. + \frac{4b^4 q_0}{D_{22} n^5 \pi^5} \right] \sin \frac{n\pi y}{b}
\end{aligned}$$

Approximate techniques based on variational principles can be utilized to obtain solutions for other boundary conditions. One such approach is discussed in detail in the next section on anisotropic plates.

o Anisotropic Plates

We now give consideration to the general class of symmetric laminates in which bending-twisting coupling exists ($D_{16}, D_{26} \neq 0$). Most engineering laminates

are in this category. In particular, balanced angle-ply laminates, $[\pm\theta]_s$, are classical examples. For such laminates the inplane shear coupling stiffnesses vanish ($A_{16} = A_{26} = 0$), while the bending-twisting stiffness terms are nonvanishing.

The inclusion of the bending-twisting terms in the governing equations increases the complexity of the analysis by a significant amount. One approach which is sometimes taken in the analysis of such plates is simply to neglect these coupling terms. The accuracy of such an assumption is investigated in this section. In particular, a number of approximate solutions to the anisotropic plate equation are compared to results obtained by neglecting the bending-twisting coupling terms. Such comparison allows us to assess the error introduced by treating such laminates as orthotropic plates.

For a symmetric laminate under static transverse load, eq. (2.3.14) is applicable. For a simply-supported plate the following boundary conditions are applicable:

at $x = 0, a$

$$w = M_{xx} = -D_{11} \frac{\partial^2 w}{\partial x^2} - 2D_{16} \frac{\partial^2 w}{\partial x \partial y} - D_{12} \frac{\partial^2 w}{\partial y^2} = 0 \quad (2.3.103)$$

at $y = 0, b$

$$w = M_{yy} = -D_{12} \frac{\partial^2 w}{\partial x^2} - 2D_{26} \frac{\partial^2 w}{\partial x \partial y} - D_{22} \frac{\partial^2 w}{\partial y^2} = 0 \quad (2.3.104)$$

We obtained a solution to this problem for the specially orthotropic case in the form of a double Fourier sine series. This was easily accomplished by also expanding the surface load, $q(x,y)$, in a double Fourier sine series. The assumed form for the solution satisfied the simply-supported boundary conditions, and when the series was substituted into the governing equations, the resulting equation contained only double sine terms. By comparing like double sine terms, the coefficients in the assumed series were determined and the problem was solved.

For the case of anisotropic plates, this approach will not work. In particular, the double sine terms will not satisfy the moment boundary conditions in eqs. (2.3.103, 104) or the governing equation, eq. (2.3.14). Difficulties arise because the bending-twisting coupling terms in the bending moment relations and the governing equation contain odd order partial derivatives with respect to both x and y , while the remaining orthotropic terms contain only even order derivatives in x and y . Thus, if a solution is assumed in the form of a double sine series, the odd order derivatives will yield double cosine terms rather than double sine terms. Such results produce nonvanishing bending moments at the boundaries and like double sine terms in the governing equation cannot be compared to determine the coefficients in the assumed series. Thus, other approaches must be utilized.

We now seek an approximate solution to eq. (2.3.14) by the Galerkin method (reference 2.3.9) subject to the boundary conditions given by eqs. (2.3.103,104). This is a widely used method for approximate solution of boundary value problems.

The method is based on the selection of a series of functions for which we attempt to determine the coefficients such that the resulting series provides an approximate solution to the problem. In the application of the Galerkin method we choose these functions so that they satisfy the displacement boundary conditions, but not necessarily the force boundary conditions, are linearly independent, and are the first n functions of a system of functions which must be complete over the region of the plate. Thus, it is assumed that

$$w = \sum_{m=1}^M \sum_{n=1}^N A_{mn} \phi_{mn}(x,y) \quad (2.3.105)$$

If the function ϕ_{mn} satisfies the conditions just discussed, then convergence to the exact solution for deflection is guaranteed (reference 2.3.9). For cases where only the displacement boundary conditions are satisfied, convergence of the moments, or any output involving derivatives of the displacement, is very slow or sometimes appears to be nonexistent (e.g. see reference 2.3.10). The coefficients A_{mn} , for the case of simple-supports, are chosen such that

$$\begin{aligned} & \int_0^b \int_0^a \left[D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} \right. \\ & + 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w}{\partial y^4} - q \left. \right] \phi_{mn} \, dx \, dy \\ & + \int_0^b \left[M_{xx}(0,y) \frac{\partial \phi_{mn}}{\partial x}(0,y) - M_{xx}(a,y) \frac{\partial \phi_{mn}}{\partial x}(a,y) \right] dy \\ & + \int_0^a \left[M_{yy}(x,0) \frac{\partial \phi_{mn}}{\partial y}(x,0) - M_{yy}(x,b) \frac{\partial \phi_{mn}}{\partial y}(x,b) \right] dx = 0 \quad (2.3.106) \end{aligned}$$

The integrals along the plate edges are a result of the moment boundary conditions. If the functions ϕ_{mn} in eq. (2.3.105) are chosen such that the moment boundary conditions are exactly satisfied, then the boundary integrals vanish.

For the case under consideration, we assume that

$$\phi_{mn} = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (2.3.107)$$

These functions satisfy the displacement boundary conditions in eqs. (2.3.103,104), but as previously discussed, do not satisfy the moment boundary conditions.

Substituting eqs. (2.3.105) and (2.3.107) into eq. (2.3.106) and integrating the results, one obtains a set of $M \times N$ algebraic equations. These equations divide into two sets, one corresponding to $m+n$ odd and a second set corresponding to $m+n$ even. For a uniform load, $q = q_0$, the resulting equations are of the form

$$\begin{aligned} & \left[D_{11} \left(\frac{m}{a} \right)^4 + 2(D_{12} + 2D_{66}) \left(\frac{mn}{ab} \right)^2 + D_{22} \left(\frac{n}{b} \right)^4 \right] A_{mn} \\ & - 32 \frac{mn}{\pi^2 ab} \sum_{k=1}^M \sum_{\ell=1}^N B_{k\ell} \left[\frac{(m^2 + k^2)}{a^2} D_{16} + \frac{(n^2 + \ell^2)}{b^2} D_{26} \right] A_{k\ell} \\ & = q_{mn} \quad (m+n \text{ even}) \end{aligned} \quad (2.3.108)$$

where

$$\begin{aligned} B_{k\ell} &= \frac{k\ell}{(m^2 - k^2)(n^2 - \ell^2)} \quad \begin{cases} m+k \text{ odd} \\ n+\ell \text{ odd} \end{cases} \\ &= 0 \quad \begin{cases} m+k \text{ even} \\ n+\ell \text{ even} \end{cases} \end{aligned}$$

and q_{mn} is given by eq. (2.3.102). Since the right-hand side of eq. (2.3.108) only exists for m, n even, these relationships need only be generated for $m+n$ even.

Let us now consider symmetric angle-ply laminates of the class $[\pm\theta]_{N_s}$ where, as previously, N denotes the number of repeating $\pm\theta$ units above the centerline. In determining the bending stiffness terms, it should be noted that

$$Q_{11}(\theta) = Q_{11}(-\theta), \quad Q_{12}(\theta) = Q_{12}(-\theta), \quad Q_{22}(\theta) = Q_{22}(-\theta)$$

$$Q_{66}(\theta) = Q_{66}(-\theta)$$

Thus,

$$D_{11} = \frac{h^3 Q_{11}(\theta)}{12}, \quad D_{12} = \frac{h^3 Q_{12}(\theta)}{12}, \quad D_{22} = \frac{h^3 Q_{22}(\theta)}{12}$$

$$D_{66} = \frac{h^3 Q_{66}(\theta)}{12}$$

Furthermore

$$Q_{16}(\theta) = -Q_{16}(-\theta), \quad Q_{26}(\theta) = -Q_{26}(-\theta)$$

Using these relations it can be shown that

$$D_{16} = \frac{Q_{16}(\theta)}{16N}, \quad D_{26} = \frac{Q_{26}(\theta)}{16N} \quad (2.3.109)$$

Thus, the bending-twisting coupling coefficients D_{16}, D_{26} decrease with increasing N .

The effect of bending-twisting coupling on maximum deflection is illustrated in figure 2.3.13 for angle-ply laminates under uniform transverse load q_0 . Ply properties are those utilized in Section 2.3.2.2. The orthotropic solution ($N \rightarrow \infty$) is obtained by neglecting the D_{16}, D_{26} terms in eq. (2.3.14). Thus, for this case the solution reduces to that of a special orthotropic plate as given by eq. (2.3.95). The maximum departure from orthotropic plate theory occurs for the $[+45]_8$ laminate where the anisotropic plate deflection is approximately 19% greater than the orthotropic plate deflection.

The effect of anisotropy ratio, E_1/E_2 , is illustrated in table 2.3.2. As in the previous section, $\nu_{12} = 0.3$ and $G_{12}/E_2 = 0.382$.

Convergence of deflections is rather rapid, while, as previously discussed, difficulties are encountered with convergence of the moments. For further discussion of convergence, see reference 2.3.10. Results in figure 2.3.13 and in table 2.3.2 are for $M=N=9$.

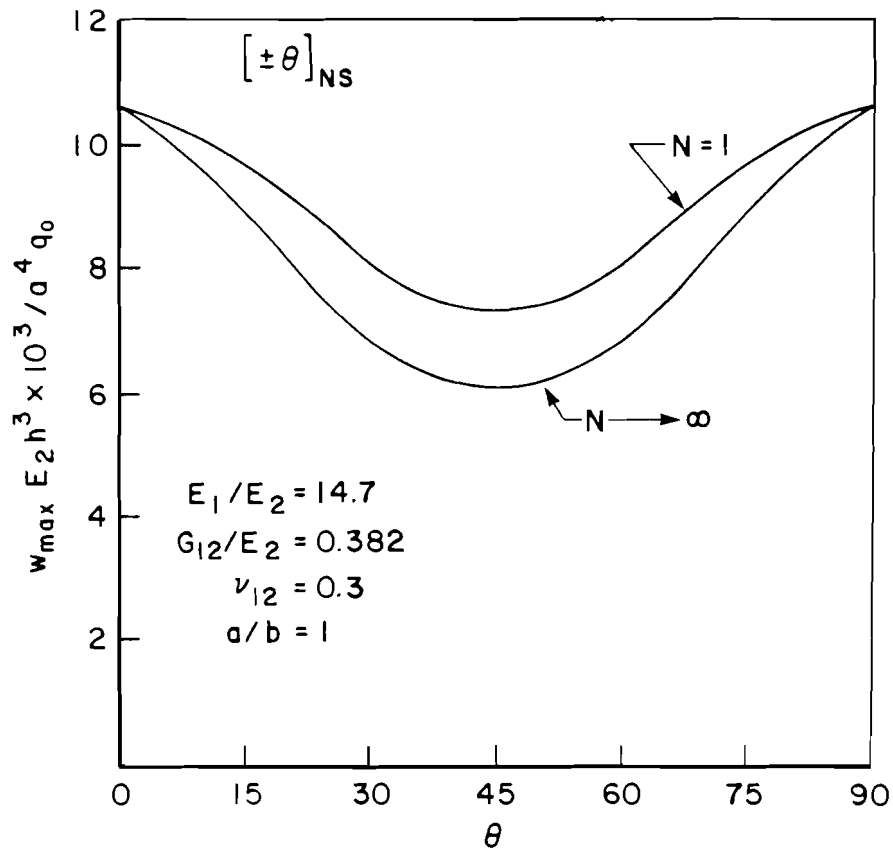


FIGURE 2.3.13. BENDING OF SYMMETRIC ANGLE-PLY PLATES UNDER UNIFORM TRANSVERSE LOAD

TABLE 2.3.2 EFFECT OF ANISOTROPY RATIO ON BENDING OF $[\pm 45]_{NS}$ LAMINATES

N	$w_{\max} E_2 h^3 \times 10^3 / a^4 q_0$		
	$E_1/E_2=5$	$E_1/E_2=10$	$E_1/E_2=15$
1	17.35	10.08	7.134
2	16.19	8.993	6.227
3	16.00	8.832	6.098
8	15.85	8.709	6.004

Although laminates of the $[\pm\theta]_{Ns}$ class have practical value, many engineering applications require composites containing 0° and 90° layers. Let us now consider a quasi-isotropic laminate of the class $[\pm 45/0/90]_{Ns}$. The following bending-twisting coupling terms are calculated for such laminates:

$$\begin{aligned} D_{16} &= \frac{3Q_{16}(\theta)}{128}, \quad D_{26} = \frac{3Q_{26}(\theta)}{128} \quad (N = 1) \\ D_{16} &= \frac{5Q_{16}(\theta)}{512}, \quad D_{26} = \frac{5Q_{26}(\theta)}{512} \quad (N = 2) \end{aligned} \quad (2.3.110)$$

If the stacking sequence is reversed such that the laminate is of the class $[0/90/\pm 45]_{Ns}$, then eqs. in (2.3.110) become

$$\begin{aligned} D_{16} &= \frac{Q_{16}(\theta)}{128}, \quad D_{26}(\theta) = \frac{Q_{26}(\theta)}{128} \quad (N = 1) \\ D_{16} &= \frac{3Q_{16}(\theta)}{512}, \quad D_{26}(\theta) = \frac{3Q_{26}(\theta)}{512} \quad (N = 2) \end{aligned} \quad (2.3.111)$$

Comparing these equations with eq. (2.3.109) for $N = 1$, we see that

$$\frac{D_{16}[\pm\theta/0/90]_s}{D_{16}[\pm\theta]_s} = \frac{D_{26}[\pm\theta/0/90]_s}{D_{16}[\pm\theta]_s} = \frac{3}{8}$$

$$\frac{D_{16}[0/90/\pm\theta]_s}{D_{16}[\pm\theta]_s} = \frac{D_{26}[0/90/\pm\theta]_s}{D_{16}[\pm\theta]_s} = \frac{1}{8}$$

Thus, for many practical laminates, the effect of bending-twisting coupling on the bending of anisotropic plates under lateral loads can be neglected without introducing serious error.

o Bending of Unsymmetric Laminates

We now turn our attention to laminated plates which do not display symmetry relative to the midplane. For specially orthotropic laminates which are symmetric, solutions were obtained with little more difficulty than for homogeneous isotropic plates. Laminates exhibiting bending-twisting coupling terms, however, prove to be more formidable, as closed form solutions cannot be obtained. Unsymmetric laminates now introduce further complexity in the form of bending-extensional coupling.

In this section the effect of bending-extensional coupling is ascertained by comparing solutions for deflection under lateral loads to solutions obtained by neglecting the B_{ij} terms in the governing equations. This allows us to establish

guidelines for determining the circumstances under which the general laminated plate theory must be used in order to obtain reasonably accurate solutions. There are two special classes of unsymmetric laminates of practical interest for which closed form solutions can be generated.

First we consider unsymmetric laminates of the class $[0/90]_{NT}$, where N is the number of repeating $0/90$ units. Since the layers are orthotropic with respect to the x, y axes of the plate, all of the shear coupling terms vanish, i.e. $A_{16} = A_{26} = B_{16} = B_{26} = D_{16} = D_{26} = 0$. Furthermore, since the laminate is homogeneous with respect to Q_{12} and Q_{66}

$$A_{12} = h\bar{Q}_{12}(0^\circ), \quad A_{66} = h\bar{Q}_{66}(0^\circ), \quad B_{12} = B_{66} = 0,$$

$$D_{12} = A_{12} \frac{h^2}{12}, \quad D_{66} = A_{66} \frac{h^2}{12}$$

Since the A_{ij} stiffness terms are independent of stacking sequence

$$A_{22} = A_{11} = [\bar{Q}_{11}(0^\circ) + \bar{Q}_{11}(90^\circ)] \frac{h}{2}$$

Let us now determine B_{11} , B_{22} , D_{11} and D_{22} for the case $N = 1$.

$$\begin{aligned} B_{11} &= \int_{-h/2}^{h/2} \bar{Q}_{11}(0^\circ) z \, dz = \int_{-h/2}^0 \bar{Q}_{11}(90^\circ) z \, dz \\ &+ \int_0^{h/2} \bar{Q}_{11}(0^\circ) z \, dz = \frac{z^2}{2} \bar{Q}_{11}(90^\circ) \Big|_{-h/2}^0 \\ &+ \frac{z^2}{2} \bar{Q}_{11}(0^\circ) \Big|_0^{h/2} = [\bar{Q}_{11}(0^\circ) - \bar{Q}_{11}(90^\circ)] \frac{h^2}{8} \end{aligned}$$

Since B_{22} is determined by the same integration with the plies reversed

$$B_{22} = -B_{11} = [\bar{Q}_{11}(90^\circ) - \bar{Q}_{11}(0^\circ)] \frac{h^2}{8}$$

These results can be generalized for this class of laminates to the relationship

$$B_{11} = -B_{22} = [\bar{Q}_{11}(0^\circ) - \bar{Q}_{11}(90^\circ)] \frac{h^2}{8N} \quad (2.3.112)$$

Thus, the coupling coefficients will decrease with increasing values of N.

$$\begin{aligned} D_{11} &= \int_{-h/2}^{h/2} \bar{Q}_{11} z^2 dz = \int_{-h/2}^0 \bar{Q}_{11}(90^\circ) z^2 dz \\ &+ \int_0^{h/2} \bar{Q}_{11}(0^\circ) z^2 dz = \frac{z^3}{3} \bar{Q}_{11}(90^\circ) \Big|_{-h/2}^0 \\ &+ \frac{z^3}{3} \bar{Q}_{11}(0^\circ) \Big|_0^{h/2} = [\bar{Q}_{11}(0^\circ) + \bar{Q}_{11}(90^\circ)] \frac{h^3}{24} = A_{11} \frac{h^2}{12} \end{aligned}$$

It can easily be seen from this result that the same relationship will be obtained for D_{22} . Furthermore, these same results will be obtained for all values of N. Thus, for this class of laminates

$$D_{11} = D_{22} = A_{11} \frac{h^2}{12} \quad (2.3.113)$$

The stress function, transverse displacement formulation of the problem, eqs. (2.3.17,19), will be utilized for obtaining a solution to the bending of this class of laminates under lateral loading. In order to use this formulation we must first develop the A^* , B^* , and D^* matrices in eq. (2.3.16).

$$A_{11}^* = A_{22}^* = \frac{A_{11}}{A_{11}^2 - A_{12}^2}, \quad A_{12}^* = \frac{-A_{12}}{A_{11}^2 - A_{12}^2},$$

$$A_{66}^* = \frac{1}{A_{66}}, \quad A_{16}^* = A_{26}^* = 0$$

$$[B^*] = -[A_{11}^*][B]$$

$$\begin{aligned} &= - \begin{bmatrix} A_{11}^* & A_{12}^* & 0 \\ A_{12}^* & A_{11}^* & 0 \\ 0 & 0 & A_{66}^* \end{bmatrix} \begin{bmatrix} B_{11} & 0 & 0 \\ 0 & -B_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -A_{11}^* B_{11} & A_{12}^* B_{11} & 0 \\ -A_{12}^* B_{11} & A_{11}^* B_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Thus

$$B_{22}^* = -B_{11}^*, \quad B_{21}^* = -B_{12}^*, \quad B_{16}^* = B_{26}^* = 0 \quad (2.3.114)$$

$$\begin{aligned} [B] [A^*] [B] &= \begin{bmatrix} B_{11} & 0 & 0 \\ 0 & -B_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11}^* B_{11} & -A_{12}^* B_{11} & 0 \\ A_{12}^* B_{11} & -A_{11}^* B_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} A_{11}^* B_{11}^2 & -A_{12}^* B_{11}^2 & 0 \\ -A_{12}^* B_{11}^2 & A_{11}^* B_{11}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ [D^*] &= \begin{bmatrix} (D_{11} - A_{11}^* B_{11}^2) & (D_{12} + A_{12}^* B_{11}^2) & 0 \\ (D_{12} + A_{12}^* B_{11}^2) & (D_{11} - A_{11}^* B_{11}^2) & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \end{aligned} \quad (2.3.115)$$

From this matrix we see that $D_{16}^* = D_{26}^* = 0$, $D_{22}^* = D_{11}^*$, $D_{66}^* = D_{66}$.

Equations (2.3.17,19) can now be written in the following manner

$$\begin{aligned} D_{11}^* \frac{\partial^4 w}{\partial x^4} + 2(D_{12}^* + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{11}^* \frac{\partial^4 w}{\partial y^4} \\ - B_{12}^* \left(\frac{\partial^4 \phi}{\partial x^4} - \frac{\partial^4 \phi}{\partial y^4} \right) = q \end{aligned} \quad (2.3.116)$$

$$\begin{aligned}
& A_{11}^* \frac{\partial^4 \phi}{\partial x^4} + (2A_{12}^* + A_{66}^*) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + A_{11}^* \frac{\partial^4 \phi}{\partial y^4} \\
& + B_{12}^* \left(\frac{\partial^4 w}{\partial x^4} - \frac{\partial^4 w}{\partial y^4} \right) = 0
\end{aligned} \tag{2.3.117}$$

We now consider the following simple support boundary conditions in conjunction with the constitutive eqs. (2.3.16) and the displacement relations, eqs. (2.3.22,23):

$$\begin{aligned}
& \text{at } x = 0, a \\
& N_{xx} = \frac{\partial^2 \phi}{\partial y^2} = 0, \quad v^0 = A_{11}^* \int \frac{\partial^2 \phi}{\partial x^2} dy + A_{12}^* \frac{\partial \phi}{\partial y} \\
& + B_{12}^* \int \frac{\partial^2 w}{\partial x^2} dy - B_{11}^* \frac{\partial w}{\partial y} + g(x) = 0 \\
& w = M_{xx} = A_{12}^* B_{11} \frac{\partial^2 \phi}{\partial x^2} + A_{11}^* B_{11} \frac{\partial^2 \phi}{\partial y^2} - D_{11}^* \frac{\partial^2 w}{\partial x^2} \\
& - D_{12}^* \frac{\partial^2 w}{\partial y^2} = 0
\end{aligned} \tag{2.3.118}$$

$$\begin{aligned}
& \text{at } y = 0, b \\
& N_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 0, \quad u^0 = A_{12}^* \frac{\partial \phi}{\partial x} + A_{11}^* \int \frac{\partial^2 \phi}{\partial y^2} dx \\
& - B_{11}^* \frac{\partial w}{\partial x} - B_{12}^* \int \frac{\partial^2 w}{\partial y^2} dx + f(y) = 0 \\
& w = M_{yy} = -A_{11}^* B_{11} \frac{\partial^2 \phi}{\partial x^2} - A_{12}^* B_{11} \frac{\partial^2 \phi}{\partial y^2} - D_{12}^* \frac{\partial^2 w}{\partial x^2} \\
& - D_{11}^* \frac{\partial^2 w}{\partial y^2} = 0
\end{aligned} \tag{2.3.119}$$

We again consider transverse loads, q_0 , which can be expanded in a double Fourier sine series. A solution to eqs. (2.3.116,117) which satisfies the boundary conditions, eqs. (2.3.118,119) is of the form

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (2.3.120)$$

$$\phi = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (2.3.121)$$

$$g(x) = f(y) = 0 \quad (2.3.122)$$

Substituting eqs. (2.3.120,121) into eqs. (2.3.116,117), equating like Fourier terms, and solving the resulting two simultaneous algebraic equations for the Fourier coefficients, we obtain

$$A_{mn} = [A_{11}^* (m^4 + n^4 \eta^4) + (2A_{12}^* + A_{66}^*) m^2 n^2 \eta^2] \frac{a^4 q_{mn}}{\pi^4 D_{mn}} \quad (2.3.123)$$

$$B_{mn} = -B_{12}^* (m^4 - n^4 \eta^4) \frac{a^4 q_{mn}}{\pi^4 D_{mn}}$$

where

$$\eta = \frac{a}{b}$$

$$D_{mn} = [D_{11}^* (m^4 + n^4 \eta^4) + 2(D_{12}^* + 2D_{66}^*)] [A_{11}^* (m^4 + n^4 \eta^4) + (2A_{12}^* + A_{66}^*) m^2 n^2 \eta^2] + B_{12}^{*2} (m^4 - n^4 \eta^4)^2$$

Displacements can be determined from eqs. (2.3.22,23) in conjunction with eqs. (2.3.120-123). It can easily be seen that the displacements will take the form

$$u^0 = q_{mn} C_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (2.3.124)$$

$$v^0 = q_{mn} E_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$

where C_{mn} and E_{mn} will be functions of the stiffness coefficients and plate dimensions. Equations (2.3.124) exactly satisfy the inplane displacement boundary conditions as prescribed by eqs. (2.3.118,119). Force resultants can be determined from eq. (2.3.15), while the moments can be calculated from eq. (2.3.16).

For the case where B_{11} vanishes, B_{mn} also vanishes, $D_{ij}^* = D_{ij}$, and A_{mn} reduces to the Fourier coefficient for an orthotropic plate as given by eq. (2.3.95). Thus, as anticipated, the orthotropic plate solution becomes the limiting case as the number of layers is increased.

The effect of coupling on this class of bidirectional composites is illustrated in figures 2.3.14 and figure 2.3.15 for uniform transverse load $q = q_0$. For this case the Fourier coefficients are given by eq. (2.3.102). Ply properties are those utilized throughout Section 2.3 (e.g. see Sec. 2.3.2.2). The effect of anisotropy ratio, E_1/E_2 , is illustrated in figure 2.3.14 (as previously $\nu_{12} = 0.3$, $G_{12}/E_2 = 0.382$). In figure 2.3.15 the effect of plate aspect ratio is illustrated. The asymptotic result corresponds to cylindrical bending (CB).

The effect of bending-extensional coupling can be very significant. However, the orthotropic solution is rapidly approached as the number of plies is increased. It is interesting to note how rapidly the center deflection in figure 2.3.15 approaches the maximum deflection of a bidirectional strip. Similar results are also found in homogeneous, isotropic plates (reference 2.3.11).

Now let us consider a rectangular angle-ply plate of the unsymmetric class $[\pm\theta]_{NT}$, where N is the number of repeating $\pm\theta$ units. For this class of laminates, $\bar{Q}_{11}$, $\bar{Q}_{12}$, $\bar{Q}_{22}$, and $\bar{Q}_{66}$ are identical for $\pm\theta$ angles. Thus,

$$\begin{aligned} A_{11} &= h\bar{Q}_{11}(\theta), \quad A_{12} = h\bar{Q}_{12}(\theta), \quad A_{22} = h\bar{Q}_{22}(\theta), \\ A_{66} &= h\bar{Q}_{66}(\theta), \quad B_{11} = B_{12} = B_{22} = B_{66} = 0, \\ D_{11} &= A_{11} \frac{h^2}{12}, \quad D_{12} = A_{12} \frac{h^2}{12}, \quad D_{22} = A_{22} \frac{h^2}{12}, \\ D_{66} &= A_{66} \frac{h^2}{12} \end{aligned}$$

Since $\bar{Q}_{16}(\theta) = -Q_{16}(-\theta)$ and $\bar{Q}_{26}(\theta) = -Q_{26}(-\theta)$, then $A_{16} = A_{26} = 0$. Let us now determine B_{16} , B_{26} , D_{16} , and D_{26} . These stiffness terms can be determined directly from eqs. (2.3.112) and (2.3.113) by simply substituting $\bar{Q}_{16}(\theta)$, $\bar{Q}_{26}(\theta)$ for $\bar{Q}_{11}(0^\circ)$ and $Q_{16}(-\theta)$, $Q_{26}(-\theta)$ for $\bar{Q}_{11}(90^\circ)$ with the result

$$\begin{aligned} B_{16} &= [\bar{Q}_{16}(\theta) - Q_{16}(-\theta)] \frac{h^2}{8N} = [\bar{Q}_{16}(\theta) - (-Q_{16}(\theta))] \frac{h^2}{8N} \\ &= \frac{h^2}{4N} Q_{16}(\theta) \end{aligned} \quad (2.3.125)$$

and similarly,

$$B_{26} = \frac{h^2}{4N} Q_{26}(\theta) \quad (2.3.126)$$

$$D_{16} = [Q_{16}(\theta) + Q_{16}(-\theta)] \frac{h^3}{24} = [Q_{16}(\theta) + (-Q_{16}(\theta))] \frac{h^3}{24} = 0 \quad (2.3.127)$$

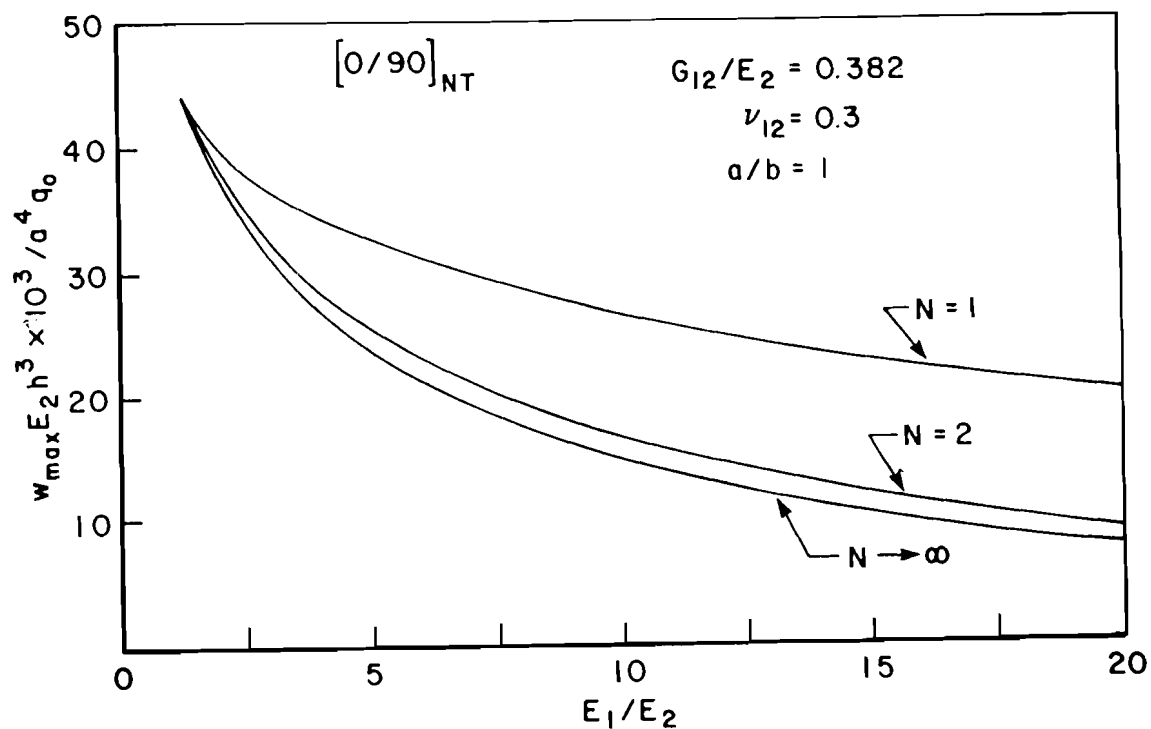


FIGURE 2.3.14. EFFECT OF ANISOTROPY RATIO ON THE BENDING OF UNSYMMETRIC BIDIRECTIONAL PLATES UNDER UNIFORM TRANSVERSE LOAD

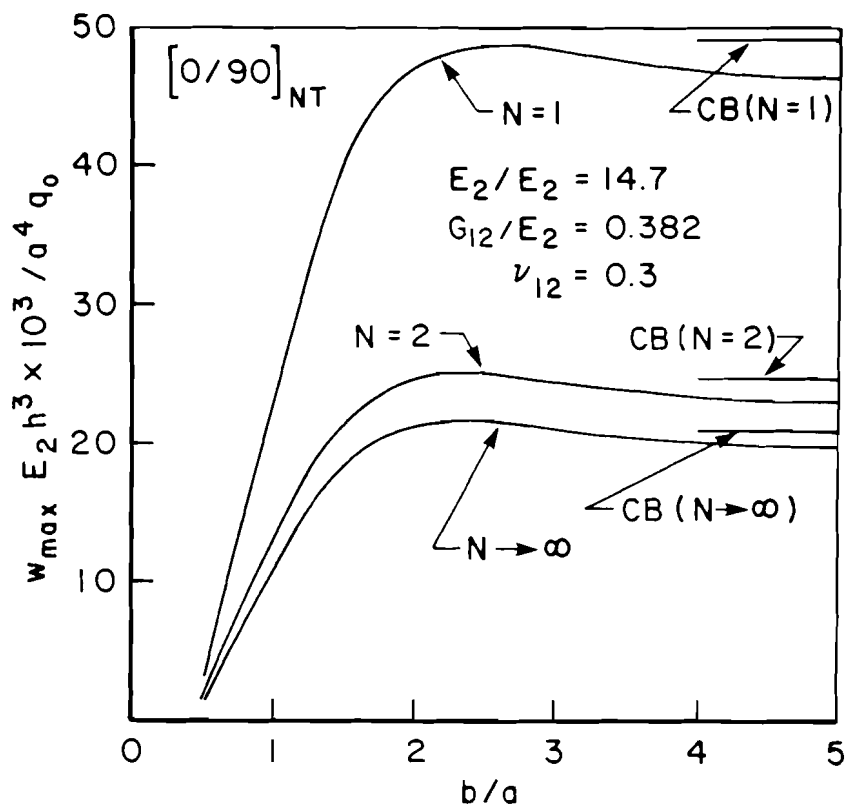


FIGURE 2.3.15. EFFECT OF ASPECT RATIO ON THE BENDING OF UNSYMMETRIC BIDIRECTIONAL PLATES UNDER UNIFORM TRANSVERSE LOAD

and similarly,

$$D_{26} = 0 \quad (2.3.128)$$

Thus, both the inplane stiffnesses, A_{ij} , and the bending stiffnesses, D_{ij} , are orthotropic for this class of laminates. The complexity, however, is introduced through the twisting-extensional coupling coefficients B_{16} and B_{26} . As in the case of the unsymmetric bidirectional laminates, the coupling coefficients also dissipate for this class of laminates with increasing N .

The displacement formulation, eqs. (2.3.9-11), will be utilized in conjunction with this class of laminates to obtain bending deflection under transverse loading. These equations now take the following form:

$$\begin{aligned} A_{11} \frac{\partial^2 u^0}{\partial x^2} + A_{66} \frac{\partial^2 u^0}{\partial y^2} + (A_{12} + A_{66}) \frac{\partial^2 v^0}{\partial x \partial y} - 3B_{16} \frac{\partial^3 w}{\partial x^2 \partial y} \\ - B_{26} \frac{\partial^3 w}{\partial y^3} = 0 \end{aligned} \quad (2.3.129)$$

$$\begin{aligned} (A_{12} + A_{66}) \frac{\partial^2 u^0}{\partial x \partial y} + A_{66} \frac{\partial^2 v^0}{\partial x^2} + A_{22} \frac{\partial^2 v^0}{\partial y^2} - B_{16} \frac{\partial^3 w}{\partial x^3} \\ - 3B_{26} \frac{\partial^3 w}{\partial x \partial y^2} = 0 \end{aligned} \quad (2.3.130)$$

$$\begin{aligned} D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} \\ - B_{16} \left(3 \frac{\partial^3 u^0}{\partial x^2 \partial y} + \frac{\partial^3 v^0}{\partial x^3} \right) - B_{26} \left(\frac{\partial^3 u^0}{\partial y^3} + 3 \frac{\partial^3 v^0}{\partial x \partial y^2} \right) = q \end{aligned} \quad (2.3.131)$$

We now consider the following simple support boundary conditions in conjunction with the constitutive relations, eq. (2.3.1):

at $x = 0, a$

$$\begin{aligned} u^0 &= 0 \\ N_{xy} &= A_{66} \left(\frac{\partial^2 v^0}{\partial x^2} + \frac{\partial u^0}{\partial y} \right) - B_{16} \frac{\partial^2 w}{\partial x^2} - B_{26} \frac{\partial^2 w}{\partial y^2} = 0 \end{aligned} \quad (2.3.132)$$

$w = 0$

$$M_{xx} = B_{16} \left(\frac{\partial v^0}{\partial x} + \frac{\partial u^0}{\partial y} \right) - D_{11} \frac{\partial^2 w}{\partial x^2} - D_{12} \frac{\partial^2 w}{\partial y^2} = 0$$

at $y = 0, b$

$$v^0 = 0$$

$$N_{xy} = A_{66} \left(\frac{\partial v^0}{\partial x} + \frac{\partial u^0}{\partial y} \right) - B_{16} \frac{\partial^2 w}{\partial x^2} - B_{26} \frac{\partial^2 w}{\partial y^2} = 0 \quad (2.3.133)$$

$w = 0$

$$M_{yy} = B_{26} \left(\frac{\partial^2 v^0}{\partial x^2} + \frac{\partial u^0}{\partial y} \right) - D_{12} \frac{\partial^2 w}{\partial x^2} - D_{22} \frac{\partial^2 w}{\partial y^2} = 0$$

The transverse load is again represented by a double Fourier sine series. The following displacement field satisfies eqs. (2.3.129-131) and the boundary conditions, eqs. (2.3.131,132):

$$\begin{aligned} u^0 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \\ v^0 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ w &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \end{aligned} \quad (2.3.134)$$

Substituting eqs. from (2.3.134) into eqs. (2.3.129-131), equating like Fourier terms, and solving the resulting simultaneous algebraic equations for the Fourier coefficients, we obtain the following solution:

$$\begin{aligned} A_{mn} &= q_{mn} \frac{\eta^4 b^3 n}{\pi^3 D_{mn}} [(A_{66} m^2 + A_{22} n^2 \eta^2) (3B_{16} m^2 + B_{26} n^2 \eta^2) \\ &\quad - m^2 (A_{12} + A_{66}) (B_{16} m^2 + 3B_{26} n^2 \eta^2)] \end{aligned}$$

$$B_{mn} = q_{mn} \frac{n^3 b^3 m}{\pi^3 D_{mn}} [(A_{11} m^2 + A_{66} n^2 \eta^2) (B_{16} m^2 + 3B_{26} n^2 \eta^2) - n^2 R^2 (A_{12} + A_{66}) (3B_{16} m^2 + B_{26} n^2 \eta^2)] \quad (2.3.135)$$

$$C_{mn} = q_{mn} \frac{n^4 b^4}{\pi^4 D_{mn}} [(A_{11} m^2 + A_{66} n^2 \eta^2) (A_{66} m^2 + A_{22} n^2 \eta^2) - (A_{12} + A_{66})^2 m^2 n^2 \eta^2]$$

where

$$D_{mn} = [(A_{11} m^2 + A_{66} n^2 \eta^2) (A_{66} m^2 + A_{22} n^2 \eta^2) - (A_{12} + A_{66})^2 m^2 n^2 \eta^2] [D_{11} m^4 + 2(D_{12} + 2D_{66}) m^2 n^2 \eta^2 + D_{22} n^4 \eta^4] + 2m^2 n^2 \eta^2 (A_{12} + A_{66}) (3B_{16} m^2 + B_{26} n^2 \eta^2) (B_{16} m^2 + 3B_{26} n^2 \eta^2) - n^2 \eta^2 (A_{66} m^2 + A_{22} n^2 \eta^2) (3B_{16} m^2 + B_{26} n^2 \eta^2)^2 - m^2 (A_{11} m^2 + A_{66} n^2 \eta^2) (B_{16} m^2 + 3B_{26} n^2 \eta^2)^2$$

Stress and moment resultants can be determined from eqs. (2.3.1).

For the case where B_{16} , B_{26} vanish, A_{mn} and B_{mn} also vanish, and D_{mn} reduces to the Fourier coefficient for an orthotropic plate as given by eq. (2.3.95).

Using the same ply properties as in the case of the unsymmetric bidirectional laminates, the effect of coupling on unsymmetric angle-ply laminates is illustrated in figures 2.3.16 and 2.3.17. The loading is again uniform, with the Fourier coefficients being given by eq. (3.102). As in the case of the unsymmetric bidirectional laminates, the effect of coupling can be very significant, but dissipates rapidly with increasing values of N . Results in figure 2.3.16 are presented for various angles of orientation. As in the case of the symmetric angle-ply plates, $+45^\circ$ represents the largest effect due to bending-extensional coupling. The effect of anisotropy ratio for the $+45^\circ$ unsymmetric laminates is illustrated in figure 2.3.17. Solutions in both of these figures for $N \rightarrow \infty$ are obtained by neglecting the B_{16} and B_{26} coupling coefficients and using the orthotropic, homogeneous solution, eq. (3.95).

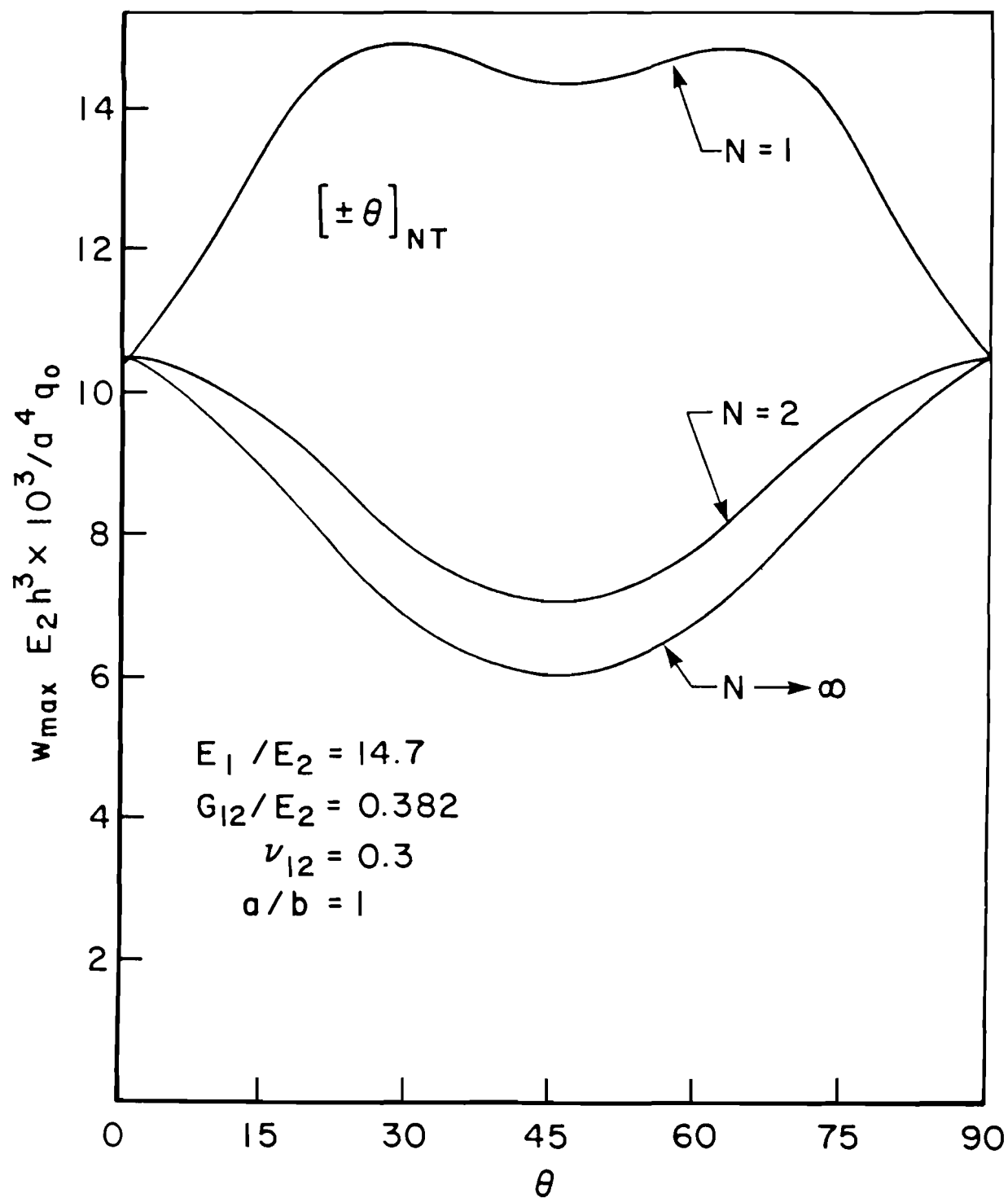


FIGURE 2.3.16 EFFECT OF ORIENTATION ON THE BENDING OF UNSYMMETRIC ANGLE-PLY PLATES UNDER UNIFORM TRANSVERSE LOAD

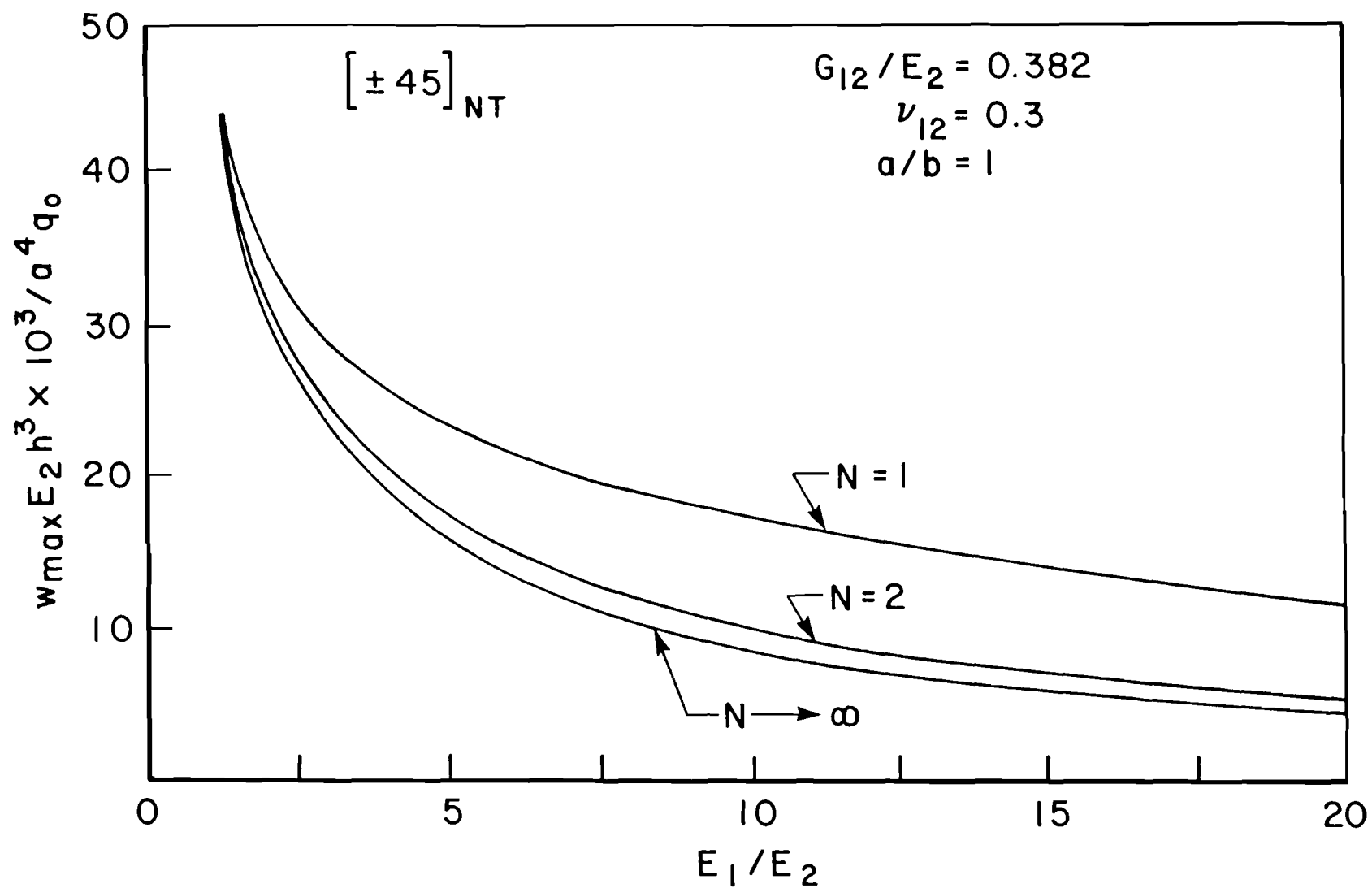


FIGURE 2.3.17. EFFECT OF ANISOTROPY RATIO ON THE BENDING OF UNSYMMETRIC $\pm 45^\circ$ LAMINATE UNDER UNIFORM TRANSVERSE LOAD

o The Finite Element Method

So far we have discussed various methods for obtaining solutions to rectangular laminated plates. With approaches such as the Galerkin method, approximate bending solutions can be obtained for any desired combination of boundary conditions. For more complicated shapes, approximate techniques such as the Galerkin method become less attractive.

The finite element method provides a viable approach to the solution of boundary value problems in the bending of laminated plates with complex geometries. This approach is a natural extension of matrix methods for skeletal structures to the analysis of continuous media.

In the finite element method, the plate is divided into discrete regions (elements) of predetermined shape, e.g. a triangle. These elements are connected at domain node points and along the boundaries. As in the case of the Galerkin method, assumed functions with undetermined coefficients are utilized to describe deformations of the plate. In the finite element method, however, the assumed functions are applied to each element individually, rather than to the entire plate domain. As in the case of the Galerkin method, the coefficients of the assumed functions are chosen such that potential energy is minimized. As a result, the finite element method also leads to a set of simultaneous algebraic equations which can be solved by classical matrix methods.

In many cases the boundaries of finite elements are straight lines. Thus, if the plate domain has curved boundaries, they are approximated by a series of straight line segments. Such a situation is illustrated in figure 2.3.18a where a typical finite element grid utilizing triangular elements is shown.

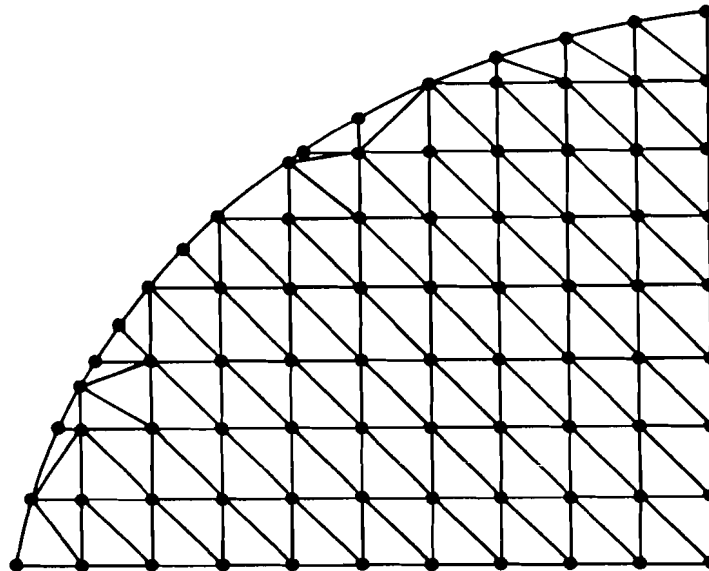


FIGURE 2.3.18a TYPICAL FINITE ELEMENT GRID FOR RECTANGULAR ELEMENT

Although the finite element technique provides a very versatile approach to solving boundary value problems, it must be applied with care. In particular, there are various levels of sophistication available in the development of a finite element. The element geometry provides various degrees of complexity. The deformation assumptions concerning a specific element provide an additional level of complexity. For example, linear displacements across an element lead to constant strain elements. In such a case continuity of displacements is provided between elements, but not continuity of stresses (force and moment resultants in the case of plate theory). Thus, in regions where large stress gradients are present, numerous elements are required in order to obtain accurate solutions. Constant strain elements allow only the displacement boundary conditions to be satisfied. Higher order elements can provide continuity of stresses between elements and, as a result, may require fewer elements in complex stress fields to obtain accurate solutions. The increased complexity of the individual elements, however, may not reduce the overall numerical calculations required.

In order to choose the proper density of elements, one must know something about the nature of the solution being sought. In addition, if one is interested in obtaining stresses in the vicinity of boundaries, then elements which do not provide for satisfaction of stress boundary conditions will not yield accurate results. Thus, it is obvious that care must be taken in applying finite element techniques if accurate solutions are desired.

The finite element method can also be applied to other classical plate problems such as vibration and buckling. In such cases the solution ends up in the form of a classical eigenvalue problem.

2.3.2.3 Failure.

Using ply stresses in conjunction with a failure criterion, we can make a prediction of first failure in a laminate under bending loads. Failure of laminates under inplane loading was discussed in detail under Section 3.7 of Vol. I. Subsequent failure after first-ply failure becomes much more complicated in the case of bending loads, as will be shown in this section. Further complications arise in the case of bending due to the fact that failure can also be induced by interlaminar stresses.

o Initial Failure

Under bending loads ply stress distributions are linear through-the-thickness with the maximum stress occurring at the ply interfaces. Thus, these ply interfaces, or outer surfaces, provide the location for application of the failure criterion. As an example, let us consider the $[0/90]_s$ laminate under cylindrical bending as discussed in Section 2.3.2.2. Let us assume the following unidirectional strength properties:

$$\begin{aligned}\sigma_L^{cu} &= \sigma_L^{tu} = 250,000 \text{ psi} \\ \sigma_T^{cu} &= 20,000 \text{ psi}, \quad \sigma_T^{tu} = 7,000 \text{ psi} \\ \tau_L^u &= 12,000 \text{ psi}\end{aligned}\tag{2.3.136}$$

For simplicity we will utilize the maximum stress failure criterion. Thus,

$$\begin{aligned}
 -\sigma_L^{cu} &\leq \sigma_{11}^k \leq \sigma_L^{tu} \\
 -\sigma_T^{cu} &\leq \sigma_{22}^k \leq \sigma_L^{tu} \\
 |\sigma_{12}^k| &\leq \tau_L^u
 \end{aligned} \tag{2.3.137}$$

Since the 0° tensile and compression strength are assumed equal, simultaneous tension and compression failure at the outer surfaces will occur for this symmetric laminate under transverse loading. Using eq. (2.3.65), we see that

$$\sigma_{11}(0^\circ) = \sigma_{xx}(0^\circ) = q_0 \frac{a^2}{h^2} (6.80) \bar{x} (1-\bar{x}) \bar{z} \tag{2.3.138}$$

The maximum stress will occur at $\bar{x} = \bar{z} = 0.5$. For the case where $a/h = 25$,

$$\sigma_{11}(0^\circ) = (25)^2 (6.80) (0.5) (1-0.5) (0.5) q_0 = 531 q_0$$

Using this result in conjunction with the failure criterion, we find that fiber failure in the 0° plies occurs at $q_0 = 471$ psi. We now need to consider transverse failure of this ply. In the outer 0° ply from eq. (2.3.64)

$$\begin{aligned}
 \sigma_{22}(0^\circ) &= \sigma_{yy}(0^\circ) = q_0 \frac{a^2}{h^2} (0.139) \bar{x} (1-\bar{x}) \bar{z} \\
 \sigma_{22}(0^\circ) &= (25)^2 (0.139) (0.5) (1-0.5) (0.5) q_0 = 10.9 q_0
 \end{aligned} \tag{2.3.139}$$

and transverse failure occurs at $q_0 = 642$ psi. Since transverse tension and compression differ, transverse failure in the lower 0° ply will occur at a higher load, since in the lower ply

$$\sigma_{22}(0^\circ) = -10.9 q_0$$

which yields failure at $q_0 = 1,835$ psi.

For the center 90° plies, tensile failure will occur at the interface $\bar{z} = 0.25$. From eq. (3.66)

$$\begin{aligned}
 \sigma_{22}(0^\circ) &= \sigma_{xx}(90^\circ) = q_0 \frac{a^2}{h^2} (0.462) \bar{x} (1-\bar{x}) \bar{z} \\
 \sigma_{22}(0^\circ) &= (25)^2 (0.462) (0.5) (1-0.5) (0.25) q_0 = 18.0 q_0
 \end{aligned}$$

Tensile failure in the 90° ply will occur at $q_0 = 389$ psi. For compression failure in the 90° ply, we consider the lower interface at $\bar{z} = -0.25$ with the result

$$\sigma_{22}(0^\circ) = -18.0 q_0$$

and compression failure will occur at $q_0 = 1,111$ psi. Fiber failure in the 90° ply is determined by applying eq. (2.3.64) at $\bar{z} = +0.25$, since 0° tension and compression strengths are assumed equal. For the tension side

$$\sigma_{11}(90^\circ) = \sigma_{yy}(90^\circ) = q_0 \frac{a^2}{h^2} (0.139) \bar{x} (1-\bar{x}) \bar{z}$$

$$\sigma_{11}(90^\circ) = (25)^2 (0.139) (0.5) (1-0.5) (0.25) q_0 = 5.43 q_0$$

Simultaneous tension and compression failure will occur at $q_0 = 46,041$ psi. Thus, the critical failure stress is on the tension side of the 90° plies with first-ply failure being a matrix failure at $\bar{x} = 0.5$, $\bar{z} = 0.25$ with

$$q_0 (\text{critical}) = 389 \text{ psi} \quad (2.3.140)$$

Unlike the case of inplane loading, all plies of like orientation will not necessarily fail simultaneously under bending load, even when tensile and compression strengths are identical. As an example, consider the same cylindrical bending problem for a laminate with the stacking geometry $[0/90]_{2s}$. The normalized bending stress distribution for $\bar{\sigma}_{xx}$ is shown in figure 2.3.18(b). It is easily seen that the maximum stress in the outer two 0° and 90° plies is larger than in the two corresponding inner plies.

o Subsequent Failures

As discussed in Section 3.7.2 of Volume I, progressive failure after first-ply failure is one of the active research areas in composite materials today. Of particular interest are initial and subsequent failures in the matrix modes. A simplified approach to first-ply failure was discussed in Section 3.7.2 in which the matrix mode properties E_T (modulus transverse to the fibers) and G_L (inplane shear modulus relative to the fiber direction) are set equal to zero for subsequent loading. Difficulties are encountered, however, with such an approach under bending loads. In particular, since the stresses are linear through-the-thickness of the ply, failure is initiated at the point of maximum stress (which is at the ply interface) rather than across the entire ply. This difficulty could be overcome by assuming failure of the entire ply when the failure criterion is satisfied at the ply interface. Another approach might be to apply the failure criterion to the average stress across the ply thickness, rather than at the point of maximum stress. Further difficulties arise for cases in which first-ply failure is not symmetric relative to the laminate midplane. This was the case in the example of the previous section in which first-ply failure occurred in the tensile side of the 90° plies. If it is assumed that the compression side of the 90° plies remain intact, then an unsymmetric laminate is produced for subsequent loading if a stiffness reduction approach is utilized. Thus, subsequent failure analysis beyond first-ply failure for laminates under bending loads is not at all straightforward.

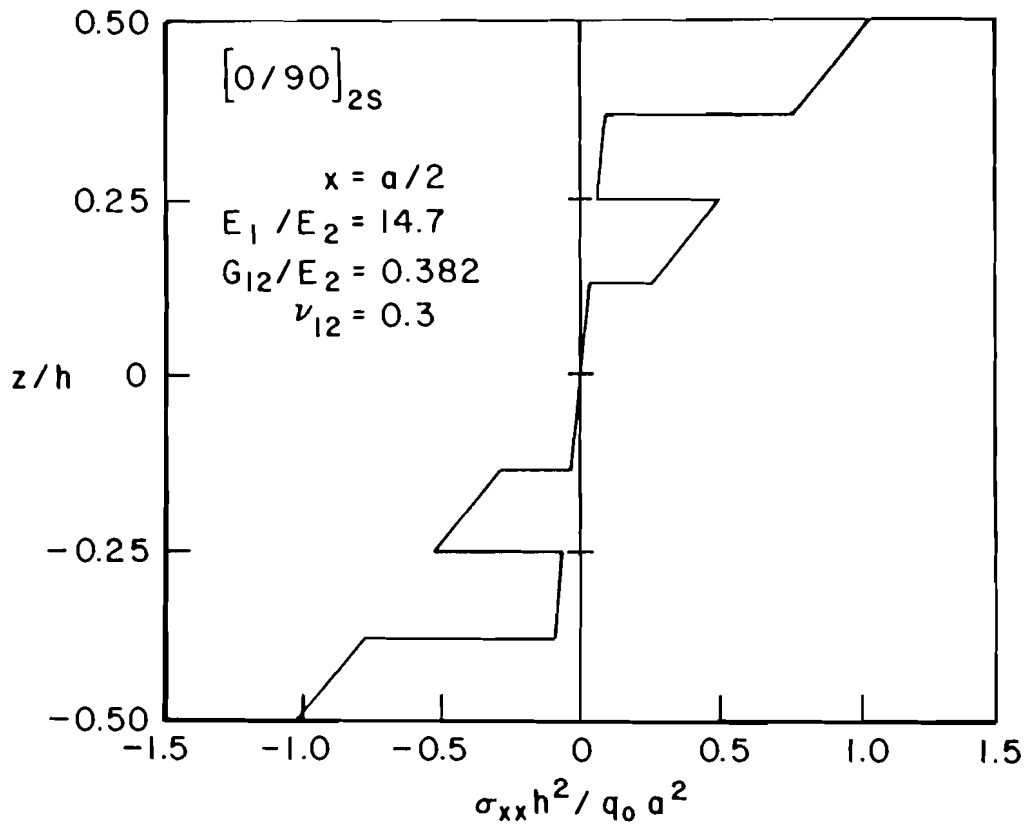


FIGURE 2.3.18(b). INPLANE NORMAL STRESS DISTRIBUTION AT $X=A/2$ FOR A BIDIRECTIONAL LAMINATE SUBJECTED TO CYLINDRICAL BENDING UNDER UNIFORM TRANSVERSE LOAD

o Interlaminar Failure

Under bending loads we must consider the possibility of interlaminar failure. As in the case of inplane loads a failure criterion must be chosen. For interlaminar failure maximum stress would seem to be an obvious criterion, as little interaction with inplane loads is anticipated. As a first estimate we might expect the interlaminar normal strength to be equivalent to the transverse unidirectional strength and the interlaminar shear strength to be the same as the unidirectional inplane shear strength. Thus, the interlaminar failure criterion becomes

$$\begin{aligned}
 -\sigma_T^{cu} &\leq \sigma_{zz} \leq \sigma_T^{tu} \\
 |\tau_{xz}| &\leq \tau_L^u, \quad |\tau_{yz}| \leq \tau_L^u
 \end{aligned}
 \tag{2.3.141}$$

As an example let us again consider the $[0/90]_s$ laminate subjected to cylindrical bending. From figure 2.3.5 it is easily seen that the maximum value of σ_{zz} occurs at the outer surface where $\sigma_{zz} = q_0$. Thus, a peel failure would occur near the outer surface when $q_0 = 7,000$ psi. If the load is reversed, i.e. $q = -q_0$, then a crushing failure would occur near the outer surface when $q_0 = -20,000$ psi. These values of q_0 are far in excess of the loads required to produce first-ply failure due to inplane bending stresses.

For interlaminar shear failure, figure 2.3.4 and eq. (2.3.68) reveal that the maximum interlaminar shear stress occurs at $\bar{x} = \bar{z} = 0$. From eq. (3.68) with $a/h = 25$

$$\tau_{xz}(0,0) = (25)(0.231)(2.82)q_0 = 16.3 q_0$$

Using the shear strength from eq. (2.3.136), an interlaminar shear failure would be anticipated for $q_0 = 736$ psi. This is in excess of the critical value of q_0 given in eq. (2.3.140). Thus, first-ply failure due to inplane bending stresses would occur prior to delamination due to interlaminar shear stress.

2.3.3 Vibrations.

We now turn our attention to free vibration of laminated plates. The governing equations developed for static bending will now have to be modified to account for inertia effects. After developing the basic equations, the various complexities of solutions to determining fundamental flexural vibration frequencies of a laminated plate will be considered.

2.3.3.1 Governing Equations.

The governing equations in terms of force and moment resultants must now be modified to account for inertia. For a general laminated plate eqs. (2.3.3-6) now become (see reference 2.3.12)

$$\begin{aligned} \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= P \frac{\partial^2 u^0}{\partial t^2} - R \frac{\partial^3 w}{\partial x \partial t^2} \\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} &= P \frac{\partial^2 v^0}{\partial t^2} - R \frac{\partial^3 w}{\partial y \partial t^2} \end{aligned} \quad (2.3.142)$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_{xx} = R \frac{\partial^2 u^0}{\partial t^2} - I \frac{\partial^3 w}{\partial x \partial t^2} \quad (2.3.143)$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_{yy} = R \frac{\partial^2 v^0}{\partial t^2} - I \frac{\partial^3 w}{\partial y \partial t^2} \quad (2.3.144)$$

$$\frac{\partial Q_{xx}}{\partial x} + \frac{\partial Q_{yy}}{\partial y} + q = P \frac{\partial^2 w}{\partial t^2} \quad (2.3.145)$$

where

$$(P, R, I) = \int_{-h/2}^{h/2} \rho^k(1, z, z^2) dz$$

and ρ^k is the density of the kth ply. Note that R introduces a new dynamic coefficient not found in homogeneous plates. As in the case of the B_{ij} stiffness terms, R vanishes for symmetric laminates.

Combining eqs. (2.3.143-145) in a similar manner as in the static case, eq. (2.3.145) becomes

$$\begin{aligned} \frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + q &= R \frac{\partial^2}{\partial t^2} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) \\ - I \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + P \frac{\partial^2 w}{\partial t^2} & \end{aligned} \quad (2.3.146)$$

The displacement formulation of the governing equations will be identical to eqs. (2.3.9-11) with the right-hand side modified to be compatible with eqs. (2.3.142-146).

2.3.3.2 Cylindrical Bending.

For the case of free vibration ($q = 0$) under conditions of cylindrical bending, eqs. 3.32-34) become

$$A_{11} \frac{\partial^2 u^0}{\partial x^2} + A_{16} \frac{\partial^2 v^0}{\partial x^2} - B_{11} \frac{\partial^3 w}{\partial x^3} = P \frac{\partial^2 u^0}{\partial t^2} - R \frac{\partial^3 w}{\partial x \partial t^2} \quad (2.3.147)$$

$$A_{16} \frac{\partial^2 u^0}{\partial x^2} + A_{66} \frac{\partial^2 v^0}{\partial x^2} - B_{16} \frac{\partial^3 w}{\partial x^3} = P \frac{\partial^2 v^0}{\partial t^2} \quad (2.3.148)$$

$$\begin{aligned} D_{11} \frac{\partial^4 w}{\partial x^4} - B_{11} \frac{\partial^3 u^0}{\partial x^3} - B_{16} \frac{\partial^3 v^0}{\partial x^3} &= - R \frac{\partial^3 u^0}{\partial x \partial t^2} + I \frac{\partial^4 w}{\partial x^2 \partial t^2} \\ - P \frac{\partial^2 w}{\partial t^2} & \end{aligned} \quad (2.3.149)$$

In cases where each ply is made of the same material, but not necessarily in identical orientations, R will vanish. Even in the case of hybrid laminates in unsymmetric layups relative to the different materials, these dynamic terms contribute very little to the problem of flexural vibration. Furthermore, the inplane inertia terms (those terms involving time derivatives of u^0 and v^0) contribute little to the first flexural mode of vibration. For thin plates the rotary inertia term (terms involving time derivatives of the curvatures) are also neglected. Under these conditions eqs. (2.3.147,148) reduce to the static eqs. (2.3.32,33), respectively, and eq. (2.3.149) becomes

$$D_{11} \frac{\partial^4 w}{\partial x^4} - B_{11} \frac{\partial^3 u^0}{\partial x^3} - B_{16} \frac{\partial^3 v^0}{\partial x^3} + P \frac{\partial^2 w}{\partial t^2} = 0 \quad (2.3.150)$$

A solution to eqs. (2.3.32,33,150) is assumed in the form

$$u^0 = e^{i\omega t} U(x)$$

$$v^0 = e^{i\omega t} V(x) \quad (2.3.151)$$

$$w = e^{i\omega t} W(x)$$

Substituting these relationships into eqs. (2.3.32,33,150), we obtain the following results

$$A_{11} \frac{d^2 U}{dx^2} + A_{16} \frac{d^2 V}{dx^2} - B_{11} \frac{d^3 W}{dx^3} = 0 \quad (2.3.152)$$

$$A_{16} \frac{d^2 U}{dx^2} + A_{66} \frac{d^2 V}{dx^2} - B_{16} \frac{d^3 W}{dx^3} = 0 \quad (2.3.153)$$

$$D_{11} \frac{d^4 W}{dx^4} - B_{11} \frac{d^3 U}{dx^3} - B_{16} \frac{d^3 V}{dx^3} - P\omega^2 W = 0 \quad (2.3.154)$$

Thus, eqs. (2.3.151) yield a steady-state solution from which the natural vibration frequencies can be obtained.

Equations (2.3.152-154) can be uncoupled using the same procedure as for the static case, eqs. (2.3.35-37), with the result

$$\frac{d^2 U}{dx^2} = \frac{B}{A} \frac{d^3 W}{dx^3} \quad (2.3.155)$$

$$\frac{d^2 V}{dx^2} = \frac{C}{A} \frac{d^3 W}{dx^3} \quad (2.3.156)$$

$$\frac{d^4 W}{dx^4} - \frac{AP\omega^2}{D} W = 0 \quad (2.3.157)$$

where A, B, C, and D are as defined in eqs. (2.3.35-37).

We now consider simple-support boundary conditions as given by eqs. (2.3.45). The functions U, V, and W must satisfy these relationships. In particular,

at $x = 0, a$

$$N_{xx} = A_{11} \frac{dU}{dx} + A_{12} \frac{dV}{dx} - B_{11} \frac{d^2W}{dx^2} = 0 \quad (2.3.158)$$

$$N_{xy} = A_{16} \frac{dU}{dx} + A_{66} \frac{dV}{dx} - B_{16} \frac{d^2W}{dx^2} = 0$$

$$W = 0$$

$$M_{xx} = B_{11} \frac{dU}{dx} + B_{16} \frac{dV}{dx} - D_{11} \frac{d^2W}{dx^2} = 0$$

The following displacements satisfy the governing eqs. (2.3.155-157) and the boundary conditions, eqs. (2.3.158)

$$U(x) = U_m \cos \frac{m\pi x}{a} \quad (2.3.159)$$

$$V(x) = V_m \cos \frac{m\pi x}{a}$$

$$W(x) = W_m \sin \frac{m\pi x}{a}$$

where m is an integer. Substituting these displacement functions into the governing eqs. (2.3.152-154), we obtain the following linear equations in matrix form:

$$\begin{bmatrix} 1 & 0 & \frac{m\pi B}{aA} \\ 0 & 1 & \frac{m\pi C}{aA} \\ 0 & 0 & F_m \end{bmatrix} \begin{bmatrix} U_m \\ V_m \\ W_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (2.3.160)$$

where

$$F_m = \frac{m^4 \pi^4}{a^4} - \frac{AP}{D} \omega^2$$

A nontrivial solution to eq. (2.3.160) occurs only if F_m vanishes with the result

$$\omega_m = \frac{m^2 \pi^2}{a^2} \sqrt{\frac{D}{AP}} \quad (2.3.161)$$

This relationship can be written in the form

$$\omega_m = \omega_m' \sqrt{1-k}$$

where ω_m' is the flexural vibration frequency of a laminated strip in which B_{ij} is neglected. It is defined by the relationship

$$\omega_m' = \frac{m^2 \pi^2}{a^2} \sqrt{\frac{D_{11}}{P}} \quad (2.3.162)$$

and k is as defined in eq. (2.3.54). As previously discussed, k is always positive; thus, bending-extensional coupling reduces flexural vibration frequencies. It will be shown in later sections that the magnitude of this decrease depends on individual ply properties and the number of plies in the laminate.

2.3.3.3 Specially Orthotropic Plates.

Consideration is now given to symmetric laminates, $B_{ij} = 0$, in which the bending-twisting coupling terms vanish, $D_{16} = D_{26} = 0$. For such laminates eq. (2.3.14) under dynamic conditions, in the absence of transverse loads ($q = 0$) and rotary inertia, becomes

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} + P \frac{\partial^2 w}{\partial t^2} = 0 \quad (2.3.163)$$

A steady-state solution to this equation is assumed in the form

$$w = e^{i\omega t} W(x, y) \quad (2.3.164)$$

Substituting this relationship into eq. (2.3.163), we obtain the time independent equation

$$D_{11} \frac{\partial^4 W}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 W}{\partial y^4} - P\omega^2 W = 0 \quad (2.3.165)$$

For simple-support boundary conditions W must satisfy eqs. (2.3.87,88), i.e.

at $x = 0, a$

$$W = M_x = -D_{11} \frac{\partial^2 W}{\partial x^2} - D_{12} \frac{\partial^2 W}{\partial y^2} = 0 \quad (2.3.166)$$

at $y = 0, b$

$$W = M_y = -D_{12} \frac{\partial^2 W}{\partial x^2} - D_{22} \frac{\partial^2 W}{\partial y^2} = 0 \quad (2.3.167)$$

A solution to eq. (2.3.158) which satisfies the boundary conditions, eqs. (2.3.166,167), is of a form similar to the static transverse bending problem, eq. (2.3.94), i.e.

$$W(x, y) = W_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (2.3.168)$$

where m and n are integers. Substituting this function into eq. (2.3.165) and solving for the frequency, we obtain the result

$$\omega_{mn} = \frac{\pi^2}{a^2 \sqrt{P}} \sqrt{D_{11}m^4 + 2(D_{12}+2D_{66})m^2n^2R^2 + D_{22}n^4R^4} \quad (2.3.169)$$

where again η is the aspect ratio a/b . Different natural frequencies are obtained for each combination of m and n . The lowest or fundamental frequency is obtained for $m = n = 1$ with the result

$$\omega_{11} = \frac{\pi^2}{a^2 \sqrt{P}} \sqrt{D_{11} + 2(D_{12}+2D_{66})\eta^2 + D_{22}\eta^4} \quad (2.3.170)$$

Note that as $b \rightarrow \infty$, $\eta \rightarrow 0$, and eq. (2.3.169) reduces to eq. (2.3.162), which is the orthotropic solution for cylindrical bending.

It should also be noted that for a square plate of the class $[0/90]_{Ns}$, the fundamental vibration frequency is independent of stacking sequence. In particular, for this class of laminates, as previously discussed

$$\begin{aligned} D_{11} &= [(1+3/4N)\bar{Q}_{11}(0^\circ) + (1-3/4N)\bar{Q}_{11}(90^\circ)] \frac{h^3}{24} \\ D_{22} &= [(1+3/4N)\bar{Q}_{11}(90^\circ) + (1-3/4N)\bar{Q}_{11}(0^\circ)] \frac{h^3}{24} \\ D_{12} &= \frac{h^3}{12} \bar{Q}_{12}(0^\circ), \quad D_{66} = \frac{h^3}{12} \bar{Q}_{66}(0^\circ) \end{aligned} \quad (2.3.171)$$

Substituting these relationships into eq. (2.3.170), we obtain the result

$$\omega_{11} = \frac{\pi^2}{2a^2 \sqrt{3P}} \sqrt{\bar{Q}_{11}(0^\circ) + \bar{Q}_{11}(90^\circ) + 2[\bar{Q}_{12}(0^\circ) + 2\bar{Q}_{66}(0^\circ)]} \quad (2.3.172)$$

Thus, the fundamental frequency is independent of N . Similar results are noted in table 2.3.1 for the bending of the same class of laminates under uniform transverse load. The value of N has little effect on the maximum deflection. This conclusion will obviously be different for other values of η .

2.3.3.4 Anisotropic Plates.

The flexural vibration frequencies for symmetric laminates, $B_{ij} = 0$, in which the bending-twisting coupling terms D_{16} , D_{26} do not vanish are now given consideration. The free vibration analysis of such laminates introduces the same complexities as encountered with the transverse bending problem. In particular, numerical difficulties are introduced due to the existence of odd order partial derivatives in x and y for both the governing partial differential equation and moment boundary conditions. Thus, the double sine solution assumed for the orthotropic case will yield both double sine and double cosine terms when substituted into either the governing equation or moment boundary conditions.

This difficulty is overcome by using the Galerkin method to obtain an approximate solution. These approximate solutions are used to ascertain the effect of bending-twisting coupling on the fundamental vibration frequencies for laminated plates. This is accomplished by comparing the anisotropic plate solution to orthotropic plate solutions in which the D_{16} , D_{26} coupling terms are neglected. For anisotropic laminates, eq. (2.3.14) under dynamic conditions with rotary inertia neglected and no transverse loads ($q = 0$) becomes

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w}{\partial y^4} + P \frac{\partial^2 w}{\partial t^2} = 0 \quad (2.3.173)$$

Time dependence is removed from this equation by assuming a solution of the form as given by eq. (2.3.164) with the result

$$D_{11} \frac{\partial^4 W}{\partial x^4} + 4D_{16} \frac{\partial^4 W}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4 W}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4 W}{\partial x \partial y^3} + D_{22} \frac{\partial^4 W}{\partial y^4} - P\omega^2 W = 0 \quad (2.3.174)$$

For simple-support boundary conditions, W must satisfy eq (2.3.103, 104). In particular,

at $x = 0, a$

$$W = 0$$

$$M_{xx} = -D_{11} \frac{\partial^2 W}{\partial x^2} - 2D_{16} \frac{\partial^2 W}{\partial x \partial y} - D_{12} \frac{\partial^2 W}{\partial y^2} = 0 \quad (2.3.175)$$

at $y = 0, b$

$$W = 0$$

$$M_{yy} = -D_{12} \frac{\partial^2 W}{\partial x^2} - 2D_{26} \frac{\partial^2 W}{\partial x \partial y} - D_{22} \frac{\partial^2 W}{\partial y^2} = 0 \quad (2.3.176)$$

An approximate solution to eq. (2.3.174) which satisfies the governing eq. (2.3.175) and the displacement boundary conditions in eq. (2.3.175,176) is sought by the Galerkin method (reference 2.3.9). Details of this method were discussed in Section 2.3.2.2 under the topic of anisotropic plates. The solution sought is of the same form as eq. (2.3.105), i.e.

$$W(x,y) = \sum_{m=1}^M \sum_{n=1}^N W_{mn} \phi_{mn}(x,y) \quad (2.3.177)$$

and the Galerkin integrals in eq. (2.3.106) now become

$$\int_0^b \int_0^a \left[D_{11} \frac{\partial^4 W}{\partial x^4} + 4D_{16} \frac{\partial^4 W}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4 W}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4 W}{\partial x \partial y^3} + D_{22} \frac{\partial^4 W}{\partial y^4} - P\omega^2 W \right] \phi_{mn} dx dy$$

$$+ B_1 + B_2 = 0 \quad (2.3.178)$$

where B_1 and B_2 correspond to the boundary integrals in eq. (2.3.106). The function ϕ_{mn} is chosen to be identical with the double sine function of eq. (2.3.107). Equation (2.3.178) leads to two sets of homogeneous algebraic equations. One set corresponds to $m+n$ odd and the other set to $m+n$ even. Natural vibration frequencies correspond to values of ω such that the determinant of the coefficient matrix in each of the two sets of equations vanish.

For angle-ply laminates of the class $[\pm\theta]_{N_s}$, the largest effect of D_{16} , D_{26} bending-twisting coupling terms is for $\theta = 45^\circ$. The effect of anisotropy ratio on the fundamental vibration frequency is illustrated in figure 2.3.19 (as previously $\nu_{12} = 0.3$, $G_{12}/E_2 = 0.382$) for square laminated plates of the class $[\pm 45]_{N_s}$. As shown in eq. (2.3.109), D_{16} and D_{26} dissipate with increasing values of N for this class of laminates. The orthotropic ($N \rightarrow \infty$) solution is obtained by neglecting the D_{16} and D_{26} coupling terms in eq. (2.3.174). For this case eq. (2.3.174) reduces to eq. (2.3.165), and the fundamental vibration frequency is given by eq. (2.3.170). Convergence of the solution is rather rapid. Results in figure 2.3.19 correspond to $M = N = 9$ in conjunction with eq. (2.3.177).

The results in figure 2.3.19 indicate that the orthotropic solution is rapidly recovered with increasing values of N . Thus, for many practical laminates, the D_{16} and D_{26} coupling terms can be neglected for the determination of fundamental vibration frequencies without serious error.

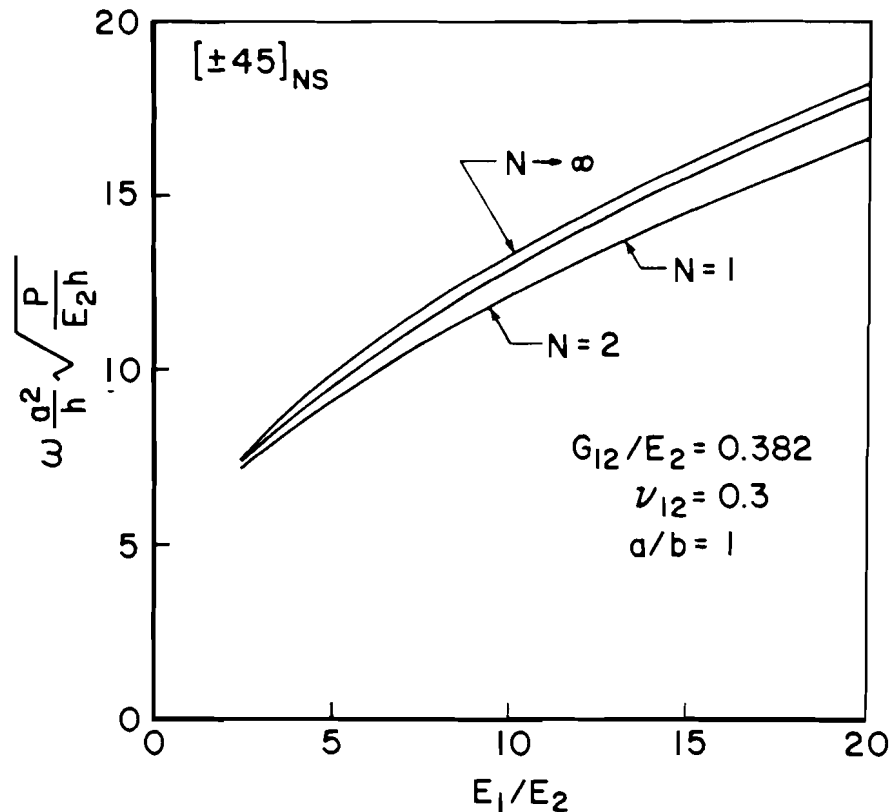


FIGURE 2.3.19. EFFECT OF ANISOTROPIC RATIO ON THE FUNDAMENTAL VIBRATION FREQUENCIES OF SYMMETRIC $+45^\circ$ SYMMETRIC LAMINATES

2.3.3.5 Vibration of Unsymmetric Laminates.

The effects of bending-extensional coupling on fundamental vibration frequencies is now assessed by obtaining closed form solutions to the free vibration of a special class of unsymmetric laminates and comparing the results to the same problem with the B_{ij} bending-extensional terms neglected. Closed form solutions can be obtained for the same two classes of unsymmetric laminates considered in Section 2.3.2.2 for bending under transverse load. It was shown in Section 2.3.3.1 that under conditions of cylindrical bending the effect of bending-extensional coupling was to reduce flexural vibration frequencies. Similar results will be shown in this section for rectangular unsymmetric laminates. The same trends are encountered for both classes of unsymmetric laminates which yield closed form solutions. Thus, it will only be necessary to consider one class of unsymmetric laminates in order to illustrate the effects of bending-extensional coupling.

Consider a laminate of the class $[+\theta]_{NT}$. For the purpose of determining fundamental flexural vibration frequencies, it will not be necessary to include inplane or rotary inertia terms. For this case eqs. (2.3.129,130) are still applicable to the present dynamic problem. Combining eqs. (2.3.131) and (2.3.146) with no transverse load ($q = 0$) and inplane and rotary inertia terms neglected, we obtain the result

$$\begin{aligned}
& D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} - B_{16} \left(3 \frac{\partial^3 u^0}{\partial x^2 \partial y} \right. \\
& \left. + \frac{\partial^3 v^0}{\partial x^3} \right) - B_{26} \left(\frac{\partial^3 u^0}{\partial y^3} + 3 \frac{\partial^3 v^0}{\partial x \partial y^2} \right) + P \frac{\partial^2 w}{\partial t^2} = 0
\end{aligned} \tag{2.3.179}$$

Time dependence can be removed by assuming solutions of the form

$$\begin{aligned}
u^0 &= e^{i\omega t} U(x, y) \\
v^0 &= e^{i\omega t} V(x, y) \\
w &= e^{i\omega t} W(x, y)
\end{aligned} \tag{2.3.180}$$

Substituting these functions into eqs. (2.3.129, 130, 179), we obtain the following equations

$$\begin{aligned}
& A_{11} \frac{\partial^2 U}{\partial x^2} + A_{66} \frac{\partial^2 U}{\partial y^2} + (A_{12} + A_{66}) \frac{\partial^2 V}{\partial x \partial y} - 3B_{16} \frac{\partial^3 W}{\partial x^2 \partial y} \\
& - B_{26} \frac{\partial^3 W}{\partial y^3} = 0
\end{aligned} \tag{2.3.181}$$

$$\begin{aligned}
& (A_{12} + A_{66}) \frac{\partial^2 U}{\partial x \partial y} + A_{66} \frac{\partial^2 V}{\partial x^2} + A_{22} \frac{\partial^2 V}{\partial y^2} - B_{16} \frac{\partial^3 W}{\partial x^3} \\
& - 3B_{26} \frac{\partial^3 W}{\partial x \partial y^2} = 0
\end{aligned} \tag{2.3.182}$$

$$\begin{aligned}
& D_{11} \frac{\partial^4 W}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 W}{\partial y^4} - B_{16} \left(3 \frac{\partial^3 U}{\partial x^2 \partial y} \right. \\
& \left. + \frac{\partial^3 V}{\partial x^3} \right) - B_{26} \left(\frac{\partial^3 U}{\partial y^3} + 3 \frac{\partial^3 V}{\partial x \partial y^2} \right) - P\omega^2 W = 0
\end{aligned} \tag{2.3.183}$$

The same simple-support boundary conditions, eqs. (2.3.132,133), used in the static bending problem are also applied to the flexural vibration problem. Thus, U , V , and W must satisfy the following boundary conditions:

at $x = 0, a$

$$\begin{aligned} U &= 0 \\ N_{xy} &= A_{66} \left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} \right) - B_{16} \frac{\partial^2 W}{\partial x^2} - B_{26} \frac{\partial^2 W}{\partial y^2} = 0 \\ W &= 0 \\ M_{xx} &= B_{16} \left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} \right) - D_{11} \frac{\partial^2 W}{\partial x^2} - D_{12} \frac{\partial^2 W}{\partial y^2} = 0 \end{aligned} \quad (2.3.184)$$

at $y = 0, b$

$$\begin{aligned} V &= 0 \\ N_{xy} &= A_{66} \left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} \right) - B_{16} \frac{\partial^2 W}{\partial x^2} - B_{26} \frac{\partial^2 W}{\partial y^2} = 0 \\ W &= 0 \\ M_{yy} &= B_{26} \left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} \right) - D_{12} \frac{\partial^2 W}{\partial x^2} - D_{22} \frac{\partial^2 W}{\partial y^2} = 0 \end{aligned} \quad (2.3.185)$$

A solution to eqs. (2.3.181-183) which satisfies the boundary conditions, eqs. (2.3.184,185), is of the same form as the analogous static bending solution, eqs. (2.3.134). Thus

$$\begin{aligned} U &= U_{mn} \sin \frac{m\pi x}{a} \cos \frac{m\pi y}{b} \\ V &= V_{mn} \cos \frac{m\pi x}{a} \sin \frac{m\pi y}{b} \\ W &= W_{mn} \sin \frac{m\pi x}{a} \sin \frac{m\pi y}{b} \end{aligned} \quad (2.3.186)$$

Substituting these functions into eqs. (2.3.181-183), we obtain the following homogeneous algebraic equations in matrix form

$$\begin{bmatrix} A_{mn} & B_{mn} & C_{mn} \\ B_{mn} & D_{mn} & E_{mn} \\ C_{mn} & E_{mn} & F_{mn} - \lambda_{mn} \end{bmatrix} \begin{bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (2.3.187)$$

where

$$A_{mn} = A_{11}m^2 + A_{66}n^2\eta^2$$

$$B_{mn} = (A_{12}+A_{66})mn\eta$$

$$C_{mn} = -\frac{n\eta\pi}{a} (3B_{16}m^2 + B_{26}n^2\eta^2)$$

$$D_{mn} = A_{66}m^2 + A_{22}n^2\eta^2$$

$$E_{mn} = -\frac{m\pi}{a} (B_{16}m^2 + 3B_{26}n^2\eta^2)$$

$$F_{mn} = \frac{\pi^2}{a^2} [D_{11}m^4 + 2(D_{12}+2D_{66})m^2n^2\eta^2 + D_{22}n^4\eta^4]$$

$$\lambda_{mn} = \frac{P\omega^2 a^2}{\pi^2}$$

where $\eta = a/b$. In order to obtain a nontrivial solution to eq. (3.187), λ_{mn} must be chosen such that the determinant of the coefficient matrix vanishes. Thus,

$$\begin{vmatrix} A_{mn} & B_{mn} & C_{mn} \\ B_{mn} & D_{mn} & E_{mn} \\ C_{mn} & E_{mn} & F_{mn} - \lambda_{mn} \end{vmatrix} = 0$$

Solving this relationship for λ_{mn} , we obtain the following frequency equation

$$\begin{aligned} \omega_{mn} = & \frac{\pi^2}{a^2 \sqrt{P}} \{ D_{11}m^4 + 2(D_{12}+2D_{66})mnR^2 + D_{22}n^4\eta^4 \\ & - \frac{1}{J_{3mn}} [m^2(B_{16}m^2 + 3B_{26}n^2\eta^2)J_{1mn} + n^2R^2(3B_{16}m^2 \\ & + B_{26}n^2\eta^2)J_{2mn}] \}^{1/2} \end{aligned} \quad (2.3.188)$$

where

$$J_{1mn} = (A_{11}m^2 + A_{66}n^2\eta^2)(B_{16}m^2 + 3B_{26}n^2\eta^2) - n^2R^2(A_{12}+A_{66})(3B_{16}m^2 + B_{26}n^2\eta^2)$$

$$J_{2mn} = (A_{66}m^2 + A_{22}n^2\eta^2)(3B_{16}m^2 + B_{26}n^2\eta^2) - m^2(A_{12}+A_{66})(B_{16}m^2 + 3B_{26}n^2\eta^2)$$

$$J_{3mn} = (A_{11}m^2 + A_{66}n^2R^2)(A_{66}m^2 + A_{22}n^2\eta^2) - (A_{12}+A_{66})^2m^2n^2\eta^2$$

When coupling is neglected ($B_{16} = B_{26} = 0$), eq. (2.3.188) reduces to eq. (2.3.169), which is the solution for a homogeneous orthotropic plate. As shown in eqs. (2.3.125) and (2.3.126), the B_{16} and B_{26} coupling coefficients dissipate with increasing values of N for the class of laminates under consideration, and the orthotropic solution is recovered in the limit ($N \rightarrow \infty$).

The influence of bending-extensional coupling is illustrated in figure 2.3.20 where the fundamental vibration frequency is plotted as a function of the anisotropy ratio, E_1/E_2 , for $[\pm 45]_{NT}$ square plates. Again $\nu_{12} = 0.3$ and $G_{12}/E_2 = 0.382$. Note that the effects of coupling dissipate quite rapidly with increasing values of N . Equation (2.3.169) is used for the case $N \rightarrow \infty$.

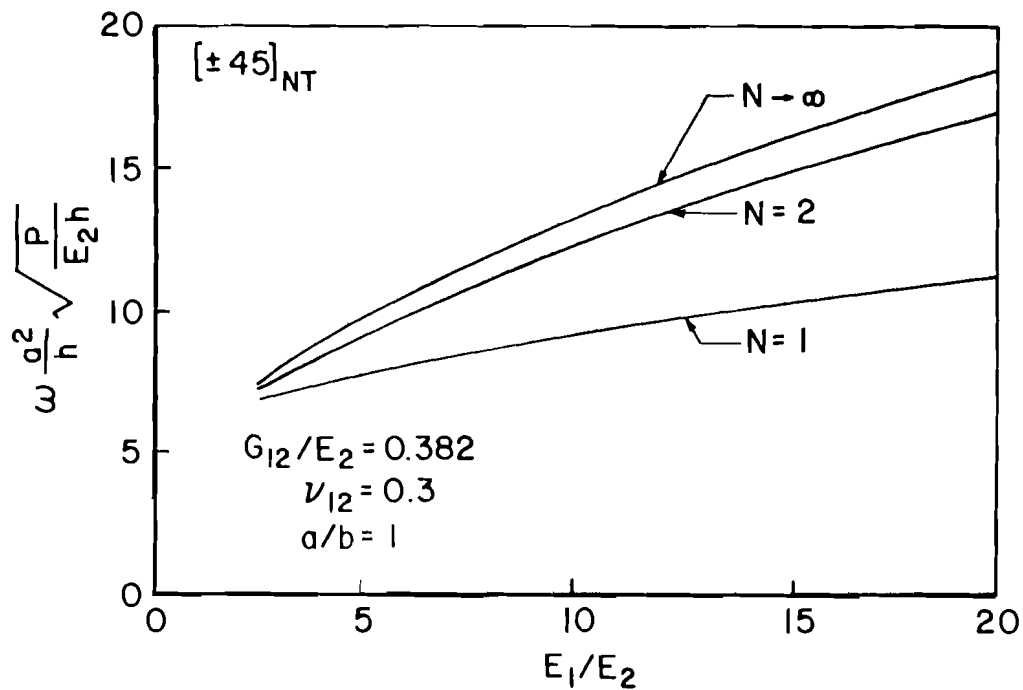


FIGURE 2.3.20 EFFECT OF ANISOTROPY RATIO ON THE FUNDAMENTAL VIBRATION FREQUENCIES OF UNSYMMETRIC $\pm 45^\circ$ LAMINATE

2.3.4 Stability.

If large inplane loads are applied to plates initially in a flat configuration, buckling may result, i.e. the inplane loads may be sufficiently large so that the plate is held in equilibrium in a slightly deflected shape. In linear buckling analysis such loads are referred to as critical buckling loads. As in the case of free vibration analysis, the magnitude of the resulting deflections are indeterminate with only the mode shapes being determinate. We now consider critical buckling calculations for laminated plates of varying complexity. The analogy between solution techniques for critical buckling loads and free vibration frequencies will become apparent.

2.3.4.1 Governing Equations.

The effect of inplane loads on plate bending must be included in the equilibrium equations. In particular, eq. (2.3.6) must be modified to include inplane force effects with the result

$$\frac{\partial Q_{xx}}{\partial x} + \frac{\partial Q_{yy}}{\partial y} + q + N_{xx}^i \frac{\partial^2 w}{\partial x^2} + 2N_{xy}^i \frac{\partial^2 w}{\partial x \partial y} + N_{yy}^i \frac{\partial^2 w}{\partial y^2} = 0 \quad (2.3.189)$$

where the superscript i denotes prebuckled values of the inplane forces. Differentiating eqs. (2.3.4,5) with respect to x and y , respectively, and substituting the results into eq. (2.3.189), we obtain

$$\begin{aligned} \frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + q + N_{xx}^i \frac{\partial^2 w}{\partial x^2} + 2N_{xy}^i \frac{\partial^2 w}{\partial x \partial y} \\ + N_{yy}^i \frac{\partial^2 w}{\partial y^2} = 0 \end{aligned} \quad (2.3.190)$$

With the modification to eq. (2.3.6), the third equilibrium equation under the displacement formulation, eq. (2.3.11), now becomes, in the absence of thermal force and moment effects,

$$\begin{aligned} D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3} \\ + D_{22} \frac{\partial^4 w}{\partial y^4} - B_{11} \frac{\partial^3 u^0}{\partial x^3} - 3B_{16} \frac{\partial^3 u^0}{\partial x^2 \partial y} - (B_{12} + 2B_{66}) \frac{\partial^3 u^0}{\partial x \partial y^2} \\ - B_{26} \frac{\partial^3 u^0}{\partial y^3} - B_{16} \frac{\partial^3 v^0}{\partial x^3} - (B_{12} + 2B_{66}) \frac{\partial^3 v^0}{\partial x^2 \partial y} - 3B_{26} \frac{\partial^3 v^0}{\partial x \partial y^2} \\ - B_{22} \frac{\partial^3 v^0}{\partial y^3} = q + N_{xx}^i \frac{\partial^2 w}{\partial x^2} + 2N_{xy}^i \frac{\partial^2 w}{\partial x \partial y} + N_{yy}^i \frac{\partial^2 w}{\partial y^2} \end{aligned} \quad (2.3.191)$$

For the stress function/transverse displacement formulation, eq. (2.3.17) becomes

$$\begin{aligned}
& D_{11}^* \frac{\partial^4 w}{\partial x^4} + 4D_{16}^* \frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12}^* + 2D_{66}^*) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26}^* \frac{\partial^4 w}{\partial x \partial y^3} \\
& + D_{22}^* \frac{\partial^4 w}{\partial y^4} + B_{21}^* \frac{\partial^4 \phi}{\partial x^4} + (2B_{26}^* - B_{61}^*) \frac{\partial^4 \phi}{\partial x^3 \partial y} + (B_{11}^* + B_{22}^* - B_{66}^*) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} \\
& + (2B_{16}^* - B_{62}^*) \frac{\partial^4 \phi}{\partial x \partial y^3} + B_{12}^* \frac{\partial^4 \phi}{\partial y^4} = q + N_x^i \frac{\partial^2 w}{\partial x^2} \\
& + 2N_{xy}^i \frac{\partial^2 w}{\partial x \partial y} + N_{yy}^i \frac{\partial^2 w}{\partial y^2}
\end{aligned} \tag{2.3.192}$$

Obviously eq. (2.3.9,10) in the displacement formulation and eq. (2.3.19) in the stress function/transverse displacement formulation remain unchanged in conjunction with the stability analysis.

2.3.4.2 Cylindrical Bending.

For buckling calculations under conditions of cylindrical bending with $q = 0$, eqs. (2.3.32,33) remain unchanged while eq. (2.3.34) becomes

$$D_{11} \frac{d^4 w}{dx^4} - B_{11} \frac{d^3 u^0}{dx^3} - B_{16} \frac{d^3 v^0}{dx^3} = N_{xx}^i \frac{d^2 w}{dx^2} \tag{2.3.193}$$

Uncoupling the governing equations in the same manner as for the static case, eqs. (2.3.35-37), we obtain

$$\frac{d^2 u^0}{dx^2} = \frac{B}{A} \frac{d^3 w}{dx^3} \tag{2.3.194}$$

$$\frac{d^2 v^0}{dx^2} = \frac{C}{A} \frac{d^3 w}{dx^3} \tag{2.3.195}$$

$$\frac{d^4 w}{dx^4} = \frac{A}{D} N_{xx}^i \frac{d^2 w}{dx^2} \tag{2.3.196}$$

where A, B, C, and D are as defined in eqs. (2.3.35-37). Now consider the initial inplane loading $N_{xx}^i = -N_0 = \text{constant}$ where $N > 0$. Equation (2.3.196) now becomes

$$\frac{d^4 w}{dx^4} + \frac{A}{D} N_0 \frac{d^2 w}{dx^2} = 0 \tag{2.3.197}$$

Let us assume the same simple-support boundary conditions as in the static bending and free vibration cases. Thus, u^0 , v^0 , and w must satisfy eqs. (2.3.132,133). A solution to eqs. (2.3.194,195 and 197) which satisfies the boundary conditions, eqs. (2.3.132,133), is of the same form as eq. (2.3.159) for the free vibration case. Thus,

$$u^0 = A_m \cos \frac{m\pi x}{a}$$

$$v^0 = B_{mn} \cos \frac{m\pi x}{a} \quad (2.3.198)$$

$$w = C_{mn} \sin \frac{m\pi x}{a}$$

where m is an integer. Substituting these displacement functions into the governing eqs. (2.3.194,195,197), we obtain the following linear equations in matrix notation:

$$\begin{bmatrix} 1 & 0 & \frac{m\pi B}{aA} \\ 0 & 1 & \frac{m\pi C}{aA} \\ 0 & 0 & F_m \end{bmatrix} \begin{bmatrix} A_m \\ B_m \\ C_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (2.3.199)$$

where

$$F_m = \frac{m^2 \pi^2}{a^2} - \frac{A}{D} N$$

In order to avoid a trivial solution, F_m must vanish. Thus,

$$N_0 = \frac{D m^2 \pi^2}{A a^2} \quad (2.3.200)$$

The critical value of N_0 , or the buckling load, is the lowest value which satisfies eq. (2.3.200). This occurs for $m = 1$. Thus,

$$N_{0cr} = \frac{D \pi^2}{A a^2} \quad (2.3.201)$$

This relationship can also be written in the form

$$N_{0cr} = N'_{0cr} (1-k) \quad (2.3.202)$$

where N'_{0cr} is the critical buckling load for a laminated strip which is symmetric or the B_{ij} coupling terms are neglected. It is defined by the relationship

$$N'_{0cr} = D_{11} \frac{\pi^2}{a^2} \quad (2.3.203)$$

and k is as defined in eq. (2.3.54). As previously discussed, k is always positive; thus, bending-extensional coupling reduces the critical buckling load. It will be shown later that stacking sequence and anisotropy ratio, E_1/E_2 , have significant influence on the magnitude of this reduction.

Let us now consider a second boundary value problem in conjunction with cylindrical bending. In this example we will introduce the concept of thermal buckling. Consider a laminated strip clamped at the edges. In this case the boundary conditions become

at $x = 0, a$

(2.3.204)

$$u^0 = v^0 = w = \frac{dw}{dx} = 0$$

Let us assume that the plate is subjected to a uniform temperature distribution. The plate will expand against the constrained boundaries, and if the temperature change becomes large enough, the plate will buckle. In order to assure a uniaxial state-of-stress in the prebuckled state, we assume that the plate is allowed to slip in the y direction along the edges such that $N_{yy} = 0$ and $\epsilon_y^0 = \text{constant}$. Under these conditions the constitutive relations, (reference 2.3.1), for N_{xx} and N_{yy} become

$$N_{xx} = A_{11} \frac{du^0}{dx} + A_{12} \epsilon_y^0 + A_{16} \frac{dv^0}{dx} - B_{11} \frac{d^2 w}{dx^2} - N_{xx}^T \quad (2.3.205)$$

$$N_{yy} = 0 = A_{12} \frac{du^0}{dx} + A_{22} \epsilon_y^0 + A_{26} \frac{dv^0}{dx} - B_{12} \frac{d^2 w}{dx^2} - N_{yy}^T$$

If we assume the prebuckled configuration is flat, then a solution which will satisfy eqs. (2.3.194-196) is of the form $u^0 = w = 0$ and $v^0 = y \epsilon_y^0$. Equations (2.3.205) reduce to

$$N_{xx} = A_{12} \epsilon_y^0 - N_{xx}^T$$

$$0 = A_{22} \epsilon_y^0 - N_{yy}^T$$

Solving these relations, one obtains

$$N_{xx}^i = \frac{-(A_{22} N_{xx}^T - A_{12} N_{yy}^T)}{A_{22}} = -N_{xx}^T \quad (2.3.206)$$

and eq. (3.196) becomes

$$\frac{d^4 w}{dx^4} - \frac{A}{D} N_{xx}^T \frac{d^2 w}{dx^2} = 0 \quad (2.3.207)$$

A solution to eqs. (2.3.194,195,207) which satisfies the boundary conditions, eq. (2.3.204), is of the form $u^0 = A_m \sin \frac{2m\pi x}{a}$

$$v^0 = B_m \sin \frac{2m\pi x}{a} \quad (2.3.208)$$

$$w = C_m (1 - \cos \frac{2m\pi x}{a})$$

where again m is an integer. Substituting these relations into eqs. (2.3.194,195,207), we obtain three homogeneous algebraic equations which can be written in a matrix form similar to eq. (2.3.199). Again a trivial solution is

avoided if the determinant of the coefficient matrix vanishes. This requires that for the present case

$$\frac{4m^2\pi^2}{a^2} - \frac{A}{D} N^T = 0$$

or

$$N^T = \frac{4Dm^2\pi^2}{Aa^2} \quad (2.3.209)$$

It can be seen from the definition of thermal force resultants that N_{xx}^T and N_{yy}^T can be written in the form

$$N_{xx}^T = T_{xx}\Delta T, \quad N_{yy}^T = T_{yy}\Delta T \quad (2.3.210)$$

where

$$T_{xx} = \sum_{i=1}^L \frac{L}{L} (\bar{Q}_{11}^i \bar{\alpha}_1^i + \bar{Q}_{12}^i \bar{\alpha}_2^i + \bar{Q}_{16}^i \bar{\alpha}_6^i) h_i$$

$$T_{yy} = \sum_{i=1}^L \frac{L}{L} (\bar{Q}_{12}^i \bar{\alpha}_1^i + \bar{Q}_{22}^i \bar{\alpha}_2^i + \bar{Q}_{26}^i \bar{\alpha}_6^i) h_i$$

and $\bar{\alpha}_1^i$ and $\bar{\alpha}_2^i$ are the thermal expansion coefficients of the i th layer relative to the x and y axes of the plate, respectively, $\bar{\alpha}_6^i$ is the shear coupling thermal expansion coefficient relative to the x - y plane, and h_i is the thickness of the i th layer. Combining eqs. (2.3.209,210) we obtain

$$\Delta T = \frac{4A_{22}Dm^2\pi^2}{(A_{22}T_{xx} - A_{12}T_{yy})Aa^2}$$

The lowest value of ΔT occurs for $m = 1$. Thus,

$$\Delta T_{cr} = \frac{4A_{22}D\pi^2}{(A_{22}T_{xx} - A_{12}T_{yy})Aa^2} \quad (2.3.211)$$

This relationship can also be written in the form

$$\Delta T_{cr} = \frac{4A_{22}D_{11}\pi^2}{(A_{22}T_{xx} - A_{12}T_{yy})a^2} (1 - k) \quad (2.3.212)$$

where the coefficient is the critical temperature change for a symmetric laminate or a laminate in which the B_{ij} stiffness terms are neglected. Again, bending-extensional coupling reduces the critical buckling parameter.

As a numerical example, consider a graphite-epoxy composite with the following ply properties

$$E_1 = 25 \times 10^6 \text{ psi}, E_2 = 1.7 \times 10^6 \text{ psi}, \nu_{12} = 0.3,$$

$$G_{12} = 0.65 \times 10^6 \text{ psi}, \alpha_1 = -0.4 \times 10^{-6}/^\circ\text{F}, \alpha_2 = 30 \times 10^{-6}/^\circ\text{F}$$

The elastic properties are the same as those used throughout Chapter 2. Note that unidirectional graphite has a negative thermal expansion coefficient parallel to the fibers. This is because graphite fibers have a negative coefficient of thermal expansion. Thus, buckling will occur at negative value of ΔT for a 0° composite. Critical values of ΔT , utilizing eq. (2.3.21), are illustrated in figure 2.3.21 for $[\pm\theta]_s$ angle-ply laminates.

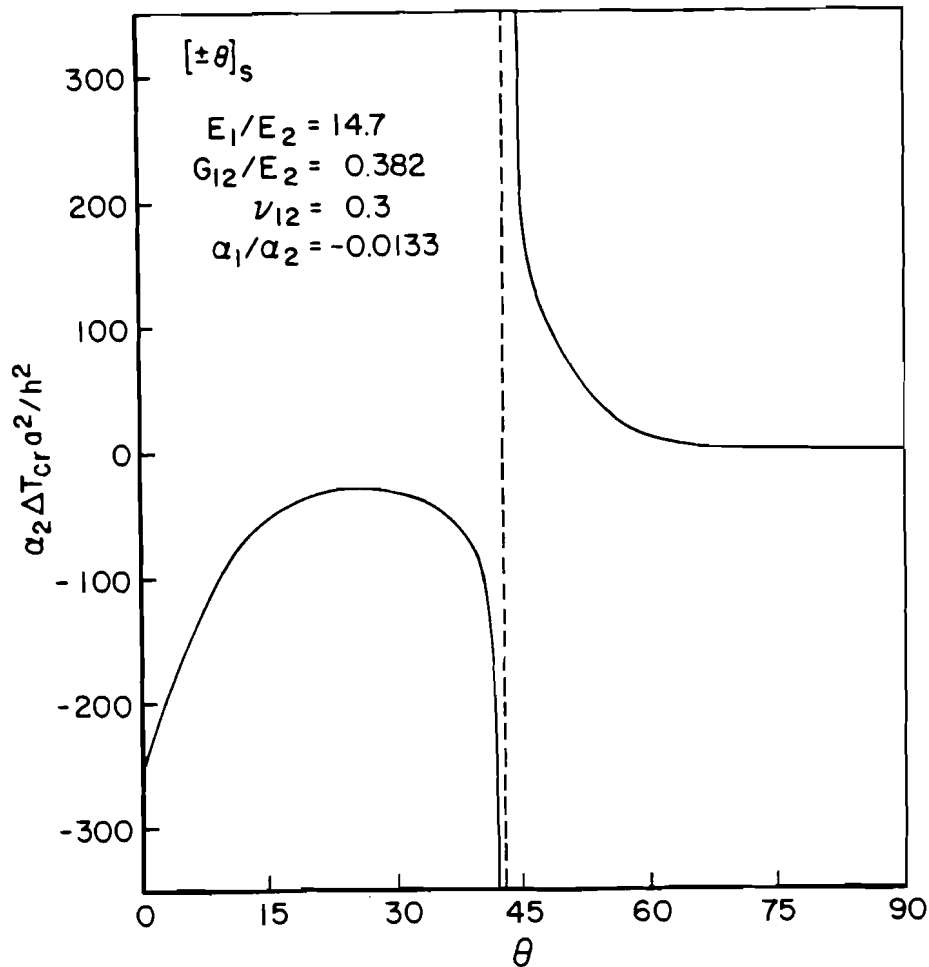


FIGURE 2.3.21 THERMAL BUCKLING FOR CLAMPED SYMMETRIC ANGLE-PLY LAMINATES UNDER CYLINDRICAL BENDING

2.3.4.3 Specially Orthotropic Plates.

We now consider symmetric laminates, $B_{1j} = 0$, in which the bending-twisting coupling terms D_{16} , D_{26} can be neglected. For this case the governing eq. (2.3.191), in the absence of transverse load ($q = 0$), becomes

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} = N_{xx}^i \frac{\partial^2 w}{\partial x^2} + 2N_{xy}^i \frac{\partial^2 w}{\partial x \partial y} + N_{yy}^i \frac{\partial^2 w}{\partial y^2} \quad (2.3.213)$$

For the case where the initial loading is of the form

$$N_{xx}^i = -k_1 N_0, \quad N_{yy}^i = k_2 N_0, \quad N_{xy}^i = 0, \quad N_0 > 0 \quad (2.3.214)$$

where k_1 , k_2 , and N are constants, eq. (2.3.213) will contain only even order derivatives in both x and y . Under such conditions a solution can be obtained in terms of double sine functions. The simple-support boundary conditions, eqs. (2.3.87,88) will also be satisfied by a double sine function. Thus, we assume

$$w = A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (2.3.215)$$

Substituting this relationship into eq. (2.3.213) with eq. (2.3.214) taken into account, we obtain

$$N_0 = \frac{\pi^2 [D_{11} m^4 + 2(D_{12} + 2D_{66}) m^2 n^2 R^2 + D_{22} n^4 R^4]}{a^2 (k_1 m^2 + k_2 n^2 R^2)} \quad (2.3.216)$$

where $R = a/b$. For the case of a square plate under uniaxial load ($k_1 = 1$, $k_2 = 0$), the lowest value of N_0 occurs for $m = n = 1$ and the critical buckling load becomes

$$N_{0cr} = \frac{\pi^2}{a^2} [D_{11} + 2(D_{12} + 2D_{66}) + D_{22}] \quad (2.3.217)$$

As in the case of fundamental vibration frequency, the critical buckling load will be independent of stacking sequence for the $[0/90]_{Ns}$ class of laminates. The bending stiffnesses for this class of laminates were previously derived in terms of 0° ply stiffnesses and displayed in eq. (2.3.171). Substituting eq. (2.3.171) into eq. (2.3.217) we obtain the result

$$N_{0cr} = \frac{\pi^2}{a^2} [\bar{Q}_{11}(0^\circ) + \bar{Q}_{22}(0^\circ) + 2(\bar{Q}_{12}(0^\circ) + 2\bar{Q}_{66}(0^\circ))] \quad (2.3.218)$$

and N_{0cr} is independent of N .

The dependence of the critical buckling load on aspect ratio R is displayed in figure 2.3.22 for a $[0/90]_s$ laminate under uniaxial compression ($k_1 = 1, k_2 = 0$). Note the change in mode shape with increasing aspect ratio. The ply properties are the same as those utilized throughout this chapter.

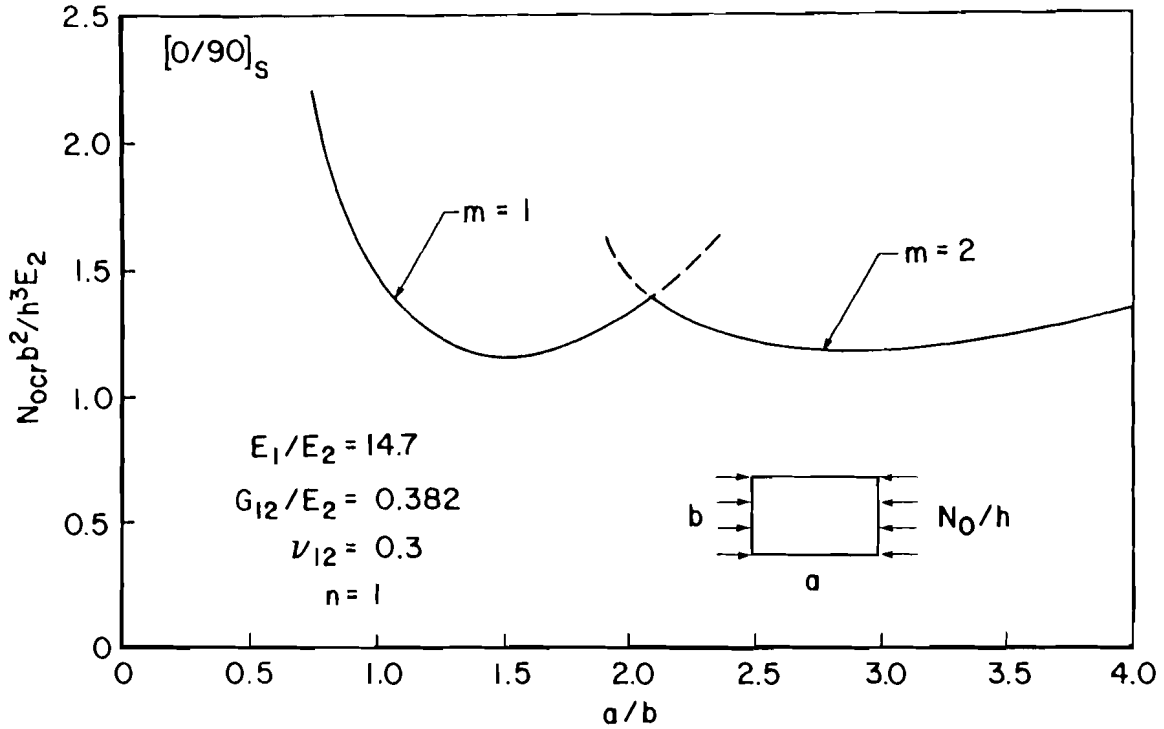


FIGURE 2.3.22 EFFECT OF ASPECT RATIO ON CRITICAL BUCKLING FOR A SYMMETRIC BIDIRECTIONAL LAMINATE UNDER UNIAXIAL COMPRESSION

2.3.4.4 Anisotropic Plates.

Consideration is now given to symmetric laminates, $B_{1j} = 0$ in which the bending-twisting coupling stiffness terms D_{16}, D_{26} do not vanish. For this class of laminates, the governing eq. (2.3.191), in the absence of transverse load ($q = 0$), reduces to

$$\begin{aligned}
 D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3} \\
 + D_{22} \frac{\partial^4 w}{\partial y^4} = N_{xx}^i \frac{\partial^2 w}{\partial x^2} + 2N_{xy}^i \frac{\partial^2 w}{\partial x \partial y} + N_{yy}^i \frac{\partial^2 w}{\partial y^2}
 \end{aligned}
 \quad (2.3.219)$$

Again numerical difficulties arise because of the existence of the D_{16} , D_{26} bending-twisting coupling terms which induce odd order derivatives in both x and y . Because of these odd order derivatives, a double sine function will not be a solution to eq. (2.3.219). The same odd order derivatives also arise in conjunction with moment boundary conditions, eqs. (2.3.103,104). Once again the Galerkin method (reference 2.3.9) is employed to obtain an approximate solution to critical buckling loads and assess the importance of the D_{16} , D_{26} coupling terms in stability analysis. In this method a solution is sought in the form given by eq. (2.3.105). Simple-support boundary conditions, eqs. (2.3.103,104), are assumed. The initial inplane loads are of the form

$$N_{xx}^i = -k_1 N_0, \quad N_{yy}^i = -k_2 N_0, \quad N_{xy}^i = k_3 N_0, \quad N_0 > 0 \quad (2.3.220)$$

where k_1 , k_2 , k_3 , and N_0 are constants. Under these conditions the Galerkin integrals in eq. (2.3.106) become

$$\begin{aligned} & \int_0^b \int_0^a \left[D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} \right. \\ & + 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3} + D_{22} \frac{\partial^4 w}{\partial y^4} - N_0 \left(k_1 \frac{\partial^2 w}{\partial x^2} - 2k_3 \frac{\partial^2 w}{\partial x \partial y} \right. \\ & \left. \left. + k_2 \frac{\partial^2 w}{\partial y^2} \right) \right] \phi_{mn} \, dx \, dy + B_1 + B_2 = 0 \end{aligned} \quad (2.3.221)$$

where B_1 and B_2 correspond to the boundary integrals in eq. (2.3.106). As previously discussed, the function ϕ_{mn} need only satisfy the displacement boundary conditions in eqs. (2.3.103,104). The double sine function given by eq. (2.3.107) is again chosen to represent ϕ_{mn} . Equation (2.3.107), in conjunction with eq. (2.3.221), leads to two sets of homogeneous algebraic equations. One set corresponds to $m+n$ odd and the other set to $m+n$ even. A nontrivial solution to each set of these equations is obtained by choosing N_0 such that the determinant of the coefficient matrix vanishes. This procedure yields $M \times N$ values of N_0 . The lowest value corresponds to the critical buckling load N_{0cr} .

Numerical results are illustrated in figure 2.3.23 for a $[+45]_{Ns}$ class of laminates. This represents a class of laminates for which bending-twisting coupling is most severe. Critical buckling loads are shown for increasing anisotropy ratios, E_1/E_2 , in conjunction with square plates under uniaxial compression ($k_1 = 1$, $k_2 = k_3 = 0$). As has been used throughout this chapter, $\nu_{12} = 0.3$, $G_{12}/E_2 = 0.382$. For this class of laminates, it was shown in eq. (3.109) that the D_{16} , D_{26} coupling terms decrease with increasing values of N . Thus, for the case $N \rightarrow \infty$ the critical buckling load is obtained by applying eq. (2.3.216) for an orthotropic plate. Convergence is rather rapid, and the results in figure 2.3.23 are for $M = N = 9$ in conjunction with eq. (2.3.105).

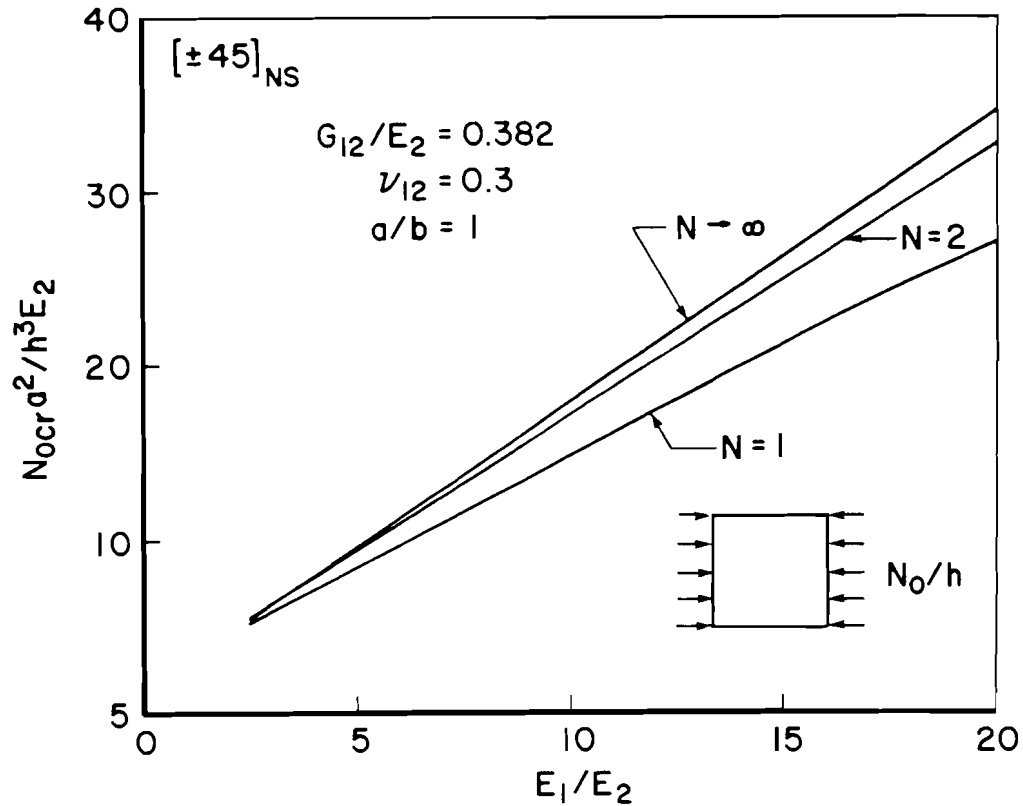


FIGURE 2.3.23 EFFECTS OF ANISOTROPY RATIO ON THE CRITICAL BUCKLING LOAD OF A SQUARE ANGLE-PLY PLATE UNDER UNIAXIAL COMPRESSION

Numerical results in figure 2.3.24 illustrate an interesting phenomenon associated with shear buckling ($k_1 = k_2 = 0$, $k_3 = \pm 1$) of anisotropic plates. In particular, positive and negative shear produce different critical buckling loads. One way to understand this effort is to consider the stress state at $\pm 45^\circ$ due to the applied shear stress. The normal stresses in these directions are equal in magnitude and opposite in sign. Since at these rotate coordinates the bending stiffnesses in the two directions differ significantly from each other, the relative orientation of the larger stiffness and of the compressive stress is of great importance. Hence, the difference between resistance to shear buckling for positive and negative shear forces. The orthotropic solution ($N \rightarrow \infty$) is approached for increasing values of N . Ply properties are those which have been used throughout Chapter 2. Results are again for $M = N = 9$.

The results in figures 2.3.23,24 indicate that the orthotropic solution is rapidly recovered with increasing values of N . Thus, for many practical laminates the bending-twisting coupling terms D_{16} and D_{26} can be neglected for the determination of critical buckling loads without introducing serious error.

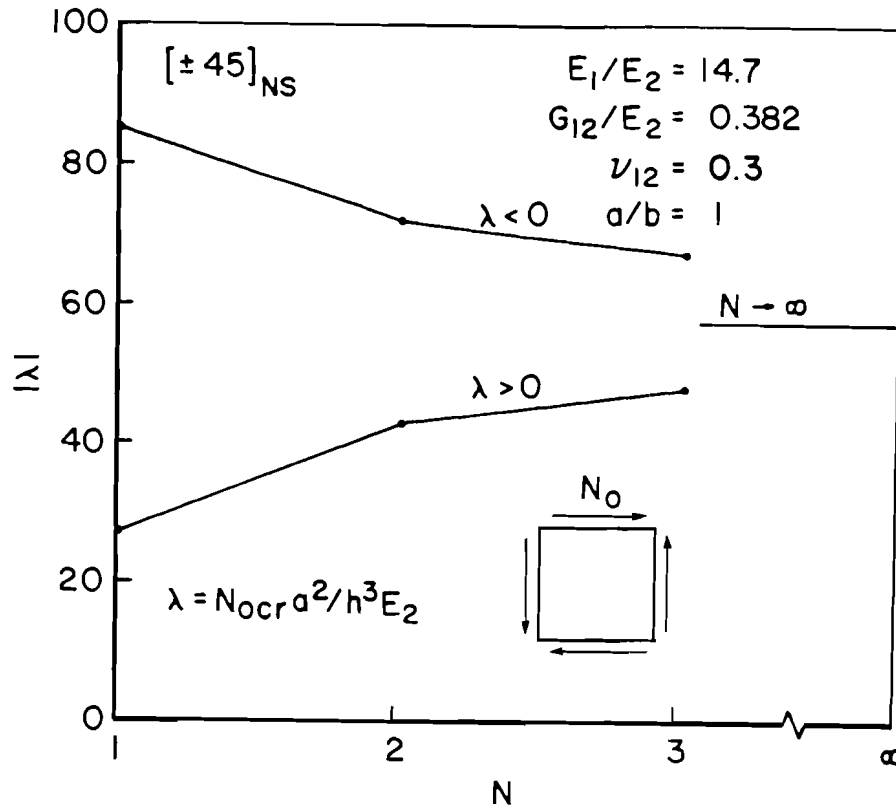


FIGURE 2.3.24 EFFECT OF NUMBER OF LAYERS ON THE SHEAR BUCKLING OF AN ANGLE-PLY LAMINATE

2.3.4.5 Buckling of Unsymmetric Laminates.

In Section 2.3.4.1 it was shown that bending-extensional coupling reduced critical buckling loads for unsymmetric laminates subjected to cylindrical bending. Such reduction will now be considered in more detail for rectangular plates of the class $[\pm\theta]_{NT}$ subjected to biaxial inplane loading. The initial inplane loading is assumed to be of the form

$$N_{xx}^i = -k_1 N_0, N_{yy}^i = -k_2 N_0, N_{xy}^i = 0, N_0 > 0 \quad (2.3.222)$$

where k_1 , k_2 , and N_0 are constants. For the $[\pm\theta]_{NT}$ class of laminates, it was shown that $A_{16} = A_{26} = D_{16} = D_{26} = 0$ and that B_{16} and B_{26} were the only nonvanishing coupling stiffness terms, e.g. see eqs. (2.3.125-128). Under these conditions eqs. (2.3.9,10) reduce to eqs. (2.3.129,130), respectively, and eq. (2.3.191), in the absence of transverse load ($q = 0$), becomes

$$\begin{aligned} D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} - B_{16} \left(3 \frac{\partial^3 u^0}{\partial x^2 \partial y} \right. \\ \left. + \frac{\partial^3 v^0}{\partial x^3} \right) - B_{26} \left(\frac{\partial^3 u^0}{\partial y^3} + 3 \frac{\partial^3 v^0}{\partial x \partial y^2} \right) + N_0 \left(k_1 \frac{\partial^2 w}{\partial x^2} \right. \\ \left. + k_2 \frac{\partial^2 w}{\partial y^2} \right) = 0 \end{aligned} \quad (2.3.223)$$

The boundary conditions are assumed to be the simple-support conditions given by eqs. (2.3.132,133). A solution to eqs. (2.3.129,130,223) which satisfies the boundary conditions is of the same form as eq. (2.3.134), i.e.

$$u^0 = A_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$

$$v^0 = B_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$w = C_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

Substituting these functions into eqs. (2.3.129,130,223) we obtain a matrix equation identical to eq. (2.3.187) with the exception of λ_{mn} which now becomes

$$\lambda_{mn} = N_0 (k_1 m^2 + k_2 n^2 \eta^2) \quad (2.3.224)$$

where, as before, η is the plate aspect ratio a/b .

Again a nontrivial solution is obtained only if λ_{mn} is chosen such that the determinant of the coefficient matrix vanishes. Performing this calculation we obtain the result

$$\begin{aligned} N_0 = & \frac{\pi^2}{a^2 (k_1 m^2 + k_2 n^2 \eta^2)} \{ D_{11} m^4 + 2(D_{12} + 2D_{66}) m^2 n^2 \eta^2 \\ & + D_{22} n^4 \eta^4 - \frac{1}{J_3} [m^2 (B_{16} m^2 + 3B_{26} n^2 \eta^2) J_1 \\ & + n^2 \eta^2 (3B_{16} m^2 + B_{26} n^2 \eta^2) J_2] \} \end{aligned} \quad (2.3.225)$$

where J_1 , J_2 , and J_3 are defined in eq. (2.3.189). Values of m and n which yield the lowest value of N_0 correspond to the critical buckling load N_{0cr} .

Numerical results are shown in figure 2.3.25 for a $[+45]_{NT}$ square laminate under uniaxial compression ($k_1 = 1$, $k_2 = 0$). Critical buckling load is plotted as a function of anisotropy ratio, E_1/E_2 . As in static bending and fundamental vibration frequencies, coupling can have a severe effect on critical buckling loads. The effect of coupling dissipates rapidly, however, with increasing values of N . The orthotropic solution ($N \rightarrow \infty$) is obtained from eq. (2.3.217). If B_{16} and B_{26} are neglected, eq. (2.3.225) reduces to eq. (2.3.216). As shown in eqs. (2.3.125,126), the values of B_{16} and B_{26} decrease with increasing values of N .

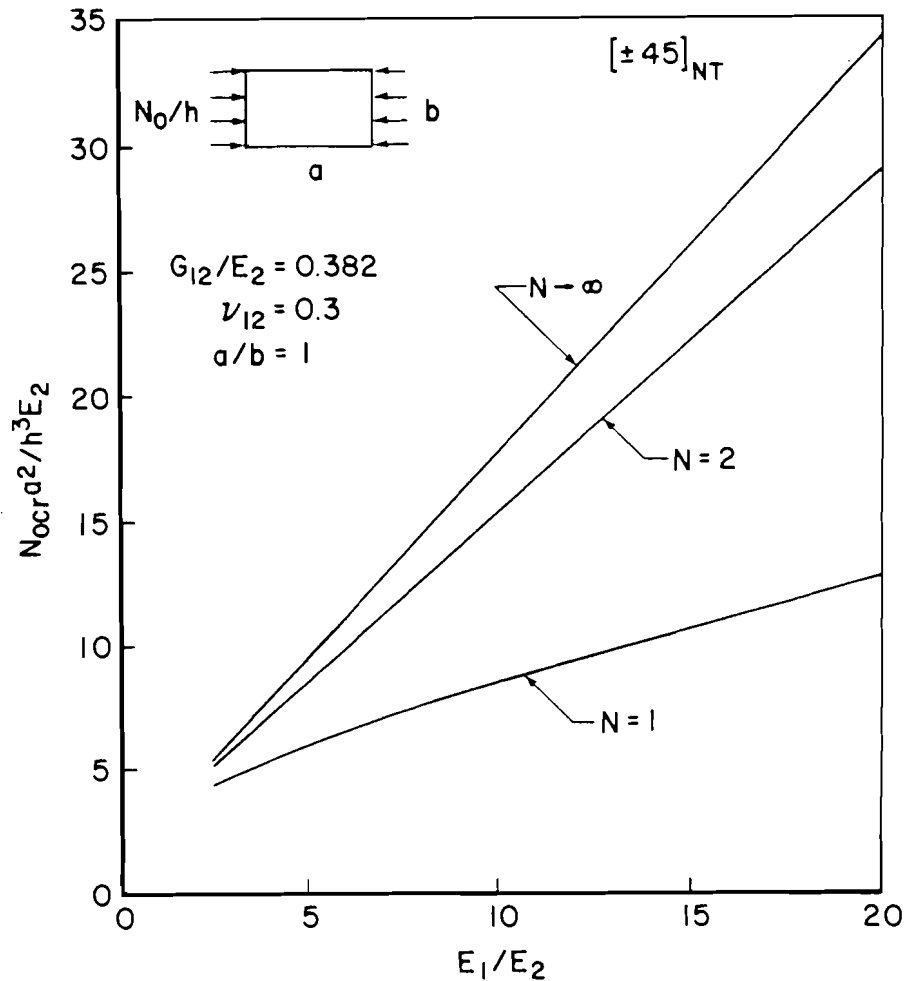


FIGURE 2.3.25 EFFECT OF ANISOTROPY RATIO ON CRITICAL BUCKLING LOAD FOR UNSYMMETRIC SQUARE ANGLE-PLY LAMINATES UNDER UNIAXIAL COMPRESSION

2.3.5 Transverse Shear Deformation.

For many practical laminates, the ratio of inplane Young's moduli to interlaminar shear moduli can be very large compared to homogeneous isotropic plates. Thus, shear deformation may become a significant factor in laminated composites at lower ratios of inplane dimensions to plate thickness than for homogeneous, isotropic plates.

The objective of this section is twofold: (1) to develop a theory for laminated plates which includes the effect of transverse shear deformation; and (2) to present solutions to the theory which will allow us to assess the effect of transverse shear deformation on the behavior of laminated plates. Although the theory will be developed for laminates of arbitrary ply construction, solutions will be limited to plates of symmetric, orthotropic construction subjected to cylindrical bending in order to simplify the details. In addition, such solutions will be sufficient to illustrate the effect of shear deformation on the response of laminated plates.

2.3.5.1 Governing Equations.

The theory presented here is an extension of Mindlin's theory (reference 2.3.8) for homogeneous plates to laminates consisting of an arbitrary number of bonded anisotropic layers. Such an extension was originally due to Yang, Norris, and Stavsky (reference 2.3.12) with some later modification by Whitney and Pagano (reference 2.3.13). The basic assumptions for the shear deformation theory remain the same as for classical laminated plate theory with the one exception being the assumption concerning the interlaminar shear strains γ_{xz} , γ_{yz} being negligible. In the present theory such an assumption is not made. Thus, the displacements are now assumed to be of the form

$$\begin{aligned} u &= u^0(x, y, t) + z \psi_x(x, y, t) \\ v &= v^0(x, y, t) + z \psi_y(x, y, t) \\ w &= w(x, y, t) \end{aligned} \quad (2.3.226)$$

Time dependence is included in these equations as a general theory, including dynamic effects, thermal effects, and inplane force effects will be developed simultaneously. Using eq. (2.3.226) in conjunction with the usual strain-displacement definitions, we obtain the inplane strains

$$\begin{aligned} \epsilon_{xx} &= \epsilon_{xx}^0 + z \kappa_{xx} \\ \epsilon_{yy} &= \epsilon_{yy}^0 + z \kappa_{yy} \\ \gamma_{xy} &= \gamma_{xy}^0 + z \kappa_{xy} \end{aligned} \quad (2.3.227)$$

where the midplane strains are defined the same as in eq. (2.3.2) and

$$\kappa_{xx} = \frac{\partial \psi_x}{\partial x}, \quad \kappa_{yy} = \frac{\partial \psi_y}{\partial y}, \quad \kappa_{xy} = \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \quad (2.3.228)$$

In addition, the interlaminar shear strains are given by the relationships

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \psi_x + \frac{\partial w}{\partial x} \quad (2.3.229)$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \psi_y + \frac{\partial w}{\partial y}$$

Using eq. (2.3.227) in conjunction with the assumption of plane stress within each ply and the definition of force and moment resultants, including thermal force and moment resultants, we obtain a constitutive relation of exactly the same form as given by eq. (2.3.1). Using the definition of shear force resultants, eq. 3.7, in conjunction with eq. (2.3.229), we obtain an additional constitutive relation involving transverse shear. Since the transverse shear strain is not uniform through the thickness, a weighting factor, k , is introduced. (Following Mindlin [reference 2.3.8].) Thus, we obtain

$$\begin{bmatrix} Q_y \\ Q_x \end{bmatrix} = k \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{bmatrix} \psi_y + \frac{\partial w}{\partial y} \\ \psi_x + \frac{\partial w}{\partial x} \end{bmatrix} \quad (2.3.230)$$

where

$$A_{ij} = \int_{-h/2}^{h/2} C_{ij} dz \quad (i, j = 4, 5) \quad (2.3.231)$$

and C_{ij} are anisotropic stiffness terms. Values associated with the factor k will be discussed in conjunction with specific solutions.

The equations of motion in terms of stress and moment resultants, including inplane force effects, are the same as eqs. (2.3.142-145) with eq. (2.3.145) being modified in conjunction with eq. (2.3.189) to include inplane force effects. Thus, the governing equations are of the form

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = P \frac{\partial^2 u^0}{\partial t^2} + R \frac{\partial \psi_x}{\partial x} \quad (2.3.232)$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = P \frac{\partial^2 v^0}{\partial t^2} + R \frac{\partial \psi_y}{\partial y} \quad (2.3.233)$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_{xx} = R \frac{\partial^2 u^0}{\partial t^2} + I \frac{\partial^2 \psi_x}{\partial t^2} \quad (2.3.234)$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_{yy} = R \frac{\partial^2 v^0}{\partial t^2} + I \frac{\partial^2 \psi_y}{\partial t^2} \quad (2.3.235)$$

$$\begin{aligned} \frac{\partial Q_{xx}}{\partial x} + \frac{\partial Q_{yy}}{\partial y} + q + N_{xx}^i \frac{\partial^2 w}{\partial x^2} + 2N_{xy}^i \frac{\partial^2 w}{\partial x \partial y} \\ + N_{yy}^i \frac{\partial^2 w}{\partial y^2} = P \frac{\partial^2 w}{\partial t^2} \end{aligned} \quad (2.3.236)$$

Note that the inertia terms in eqs. (2.3.142-144) involving derivatives of w have been replaced by terms involving ψ_x and ψ_y . This due to the displacement assumptions as given by eq. (2.3.226).

Substituting the constitutive relations, eqs. (2.3.1,230) into eqs. (2.3.232-236), and taking into account eq. (2.3.228), we obtain the following governing equations:

$$\begin{aligned}
 & A_{11} \frac{\partial^2 u^0}{\partial x^2} + 2A_{16} \frac{\partial^2 u^0}{\partial x \partial y} + A_{66} \frac{\partial^2 u^0}{\partial y^2} + A_{16} \frac{\partial^2 v^0}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 v^0}{\partial x \partial y} \\
 & + A_{26} \frac{\partial^2 v^0}{\partial y^2} + B_{11} \frac{\partial^2 \psi_x}{\partial x^2} + 2B_{16} \frac{\partial^2 \psi_x}{\partial x \partial y} + B_{66} \frac{\partial^2 \psi_x}{\partial y^2} + B_{16} \frac{\partial^2 \psi_y}{\partial x^2} \\
 & + (B_{12} + B_{66}) \frac{\partial^2 \psi_y}{\partial x \partial y} + B_{26} \frac{\partial^2 \psi_y}{\partial y^2} - \frac{\partial N_{xx}^T}{\partial x} - \frac{\partial N_{xy}^T}{\partial y} = P \frac{\partial^2 u^0}{\partial t^2} + I \frac{\partial^2 \psi_x}{\partial t^2} \quad (2.3.237)
 \end{aligned}$$

$$\begin{aligned}
 & A_{16} \frac{\partial^2 u^0}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 u^0}{\partial x \partial y} + A_{26} \frac{\partial^2 u^0}{\partial y^2} + A_{66} \frac{\partial^2 v^0}{\partial x^2} + 2A_{26} \frac{\partial^2 v^0}{\partial x \partial y} \\
 & + A_{22} \frac{\partial^2 v^0}{\partial y^2} + B_{16} \frac{\partial^2 \psi_x}{\partial x^2} + (B_{12} + B_{66}) \frac{\partial^2 \psi_x}{\partial x \partial y} + B_{26} \frac{\partial^2 \psi_x}{\partial y^2} + B_{66} \frac{\partial^2 \psi_y}{\partial x^2} \\
 & + 2B_{26} \frac{\partial^2 \psi_y}{\partial x \partial y} + B_{22} \frac{\partial^2 \psi_y}{\partial y^2} - \frac{\partial N_{xy}^T}{\partial x} - \frac{\partial N_{yy}^T}{\partial y} = P \frac{\partial^2 v^0}{\partial x^2} + R \frac{\partial^2 \psi_y}{\partial t^2} \quad (2.3.238)
 \end{aligned}$$

$$\begin{aligned}
 & B_{11} \frac{\partial^2 u^0}{\partial x^2} + 2B_{16} \frac{\partial^2 u^0}{\partial x \partial y} + B_{66} \frac{\partial^2 u^0}{\partial y^2} + B_{16} \frac{\partial^2 v^0}{\partial x^2} + (B_{12} + B_{66}) \frac{\partial^2 v^0}{\partial x \partial y} \\
 & + B_{26} \frac{\partial^2 v^0}{\partial y^2} + D_{11} \frac{\partial^2 \psi_x}{\partial x^2} + 2D_{16} \frac{\partial^2 \psi_x}{\partial x \partial y} + D_{66} \frac{\partial^2 \psi_x}{\partial y^2} + D_{16} \frac{\partial^2 \psi_y}{\partial x^2} \\
 & + (D_{12} + D_{66}) \frac{\partial^2 \psi_y}{\partial x \partial y} + D_{26} \frac{\partial^2 \psi_y}{\partial y^2} - k \left[A_{55} \left(\psi_x + \frac{\partial w}{\partial x} \right) + A_{45} \left(\psi_y \right. \right. \\
 & \left. \left. + \frac{\partial w}{\partial y} \right) \right] - \frac{\partial M_{xx}^T}{\partial x} - \frac{\partial M_{xy}^T}{\partial y} = R \frac{\partial^2 u^0}{\partial y^2} + I \frac{\partial^2 \psi_x}{\partial t^2} \quad (2.3.239)
 \end{aligned}$$

$$\begin{aligned}
& B_{16} \frac{\partial^2 u^0}{\partial x^2} + (B_{12} + B_{66}) \frac{\partial^2 u^0}{\partial x \partial y} + B_{26} \frac{\partial^2 u^0}{\partial y^2} + B_{66} \frac{\partial^2 v^0}{\partial x^2} + 2B_{26} \frac{\partial^2 v^0}{\partial x \partial y} \\
& + B_{22} \frac{\partial^2 v^0}{\partial y^2} + D_{16} \frac{\partial^2 \psi_x}{\partial x^2} + (D_{12} + D_{66}) \frac{\partial^2 \psi_x}{\partial x \partial y} + D_{26} \frac{\partial^2 \psi_x}{\partial y^2} + D_{66} \frac{\partial^2 \psi_y}{\partial x^2} \\
& + 2D_{26} \frac{\partial^2 \psi_y}{\partial x \partial y} + D_{22} \frac{\partial^2 \psi_y}{\partial y^2} - k \left[A_{45} \left(\psi_x + \frac{\partial w}{\partial x} \right) + A_{44} \left(\psi_y + \frac{\partial w}{\partial y} \right) \right] \\
& - \frac{\partial M_{xy}^T}{\partial x} - \frac{\partial M_{yy}^T}{\partial y} = R \frac{\partial^2 v^0}{\partial t^2} + I \frac{\partial^2 \psi_y}{\partial t^2} \quad (2.3.240)
\end{aligned}$$

$$\begin{aligned}
& k \left[A_{55} \left(\frac{\partial \psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + A_{45} \left(\frac{\partial \psi_y}{\partial x} + \frac{\partial \psi_x}{\partial y} + 2 \frac{\partial^2 w}{\partial x \partial y} \right) + A_{44} \left(\frac{\partial \psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) \right] \\
& + q + N_{xx}^i \frac{\partial^2 w}{\partial x^2} + 2N_{xy}^i \frac{\partial^2 w}{\partial x \partial y} + N_{yy}^i \frac{\partial^2 w}{\partial y^2} = P \frac{\partial^2 w}{\partial t^2} \quad (2.3.241)
\end{aligned}$$

For symmetric laminates eqs. (2.3.237,238) uncouple from eqs. (2.3.239-241). The proper boundary conditions which will assure a unique solution can be obtained by using Mindlin's procedure (reference 2.3.8) with the result that one member of each pair of the following five quantities must be prescribed along the boundary:

$$u_n^0, N_n; \quad u_s^0, N_{ns}; \quad \psi_n, M_n; \quad \psi_s, M_{ns}; \quad w, Q_n \quad (2.3.242)$$

where n and s denote coordinates normal and tangential to the plate edges, respectively. Thus, for the general case of a laminated plate which includes shear deformation, five boundary conditions along each edge must be prescribed.

2.3.5.2 Bending Under Transverse Load.

We now consider cylindrical bending of a special class of symmetric laminates. In particular, we consider symmetric laminates, $B_{ij} = 0$, composed of orthotropic layers such that the various axes of material symmetry are parallel to the x - y axes of the plate. In addition, the usual assumptions of cylindrical bending are applied, i.e. the plate is of infinite extent in the y direction, uniformly supported along the edges $x = 0, a$, and subjected to the transverse surface load $q = q(x)$.

Since the laminates are assumed to be symmetric, only eqs. (2.3.239-241) need be considered. Furthermore, since each layer is assumed to be orthotropic with respect to the x - y axes of the plate, the bending-twisting coupling terms D_{16} , D_{26} will vanish. Under these conditions we have the following displacement functions

$$u^0 = v^0 = \psi_y = 0, \quad \psi_x = \psi_x(x), \quad w = w(x) \quad (2.3.243)$$

and in the absence of thermal effects, inertia terms, and inplane load effects, eqs. (2.3.239-241) reduce to

$$D_{11} \frac{d^2 \psi_x}{dx^2} - k A_{55} \left(\psi_x + \frac{dw}{dx} \right) = 0 \quad (2.3.244)$$

$$k A_{55} \left(\frac{d\psi_x}{dx} + \frac{d^2 w}{dx^2} \right) + q = 0 \quad (2.3.245)$$

These equations can be uncoupled by differentiating eq. (2.3.244) with the result

$$k A_{55} \left(\frac{d\psi_x}{dx} + \frac{d^2 w}{dx^2} \right) = D_{11} \frac{d^3 \psi_x}{dx^3} \quad (2.3.246)$$

Substituting this result into eq. (2.3.245), we obtain the expression

$$D_{11} \frac{d^3 \psi_x}{dx^3} + q = 0 \quad (2.3.247)$$

Integrating this equation and substituting the result into eq. (2.3.245), we obtain a differential equation for w which can be directly integrated.

Let us consider the uniform load $q = q_0 = \text{constant}$ in conjunction with the simple-support boundary conditions of $w = M_x = 0$. The expression for the bending moment is obtained by combining the constitutive eqs. (2.3.1) with eq. (2.3.228), from which one obtains

$$M_x = D_{11} \frac{d\psi_x}{dx} \quad (2.3.248)$$

and the simple-support boundary conditions are as follows:

at $x = 0, a$

$$w = \frac{d\psi_x}{dx} = 0 \quad (2.3.249)$$

Integrating eq. (2.3.247) and applying the boundary conditions in eqs. (2.3.249), we obtain

$$\psi_x = \frac{q_0 x^2}{12 D_{11}} (3a - 2x) + c_1 \quad (2.3.250)$$

Substituting this relationship into eq. (2.3.245) and integrating, we obtain

$$w = q_0 \left[\frac{x^3}{24 D_{11}} (x - 2a) - \frac{x^2}{2 k A_{55}} \right] - c_1 x + c_2 \quad (2.3.251)$$

Applying the boundary conditions to the displacement, w , from eq. (2.3.249), we obtain the following relationship

$$w = \frac{q_0 x}{24D_{11}} \left[x^3 - 2ax^2 + a^3 + \frac{12D_{11}}{kA_{55}} (a - x) \right] \quad (2.3.252)$$

The maximum deflection occurs at the center and can be written in the form

$$w_{\max} = w'_{\max} (1 + S) \quad (2.3.253)$$

where $w'_{\max}$ is the maximum deflection for a plate in which shear deformation is neglected and S is a shear deformation term. These expressions are given as follows:

$$w'_{\max} = \frac{5q_0 a^4}{384D_{11}}, \quad S = \frac{48D_{11}}{5kA_{55}a^2} \quad (2.3.254)$$

Thus, shear deformation increases the maximum deflection. Let us examine the shear deformation term, S , in more detail. The bending stiffness D_{11} can be written in terms of an effective bending stiffness, $\bar{Q}_{11}^b$, having the same units as modulus, and A_{55} can be written in terms of an average interlaminar shear modulus, $\bar{G}_{13}$, i.e.

$$\bar{Q}_{11}^b = \frac{12}{h^3} D_{11}, \quad \bar{G}_{13} = \frac{A_{55}}{h} \quad (2.3.255)$$

The shear factor S in eq. (2.3.254) can now be written in the form

$$S = \frac{4}{5k} \left(\frac{\bar{Q}_{11}^b}{\bar{G}_{13}} \right) \left(\frac{h}{a} \right)^2 \quad (2.3.256)$$

Thus, the magnitude of the shear deformation term depends on the modulus ratio $\bar{Q}_{11}^b/\bar{G}_{13}$ and the span-to-depth ratio h/a . It should also be noted that shear deformation increases the maximum deflection.

As a numerical example let us consider a $[0/90]_s$ laminate with the following ply properties:

$$\begin{aligned} E_2 &= 1.7 \times 10^6 \text{ psi}, & G_{12} &= 0.65 \times 10^6 \text{ psi}, \\ G_{13} &= 0.55 \times 10^6 \text{ psi}, & \nu_{12} &= 0.3 \end{aligned} \quad (2.3.257)$$

Two values of E_1 will be considered.

$$E_1 = 10 \times 10^6 \text{ psi}, \quad E_1 = 25 \times 10^6 \text{ psi} \quad (2.3.258)$$

The following $\bar{Q}_{ij}$ (0°) are determined.

For $E_1 = 10 \times 10^6$ psi:

$$[\bar{Q}(0^\circ)] = \begin{bmatrix} 10.09 \times 10^6 & 5.146 \times 10^5 & 0 \\ & 1.72 \times 10^6 \text{ psi} & 0 \\ \text{SYM} & & 0.65 \times 10^6 \end{bmatrix}$$

For $E_1 = 25 \times 10^6$ psi:

$$[\bar{Q}(0^\circ)] = \begin{bmatrix} 25.15 \times 10^6 & 5.131 \times 10^5 & 0 \\ & 1.71 \times 10^6 \text{ psi} & 0 \\ \text{SYM} & & 0.65 \times 10^6 \end{bmatrix}$$

It has been shown previously that

$$D_{11} = [7\bar{Q}_{11}(0^\circ) + \bar{Q}_{11}(90^\circ)] \frac{h^3}{96}$$

Thus for $E_1 = 10 \times 10^6$ psi:

$$D_{11} = \frac{[(7)(10.09) + 1.72] h^3 \times 10^6}{96} = 0.7536 h^3 \times 10^6 \text{ lbs.}$$

and

$$\bar{Q}_{11}^b = \frac{12D_{11}}{h^3} = (12)(3.0145)(10^6) = 9.043 \times 10^6 \text{ psi}$$

For $E_2 = 25 \times 10^6$ psi:

$$D_{11} = \frac{[(7)(25.15) + 1.71] h^3 \times 10^6}{96} = 1.852 h^3 \times 10^6 \text{ lbs.}$$

and

$$\bar{Q}_{11}^b = \frac{12D_{11}}{h^3} = 22.22 \times 10^6 \text{ psi}$$

For both values of E_1

$$\begin{aligned} A_{55} &= [G_{12}(0^\circ) + G_{13}(0^\circ)] \frac{h}{2} \\ &= (0.65 + 0.55) \frac{h}{2} \times 10^6 = 0.60 h \times 10^6 \text{ lb/in.} \end{aligned}$$

and

$$\bar{G}_{13} = \frac{A_{55}}{h} = 0.60 \times 10^6 \text{ psi}$$

Thus for $E_1 = 10 \times 10^6 \text{ psi}$:

$$\frac{\bar{Q}_{11}^b}{\bar{G}_{13}} = \frac{9.043 \times 10^6}{0.60 \times 10^6} = 15.1 \quad (2.3.259)$$

and for $E_1 = 25 \times 10^6 \text{ psi}$:

$$\frac{\bar{Q}_{11}^b}{\bar{G}_{13}} = \frac{22.22 \times 10^6}{0.60 \times 10^6} = 37.0 \quad (2.3.260)$$

Results are shown in figure 2.3.26 for both of these ratios of $\bar{Q}_{11}^b/\bar{G}_{13}$ as a function of a/h . The straight line is the solution from classical laminated plate theory (CPT) in which shear deformation is neglected, eq. (2.3.254). These results are based on $k = 5/6$, which is a value derived for homogeneous isotropic plates by Reissner (reference 2.3.14) and has been shown to be reasonable for laminated plates (reference 2.3.13). Other values used for homogeneous isotropic materials include $2/3$ used by Timoshenko (reference 2.3.15) for beams, and $\pi^2/12$ used by Mindlin in conjunction with wave propagation analysis for plates (reference 2.3.8).

It should also be noted that the stresses for these laminates will be determined by the first derivative of ψ_x .

$$\frac{d\psi_x}{dx} = \frac{q_0 x}{2D_{11}} (a - x)$$

This is of exactly the same form as κ_{xx} in eq. (2.3.59) which is the function that determines the stresses in conjunction with cylindrical bending for CPT. Thus, the inclusion of shear deformation, in the form chosen here, will not change the determination of ply stresses for the case of cylindrical bending. Although not demonstrated here, this is a general conclusion, i.e. higher order terms must be included in the displacement formulation, eq. (2.3.226), if significant improvement in stress analysis is desired for thick laminated plates.

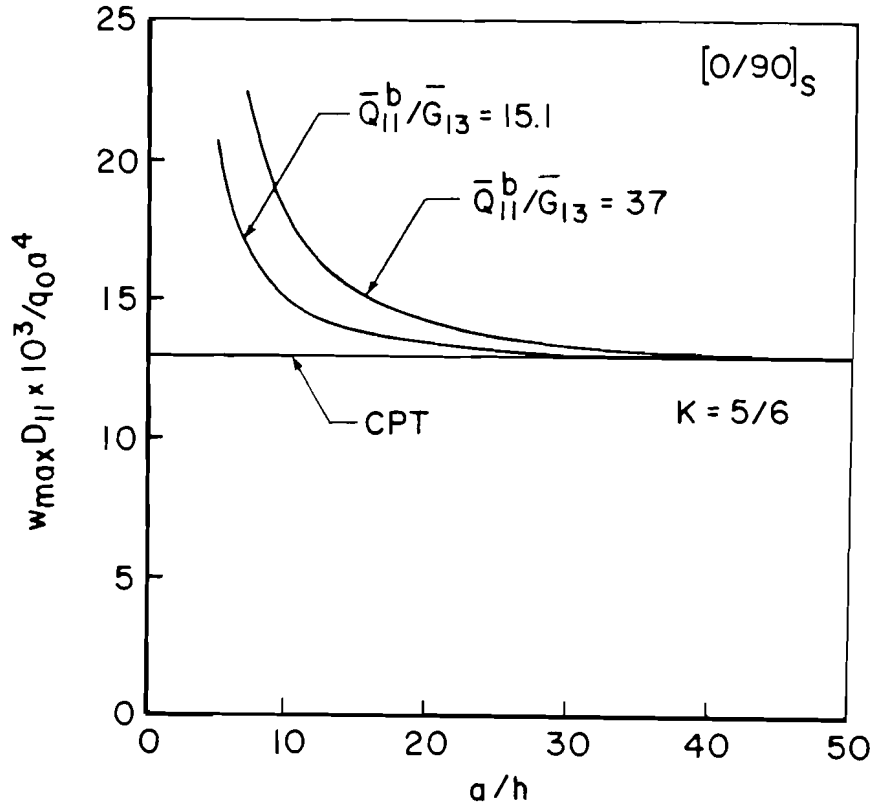


FIGURE 2.3.26 EFFECT OF SPAN-TO-DEPTH RATIO ON THE BENDING OF A BIDIRECTIONAL LAMINATED PLATE SUBJECTED TO CYLINDRICAL BENDING UNDER A UNIFORM TRANSVERSE LOAD $q=q_0$

2.3.5.3 Vibrations.

Natural vibration frequencies, ω , can be obtained for the symmetric, specially orthotropic class of laminates, under the conditions of cylindrical bending, discussed in Section 2.3.5.2 by assuming displacements of the form

$$u^0 = v^0 = \psi_y = 0 \quad (2.3.261)$$

$$\psi_x = \psi_x(x) e^{i\omega t} \quad (2.3.262)$$

$$w = W(x) e^{i\omega t} \quad (2.3.263)$$

Substituting these displacement functions into eqs. (2.3.237-241) in the absence of thermal effects, transverse load ($q = 0$), inplane force effects, and rotary inertia, we obtain the following governing equations

$$D_{11} \frac{d^2 \psi_x}{dx^2} - k A_{55} \left(\psi_x + \frac{dW}{dx} \right) = 0 \quad (2.3.264)$$

$$k A_{55} \left(\frac{d\psi}{dx} + \frac{d^2 W}{dx^2} \right) + P \omega^2 W = 0 \quad (2.3.265)$$

Using the same simple-support boundary conditions as given by eq. (2.3.249), we now require that

at $x = 0, a$

$$W = \frac{d\psi}{dx} = 0 \quad (2.3.266)$$

A solution to eqs. (2.3.264,265) which satisfies the boundary conditions in eq. (2.3.266) is of the form:

$$\psi_x = A_m \cos \frac{m\pi x}{a} \quad (2.3.267)$$

$$W = B_m \sin \frac{m\pi x}{a} \quad (2.3.268)$$

where m and n are integers. Substituting these functions into eqs. (2.3.264-265), we obtain two homogeneous, algebraic equations. A nontrivial solution is obtained only if the determinant of the coefficient matrix vanishes, i.e.

$$\begin{vmatrix} \left(\frac{D_{11} m^2 \pi^2}{a^2} + kA_{55} \right) & kA_{55} \frac{m\pi}{a} \\ kA_{55} \frac{m\pi}{a} & \left(kA_{55} \frac{m^2 \pi^2}{a^2} - P \omega^2 \right) \end{vmatrix} = 0 \quad (2.3.269)$$

Solving this relationship we obtain a solution for flexural vibration frequencies which can be written in the form

$$\omega_m = \omega_m^i \sqrt{1 - \frac{1}{1 + 48/5 m^2 \omega^2 S}} \quad (2.3.270)$$

where S is the shear deformation factor defined in eqs. (2.3.254,256) and ω_m^i is the flexural vibration frequency for the same class of laminates with shear deformation neglected and is defined by eq. (2.3.162). It can be seen from eq. (2.3.270) that shear deformation decreases flexural vibration frequencies. The amount of reduction again depends on the ratios $\bar{Q}_{11}^p / \bar{G}_{13}$ and a/h , i.e. as $S \rightarrow 0$, $\omega_m \rightarrow \omega_m^i$. It is also obvious from eqs. (2.3.270,271) that the fundamental vibration frequency occurs for $m = 1$.

Results are shown in figure 2.3.27 for a $[0/90]_S$ laminate with the ply properties given in eqs. (2.3.257,258) and $k = 5/6$.

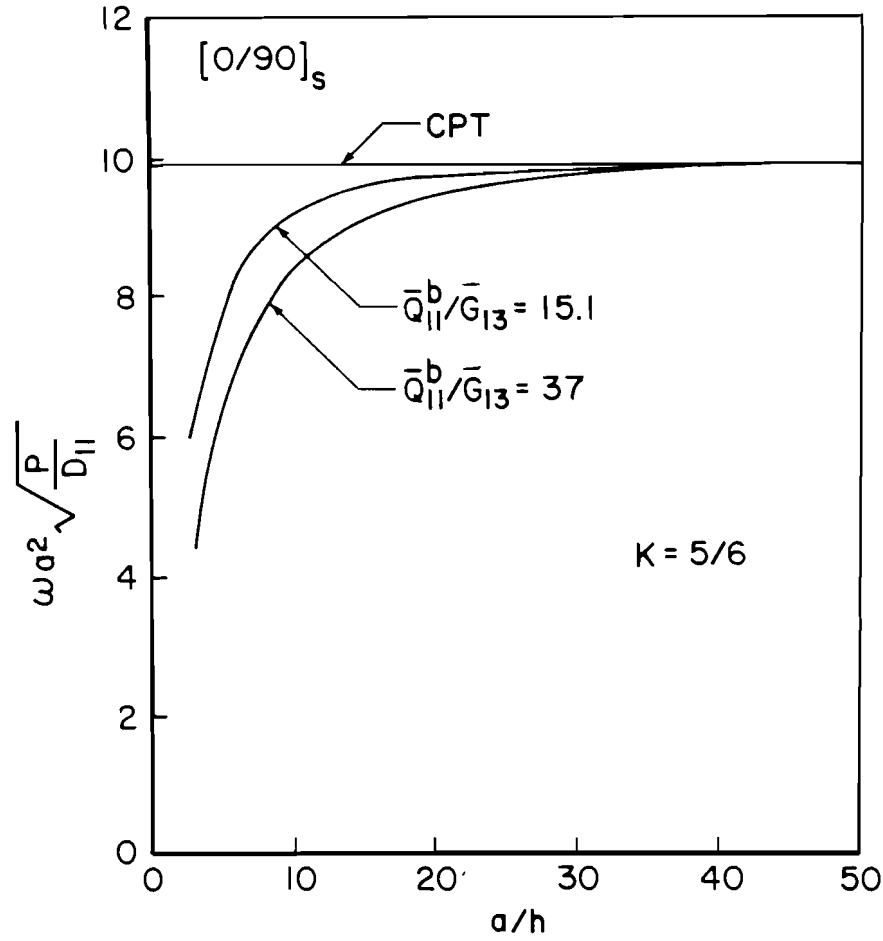


FIGURE 2.3.27 EFFECT OF SPAN-TO-DEPTH RATIO ON THE FUNDAMENTAL FLEXURAL VIBRATION FREQUENCY OF A BIDIRECTIONAL LAMINATED STRIP SUBJECTED TO CYLINDRICAL

2.3.5.4 Stability.

Now we consider the effect of shear deformation on critical buckling loads. For symmetric, specially orthotropic laminates subjected to cylindrical bending, as discussed in Sections 2.3.5.2 and 2.3.5.3, the displacement field displayed in eq. (2.3.243) is applicable. The governing equations in the absence of inertia terms, thermal effects, and transverse load ($q = 0$), reduce to

$$D_{11} \frac{d^2 \psi_x}{dx^2} - k A_{55} \left(\psi_x + \frac{dw}{dx} \right) = 0 \quad (2.3.271)$$

$$k A_{55} \left(\frac{d\psi_x}{dx} + \frac{d^2 w}{dx^2} \right) + N_{xx}^i \frac{d^2 w}{dx^2} = 0 \quad (2.3.272)$$

We consider the initial inplane load $N_{xx}^i = -N_0 = \text{constant}$ where $N_0 > 0$. For the simple-support boundary conditions used in the static case, eq. (2.3.249), a solution to eqs. (2.3.271,272) is of the same form as eqs. (2.3.267,268), i.e.

$$\psi_x = A_m \cos \frac{m\pi x}{a} \quad (2.3.273)$$

$$w = B_m \sin \frac{m\pi x}{a} \quad (2.3.274)$$

Substituting these functions into eqs. (2.3.271,272), we obtain two homogeneous, algebraic equations. Setting the determinant of the coefficient matrix equal to zero, we obtain a nontrivial solution, i.e.

$$\begin{vmatrix} \left(D_{11} + \frac{kA_{55}a^2}{m^2\pi^2} \right) & \frac{kA_{55}a}{m\pi} \\ \frac{kA_{55}D_{11}}{m\pi} & (kA_{55} - N_0) \end{vmatrix} = 0 \quad (2.3.275)$$

Solving this relationship for N_0 we obtain the expression

$$N_0 = \left(\frac{kA_{55}D_{11}}{D_{11} + \frac{kA_{55}a^2}{m^2\pi^2}} \right) \quad (2.3.276)$$

The lowest value of N_0 , and hence the critical buckling load, is obtained for $m = 1$. We now write this result in the following form:

$$N_{0cr} = N_{0cr}^i \sqrt{1 - \frac{1}{1 + 48/5\pi^2 S}} \quad (2.3.277)$$

where S is the shear deformation factor defined in eqs. (2.3.254,256), and N_{0cr}^i is the critical buckling load for the same laminate with shear deformation neglected and is given by eq. (2.3.203). It can be seen from eq. (2.3.277) that shear deformation decreases critical buckling loads. As in the case of flexural vibration frequencies, the value of this reduction depends on the ratios $\bar{Q}_{11}^b/\bar{G}_{13}$ and a/h .

Numerical results are illustrated in figure 2.3.28 for a $[0/90]_S$ laminate with the ply properties given in eqs. (2.3.257,258) and $k = 5/6$.

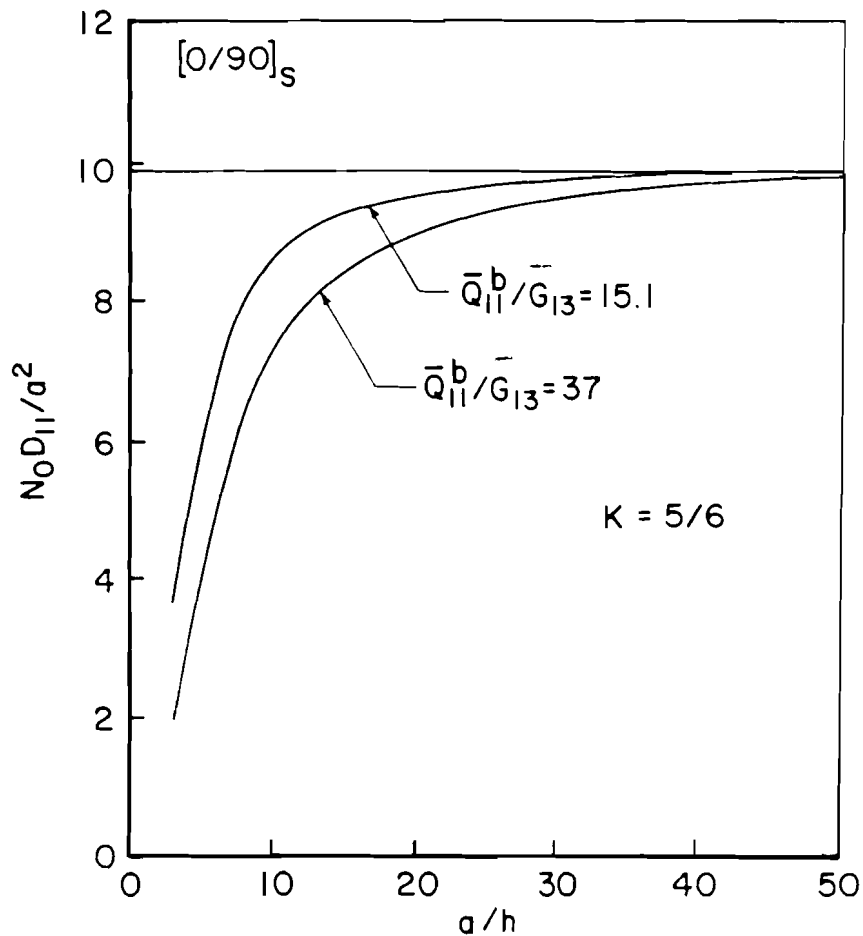


FIGURE 2.3.28 EFFECT OF SPAN-TO-DEPTH RATIO ON THE CRITICAL BUCKLING LOAD OF A BIDIRECTIONAL LAMINATED STRIP SUBJECTED TO CYLINDRICAL BENDING UNDER UNIAXIAL INPLANE COMPRESSION

2.3.6 Sandwich Plates.

No discussion on layered plate analysis is complete without considering sandwich plates. In this section the governing equations for sandwich plates are derived. Bending under lateral loads, flexural vibration frequencies and buckling under inplane loads are considered.

2.3.6.1 Governing Equations.

We now derive the governing equations for a sandwich plate consisting of a core material of thickness h , having face sheets bonded to the top and bottom surfaces. The face sheets are assumed to be laminates with the top plate having thickness h_1 and the bottom surface having thickness h_2 . The core material is assumed to be of low density, and its primary function is to stabilize the face sheets. The coordinate system is assumed to be at the center of the core, as shown in figure 2.3.29. The following specific assumptions are made:

1. The core material is transversely isotropic with respect to an axis normal to the plane of the plate and of much greater thickness than the face sheets, i.e. $h \gg h_1, h_2$.
2. The inplane stresses σ_{xx} , σ_{yy} , and τ_{xy} of the core are assumed to be negligible.
3. The inplane core displacements u_c and v_c are assumed to be linear functions of the z coordinate.
4. The face sheets are constructed of an arbitrary number of layers of orthotropic sheets bonded together. However, the orthotropic axes of material symmetry of an individual layer need not coincide with the x - y axes of the plate.
5. Inplane strains ϵ_{xx} , ϵ_{yy} , and γ_{xy} are uniform through-the-thickness of the face sheets.
6. The plate displacements are small compared to the plate thickness.
7. The plate inplane strains are small compared to unity.
8. The transverse displacement, w , of the plate is independent of the z coordinate, i.e. $\epsilon_{zz} = 0$.
9. The core and each ply of the face sheets obey Hooke's law.
10. The plate has constant thickness.

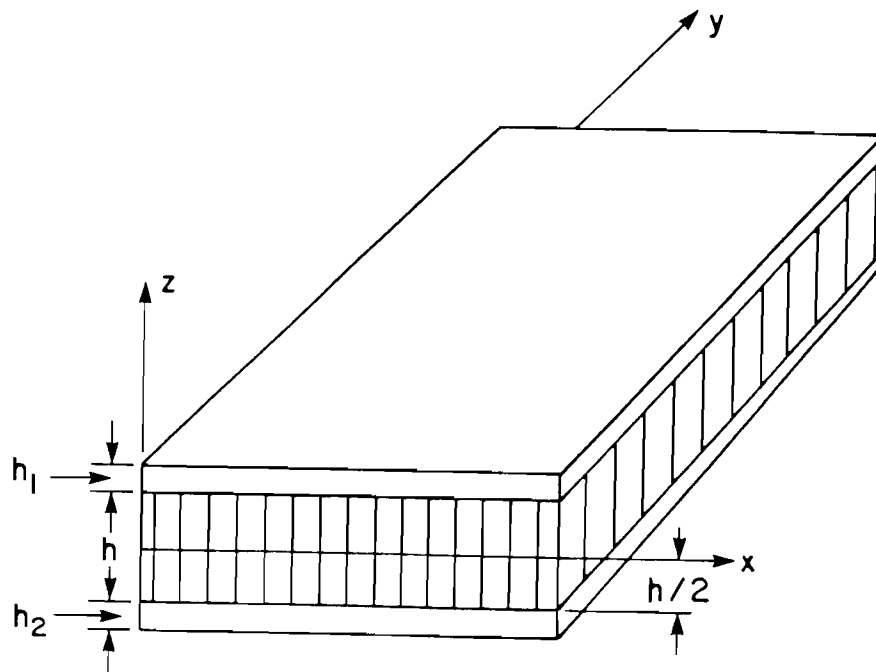


FIGURE 2.3.29 NOMENCLATURE FOR SANDWICH PLATES

o Strain-Displacement Relations

Based on assumption 2, the core displacements are of the form

$$u_c = u^0(x, y) + z \psi_x(x, y)$$

$$v_c = v^0(x, y) + z \psi_y(x, y)$$

From assumption 5 and the requirement that the displacements at the interfaces between the core and the face sheets be continuous, we obtain the face sheet displacements

$$u_1 = u^0 + \frac{h}{2} \psi_x \quad (2.3.278)$$

$$v_1 = v^0 + \frac{h}{2} \psi_y \quad (2.3.279)$$

$$u_2 = u^0 - \frac{h}{2} \psi_x \quad (2.3.280)$$

$$v_2 = v^0 - \frac{h}{2} \psi_y \quad (2.3.281)$$

From assumption 8

$$w = w(x, y) \quad (2.3.282)$$

Thus the sandwich plate theory is based on essentially the same variables as in the shear deformation plate theory.

Strains are now of the form

$$\epsilon_{xxc} = \epsilon_{xx}^0 + z \kappa_{xx} \quad (2.3.283)$$

$$\epsilon_{yyc} = \epsilon_{yy}^0 + z \kappa_{yy} \quad (2.3.284)$$

$$\gamma_{xyc} = \gamma_{xy}^0 + z \kappa_{xy} \quad (2.3.285)$$

$$\gamma_{xzc} = \psi_y + \frac{\partial w}{\partial y} \quad (2.3.286)$$

$$\gamma_{yzc} = \psi_x + \frac{\partial w}{\partial x} \quad (2.3.287)$$

$$\epsilon_{xx1} = \epsilon_{xx}^0 + \frac{h}{2} \kappa_{xx}, \quad \epsilon_{xx2} = \epsilon_{xx}^0 - \frac{h}{2} \kappa_{xx} \quad (2.3.288)$$

$$\epsilon_{yy1} = \epsilon_{yy}^0 + \frac{h}{2} \kappa_{yy}, \quad \epsilon_{yy2} = \epsilon_{yy}^0 - \frac{h}{2} \kappa_{yy} \quad (2.3.289)$$

$$\gamma_{xy1} = \gamma_{xy}^0 + \frac{h}{2} \kappa_{xy}, \quad \gamma_{xy2} = \gamma_{xy}^0 - \frac{h}{2} \kappa_{xy} \quad (2.3.290)$$

where ϵ_{xx}^0 , ϵ_{yy}^0 , and γ_{xy}^0 are defined in eq. (2.3.2) and κ_{xx} , κ_{yy} , and κ_{xy} are defined in eq. (2.3.227). The subscripts c, 1, and 2 refer to the core, top face sheet, and lower face sheet, respectively.

o Constitutive Relations

Plate constitutive relations are derived by considering force and moment resultants similar to the ones defined for classical plate theory, i.e.

$$\begin{aligned} (N_{xx}, N_{yy}, N_{xy}) &= \int_{-(h/2+h_2)}^{-h/2} (\sigma_{xx}, \sigma_{yy}, \tau_{xy}) dz \\ &+ \int_{h/2}^{(h/2+h_1)} (\sigma_{xx}, \sigma_{yy}, \tau_{xy}) dz \end{aligned} \quad (2.3.291)$$

$$\begin{aligned} (M_{xx}, M_{yy}, M_{xy}) &= \int_{-(h/2+h_2)}^{-h} (\sigma_{xx}, \sigma_{yy}, \tau_{xy}) z dz \\ &+ \int_{h/2}^{(h/2+h_1)} (\sigma_{xx}, \sigma_{yy}, \tau_{xy}) z dz \end{aligned} \quad (2.3.292)$$

$$(Q_{xx}, Q_{yy}) = \int_{-h/2}^{h/2} (\tau_{xz}, \tau_{yz}) dz \quad (2.3.293)$$

For the face sheets a state of plane stress is assumed to exist in each ply. Thus, eq. (2.4.22) from Volume I is applicable to the face sheets, i.e.

$$\{\sigma\} = [\bar{Q}] \{\epsilon\} - \{\Gamma\} \Delta T \quad (2.3.294)$$

and for the core

$$\begin{bmatrix} \tau_{yz} \\ \tau_{xz} \end{bmatrix} = \begin{bmatrix} G_{23} & 0 \\ 0 & G_{13} \end{bmatrix} \begin{bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} \quad (2.3.295)$$

Substituting these relations along with the strain-displacement relations, eqs. (2.3.283-290), into the force and moment resultants, eqs. (2.3.291-293), we obtain the constitutive relations which are of a form similar to eq. (2.3.1).

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & | & B \\ \hline C & | & D \end{bmatrix} \begin{Bmatrix} \epsilon^0 \\ \kappa \end{Bmatrix} - \begin{Bmatrix} N^T \\ M^T \end{Bmatrix} \quad (2.3.296)$$

where

$$\begin{aligned}\{A\} &= \int_{-(h/2+h_2)}^{h/2} \{\bar{Q}\} dz + \int_{h/2}^{(h/2+h_1)} \{\bar{Q}\} dz \\ &= \{A^{(1)}\} + \{A^{(2)}\}\end{aligned}\quad (2.3.297)$$

$$\begin{aligned}\{B\} &= -\frac{h}{2} \int_{-(h/2+h_2)}^{-h/2} \{\bar{Q}\} dz + \frac{h}{2} \int_{h/2}^{(h/2+h_1)} \{\bar{Q}\} dz \\ &= \frac{h}{2} [\{A^{(1)}\} - \{A^{(2)}\}]\end{aligned}\quad (2.3.298)$$

where $A^{(1)}$ and $A^{(2)}$ are the A matrices for the top and bottom face sheets, respectively, and

$$C = \int_{-(h/2+h_2)}^{-h/2} \{\bar{Q}\} z dz + \int_{h/2}^{(h/2+h_1)} \{\bar{Q}\} z dz \quad (2.3.299)$$

$$D = -\frac{h}{2} \int_{-(h/2+h_2)}^{-h/2} \{\bar{Q}\} z dz + \frac{h}{2} \int_{h/2}^{(h/2+h_1)} \{\bar{Q}\} z dz \quad (2.3.300)$$

$$\begin{aligned}\{N^T, M^T\} &= \int_{-(h/2+h_2)}^{h/2} \{\Gamma\} \Delta T(1, z) dz \\ &\quad + \int_{h/2}^{(h/2+h_1)} \{\Gamma\} \Delta T(1, z) dz\end{aligned}\quad (2.3.301)$$

The transverse shear relations are of the form

$$\begin{bmatrix} Q_y \\ Q_x \end{bmatrix} = h \begin{bmatrix} G_{23} & 0 \\ 0 & G_{13} \end{bmatrix} \begin{bmatrix} \psi_y + \frac{\partial w}{\partial y} \\ \psi_x + \frac{\partial w}{\partial x} \end{bmatrix} \quad (2.3.302)$$

It can easily be seen that if the top and bottom face sheets are identical that the B and C coefficients will vanish and the inplane and bending problems uncouple.

The stiffness matrices C and D can also be written in terms of the $A^{(1)}$ and $A^{(2)}$ matrices by treating the face sheets as homogeneous materials with effective stiffnesses $A_{ij}^{(k)}/h_k$. This may be a reasonable approximation since the face sheets are thin compared to the core. For such cases we have

$$\{C\} = \frac{h}{2} [(1+h_1/h)\{A^{(1)}\} - (1+h_2/h)\{A^{(2)}\}] \quad (2.3.303)$$

$$\{D\} = \frac{h^2}{4} [(1+h_1/h)\{A^{(1)}\} + (1+h_2/h)\{A^{(2)}\}] \quad (2.3.304)$$

To illustrate the order of the error introduced by this approximation, let us consider two layer face sheets. Using eqs. (2.3.299,300) we find

$$C_{ij} = \frac{h}{8} (h_1 F_{ij}^{(1)} - h_2 F_{ij}^{(2)}) \quad (2.3.305)$$

$$D_{ij} = \frac{h}{16} (h_1 F_{ij}^{(1)} + h_2 F_{ij}^{(2)}) \quad (2.3.306)$$

where

$$F_{ij}^{(k)} = (2+3h_k/h) \bar{Q}_{ij}^{(k)}(\theta_1) + (2+h_k/h) \bar{Q}_{ij}^{(k)}(\theta_2)$$

and θ_1 and θ_2 denote inner ply and outer ply orientations, respectively. If eqs. (2.3.303,304) are utilized, the results can be written in the same form as eqs. (2.3.305,306) with

$$F_{ij}^{(k)} = 2(1+h_k/h) [\bar{Q}_{ij}^{(k)}(\theta_1) + \bar{Q}_{ij}^{(k)}(\theta_2)]$$

Thus, the ratio h_k/h determines the order of the error involved in assuming homogeneous face sheets. The two-ply laminate represents a "worst" case situation. Less error is introduced for face sheets with more layers.

o Governing Equations in Terms of Displacements

For the static case the equilibrium equations in terms of force and moment resultants will be identical to eqs. (2.3.3-6). Substituting the constitutive relations, the constitutive eqs. (2.3.296), into the equilibrium equations, and taking into account the strain-displacement relations, eqs. (2.3.283-387), we obtain the governing equations in terms of the displacements.

$$\begin{aligned} & A_{11} \frac{\partial^2 u^0}{\partial x^2} + 2A_{16} \frac{\partial^2 u^0}{\partial x \partial y} + A_{66} \frac{\partial^2 u^0}{\partial y^2} + A_{16} \frac{\partial^2 v^0}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 v^0}{\partial x \partial y} \\ & + A_{26} \frac{\partial^2 v^0}{\partial y^2} + B_{11} \frac{\partial^2 \psi_x}{\partial x^2} + 2B_{16} \frac{\partial^2 \psi_x}{\partial x \partial y} + B_{66} \frac{\partial^2 \psi_x}{\partial y^2} + B_{16} \frac{\partial^2 \psi_y}{\partial x^2} \\ & + (B_{12} + B_{66}) \frac{\partial^2 \psi_y}{\partial x \partial y} + B_{26} \frac{\partial^2 \psi_y}{\partial y^2} - \frac{\partial N_{xx}^T}{\partial x^2} - \frac{\partial N_{xy}^T}{\partial y} = 0 \end{aligned} \quad (2.3.307)$$

$$\begin{aligned}
& A_{16} \frac{\partial^2 u^o}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 u^o}{\partial x \partial y} + A_{26} \frac{\partial^2 u^o}{\partial y^2} + A_{66} \frac{\partial^2 v^o}{\partial x^2} + 2A_{26} \frac{\partial^2 v^o}{\partial x \partial y} \\
& + A_{22} \frac{\partial^2 v^o}{\partial y^2} + B_{16} \frac{\partial^2 \psi_x}{\partial x^2} + (B_{12} + B_{66}) \frac{\partial^2 \psi_x}{\partial x \partial y} + B_{26} \frac{\partial^2 \psi_x}{\partial y^2} + B_{66} \frac{\partial^2 \psi_y}{\partial x^2} \\
& + 2B_{26} \frac{\partial^2 \psi_y}{\partial x \partial y} + B_{22} \frac{\partial^2 \psi_y}{\partial y^2} - \frac{\partial N_{xy}^T}{\partial x} - \frac{\partial N_{yy}^T}{\partial y} = 0
\end{aligned} \tag{2.3.308}$$

$$\begin{aligned}
& C_{11} \frac{\partial^2 u^o}{\partial x^2} + 2C_{16} \frac{\partial^2 u^o}{\partial x \partial y} + C_{66} \frac{\partial^2 u^o}{\partial y^2} + C_{16} \frac{\partial^2 v^o}{\partial x^2} + (C_{12} + C_{66}) \frac{\partial^2 v^o}{\partial x \partial y} \\
& + C_{26} \frac{\partial^2 v^o}{\partial y^2} + D_{11} \frac{\partial^2 \psi_x}{\partial x^2} + 2D_{16} \frac{\partial^2 \psi_x}{\partial x \partial y} + D_{66} \frac{\partial^2 \psi_x}{\partial y^2} + D_{16} \frac{\partial^2 \psi_y}{\partial x^2} \\
& + (D_{12} + D_{66}) \frac{\partial^2 \psi_y}{\partial x \partial y} + D_{26} \frac{\partial^2 \psi_y}{\partial y^2} - G_{13} h \left(\psi_x + \frac{\partial w}{\partial x} \right) \\
& - \frac{\partial M_{xx}^T}{\partial x} - \frac{\partial M_{xy}^T}{\partial y} = 0
\end{aligned} \tag{2.3.309}$$

$$\begin{aligned}
& C_{16} \frac{\partial^2 u^o}{\partial x^2} + (C_{12} + C_{66}) \frac{\partial^2 u^o}{\partial x \partial y} + C_{26} \frac{\partial^2 u^o}{\partial y^2} + C_{66} \frac{\partial^2 v^o}{\partial x^2} + 2C_{26} \frac{\partial^2 v^o}{\partial x \partial y} \\
& + C_{22} \frac{\partial^2 v^o}{\partial y^2} + D_{16} \frac{\partial^2 \psi_x}{\partial x^2} + (D_{12} + D_{66}) \frac{\partial^2 \psi_x}{\partial x \partial y} + D_{26} \frac{\partial^2 \psi_x}{\partial y^2} + D_{66} \frac{\partial^2 \psi_y}{\partial x^2} \\
& + 2D_{26} \frac{\partial^2 \psi_y}{\partial x \partial y} + D_{22} \frac{\partial^2 \psi_y}{\partial y^2} - G_{23} h \left(\psi_y + \frac{\partial w}{\partial y} \right) - \frac{\partial M_{xy}^T}{\partial x} - \frac{\partial M_{yy}^T}{\partial y} = 0
\end{aligned} \tag{2.3.310}$$

$$G_{13} h \left(\frac{\partial \psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + G_{23} h \left(\frac{\partial \psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) + q = 0 \tag{2.3.311}$$

Note the similarity between these equations and the static version of eqs. (2.3.237-241).

The boundary conditions for this theory will be identical to those prescribed in eq. (2.3.242).

2.3.6.2 Bending Under Transverse Load.

Let us now consider cylindrical bending of a sandwich plate under uniform transverse load $q = q_0 = \text{constant}$. For simplicity let us consider identical top and bottom face sheets. As previously, cylindrical bending implies the laminate is of infinite extent in the y -direction and uniformly supported along the edges $x = 0, a$. In addition, let us assume the face sheets contain only layers which are orthotropic with respect to the x - y axes of the plate. Under these conditions $B_{ij} = D_{ij} = A_{16} = A_{26} = D_{16} = D_{26} = 0$ and

$$u^0 = v^0 = \psi_y = 0, \quad \psi_x = \psi_x(x), \quad w = w(x)$$

and eqs. (2.3.307-311) reduce to

$$D_{11} \frac{\partial^2 \psi_x}{\partial x^2} - G_{13} h \left(\psi_x + \frac{dw}{dx} \right) = 0$$

$$G_{13} h \frac{d\psi_x}{dx} + \frac{d^2 w}{dx^2} + q_0 = 0$$

These equations are of exactly the same form as eqs. (2.3.244-245) for cylindrical bending of laminated plates in which shear deformation is included. Thus, for the simple-support boundary conditions, eq. (2.3.249), a solution for plate deflection can be obtained from eq. (2.3.252) by direct analogy, i.e.

$$w = \frac{q_0 a^4 \bar{x}}{24 D_{11}} \left[\bar{x}^3 - 2\bar{x}^2 + 1 + \frac{12 D_{11}}{G_{13} a^3 h} (1 - \bar{x}) \right] \quad (2.3.312)$$

where $\bar{x} = x/a$.

Just as in the case of classical laminated plate theory, the interlaminar shear stress distribution in a sandwich plate can be determined by integrating the equilibrium equations of elasticity in the form of eqs. (2.3.28). For the core material $\sigma_{xx} = \tau_{xy} = 0$ and the first of eqs. (2.3.26) becomes

$$\frac{\partial \tau_{xz}}{\partial z} = 0$$

which implies $\tau_{xz} = f(x, y)$ in the core. For the sandwich plate under consideration, the inplane stresses are given by

$$\sigma_x = \pm \frac{h}{2} \frac{d\psi_x}{dx} \quad (2.3.313)$$

$$\sigma_y = \pm \frac{h}{2} \frac{d\psi_x}{dx} \quad (2.3.314)$$

$$\tau_{xy} = 0 \quad (2.3.315)$$

where a + sign is associated with the upper face sheet and a - sign with the lower face sheet. Again by direct analogy, ψ_x is of the same form as eq. (2.3.250). Thus,

$$\frac{d\psi_x}{dx} = \frac{q_0 a^2 \bar{x}}{2D_{11}} (1 - \bar{x}) \quad (2.3.316)$$

Since the face sheets are identical and the core interlaminar shear stress is constant, then it is only necessary to calculate the shear distribution through the bottom face sheet to obtain the entire profile. Let us consider face sheets constructed of two-ply laminates with stacking sequence $[0/90]_T$. Using eqs. (2.3.313, 316) in conjunction with eq. (2.3.28), we obtain the following interlaminar shear stress distribution:

0° outer plies

$$\begin{aligned} \bar{\tau}_{xz} &= \frac{\bar{Q}_{11}(0^\circ)(2\bar{x}-1)}{4E_2} \int_{-(1/2+h_1/h)}^{\bar{z}} d\bar{z} \\ &= \frac{\bar{Q}_{11}(0^\circ)(2\bar{x}-1)}{8E_2} (1+2h_1/h+2\bar{z}) \\ &\quad -(1/2+h_1) \leq \bar{z} \leq (1/2-h_1/2h) \end{aligned} \quad (2.3.317)$$

where

$$\bar{\tau}_{xz} = \frac{\tau_{xz} D_{11}}{q_0 E_2 a h^2} \quad (2.3.318)$$

90° inner plies

$$\begin{aligned} \bar{\tau}_{xz} &= \frac{(2\bar{x}-1)}{8E_2} \left[\int_{-(1/2+h_1/h)}^{-(1/2+h_1/2h)} \bar{Q}_{11}(0^\circ) d\bar{z} \right. \\ &\quad \left. + \int_{-(1/2+h_1/2h)}^{\bar{z}} \bar{Q}_{11}(90^\circ) d\bar{z} \right] \end{aligned}$$

$$= \frac{(2\bar{x}-1)}{16E_2} [2h_1/h Q_{11}(0^\circ) + (1+h_1/h+2\bar{z})Q_{11}(90^\circ)] \quad (2.3.319)$$

$$-(1/2+h_1/2h) \leq \bar{z} \leq -1/2$$

Core

Since the core interlaminar shear stress is constant and must be continuous at the face sheet-core interface, $\tau_{xz}(\text{core}) = \tau_{xz}(90^\circ)$ at $\bar{z} = -1/2$. Thus, from eq. (2.3.319)

$$\bar{\tau}_{xz} = \frac{(2\bar{x}-1)h_1}{16E_2h} [2\bar{Q}_{11}(0^\circ) + \bar{Q}_{11}(90^\circ)] \quad -1/2 \leq \bar{z} \leq 1/2 \quad (2.3.320)$$

Numerical results are illustrated in figure 2.3.30 for $E_1/E_2 = 15$ and $h_1/h = 0.1$. Note that the assumption of uniform strain in the face sheets leads to a linear interlaminar shear stress distribution within each ply rather than the parabolic distribution found in classical laminated plate theory. Note that the interlaminar shear stress distribution is independent of core properties. Also note that the maximum interlaminar shear stress occurs in the core, including the face sheet-core interfaces. This is an important observation when considering interlaminar shear failure of sandwich plates.

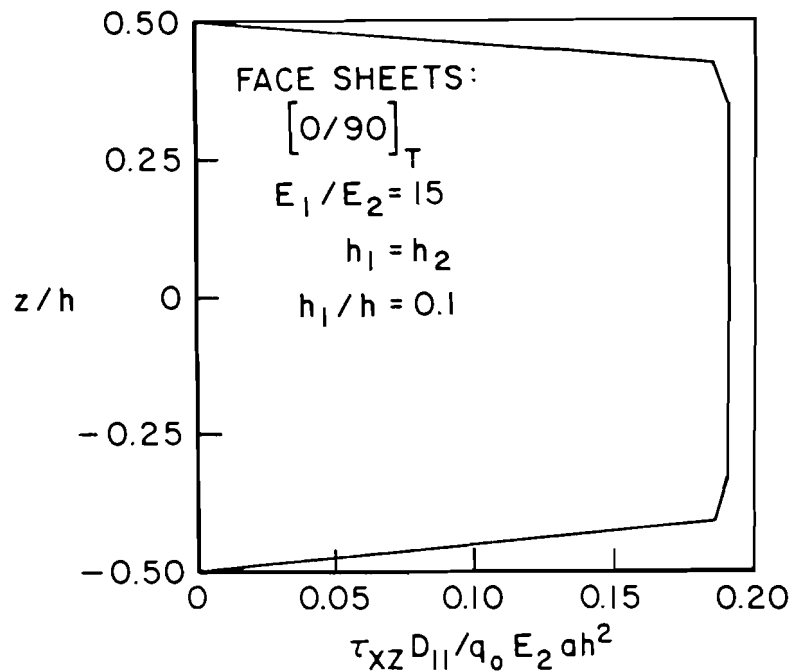


FIGURE 2.3.30 INTERLAMINAR SHEAR STRESS DISTRIBUTION FOR SYMMETRIC SANDWICH PLATE WITH 2 PLY FACE SHEETS

2.3.6.3 Flexural Vibrations.

In the absence of rotary and inplane inertia terms, flexural vibration frequencies can be obtained by including a flexural inertia term in eq. (2.3.311). In particular this equation becomes, in the absence of transverse load ($q = 0$)

$$G_{13}h \left(\frac{\partial \psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + G_{23}h \left(\frac{\partial \psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) = P \frac{\partial^2 w}{\partial t^2} \quad (2.3.321)$$

Time dependence can be removed by assuming a solution to eqs. (2.3.307-310,321) of the form

$$u^0 = e^{i\omega t} U(x,y)$$

$$v^0 = e^{i\omega t} V(x,y)$$

$$\psi_x = e^{i\omega t} \psi_x(x,y)$$

$$\psi_y = e^{i\omega t} \psi_y(x,y)$$

$$w = e^{i\omega t} W(x,y)$$

Equations (2.3.307-310) will remain unchanged except that U , V , ψ_x , ψ_y and W will be substituted for u^0 , v^0 , ψ_x , ψ_y and w , respectively. Equation (2.3.321) becomes

$$G_{13}h \left(\frac{\partial \psi_x}{\partial x} + \frac{\partial^2 W}{\partial x^2} \right) + G_{23}h \left(\frac{\partial \psi_y}{\partial y} + \frac{\partial^2 W}{\partial y^2} \right) + P\omega^2 W = 0 \quad (2.3.322)$$

For cylindrical bending of sandwich plates with identical face sheets, the governing equations reduce to the same form as eqs. (2.3.264,265), i.e.

$$D_{11} \frac{\partial^2 \psi_x}{\partial x^2} - G_{13}h \left(\psi_x + \frac{dW}{dx} \right) = 0 \quad (2.3.323)$$

$$G_{13}h \left(\frac{d\psi_x}{dx} + \frac{d^2 W}{dx^2} \right) + P\omega^2 W = 0 \quad (2.3.324)$$

By direct analogy a solution for the simple-support boundary conditions of eq. (2.3.266) is given by (2.3.270) with

$$S = \frac{48D_{11}}{5G_{13}ha^2} \quad (2.3.325)$$

Solutions for rectangular plates are of complex form and will not be discussed here.

2.3.6.4 Stability.

Buckling calculations can be performed by modifying eq. (2.3.311) to include inplane force effects. In the absence of transverse load ($q = 0$), this relationship becomes

$$\begin{aligned} G_{13}h \left(\frac{\partial \psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + G_{23}h \left(\frac{\partial \psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) + N_{xx}^i \frac{\partial^2 w}{\partial x^2} \\ + 2N_{xy}^i \frac{\partial^2 w}{\partial x \partial y} + N_{yy}^i \frac{\partial^2 w}{\partial y^2} = 0 \end{aligned} \quad (2.3.326)$$

For the case of cylindrical bending of sandwich plates with identical face sheets, the governing equations reduce to the same form as eqs. (2.3.271,272), i.e.

$$D_{11} \frac{d^2 \psi_x}{dx^2} - G_{13}h \left(\psi_x + \frac{d\psi_x}{dx} \right) = 0 \quad (2.3.327)$$

$$G_{13}h \left(\frac{d\psi_x}{dx} + \frac{d^2 w}{dx^2} \right) + N_{xx}^i \frac{d^2 w}{dx^2} = 0 \quad (2.3.328)$$

By direct analogy a solution for the simple-support boundary conditions of eq. (2.3.249) with $N_{xx}^i = -N_0 = \text{constant}$ ($N_0 > 0$) is given by eq. (2.3.277) for critical buckling load with S as defined in eq. (2.3.325).

Solutions to rectangular plates are more complex and are not considered here.

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2.4 SHELLS.

2.4.1 Introduction.

Analysis of shell structures is a complex subject. A number of first-order shell theories can be found in the technical literature. The theory presented here is based on Donnell-Vlasov type assumptions which are applicable to thin shells.

The theory will be developed for general shell geometries. Equations applicable to circular cylindrical shells will then be deduced from the general theory. Static behavior, vibrations, and buckling of circular cylindrical shells will be discussed in some detail.

2.4.2 Governing Equations.

The governing equations for the static deformation of general shell geometries will be derived. Equations for circular cylindrical shells will be obtained from the general theory. These equations will be modified later to consider vibrations and buckling of circular cylindrical shells.

2.4.2.1 Theory of Surfaces.

Some basic concepts from the theory of surfaces are necessary in order to develop the basic equations of shell analysis. In particular, as in plate theory, the shell equations are written in terms of middle surface displacements.

The undeformed middle surface can be defined in terms of two independent parameters β_1 and β_2 and by the radius vector

$$\hat{\mathbf{r}} = \hat{\mathbf{r}}(\beta_1, \beta_2) \quad (2.4.1)$$

where $\hat{\mathbf{r}}$ denotes a vector quantity. This equation provides a method for determining points on the surface. If, for example, the parameter β_1 is held constant, then eq. (2.4.1) determines a curve on the surface. These space curves are referred to as β_2 curves. In a similar manner we can define β_1 curves.

We now define the following notation

$$\hat{\mathbf{r}}_{,1} = \frac{\partial \hat{\mathbf{r}}}{\partial \beta_1}, \quad \hat{\mathbf{r}}_{,2} = \frac{\partial \hat{\mathbf{r}}}{\partial \beta_2} \quad (2.4.2)$$

The vectors $\hat{\mathbf{r}}_{,1}$ and $\hat{\mathbf{r}}_{,2}$ are tangent to the β_1 and β_2 curves, respectively. The length of these tangent vectors will be denoted by

$$A_1 = |\hat{\mathbf{r}}_{,1}|, \quad A_2 = |\hat{\mathbf{r}}_{,2}| \quad (2.4.3)$$

We now denote the following unit vectors

$$\frac{\hat{\mathbf{r}}_{,1}}{A_1} = \hat{\mathbf{i}}_1, \quad \frac{\hat{\mathbf{r}}_{,2}}{A_2} = \hat{\mathbf{i}}_2, \quad \hat{\mathbf{i}}_n = \frac{\hat{\mathbf{r}}_{,1} \times \hat{\mathbf{r}}_{,2}}{A_1 A_2} \quad (2.4.4)$$

where $\hat{i}_n$ is a unit vector of the normal to the surface and is orthogonal to the unit vectors $\hat{i}_1$ and $\hat{i}_2$.

For orthogonal curvilinear surface coordinates, the element of arch length on the surface is given by the relationship

$$ds^2 = A_1^2 d\beta_1^2 + A_2^2 d\beta_2^2 \quad (2.4.5)$$

The surface nomenclature is illustrated in figure 2.4.1. A more complete discussion on surfaces may be found in textbooks on differential geometry.

A basic shell element with the coordinate system in the middle surface is shown in figure 2.4.2. Any point in the shell can be located by a position vector of the form

$$\begin{aligned} \hat{r}(\beta_1, \beta_2, z) &= \hat{r}(\beta_1, \beta_2) + z \hat{i}_n \\ &(-h/2 \leq z \leq h/2) \end{aligned} \quad (2.4.6)$$

The distance between any two infinitesimally close points is given by

$$ds^2 = g_1 d\beta_1^2 + g_2 d\beta_2^2 + g_3 dz^2 \quad (2.4.7)$$

where

$$\begin{aligned} g_1 &= \left[A_1 \left(1 + \frac{z}{R_1} \right) \right]^2 \\ g_2 &= \left[A_2 \left(1 + \frac{z}{R_2} \right) \right]^2 \\ g_3 &= 1 \end{aligned} \quad (2.4.8)$$

In addition R_1 and R_2 are local curvatures of the β_1 and β_2 curves, respectively. The volume of the fundamental element is given by

$$dV = A_1 A_2 \left(1 + \frac{z}{R_1} \right) \left(1 + \frac{z}{R_2} \right) d\beta_1 d\beta_2 dz \quad (2.4.9)$$

It can also be shown from reference 2.4.2 that

$$\frac{\partial}{\partial \beta_1} \left[A_2 \left(1 + \frac{z}{R_2} \right) \right] = \left(1 + \frac{z}{R_1} \right) \frac{\partial A_2}{\partial \beta_1} \quad (2.4.10)$$

$$\frac{\partial}{\partial \beta_2} \left[A_1 \left(1 + \frac{z}{R_1} \right) \right] = \left(1 + \frac{z}{R_2} \right) \frac{\partial A_1}{\partial \beta_2}$$

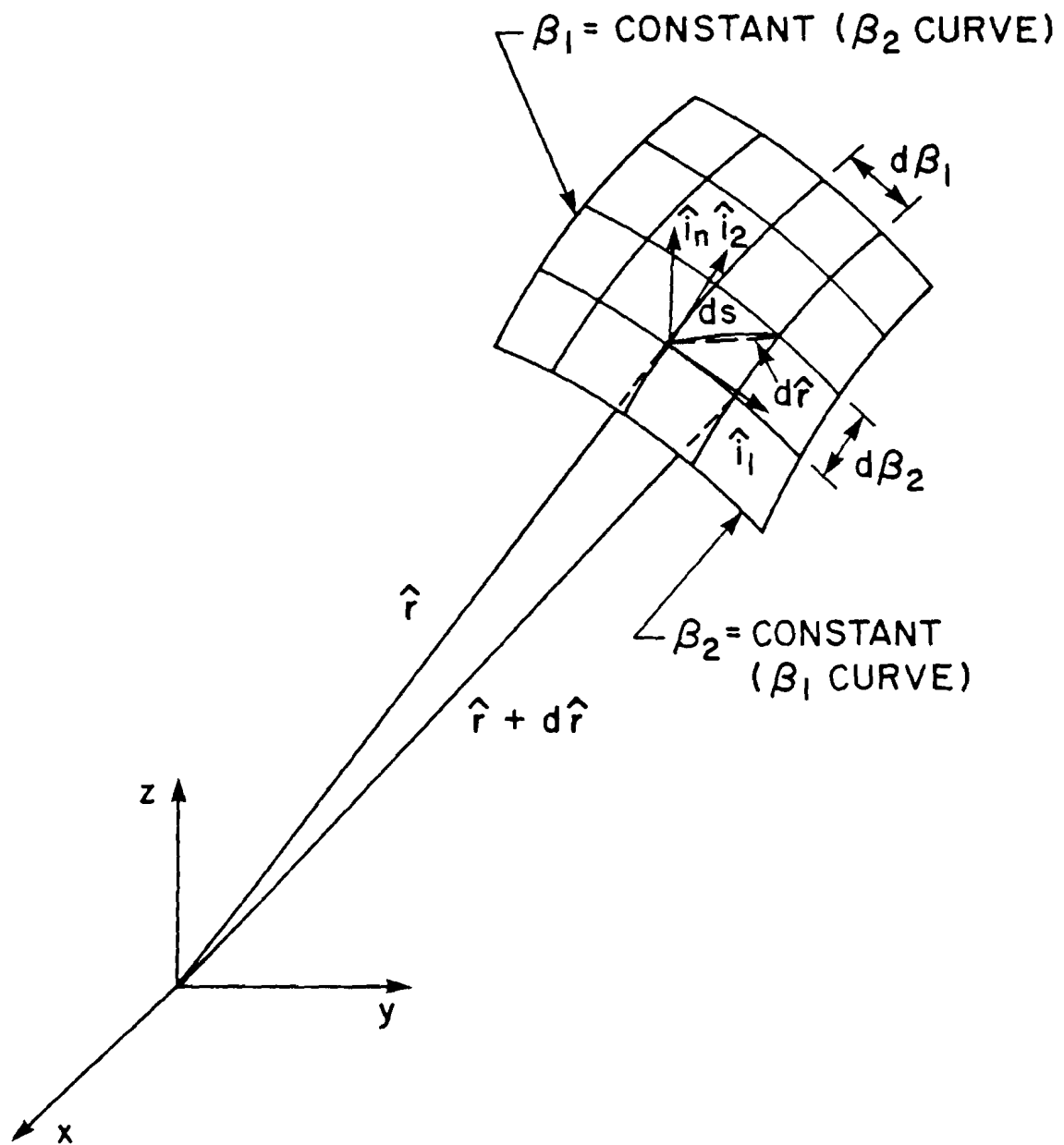


FIGURE 2.4.1 SURFACE NOMENCLATURE

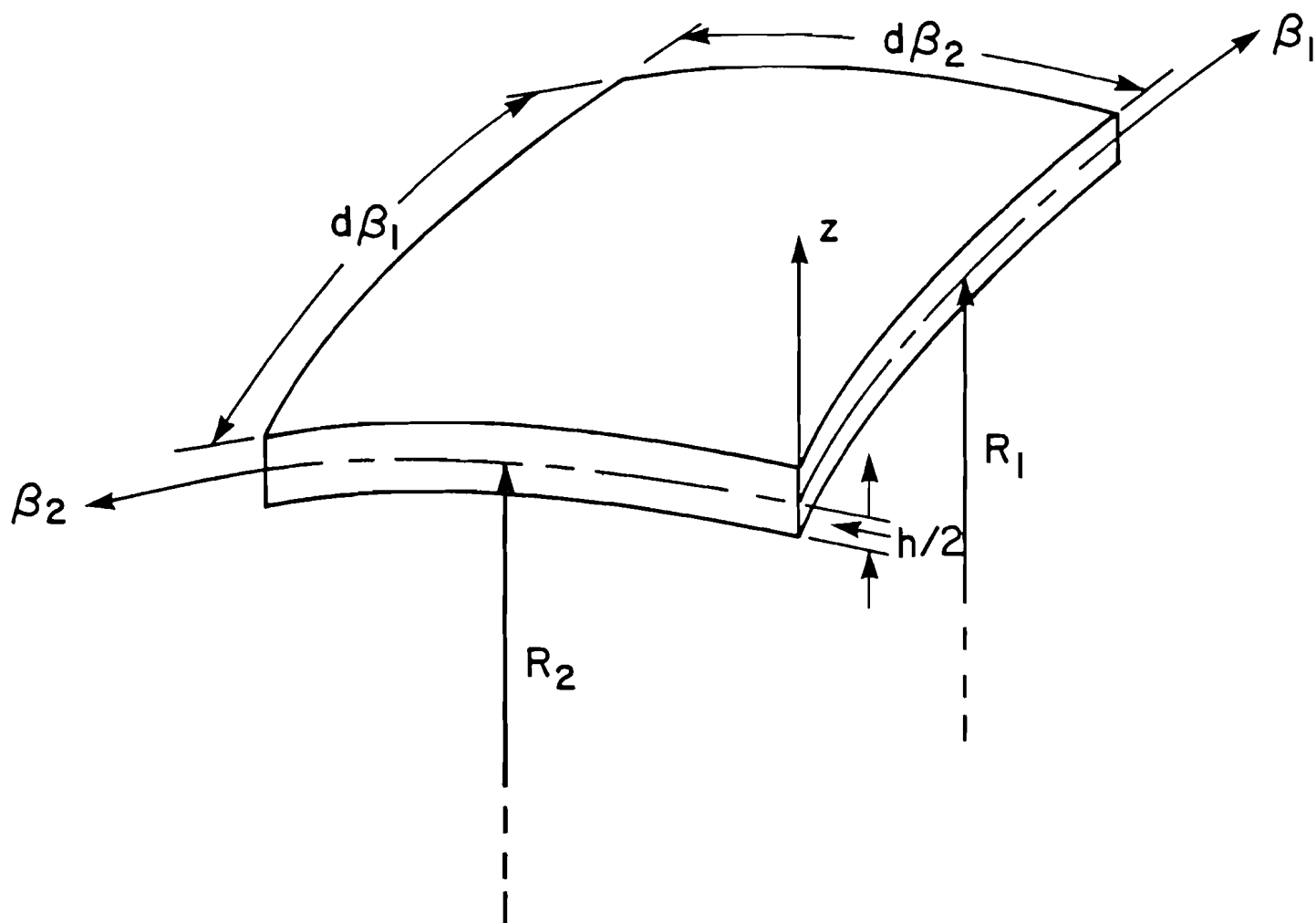


FIGURE 2.4.2 SHELL ELEMENT

Equation (2.4.10) is a consequence of the well-known Codazzi conditions (references 2.4.1-3).

2.4.2.2 Basic Assumptions.

Laminated thin shell theory is based on the following assumptions:

1. The shell is constructed of an arbitrary number of layers of orthotropic sheets bonded together. The orthotropic axes of material symmetry of an individual layer need not coincide with the β_1 - β_2 axes of the plate.
2. The shell is thin, i.e. the thickness h is much smaller than the other physical dimensions, including the smallest radius of curvature of the middle surface.
3. The displacements and strains are small, so that quantities above first order in the strain-displacement relations may be neglected.
4. Transverse shear strains $\gamma_{\beta_1 z}, \gamma_{\beta_2 z}$ are negligible.
5. Inplane displacements u and v are linear functions of the z coordinate.
6. Both the transverse normal stress σ_{zz} and strain ϵ_{zz} are negligible.
7. Each ply obeys Hooke's law.

These assumptions are consistent with Love's first approximation (reference 2.4.1) for homogeneous isotropic shells.

2.4.2.3 Strain-Displacement Relations.

The three-dimensional strain-displacement equations from classical theory of elasticity in curvilinear coordinates (reference 2.4.2) can be written in the form

$$\epsilon_{ii} = \frac{\partial}{\partial \beta_1} \left(\frac{u_i}{\sqrt{g_i}} \right) + \frac{1}{2g_i} \sum_{k=1}^3 \frac{\partial g_i}{\partial \beta_k} \frac{u_k}{\sqrt{g_k}} \quad (i = 1, 2, 3) \quad (2.4.11)$$

(no sum on i)

$$\gamma_{ij} = \frac{1}{\sqrt{g_i g_j}} \left[g_i \frac{\partial}{\partial \beta_j} \left(\frac{u_i}{\sqrt{g_i}} \right) + g_j \frac{\partial}{\partial \beta_i} \left(\frac{u_j}{\sqrt{g_j}} \right) \right] \quad (2.4.12)$$

$i, j = 1, 2, 3$
 $i \neq j$

where $\beta_3 = z$, $u_1 = u$, $u_2 = v$, and $u_3 = w$. Substituting eqs. (2.4.8) into eqs. (2.4.11,12), taking into account eqs. (2.4.10), and recognizing that $R_1 = R_1$ (α_2) and $R_2 = R_2$ (α_1), we obtain

$$\epsilon_{11} = \frac{1}{(1+z/R_1)} \left(\frac{1}{A_1} \frac{\partial u}{\partial \beta_1} + \frac{v}{A_1 A_2} \frac{\partial A_1}{\partial \beta_2} + \frac{w}{R_1} \right) \quad (2.4.13)$$

$$\epsilon_{22} = \frac{1}{(1+z/R_2)} \left(\frac{u}{A_1 A_2} \frac{\partial A_2}{\partial \beta_1} + \frac{1}{A_2} \frac{\partial v}{\partial \beta_2} + \frac{w}{R_2} \right) \quad (2.4.14)$$

$$\epsilon_{33} = \frac{\partial w}{\partial z} \quad (2.4.15)$$

$$\begin{aligned} \gamma_{12} = & \frac{A_1 (1+z/R_1)}{A_2 (1+z/R_2)} \frac{\partial}{\partial \beta_2} \left[\frac{u}{A_1 (1+z/R_1)} \right] \\ & + \frac{A_2 (1+z/R_2)}{A_1 (1+z/R_1)} \frac{\partial}{\partial \beta_1} \left[\frac{v}{A_2 (1+z/R_1)} \right] \end{aligned} \quad (2.4.16)$$

$$\gamma_{13} = \frac{1}{A_1 (1+z/R_1)} \frac{\partial w}{\partial \beta_1} + A_1 (1+z/R_1) \frac{\partial}{\partial z} \left[\frac{u}{A_1 (1+z/R_1)} \right] \quad (2.4.17)$$

$$\gamma_{23} = \frac{1}{A_2 (1+z/R_2)} \frac{\partial w}{\partial \beta_2} + A_2 (1+z/R_2) \frac{\partial}{\partial z} \left[\frac{v}{A_2 (1+z/R_2)} \right] \quad (2.4.18)$$

Applying assumption 5, we have

$$u = u^0(\beta_1, \beta_2) + z\theta_1(\beta_1, \beta_2) \quad (2.4.19)$$

$$v = v^0(\beta_1, \beta_2) + z\theta_2(\beta_1, \beta_2) \quad (2.4.20)$$

where θ_1 and θ_2 are rotations of the normal to the middle surface about the β_1 and β_2 axes, respectively. From assumption 6

$$w = w(\beta_1, \beta_2) \quad (2.4.21)$$

Substituting eqs. (2.4.19-21) into eqs. (2.4.17,18) and applying assumption 4 concerning the transverse shear strains, we find

$$\theta_1 = \frac{u^0}{R_1} - \frac{1}{A_1} \frac{\partial w}{\partial \beta_1}, \quad \theta_2 = \frac{v^0}{R_2} - \frac{1}{A_2} \frac{\partial w}{\partial \beta_2} \quad (2.4.22)$$

It is at this point that various theories differ in developing strain-displacement relations. The approach used here is that of Vlasov (reference 2.4.3). Since z/R_1 is much less than unity, we can use the following geometric series

$$\frac{1}{(1+z/R_i)} = \sum_{n=0}^{\infty} \left(\frac{z}{R_i}\right)^n, \quad i = 1, 2 \quad (2.4.23)$$

Substituting eqs. (2.4.19-23) into eqs. (2.4.13,14,16), we obtain the desired strain-displacement relations in the following form

$$\varepsilon_{11} = \varepsilon_{11}^0 + \sum_{n=1}^{\infty} \kappa_{11}(n) z^n \quad (2.4.24)$$

$$\varepsilon_{22} = \varepsilon_{22}^0 + \sum_{n=1}^{\infty} \kappa_{22}(n) z^n \quad (2.4.25)$$

$$\gamma_{12} = \gamma_{12}^0 + \sum_{n=1}^{\infty} \kappa_{12}(n) z^n \quad (2.4.26)$$

where

$$\varepsilon_{11}^0 = \frac{1}{A_1} \frac{\partial u^0}{\partial \beta_1} + \frac{v^0}{A_1 A_2} \frac{\partial A_1}{\partial \beta_2} + \frac{w}{R_1} \quad (2.4.27)$$

$$\varepsilon_{22}^0 = \frac{u^0}{A_1 A_2} \frac{\partial A_2}{\partial \beta_1} + \frac{1}{A_2} \frac{\partial v^0}{\partial \beta_2} + \frac{w}{R_2} \quad (2.4.28)$$

$$\gamma_{12}^0 = \frac{A_1}{A_2} \frac{\partial}{\partial \beta_2} \left(\frac{u^0}{A_1} \right) + \frac{A_2}{A_1} \frac{\partial}{\partial \beta_1} \left(\frac{v^0}{A_2} \right) \quad (2.4.29)$$

and if the summations in eqs. (2.4.24-26) are truncated at $n = 1$, then

$$\kappa_{11} = \frac{1}{A_1} \frac{\partial \theta_1}{\partial \beta_1} + \frac{\theta_2}{A_1 A_2} \frac{\partial A_1}{\partial \beta_2} - \frac{\varepsilon_{11}^0}{R_1} \quad (2.4.30)$$

$$\kappa_{22} = \frac{\theta_1}{A_1 A_2} \frac{\partial A_2}{\partial \beta_1} + \frac{1}{A_2} \frac{\partial \theta_2}{\partial \beta_2} - \frac{\varepsilon_{22}^0}{R_2} \quad (2.4.31)$$

$$\kappa_{12} = \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \left[\frac{A_1}{A_2} \frac{\partial}{\partial \beta_2} \left(\frac{u^0}{A_1} \right) - \frac{A_2}{A_1} \frac{\partial}{\partial \beta_1} \left(\frac{v^0}{A_2} \right) \right]$$

$$- \frac{2}{A_1 A_2} \left(\frac{\partial^2 w}{\partial \beta_1 \partial \beta_2} - \frac{1}{A_2} \frac{\partial A_1}{\partial \beta_1} \frac{\partial w}{\partial \beta_2} - \frac{1}{A_1} \frac{\partial A_1}{\partial \beta_2} \frac{\partial w}{\partial \beta_1} \right) \quad (2.4.32)$$

2.4.2.4 Constitutive Equations.

As in the case of classical plate theory, we develop constitutive equations for thin shells based on force resultants and moment resultants as illustrated in figures 2.4.3 and 2.4.4, respectively. These force and moment resultants are defined as follows:

$$\begin{pmatrix} N_{11} \\ N_{12} \\ Q_{11} \end{pmatrix} = \int_{-h/2}^{h/2} \begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{pmatrix} \left(1 + \frac{z}{R_2} \right) dz \quad (2.4.33)$$

$$\begin{pmatrix} N_{22} \\ N_{21} \\ Q_{22} \end{pmatrix} = \int_{-h/2}^{h/2} \begin{pmatrix} \sigma_{22} \\ \sigma_{12} \\ \sigma_{23} \end{pmatrix} \left(1 + \frac{z}{R_1} \right) dz \quad (2.4.34)$$

$$\begin{pmatrix} M_{11} \\ M_{12} \end{pmatrix} = \int_{-h/2}^{h/2} \begin{pmatrix} \sigma_{11} \\ \sigma_{12} \end{pmatrix} \left(1 + \frac{z}{R_2} \right) z dz \quad (2.4.35)$$

$$\begin{pmatrix} M_{22} \\ M_{21} \end{pmatrix} = \int_{-h/2}^{h/2} \begin{pmatrix} \sigma_{22} \\ \sigma_{21} \end{pmatrix} \left(1 + \frac{z}{R_1} \right) z dz \quad (2.4.36)$$

It is interesting to note that, although $\tau_{12} = \tau_{21}$, $N_{12} \neq N_{21}$ and $M_{12} \neq M_{21}$ in eqs. (2.4.33-36) unless $R_1 = R_2$.

Assumptions 4 and 5 imply that a state of plane stress exists in the kth ply of the shell. Thus,

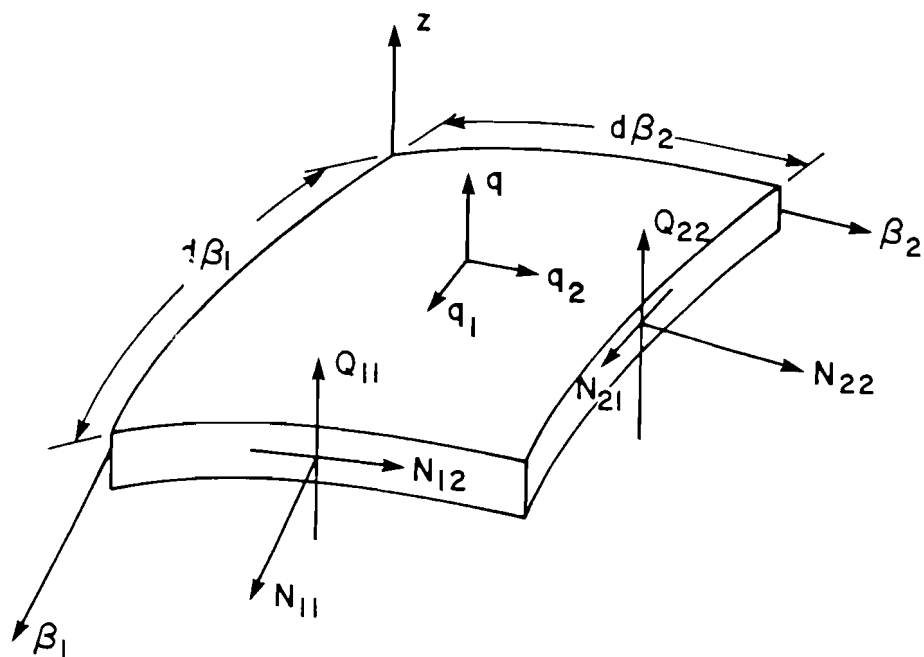


FIGURE 2.4.3 SHELL ELEMENT WITH FORCE RESULTANTS AND SURFACE STRESSES APPLIED

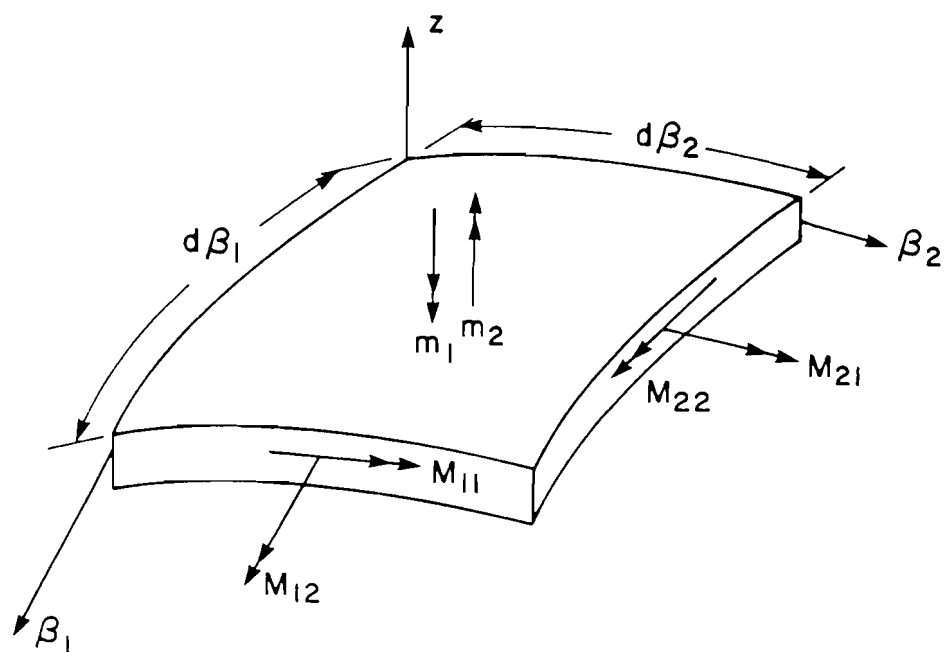


FIGURE 2.4.4 SHELL ELEMENT WITH MOMENT RESULTANTS AND SURFACE MOMENTS APPLIED

$$\begin{bmatrix} \sigma_{11}^k \\ \sigma_{22}^k \\ \tau_{12}^k \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11}^k & \bar{Q}_{12}^k & \bar{Q}_{16}^k \\ \bar{Q}_{12}^k & \bar{Q}_{22}^k & \bar{Q}_{26}^k \\ \bar{Q}_{16}^k & \bar{Q}_{26}^k & \bar{Q}_{66}^k \end{bmatrix} \begin{bmatrix} (\epsilon_{11} - \alpha_1 \Delta T) \\ (\epsilon_{22} - \alpha_2 \Delta T) \\ (\gamma_{12} - \alpha_6 \Delta T) \end{bmatrix} \quad (2.4.37)$$

If we neglect z/R_1 and z/R_2 in comparison to unity, then eqs. (2.4.33-36) can be written in the form

$$\begin{Bmatrix} N_{11}, M_{11} \\ N_{22}, M_{22} \\ N_{12}=N_{21}, M_{12}=M_{21} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_{11}^k \\ \sigma_{22}^k \\ \sigma_{12}^k \end{Bmatrix} (1, z) dz \quad (2.4.38)$$

Substituting eqs. (2.4.24-26) with $n = 1$ into eq. (2.4.41) and substituting the result into eq. (2.4.38), we obtain the shell constitutive equation

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} - \begin{bmatrix} N^T \\ M^T \end{bmatrix} \quad (2.4.39)$$

which is exactly of the same form as eq. (2.3.1) for laminated plates. The thermal force and moment resultants are the same as defined in eq. (2.3.1), i.e.

$$\{N^T, M^T\} = \int_{-h/2}^{h/2} \{\Gamma^k\} \Delta T(1, z) dz \quad (2.4.40)$$

where

$$\{\Gamma^k\} = \begin{bmatrix} \Gamma_{11}^k \\ \Gamma_{22}^k \\ \Gamma_{12}^k \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11}^k & \bar{Q}_{12}^k & \bar{Q}_{16}^k \\ \bar{Q}_{12}^k & \bar{Q}_{22}^k & \bar{Q}_{26}^k \\ \bar{Q}_{16}^k & \bar{Q}_{26}^k & \bar{Q}_{66}^k \end{bmatrix} \begin{bmatrix} \alpha_1^k \\ \alpha_2^k \\ \alpha_6^k \end{bmatrix} \quad (2.4.41)$$

2.4.2.5 Equations of Equilibrium

Integration of the equilibrium equations of classical theory of elasticity in curvilinear coordinates yields the following equations of equilibrium in terms of force and moment resultants (ref. 2.4.2)

$$\begin{aligned} \frac{\partial}{\partial \beta_1} (N_{11}A_1) + \frac{\partial}{\partial \beta_2} (N_{21}A_1) + N_{12} \frac{\partial A_1}{\partial \beta_2} - N_{22} \frac{\partial A_2}{\partial \beta_1} \\ + A_1A_2 \left(\frac{Q_{11}}{R_1} + q_1 \right) = 0 \end{aligned} \quad (2.4.42)$$

$$\begin{aligned} \frac{\partial}{\partial \beta_1} (N_{12}A_2) + \frac{\partial}{\partial \beta_2} (N_{22}A_1) + N_{21} \frac{\partial A_2}{\partial \beta_1} - N_{11} \frac{\partial A_1}{\partial \beta_2} \\ + A_1A_2 \left(\frac{Q_{22}}{R_2} + q_2 \right) = 0 \end{aligned} \quad (2.4.43)$$

$$\frac{\partial}{\partial \beta_1} (Q_{11}A_2) + \frac{\partial}{\partial \beta_2} (Q_{22}A_1) - \left(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} - q \right) A_1A_2 = 0 \quad (2.4.44)$$

$$\begin{aligned} \frac{\partial}{\partial \beta_1} (M_{11}A_2) + \frac{\partial}{\partial \beta_2} (M_{21}A_1) + M_{12} \frac{\partial A_1}{\partial \beta_2} - M_{22} \frac{\partial A_2}{\partial \beta_1} \\ - A_1A_2 (Q_{11} - m_1) = 0 \end{aligned} \quad (2.4.45)$$

$$\begin{aligned} \frac{\partial}{\partial \beta_1} (M_{12}A_2) + \frac{\partial}{\partial \beta_2} (M_{22}A_1) + M_{21} \frac{\partial A_2}{\partial \beta_1} - M_{11} \frac{\partial A_1}{\partial \beta_2} \\ - A_1A_2 (Q_{22} - m_2) = 0 \end{aligned} \quad (2.4.46)$$

where q_1 and q_2 are surface shear tractions in the β_1 and β_2 directions, respectively, q is the normal pressure on the shell surface (see figure 2.4.3), and m_1 and m_2 are moments due to the surface shear tractions. Equations (2.4.45,46) can be solved for Q_{11} and Q_{22} and the results substituted into eq. (2.4.44), reducing the number of equilibrium equations to three. Substituting eq. (2.4.39) into the modified equilibrium equations and taking into account the strain-displacement relations, eqs. (2.4.27-32), we can obtain a set of three equilibrium equations in terms of the displacements u^0 , v^0 , w , and the externally applied stresses and moments. These equations are obviously quite complex and will not be presented.

The proper boundary conditions which assure a unique solution are very similar to those of classical plate theory. One member of each pair of four quantities must be prescribed along the boundary of the shell (reference 2.4.2). On an edge of constant β_1 :

$$N_{11}, u^0; \quad N_{12} + \frac{M_{12}}{R_2}, v^0; \quad Q_{11} + \frac{1}{A_2} \frac{\partial M_{12}}{\partial \beta_2}, w; \quad M_{11}, \theta_1 \quad (2.4.47)$$

and on an edge of constant β_2 :

$$N_{22}, v^0; \quad N_{21} + \frac{M_{21}}{R_1}, u^0; \quad Q_{22} + \frac{1}{A_1} \frac{\partial M_{21}}{\partial \beta_1}, w; \quad M_{22}, \theta_2 \quad (2.4.48)$$

2.4.2.6 Circular Cylindrical Shells.

We now consider a circular cylindrical shell with a coordinate system as illustrated in figure 2.4.5. In this case

$$\beta_1 = x, \quad \beta_2 = R\theta = s, \quad A_1 = A_2 = 1$$

$$1/R_1 = 0, \quad R_2 = R \quad (2.4.49)$$

The inplane strains and curvatures, eqs. (2.4.27-31), now become

$$\epsilon_{xx}^0 = \frac{\partial u^0}{\partial x}, \quad \epsilon_{ss}^0 = \frac{\partial v^0}{\partial s} + \frac{w}{R}, \quad \gamma_{xs}^0 = \frac{\partial v^0}{\partial x} + \frac{\partial u^0}{\partial s} \quad (2.4.50)$$

$$\begin{aligned} \kappa_{xx} &= -\frac{\partial^2 w}{\partial x^2}, \quad \kappa_{ss} = -\frac{\partial^2 w}{\partial s^2} - \frac{w}{R^2}, \quad \kappa_{xs} = -2 \frac{\partial^2 w}{\partial x \partial s} \\ &- \frac{1}{R} \left(\frac{\partial u^0}{\partial s} - \frac{\partial v^0}{\partial x} \right), \quad \theta_1 = -\frac{\partial w}{\partial x}, \quad \theta_2 = \frac{v^0}{R} - \frac{\partial w}{\partial s} \end{aligned} \quad (2.4.51)$$

and the equilibrium eqs.(2.4.42-46) reduce to

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{sx}}{\partial s} + q_x = 0 \quad (2.4.52)$$

$$\frac{\partial N_{xs}}{\partial x} + \frac{\partial N_{ss}}{\partial s} + \frac{Q_{ss}}{R} + q_s = 0 \quad (2.4.53)$$

$$\frac{\partial Q_{xx}}{\partial x} + \frac{\partial Q_{ss}}{\partial s} - \frac{N_{ss}}{R} + q = 0 \quad (2.4.54)$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{sx}}{\partial s} - Q_{xx} + m_x = 0 \quad (2.4.55)$$

$$\frac{\partial M_{xs}}{\partial x} + \frac{\partial M_{yy}}{\partial s} - Q_{ss} + m_s = 0 \quad (2.4.56)$$

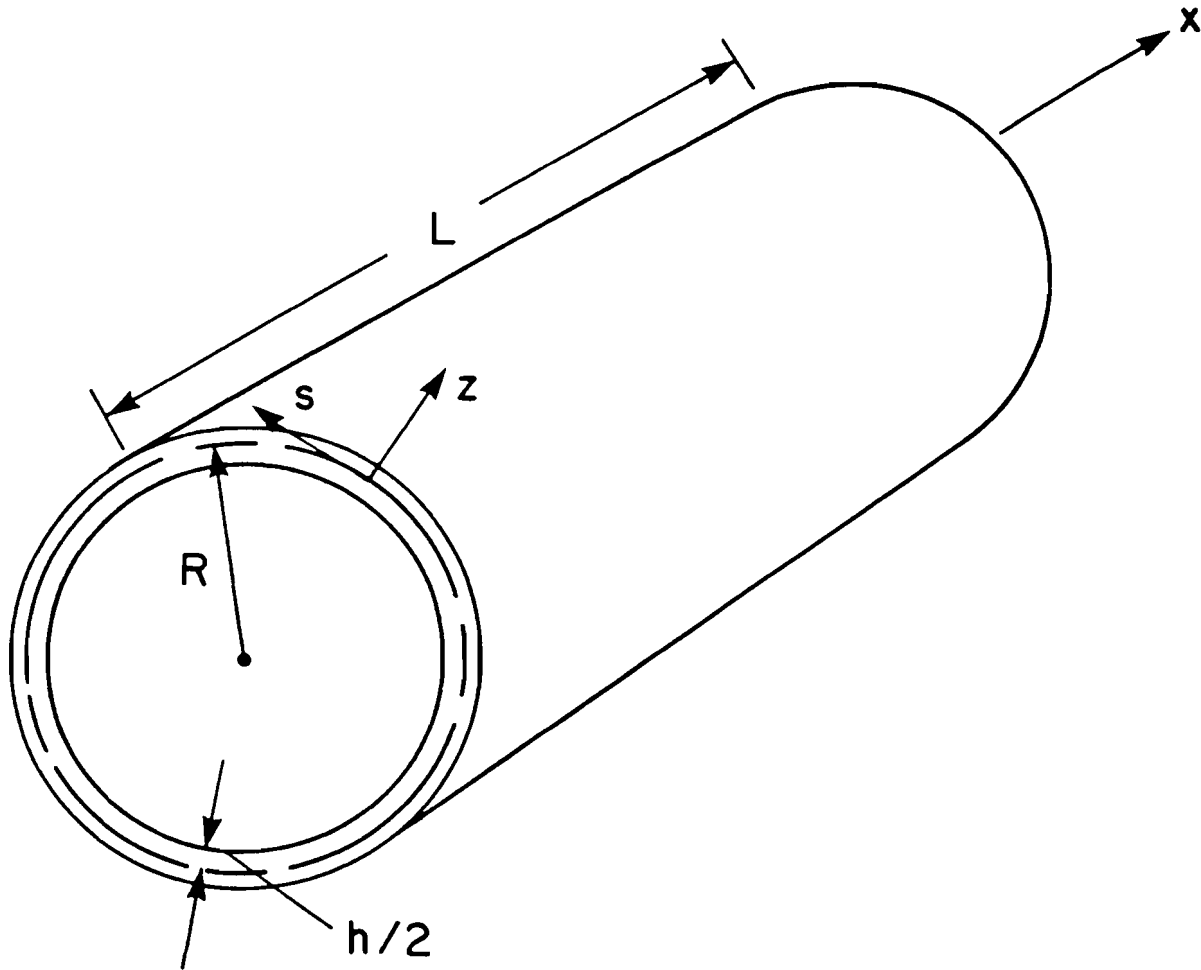


FIGURE 2.4.5 COORDINATE SYSTEM FOR CIRCULAR CYLINDRICAL SHELLS

Differentiating eqs. (2.4.55,56) with respect to x and s , respectively, solving for $\partial Q_{xx}/\partial x$ and $\partial Q_{ss}/\partial s$, and substituting the results into eq. (2.4.54), we obtain

$$\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial^2 M_{sx}}{\partial x \partial y} + \frac{\partial^2 M_{xs}}{\partial x \partial y} - \frac{N_{ss}}{R} + \frac{\partial m_x}{\partial x} + \frac{\partial m_s}{\partial s} + q = 0 \quad (2.4.57)$$

For cylindrical shells

$$q_x = \sigma_{xz}(h/2) - \sigma_{xz}(-h/2)$$

$$q_s = \sigma_{sz}(h/2) - \sigma_{sz}(-h/2)$$

$$q = \sigma_{zz}(h/2) - \sigma_{zz}(-h/2)$$

$$m_x = \frac{h}{2} [\sigma_{xz}(h/2) + \sigma_{xz}(-h/2)]$$

$$m_s = \frac{h}{2} [\sigma_{sz}(h/2) + \sigma_{sz}(-h/2)]$$

The strain-displacement relations and the equilibrium equations can be further simplified by applying the assumptions of Donnell (reference 2.4.4). The first assumption involves the transverse shear resultant in eq. (2.4.53). In particular, it is assumed that this term makes negligible contribution to equilibrium in the circumferential direction. This assumption will improve in accuracy with decreasing h/R ratios. Recognizing that the constitutive equations (2.4.39) imply $N_{xs} = N_{sx}$ and $M_{xs} = M_{sx}$, the equilibrium equations now become

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xs}}{\partial s} + q_x = 0 \quad (2.4.58)$$

$$\frac{\partial N_{xs}}{\partial s} + \frac{\partial N_{ss}}{\partial s} + q_s = 0 \quad (2.4.59)$$

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial^2 M_{xs}}{\partial x \partial s} - \frac{N_{ss}}{R} + \frac{\partial m_x}{\partial x} + \frac{\partial m_s}{\partial s} + q = 0 \quad (2.4.60)$$

It is also assumed that the middle plane displacements have negligible effect on the curvatures. Thus, the kinematics of the shell now become

$$u = u^0 - z \frac{\partial w}{\partial x} \quad (2.4.61)$$

$$v = v^0 - z \left(\frac{\partial w}{\partial s} - \frac{v^0}{R} \right) \quad (2.4.62)$$

$$\varepsilon_{xx} = \varepsilon_{xx}^0 + z \kappa_{xx} \quad (2.4.63)$$

$$\varepsilon_{ss} = \varepsilon_{ss}^0 + z \kappa_{ss} \quad (2.4.64)$$

$$\gamma_{xs} = \gamma_{xs}^0 + 2z \kappa_{xs} \quad (2.4.65)$$

where ε_{xx}^0 , ε_{ss}^0 , and γ_{xs}^0 are defined by eq. (2.4.50), and

$$\kappa_{xx} = -\frac{\partial^2 w}{\partial x^2}, \quad \kappa_{ss} = -\frac{\partial^2 w}{\partial s^2}, \quad \kappa_{xs} = -\frac{\partial^2 w}{\partial x \partial s} \quad (2.4.66)$$

Substituting eq. (2.4.39) into eqs. (2.4.58-60) and taking eqs. (2.4.50,66) into account, we obtain the governing equations in terms of displacements:

$$\begin{aligned} & A_{11} \frac{\partial^2 u^0}{\partial x^2} + 2A_{16} \frac{\partial^2 u^0}{\partial x \partial s} + A_{66} \frac{\partial^2 u^0}{\partial s^2} + A_{16} \frac{\partial^2 v^0}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 v^0}{\partial x \partial s} \\ & + A_{26} \frac{\partial^2 v^0}{\partial s^2} + \frac{A_{12}}{R} \frac{\partial w}{\partial x} + \frac{A_{26}}{R} \frac{\partial w}{\partial s} - B_{11} \frac{\partial^3 w}{\partial x^3} - 3B_{16} \frac{\partial^3 w}{\partial x^2 \partial s} \\ & - (B_{12} + 2B_{66}) \frac{\partial^3 w}{\partial x \partial s^2} - B_{26} \frac{\partial^3 w}{\partial s^3} - \frac{\partial N_{xx}^T}{\partial x} - \frac{\partial N_{xs}^T}{\partial s} + q_x = 0 \end{aligned} \quad (2.4.67)$$

$$\begin{aligned} & A_{16} \frac{\partial^2 u^0}{\partial x^2} + (A_{12} + A_{66}) \frac{\partial^2 u^0}{\partial x \partial s} + A_{26} \frac{\partial^2 u^0}{\partial s^2} + A_{66} \frac{\partial^2 v^0}{\partial x^2} + 2A_{26} \frac{\partial^2 v^0}{\partial x \partial s} \\ & + A_{22} \frac{\partial^2 v^0}{\partial s^2} + \frac{A_{26}}{R} \frac{\partial w}{\partial x} + \frac{A_{22}}{R} \frac{\partial w}{\partial s} - B_{16} \frac{\partial^3 w}{\partial x^3} - (B_{12} + 2B_{66}) \frac{\partial^3 w}{\partial x^2 \partial s} \\ & - 3B_{26} \frac{\partial^3 w}{\partial x \partial s^2} - B_{22} \frac{\partial^3 w}{\partial s^3} - \frac{\partial N_{xs}^T}{\partial x} - \frac{\partial N_{ss}^T}{\partial s} + q_s = 0 \end{aligned} \quad (2.4.68)$$

$$\begin{aligned} & \frac{A_{12}}{R} \frac{\partial u^0}{\partial x} + \frac{A_{26}}{R} \frac{\partial u^0}{\partial s} - B_{11} \frac{\partial^3 u^0}{\partial x^3} - 3B_{16} \frac{\partial^3 u^0}{\partial x^2 \partial s} - (B_{12} + 2B_{66}) \frac{\partial^3 u^0}{\partial x \partial s^2} \\ & - B_{26} \frac{\partial^3 u^0}{\partial s^3} + \frac{A_{26}}{R} \frac{\partial v^0}{\partial x} + \frac{A_{22}}{R} \frac{\partial v^0}{\partial s} - B_{16} \frac{\partial^3 v^0}{\partial x^3} - (B_{12} + 2B_{66}) \frac{\partial^3 v^0}{\partial x^2 \partial s} \end{aligned}$$

$$\begin{aligned}
& - 3B_{26} \frac{\partial^3 v^0}{\partial x \partial s^2} - B_{22} \frac{\partial^3 v^0}{\partial s^3} + D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial s} \\
& + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial s^2} + 4D_{26} \frac{\partial^4 w}{\partial x \partial s^3} + D_{22} \frac{\partial^4 w}{\partial s^4} - \frac{B_{12}}{R} \frac{\partial^2 w}{\partial x \partial s} \\
& - 4 \frac{B_{26}}{R} \frac{\partial^2 w}{\partial x \partial s} - 2 \frac{B_{22}}{R} \frac{\partial^2 w}{\partial s^2} + A_{22} \frac{w}{R^2} + \frac{\partial^2 M_{xx}^T}{\partial x^2} + 2 \frac{\partial^2 M_{xs}^T}{\partial x \partial s} \\
& + \frac{\partial M_{ss}^T}{\partial s^2} - \frac{\partial m_x}{\partial x} - \frac{\partial m_s}{\partial s} - q = 0
\end{aligned} \tag{2.4.69}$$

These are the static equations in terms of displacements for a laminated, anisotropic shell based on Vlasov-Donnell assumptions. Note that eqs. (2.4.67-69) do not uncouple for symmetric laminates due to the effect of curvature. If $1/R = 0$, these equations reduce to eqs. (2.3.9-13) for laminated plates.

2.4.3 Static Elastic Behavior of Cylindrical Shells.

We now consider axisymmetric deformations of circular cylindrical shells. Two cases are considered. The first case involves long cylinders in which end effects are not considered. In the second case the length of the zone of end effects is determined by considering boundary conditions in conjunction with a semi-infinite cylinder.

2.4.3.1 Axisymmetric Deformation of a Long Cylinder.

A circular cylindrical shell under simple inplane loading represents a problem of considerable engineering interest. Pressure vessels or laboratory tubular test specimens are two examples of practical applications of cylindrical shells.

Let us consider a circular cylindrical shell under simple inplane loading. The deformation in this case is axisymmetric, i.e.

$$u^0 = u^0(x), \quad v^0 = v^0(x), \quad w = w(x) \tag{2.4.70}$$

For the case in which the tube is clamped at both ends, but in such a manner that it is free to rotate and extend (or contract) at $x = L$, the following boundary conditions are appropriate:

at $x = 0$

$$u^0 = v^0 = w = \frac{dw}{dx} = 0 \tag{2.4.71}$$

at $x = L$

$$N_{xx} = \text{constant}, \quad N_{xs} = \text{constant}, \quad w = \frac{dw}{dx} = 0 \quad (2.4.72)$$

Let us assume that N_{xx} , N_{ss} , and N_{xs} are constant and that the curvatures vanish. In the absence of surface shear tractions, the first two equilibrium equations, (2.4.58,59), are exactly satisfied by constant force resultants. The constitutive equations (2.4.39), in the absence of thermal forces and moments, become

$$\{N\} = [A]\{\epsilon^0\} \quad (2.4.73)$$

$$\{M\} = [B]\{\epsilon^0\} \quad (2.4.74)$$

Since the force resultants are constant, eq. (2.4.73) implies that the middle plane strains are also constant. Thus, from eq. (2.4.74) the moments must also be constant. The third equilibrium equation, (2.4.60) yields

$$N_{ss} = Rq \quad (2.4.75)$$

We now write eq. (2.4.73) in the inverted form

$$\{\epsilon^0\} = [A^*]\{N\} \quad (2.4.76)$$

where $A^* = A^{-1}$ (inverse of A matrix). Using this relationship in conjunction with eq. (2.4.74), we obtain

$$\{M\} = [-(B^*)^T]\{N\} \quad (2.4.77)$$

where B^* is the same as defined in eq. (2.3.16), i.e. $B^* = A^{-1}B$, and superscript T implies transpose.

Applying the axisymmetric assumption in conjunction with the condition of constant strain, we obtain the following strain-displacement relations from eq. (2.4.50)

$$\epsilon_{xx}^0 = \frac{du^0}{dx} = c_1, \quad \epsilon_{ss}^0 = \frac{w}{R} = c_2, \quad \gamma_{xs}^0 = \frac{dv^0}{dx} = c_3 \quad (2.4.78)$$

Integrating ϵ_{xx}^0 and γ_{xs}^0 from these relations, and applying the boundary conditions, eq. (2.4.71), we obtain

$$u^0 = c_1 x, \quad v^0 = c_3 x \quad (2.4.79)$$

In addition we see from eq. (2.4.78) that

$$w = Rc_2 \quad (2.4.80)$$

This result does not satisfy the required boundary condition concerning w . If, however, the tube is of sufficient length, this is of little consequence to the results obtained some distance from the constrained end. The exact value of this distance will be given consideration in the next section.

Using eq. (2.4.77), the total torque T applied to the tube is given by

$$T = 2\pi R(RN_{xs} + M_{xs}) = 2\pi R[RN_{xs} - B_{16}^* N_x - B_{26}^* Rq]$$

or

$$N_{xs} = \frac{B_{16}^*}{R} N_x + \frac{B_{26}^*}{R} Rq + \frac{T}{2\pi R^2} \quad (2.4.81)$$

We now use the notation

$$N_x = N_0, \quad q = p = \text{internal pressure} \quad (2.4.82)$$

Combining eqs. (2.4.76,78,81,82), we obtain

$$\begin{aligned} \epsilon_{xx}^0 = c_1 &= \left(A_{11}^* + A_{16}^* \frac{B_{16}^*}{R} \right) N_0 + \left(A_{12}^* + A_{16}^* \frac{B_{26}^*}{R} \right) Rp \\ &+ \frac{A_{16}^*}{2\pi R^2} T \end{aligned} \quad (2.4.83)$$

$$\begin{aligned} \epsilon_{ss}^0 = c_2 &= \left(A_{12}^* + A_{26}^* \frac{B_{16}^*}{R} \right) N_0 + \left(A_{22}^* + A_{26}^* \frac{B_{26}^*}{R} \right) Rp \\ &+ \frac{A_{26}^*}{2\pi R^2} T \end{aligned} \quad (2.4.84)$$

$$\begin{aligned} \gamma_{xs}^0 &= \left(A_{16}^* + A_{66}^* \frac{B_{16}^*}{R} \right) N_0 + \left(A_{26}^* + A_{66}^* \frac{B_{26}^*}{R} \right) Rp \\ &+ \frac{A_{66}^*}{2\pi R^2} T \end{aligned} \quad (2.4.85)$$

Applying eq. (2.4.37), in the absence of thermal strains, in conjunction with eqs. (2.4.63-65) and eqs. (2.4.83-85), we obtain the stresses for the k th layer.

$$\begin{aligned}
\sigma_{xx}^k = & \left[\bar{Q}_{11}^k \left(A_{11}^* + A_{16}^* \frac{B_{16}^*}{R} \right) + \bar{Q}_{12}^k \left(A_{12}^* + A_{26}^* \frac{B_{16}^*}{R} \right) \right. \\
& + \left. \bar{Q}_{16}^k \left(A_{16}^* + A_{66}^* \frac{B_{16}^*}{R} \right) \right] N_0 + \left[\bar{Q}_{11}^k \left(A_{12}^* + A_{16}^* \frac{B_{26}^*}{R} \right) \right. \\
& + \left. \bar{Q}_{12}^k \left(A_{22}^* + A_{26}^* \frac{B_{26}^*}{R} \right) + \bar{Q}_{16}^k \left(A_{26}^* + A_{66}^* \frac{B_{26}^*}{R} \right) \right] R_P \\
& + (\bar{Q}_{11}^k A_{16}^* + \bar{Q}_{12}^k A_{26}^* + \bar{Q}_{16}^k A_{66}^*) \frac{T}{2\pi R^2}
\end{aligned} \tag{2.4.86}$$

$$\begin{aligned}
\sigma_{ss}^k = & \left[\bar{Q}_{12}^k \left(A_{11}^* + A_{16}^* \frac{B_{16}^*}{R} \right) + \bar{Q}_{22}^k \left(A_{12}^* + A_{26}^* \frac{B_{16}^*}{R} \right) \right. \\
& + \left. \bar{Q}_{16}^k \left(A_{16}^* + A_{66}^* \frac{B_{16}^*}{R} \right) \right] N_0 + \left[\bar{Q}_{12}^k \left(A_{12}^* + A_{16}^* \frac{B_{26}^*}{R} \right) \right. \\
& + \left. \bar{Q}_{22}^k \left(A_{22}^* + A_{26}^* \frac{B_{26}^*}{R} \right) + \bar{Q}_{26}^k \left(A_{26}^* + A_{66}^* \frac{B_{26}^*}{R} \right) \right] R_P \\
& + (\bar{Q}_{12}^k A_{16}^* + \bar{Q}_{22}^k A_{26}^* + \bar{Q}_{16}^k A_{66}^*) \frac{T}{2\pi R^2}
\end{aligned} \tag{2.4.87}$$

$$\begin{aligned}
\tau_{xs}^k = & \left[\bar{Q}_{16}^k \left(A_{11}^* + A_{16}^* \frac{B_{16}^*}{R} \right) + \bar{Q}_{26}^k \left(A_{12}^* + A_{26}^* \frac{B_{16}^*}{R} \right) \right. \\
& + \left. \bar{Q}_{66}^k \left(A_{16}^* + A_{66}^* \frac{B_{16}^*}{R} \right) \right] N_0 + \left[\bar{Q}_{16}^k \left(A_{12}^* + A_{16}^* \frac{B_{26}^*}{R} \right) \right. \\
& + \left. \bar{Q}_{26}^k \left(A_{22}^* + A_{26}^* \frac{B_{26}^*}{R} \right) + \bar{Q}_{66}^k \left(A_{26}^* + A_{66}^* \frac{B_{26}^*}{R} \right) \right] R_P \\
& + (\bar{Q}_{16}^k A_{16}^* + \bar{Q}_{26}^k A_{26}^* + \bar{Q}_{66}^k A_{66}^*) \frac{T}{2\pi R^2}
\end{aligned} \tag{2.4.88}$$

For symmetric laminates a cursory examination of eqs. (2.4.86-88) shows that components of the A^* matrix can be characterized by applying combinations of axial load, internal pressure, and pure torque. As a result, all of the engineering constants can be determined by testing a thin tube.

Some interesting phenomena can occur in tubular specimens. Consider, for example, if both ends of the tube are fixed against rotation then the boundary condition eq. (2.4.72) would become

at $x = L$

$$N_{xx} = \text{constant}, \quad v^0 = w = \frac{dw}{dx} = 0 \quad (2.4.89)$$

Under these conditions $v^0 = 0$, and the constitutive eq. (2.4.73) would take the form

$$\begin{bmatrix} N_{xx} \\ N_{ss} \\ N_{xs} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{xx}^0 \\ \epsilon_{ss}^0 \\ 0 \end{bmatrix} \quad (2.4.90)$$

From this relationship we obtain

$$\epsilon_{xx}^0 = \frac{1}{A_{11}A_{22} - A_{12}^2} (A_{22}N_{xx} - A_{12}N_{ss}) \quad (2.4.91)$$

$$\epsilon_{ss}^0 = \frac{1}{A_{11}A_{22} - A_{12}^2} (-A_{12}N_{xx} + A_{22}N_{ss}) \quad (2.4.92)$$

$$N_{xs} = \frac{1}{A_{11}A_{22} - A_{12}^2} [(A_{16}A_{22} - A_{26}A_{12})N_{xx} + (A_{26}A_{22} - A_{16}A_{12})N_{ss}] \quad (2.4.93)$$

Thus, any fully-clamped tube in which the A_{16} and A_{26} inplane stiffness terms do not vanish induces a shear stress. This can result in shear buckling under any combination of axial load and internal pressure. For example, such tubes under an axial tensile load can lead to buckling (reference 2.4.5). Thus, the free rotation at the end $x = L$, as prescribed by the boundary condition, eq. (2.4.72), prevents such buckling.

2.4.3.2 End Effects Under Axisymmetric Loading.

For simplicity let us consider a symmetrically laminated cylinder in which the shear coupling terms A_{16} and A_{26} vanish. For axisymmetric deformation, eqs. (2.4.67-69), in the absence of thermal loads and surface shear tractions, become

$$A_{11} \frac{d^2 u^0}{dx^2} + \frac{A_{12}}{R} \frac{dw}{dx} = 0 \quad (2.4.94)$$

$$A_{66} \frac{d^2 v^0}{dx^2} = 0 \quad (2.4.95)$$

$$\frac{A_{12}}{R} \frac{du^0}{dx} + D_{11} \frac{d^4 w}{dx^4} + A_{22} \frac{w}{R^2} - q = 0 \quad (2.4.96)$$

Integrating eqs. (2.4.94,95) in conjunction with the constitutive relations, eq. (2.4.39), we obtain

$$A_{11} \frac{du^0}{dx} + A_{12} \frac{w}{R} = N_x = \text{constant} = N_0 \quad (2.4.97)$$

$$A_{66} \frac{dv^0}{dx} = N_{xs} = \text{constant} = S \quad (2.4.98)$$

In order to determine the zone of end influence, we consider a semi-infinite shell clamped at $x = 0$. Thus, the boundary conditions at $x = 0$ as given by eq. (2.4.71) are applicable. Integrating eq. (2.4.98) and applying eq. (2.4.71) we find

$$v^0 = Sx \quad (2.4.99)$$

Thus for balanced ($A_{16} = A_{26} = 0$), symmetric ($B_{1j} = 0$) laminates, the solution for v^0 is unaffected by end attachments. Equation (2.4.99) gives a pure torsional mode and assumes that the cylinder is free to rotate for $x > 0$. Substituting eq. (2.4.97) into eq. (2.4.96) and letting $q = p$ (internal pressure), we obtain the differential equation

$$\frac{d^4 w}{dx^4} + \frac{(A_{11}A_{22} - A_{12}^2)}{D_{11}A_{11}R^2} w = \frac{A_{11}Rp - A_{12}N_0}{D_{11}A_{11}R} \quad (2.4.100)$$

A particular solution, w_p , to eq. (2.4.100) is of the form

$$w_p = \frac{R(A_{11}Rp - A_{12}N_0)}{(A_{11}A_{22} - A_{12}^2)} \quad (2.4.101)$$

Note that eq. (2.4.101) is identical to eq. (2.4.80) for balanced, symmetric laminates. The homogeneous solution, w_h , to eq. (2.4.100) is of the form

$$w_h = e^{-\lambda x} (c_1 \sin \lambda x + c_2 \cos \lambda x) + e^{\lambda x} (c_3 \sin \lambda x + c_4 \cos \lambda x) \quad (2.4.102)$$

where

$$\lambda^2 = \frac{1}{2R} \sqrt{\frac{A_{11}A_{22}-A_{12}^2}{D_{11}A_{11}}} \quad (2.4.103)$$

For the case of a semi-infinite cylinder, w must be bounded. Thus,

$$c_3 = c_4 = 0 \quad (2.4.104)$$

and the solution becomes

$$w = e^{-\lambda x} (c_1 \sin \lambda x + c_2 \cos \lambda x) + \frac{R(A_{11}R_p - A_{12}N_0)}{(A_{11}A_{22} - A_{12}^2)} \quad (2.4.105)$$

The coefficients c_1 and c_2 are determined by applying conditions on w from eq. (2.4.71), i.e.

$$w(0) = \frac{dw(0)}{dx} = 0 \quad (2.4.106)$$

Combining eqs. (2.4.105,106) we obtain

$$c_1 = c_2 = - \frac{R(A_{11}R_p - A_{12}N_0)}{(A_{11}A_{22} - A_{12}^2)}$$

and

$$w = \frac{R(A_{11}R_p - A_{12}N_0)}{(A_{11}A_{22} - A_{12}^2)} [1 - e^{-\lambda x} (\sin \lambda x + \cos \lambda x)] \quad (2.4.107)$$

A cursory examination of eq. (2.4.107) reveals that the value of λ determines the distance over which end effects are dissipated.

Integrating eq. (2.4.97) along with the requirement that $u^0(0) = 0$, we obtain

$$u^0 = \frac{1}{(A_{11}A_{22} - A_{12}^2)} \left[(A_{22}N_0 - A_{12}R_p)x + \frac{A_{12}}{A_{11}} (A_{12}N_0 - A_{11}R_p) (1 - e^{-\lambda x} \cos \lambda x) \right] \quad (2.4.108)$$

Using eq. (2.4.103) let us rewrite the argument of $\exp(\lambda x)$ in the form

$$\lambda x = \left[\sqrt{\frac{R}{2h}} \left(\frac{\bar{A}_{11}\bar{A}_{22} - \bar{A}_{12}^2}{\bar{D}_{11}\bar{A}_{11}} \right)^{1/4} \right] \frac{x}{R} \quad (2.4.109)$$

where $\bar{A}_{ij} = A_{ij}/h$ and $\bar{D}_{11} = D_{11}/h^3$. If w is now plotted as a function of x/R , then the distance over which the end effect dissipates will decrease with increasing values of R/h . This is illustrated in figure 2.4.6 where the transverse displacement w of a $[+45]_s$ tube subjected to an axial load N_0 is plotted as a function of x/R for three values of R/h . A similar plot is shown in figure 2.4.7 for the bending moment M_x which can be determined from the constitutive relations (2.4.39) in conjunction with eq. (2.4.107). In particular

$$M_x = -D_{11} \frac{d^2 w}{dx^2} = A_{12} N_0 \sqrt{\frac{D_{11}}{A_{11}(A_{11}A_{22} - A_{12}^2)}} e^{-\lambda x} (\cos \lambda x - \sin \lambda x) \quad (2.4.110)$$

The ply properties used in figures 2.4.6 and 2.4.7 are as follows:

$$\begin{aligned} E_1 &= 25 \times 10^6 \text{ psi}, & E_2 &= 1.7 \times 10^6 \text{ psi}, \\ \nu_{12} &= 0.3, & G_{12} &= 0.65 \times 10^6 \text{ psi} \end{aligned} \quad (2.4.111)$$

2.4.4 Vibration and Buckling of Cylindrical Shells.

Vibration and buckling analysis of circular cylindrical shells can be performed by modifying eqs. (2.4.67-69) to include inertia terms and initial inplane force effects.

In the vibration analysis the flexural vibration mode is of primary concern. For thin shells the inplane inertia terms can be neglected as a first estimate of the fundamental flexural vibration frequencies. In Donnell's approximation (reference 2.4.4) inplane force effects are reflected only in the third equilibrium equation. Thus, vibration and buckling analysis can be performed by using the following modification of eq. (2.4.69):

$$\begin{aligned} &\frac{A_{12}}{R} \frac{\partial u^0}{\partial x} + \frac{A_{26}}{R} \frac{\partial u^0}{\partial s} - B_{11} \frac{\partial^3 u^0}{\partial x^3} - 3B_{16} \frac{\partial^3 u^0}{\partial x^2 \partial s} - (B_{12} + 2B_{66}) \frac{\partial^3 u^0}{\partial x \partial s^2} \\ &- B_{26} \frac{\partial^3 u^0}{\partial s^3} + \frac{A_{26}}{R} \frac{\partial v^0}{\partial x} + \frac{A_{22}}{R} \frac{\partial v^0}{\partial s} - B_{16} \frac{\partial^3 v^0}{\partial x^3} - (B_{12} + 2B_{66}) \frac{\partial^3 v^0}{\partial x^2 \partial s} \\ &- 3B_{26} \frac{\partial^3 v^0}{\partial x \partial s^2} - B_{22} \frac{\partial^3 v^0}{\partial s^3} + D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial s} \\ &+ 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial s^2} + 4D_{26} \frac{\partial^4 w}{\partial x \partial s^3} + D_{22} \frac{\partial^4 w}{\partial s^4} - \frac{B_{12}}{R} \frac{\partial^2 w}{\partial x^2} \end{aligned}$$

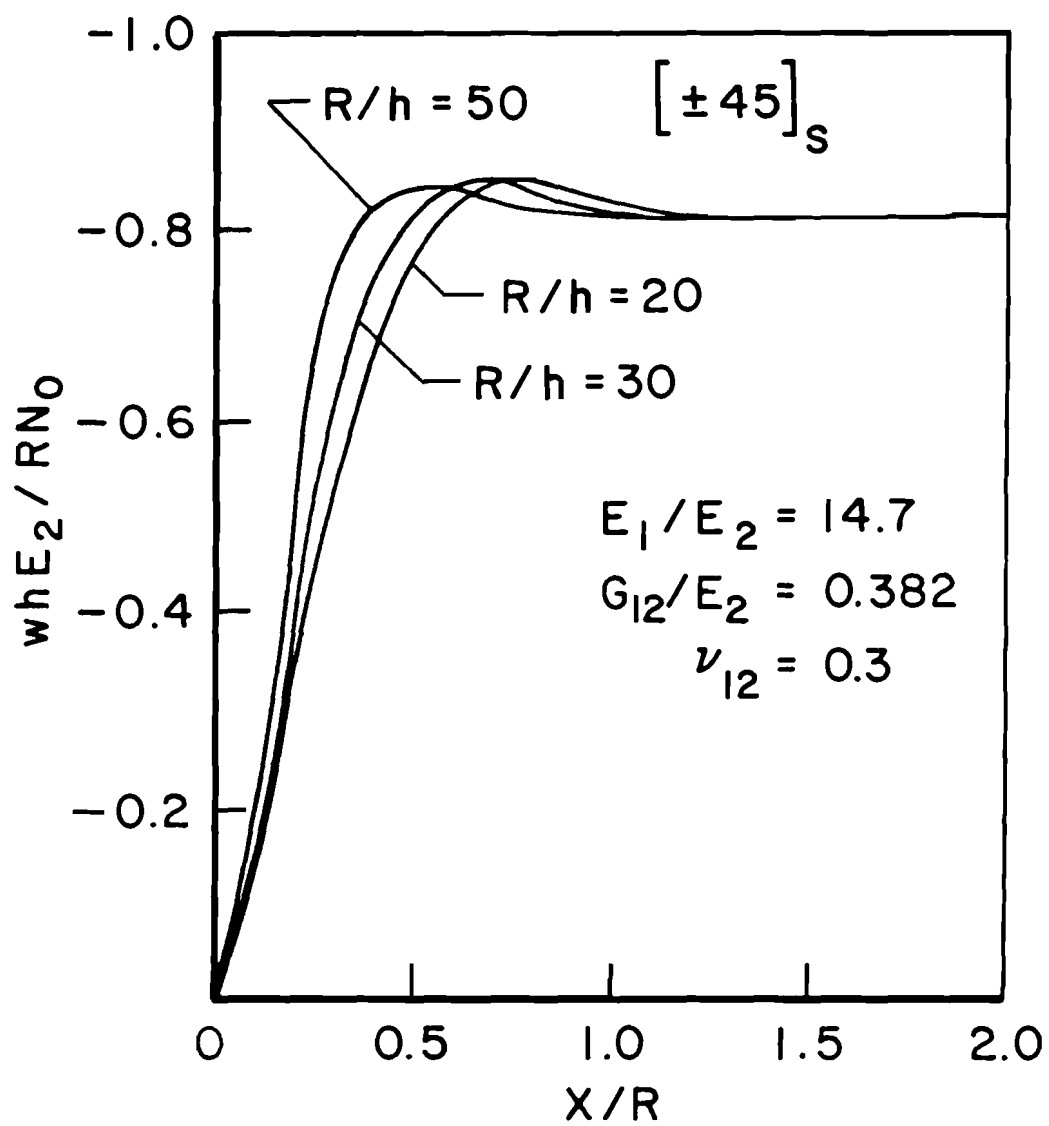


FIGURE 2.4.6 DISPLACEMENT END EFFECT FOR ANGLE-PLY CYLINDER UNDER TENSION LOADING

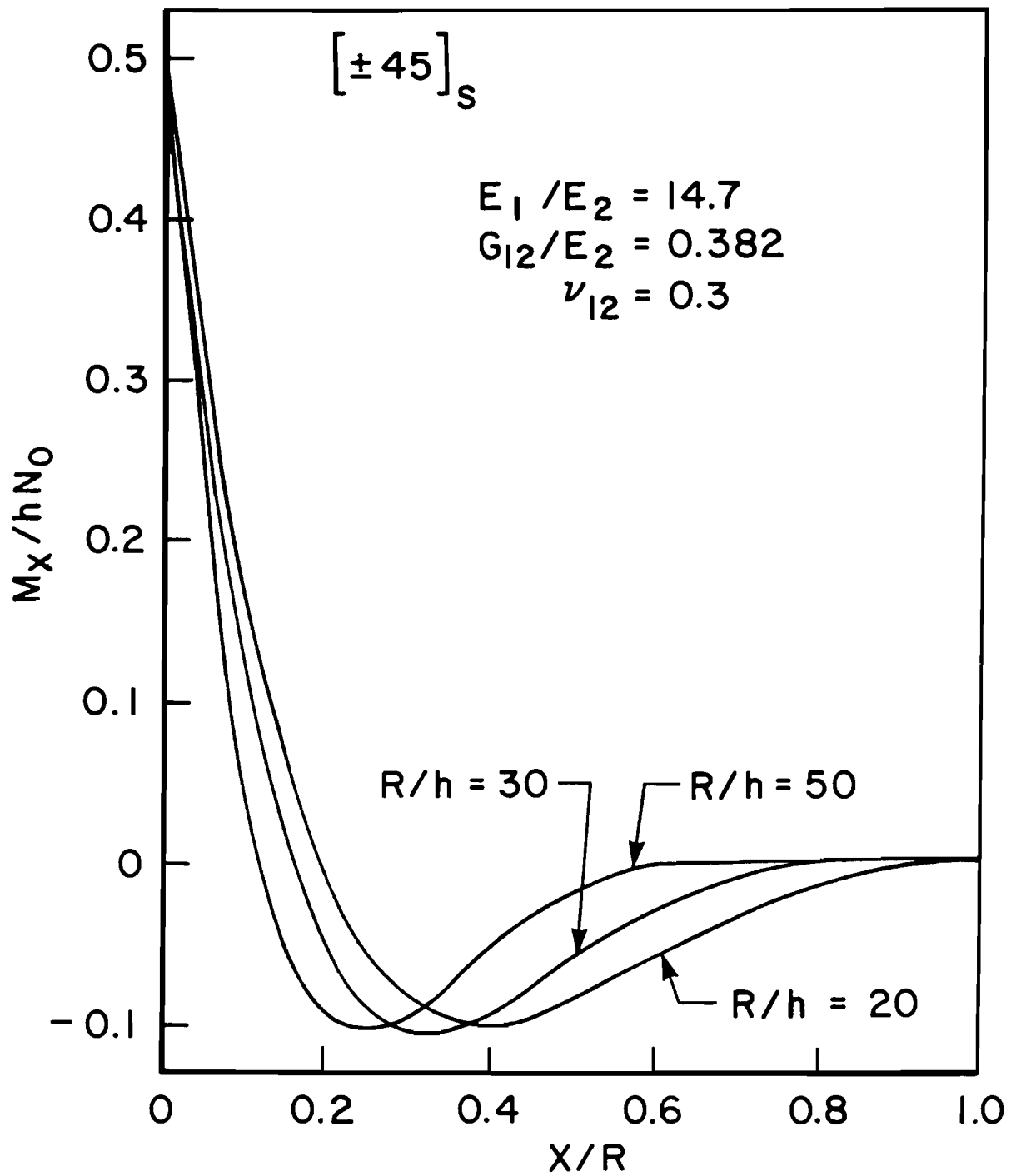


FIGURE 2.4.7 MOMENT END EFFECT FOR ANGLE-PLY CYLINDER UNDER TENSION LOADING

$$\begin{aligned}
& - 4 \frac{B_{26}}{R} \frac{\partial^2 w}{\partial x \partial s} - 2 \frac{B_{22}}{R} \frac{\partial^2 w}{\partial s^2} + A_{22} \frac{w}{R^2} + \frac{\partial M_{xx}^T}{\partial x^2} + 2 \frac{\partial M_{xs}^T}{\partial x \partial s} + \frac{\partial M_{ss}^T}{\partial s^2} \\
& - \frac{\partial m_x}{\partial x} - \frac{\partial m_s}{\partial s} = q + N_{xx}^i \frac{\partial^2 w}{\partial x^2} + 2N_{xy}^i \frac{\partial^2 w}{\partial x \partial s} + N_{ss}^i \frac{\partial^2 w}{\partial s^2} - P \frac{\partial^2 w}{\partial t^2}
\end{aligned} \tag{2.4.112}$$

where the superscript i denotes initial prebuckling loads and

$$P = \int_{-h/2}^{h/2} \rho \, dz$$

As previously, ρ denotes ply density.

2.4.4.1 Free Vibration of Orthotropic Cylinders.

Let us consider an orthotropic laminated cylinder of symmetric construction. For such shells $A_{16} = A_{26} = D_{16} = D_{26} = B_{1j} = 0$. We will determine the flexural vibration frequencies in the presence of internal pressure loading. Thus,

$$N_x^i = N_{xs}^i = 0, \quad N_{ss}^i = Rp \tag{2.4.113}$$

where p = internal pressure and eqs. (2.4.67,68,112), in the absence of surface tractions and thermal effects, become

$$A_{11} \frac{\partial^2 u^0}{\partial x^2} + A_{66} \frac{\partial^2 u^0}{\partial s^2} + (A_{12} + A_{66}) \frac{\partial^2 v^0}{\partial x \partial s} + \frac{A_{12}}{R} \frac{\partial w}{\partial x} = 0 \tag{2.4.114}$$

$$(A_{12} + A_{66}) \frac{\partial^2 u^0}{\partial x \partial s} + A_{66} \frac{\partial^2 v^0}{\partial x^2} + A_{22} \frac{\partial^2 v^0}{\partial s^2} + \frac{A_{22}}{R} \frac{\partial w}{\partial s} = 0 \tag{2.4.115}$$

$$\begin{aligned}
& \frac{A_{12}}{R} \frac{\partial u^0}{\partial x} + \frac{A_{22}}{R} \frac{\partial v^0}{\partial s} + D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial s^2} \\
& + A_{22} \frac{w}{R^2} - Rp \frac{\partial^2 w}{\partial s^2} + P \frac{\partial^2 w}{\partial t^2} = 0
\end{aligned} \tag{2.4.116}$$

We consider the following simple-support boundary conditions:

at $x = 0, L$

$$v^0 = N_{xx} = A_{11} \frac{\partial u^0}{\partial x} + A_{12} \left(\frac{\partial v^0}{\partial s} + \frac{w}{R} \right) = 0 \quad (2.4.117)$$

$$w = M_{xx} = -D_{11} \frac{\partial^2 w}{\partial x^2} - D_{12} \frac{\partial^2 w}{\partial s^2} = 0 \quad (2.4.118)$$

A solution satisfying eqs. (2.4.114-116) and the boundary conditions, eqs. (2.4.117,118) is of the form

$$u^0 = U_{mn} e^{i\omega t} \cos \frac{m\pi x}{L} \cos \frac{n\pi s}{R} \quad (2.4.119)$$

$$v^0 = V_{mn} e^{i\omega t} \sin \frac{m\pi x}{L} \sin \frac{n\pi s}{R}$$

$$w = W_{mn} e^{i\omega t} \sin \frac{m\pi x}{L} \cos \frac{n\pi s}{R}$$

Substituting these functions into eqs. (2.4.114-116), we obtain the following algebraic equations in matrix form

$$\begin{bmatrix} A_{mn} & B_{mn} & C_{mn} \\ B_{mn} & D_{mn} & E_{mn} \\ C_{mn} & E_{mn} & F_{mn} - \lambda \end{bmatrix} \begin{bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (2.4.120)$$

where

$$A_{mn} = \left(A_{11} \frac{m^2 \pi^2}{L^2} + A_{66} \frac{n^2}{R^2} \right)$$

$$B_{mn} = (A_{12} + A_{66}) \frac{mn\pi}{LR}$$

$$C_{mn} = -A_{12} \frac{m\pi}{LR}$$

$$D_{mn} = \left(A_{66} \frac{m^2 \pi^2}{L^2} + A_{22} \frac{n^2}{R^2} \right)$$

$$E_{mn} = -A_{22} \frac{n}{R^2}$$

$$F_{mn} = \left[D_{11} \frac{m^4 \pi^4}{L^4} + 2(D_{12} + 2D_{66}) \frac{m^2 n^2 \pi^2}{L^2 R^2} + D_{22} \frac{n^4}{R^4} + \frac{A_{22}}{R^2} + p \frac{n^2}{R} \right]$$

$$\lambda = p\omega^2$$

A nontrivial solution to eq. (2.4.120) is obtained only if λ is chosen such that the determinant of the coefficient matrix vanishes. Solving the resulting algebraic equation for the frequency, we obtain

$$\omega_{mn}^2 = \frac{1}{P} \left\{ F_{mn} - \frac{[E_{mn}(A_{mn}E_{mn} - 2B_{mn}C_{mn}) + D_{mn}C_{mn}^2]}{(A_{mn}D_{mn} - B_{mn}^2)} \right\} \quad (2.4.121)$$

The fundamental frequency is obtained by finding values of m and n which minimize ω . Results are shown in figure 2.4.8 where the fundamental vibration frequency is plotted as a function of anisotropy ratio, E_1/E_2 , for a $[0/90]_s$ laminate with the ply properties

$$E_2 = 1.7 \times 10^6 \text{ psi}, \quad \nu = 0.3, \quad G_{12} = 0.65 \times 10^6 \text{ psi} \quad (2.4.122)$$

Effect of increasing internal pressure is also illustrated. Additional solutions to this problem can be found in reference 2.4.6.

2.4.4.2 Axisymmetric Compression Buckling.

Now consider a laminated cylinder of symmetric ($B_{ij} = 0$), orthotropic ($A_{16} = A_{26} = D_{16} = D_{26} = 0$) construction under compression loading $N_{xx} = \text{constant} = -N_0$, where $N_0 > 0$. Under these conditions axisymmetric buckling of the cylinder may occur. Equations (2.4.94,95) are applicable, and eq. (2.4.97) can be rewritten in the following form

$$\frac{du^0}{dx} = -\frac{1}{A_{11}} (N_0 + A_{12} \frac{w}{R}) \quad (2.4.123)$$

Equation (2.4.95) is satisfied if $v^0 = 0$. For axisymmetric deformation eq. (2.4.112) becomes

$$\frac{A_{12}}{R} \frac{du^0}{dx} + D_{11} \frac{d^4 w}{dx^4} + N_0 \frac{d^2 w}{dx^2} + A_{22} \frac{w}{R^2} = 0 \quad (2.4.124)$$

Substituting eq. (2.4.123) into eq. (2.4.124), we obtain a differential equation in w .

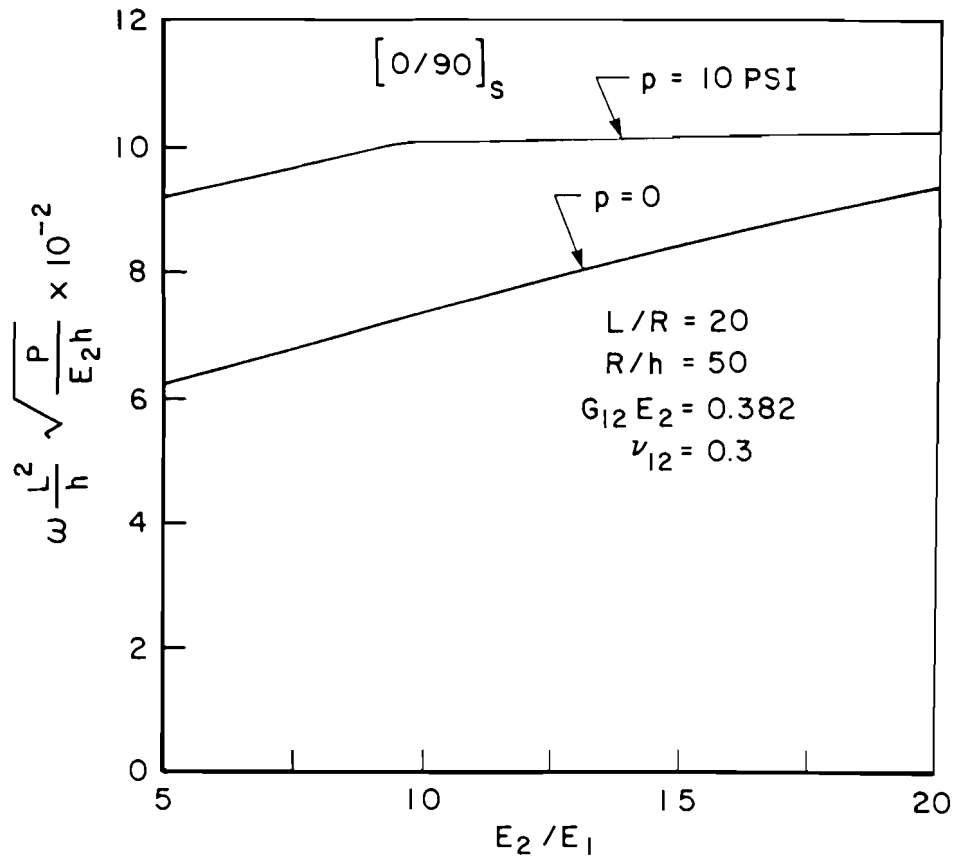


FIGURE 2.4.8. EFFECT OF ANISOTROPY RATIO ON FUNDAMENTAL VIBRATION FREQUENCIES OF A BIDIRECTIONAL CYLINDER UNDER INTERNAL PRESSURE

$$D_{11} \frac{d^4 w}{dx^4} + N_0 \frac{d^2 w}{dx^2} + \left(\frac{A_{11} A_{22} - A_{12}^2}{A_{11}} \right) \frac{w}{R^2} = \frac{A_{12}}{A_{11}} \frac{N_0}{R} \quad (2.4.125)$$

A particular solution, w_p , of this equation is of the form

$$w_p = \left(\frac{R A_{12}}{A_{11} A_{22} - A_{12}^2} \right) N_0 \quad (2.4.126)$$

Let us assume the tube is free to expand circumferentially up to the onset of buckling. For the buckled shape we apply the following simple-support boundary conditions to the homogeneous solution w_h :

at $x = 0, L$

$$w_h = M_{xx} = -D_{11} \frac{d^2 w_h}{dx^2} = 0 \quad (2.4.127)$$

The homogeneous solution is assumed to be of the form

$$w_h = W_m \sin \frac{m\pi x}{L}, \quad m = 1, 2, \dots \quad (2.4.128)$$

Substituting this relationship into eq. (2.4.125) we obtain the relationship

$$W_m \left[N_0 - D_{11} \frac{m^2 \pi^2}{L^2} - \frac{(A_{11}A_{22} - A_{12}^2)L^2}{m^2 \pi^2 R^2 A_{11}} \right] \sin \frac{m\pi x}{L} = 0 \quad (2.4.129)$$

This equation is satisfied prior to buckling by $W_m = 0$, and after buckling by the relationship

$$N_0 = \left(D_{11} \frac{m^2 \pi^2}{L^2} + \frac{E_s h L^2}{R^2 m^2 \pi^2} \right) \quad (2.4.130)$$

where

$$E_s = \frac{(A_{11}A_{22} - A_{12}^2)}{A_{11}h}$$

and is the effective inplane modulus of the laminate in the circumferential direction (see Section 3.4 of Volume I for a discussion of effective elastic constants of a laminate). The value of m which minimizes N_0 yields the critical buckling load N_{0cr} . For this form of buckling, many waves will be formed along the length of the cylinder (reference 2.4.7). Under these circumstances we consider N_0 to be a continuous function of $m\pi/L$ and rewrite eq. (2.4.130) in the form

$$N_0 = D_{11} \left(y^2 + \frac{E_s h}{R^2 D_{11} y^2} \right) \quad (2.4.131)$$

where

$$y = \frac{m\pi}{L}$$

Differentiating this function we obtain an expression for the critical buckling load, i.e.

$$\frac{dN_0}{dy} = 2D_{11} \left(y - \frac{E_s h}{R^2 D_{11} y^3} \right) = 0 \quad (2.4.132)$$

or

$$\frac{m\pi}{L} = \left(\frac{E_s h}{R^2 D_{11}} \right)^{1/4} \quad (2.4.133)$$

Substituting this relationship into eq. (2.4.130), we obtain an approximate expression for the critical buckling load.

$$N_{0cr} = \frac{2}{R} \sqrt{D_{11} E_s h} \quad (2.4.134)$$

To check the accuracy of eq. (2.4.134), we consider a $[0/90]_s$ laminate with the ply properties listed in eq. (2.4.111). In addition $R/h = 50$ and $L/R = 5$. For this case we obtain $N_{0cr} = 3348.51$ #/in from eq. (2.4.134). Solving eq. (2.4.133) for m , we find that the critical buckling load corresponds to $m = 27.28$. To calculate an exact critical buckling load, we consider the following results obtained from eq. (2.4.130).

m	N_0 (#/in)
25	3399
26	3364
27	3349.21
28	3353
29	3374

Thus, the exact solution is obtained for $m = 27$. The approximate solution is accurate to four significant figures. Now consider the same laminated tube with $L/R = 20$. Using the approximation from eq. (2.4.134), we find $N_{0cr} = 53576.260$ #/in which corresponds to $m = 109.2$ as determined from eq. (2.4.133). Using eq. (2.4.130) we find

m	N_0 (#/in)
107	53616
108	53587
109	53576.286
110	53583
111	53607

Thus, the approximate solution and the exact solution ($m = 109$) agree to six significant figures.

Based on the numerical results presented here, it appears that eq. (2.4.134) is very accurate for calculating critical buckling loads.

2.4.4.3 Buckling of Orthotropic Cylinders.

For a symmetric ($B_{ij} = 0$), orthotropic cylinder ($A_{16} = A_{26} = D_{16} = D_{26} = 0$) under combined axial load and internal pressure, eqs. (2.4.114,115) are applicable. The third equilibrium equation is obtained from eq. (2.4.112) which reduces to

$$\begin{aligned} & \frac{A_{12}}{R} \frac{\partial u^0}{\partial x} + A_{22} \frac{\partial v^0}{\partial s} + D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial s^2} \\ & + D_{22} \frac{\partial^4 w}{\partial s^4} + A_{22} \frac{w}{R^2} - N_{xx}^i \frac{\partial^2 w}{\partial x^2} - N_{ss}^i \frac{\partial^2 w}{\partial s^2} = 0 \end{aligned} \quad (2.4.135)$$

We consider the following initial loads:

$$N_{xx}^i = -k_1 N_0, \quad N_{ss}^i = R_p = k_2 N_0 \quad (2.4.136)$$

where k_1 , k_2 , and N_0 are constants and $N_0 > 0$. The boundary conditions are the same simple-support conditions as considered in the case of free-vibration, and given by eqs. (2.4.117,118).

A solution to eqs. (2.4.114,115,135) which satisfies the boundary conditions, eqs. (2.4.117,118), is of the form

$$\begin{aligned} u^0 &= U_{mn} \cos \frac{m\pi x}{L} \cos \frac{n s}{R} \\ v^0 &= V_{mn} \sin \frac{m\pi x}{L} \sin \frac{n s}{R} \\ w &= W_{mn} \sin \frac{m\pi x}{L} \cos \frac{n s}{R} \end{aligned} \quad (2.4.137)$$

Substituting these relations into eqs. (2.4.114,115,135), we obtain a set of algebraic equations which in matrix form is identical to eq. (2.4.120) with the exception of λ which is defined in the following manner:

$$\lambda = \left(k_1 \frac{m^2 \pi^2}{L^2} - k_2 \frac{n^2}{R^2} \right) N_0 \quad (2.4.138)$$

If we determine λ such that the determinant of the coefficient matrix vanishes, we obtain

$$N_0 = \frac{L^2 R^2}{(k_1 R^2 m^2 \pi^2 - k_2 L^2 n^2)} \left\{ F_{mn} - \frac{[E_{mn}(A_{mn} E_{mn} - 2B_{mn} C_{mn}) + D_{mn} C_{mn}^2]}{(A_{mn} D_{mn} - B_{mn}^2)} \right\} \quad (2.4.139)$$

where A_{mn} , B_{mn} , C_{mn} , D_{mn} , E_{mn} , and F_{mn} are defined in conjunction with eq. (2.4.120). The critical buckling load corresponds to values of m and n which minimize N_0 .

Numerical results are illustrated in figure 2.4.9 where the critical buckling load, N_{0cr} , is plotted as a function of anisotropy ratio, E_1/E_2 , for a $[0/90]_s$ laminate with the ply properties listed in eq. (2.4.122). The load conditions are combined compression, $k_1 = 1$, and internal pressure with $k_2 = 0.5$. Tube dimensions are given by $R/h = 50$ and $L/R = 20$.

2.4.4.4 Buckling of Long Cylinders.

We now consider buckling of a cylinder of arbitrary construction subjected to combined axial load, internal pressure, and shear. Response under such loading is described by eqs. (2.4.67,68, 112), in the absence of thermal and inertia terms. The initial loading is of the form

$$N_{xx}^i = k_1 N_0, \quad N_{ss}^i = R_p = k_2 N_0, \quad N_{xs}^i = k_3 N_0 \quad (2.4.140)$$

where k_1 , k_2 , k_3 , and N_0 are constants, and $N_0 > 0$.

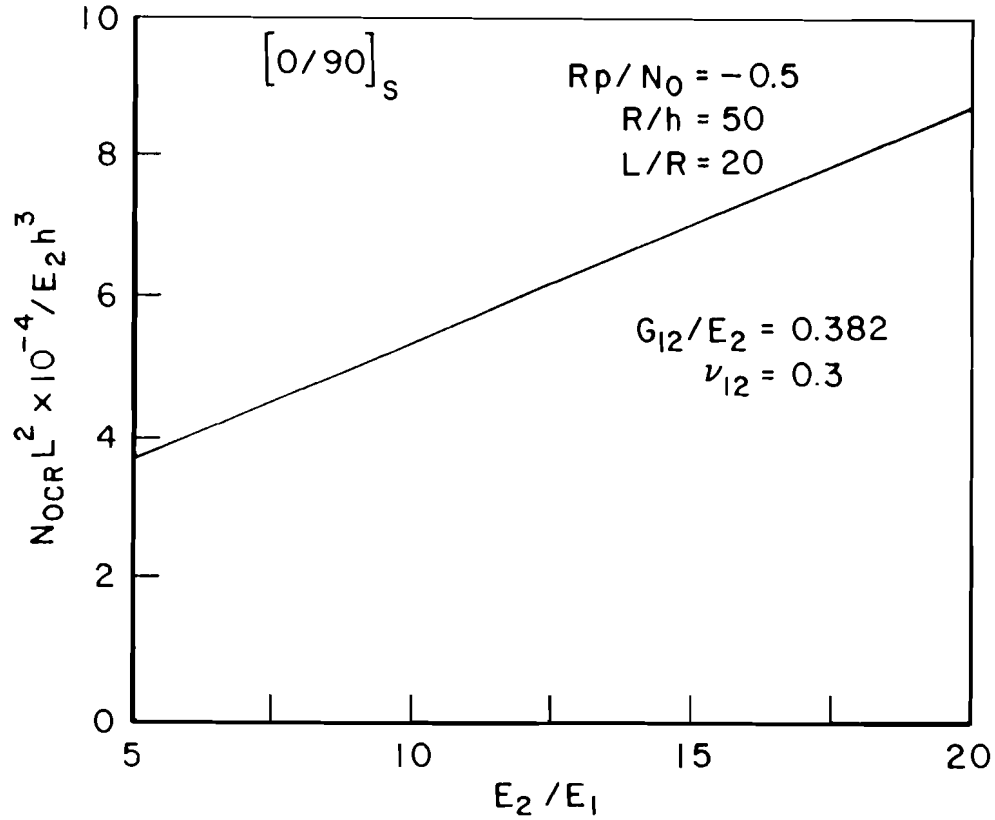


FIGURE 2.4.9 EFFECT OF ANISOTROPY RATIO ON THE CRITICAL BUCKLING LOAD OF A BIDIRECTIONAL CYLINDER UNDER COMBINED AXIAL COMPRESSION AND INTERNAL PRESSURE

A solution to eqs. (2.4.67,68,112) can be obtained in the form

$$\begin{aligned} v^o &= V_{mn} \cos\left(\frac{m\pi x}{L} \pm \frac{ns}{R}\right) \\ w &= W_{mn} \sin\left(\frac{m\pi x}{L} \pm \frac{ns}{R}\right) \\ u^o &= U_{mn} \cos\left(\frac{m\pi x}{L} \pm \frac{ns}{R}\right) \end{aligned} \quad (2.4.141)$$

If we assume that the cylinder is long so that the end constraints have little effect on critical buckling loads, we need not consider specific boundary conditions.

Substituting eqs. (2.4.141) into eqs. (2.4.67,68,112), in the absence of thermal and inertia effects, we obtain a set of algebraic equations which can be written in the same matrix form as eq. (2.4.120), where

$$\begin{aligned} A_{mn} &= \left(A_{11} \frac{m^2 \pi^2}{L^2} \pm 2A_{16} \frac{mn\pi}{LR} + A_{66} \frac{n^2}{R^2} \right) \\ B_{mn} &= \left[A_{16} \frac{m^2 \pi^2}{L^2} \pm (A_{12} + A_{66}) \frac{mn\pi}{LR} + A_{26} \frac{n^2}{R^2} \right] \end{aligned}$$

$$C_{mn} = - \left[A_{12} \frac{m\pi}{LR} \pm A_{26} \frac{n}{R^2} + B_{11} \frac{m^3 \pi^3}{L^3} \pm 3B_{16} \frac{m^2 n \pi^2}{L^2 R} \right. \\ \left. + (B_{12} + 2B_{66}) \frac{mn^2 \pi}{LR^2} \pm B_{26} \frac{n^3}{R^3} \right]$$

$$D_{mn} = \left(A_{66} \frac{m^2 \pi^2}{L^2} \pm 2A_{26} \frac{mn\pi}{LR} + A_{22} \frac{n^2}{R^2} \right)$$

$$E_{mn} = - \left[A_{26} \frac{m\pi}{LR} \pm A_{22} \frac{n}{R^2} + B_{16} \frac{m^3 \pi^3}{L^3} + (B_{12} + 2B_{66}) \frac{m^2 n \pi^2}{L^2 R} \right. \\ \left. + 3B_{26} \frac{mn^2 \pi}{LR^2} \pm B_{22} \frac{n^3}{R^3} \right]$$

$$F_{mn} = \left[D_{11} \frac{m^4 \pi^4}{L^4} \pm 4D_{16} \frac{m^3 n \pi^3}{L^3 R} + 2(D_{12} + 2D_{66}) \frac{m^2 n^2 \pi^2}{L^2 R^2} \right. \\ \left. \pm 4D_{26} \frac{mn^3 \pi}{LR^3} + D_{22} \frac{n^4}{R^4} + \frac{1}{R} \left(\frac{A_{22}}{R} + B_{12} \frac{m^2 \pi^2}{L^2} \right. \right. \\ \left. \left. \pm 4B_{26} \frac{mn\pi}{LR} + 2B_{22} \frac{n^2}{R^2} \right) \right]$$

$$\lambda = \left(k_1 \frac{m^2 \pi^2}{L^2} + 2k_3 \frac{mn\pi}{RL} - k_2 \frac{n^2}{R^2} \right) N_0$$

Setting the determinant of the coefficient matrix in eq. (2.4.120) equal to zero and solving the resulting algebraic equation for λ , we obtain

$$N_0 = \frac{L^2 R^2}{(k_1 R^2 m^2 \pi^2 + 2k_3 R L m n \pi - k_2 L^2 n^2)} \left\{ F_{mn} - \frac{[E_{mn} (A_{mn} E_{mn} - 2B_{mn} C_{mn}) + D_{mn} C_{mn}^2]}{(A_{mn} D_{mn} - B_{mn}^2)} \right\} \quad (2.4.142)$$

For the case where $N_{xs}^i = 0$ ($k_3 = 0$) and the laminate construction is symmetric ($B_{ij} = 0$), and orthotropic ($A_{16} = A_{26} = D_{16} = D_{26} = 0$), eq. (2.4.142) becomes identical to eq. (2.4.139). Thus, for this particular case the exact solution for simple-support boundary conditions is obtained even though eqs. (2.4.141) do not satisfy the boundary conditions in eqs. (2.4.117,118).

Numerical results are illustrated in figure 2.4.10 for pure shear ($k_1 = k_2 = 0$, $k_3 = \pm 1$) of a $[\pm 45]_s$ laminate with the ply properties listed in eq. (2.4.122). In this case $k_3 = -1$ corresponds to positive shear and $k_3 = +1$ corresponds to negative shear. The critical buckling load is plotted as a function of anisotropy ratio E_1/E_2 , and tube dimensions are given by $R/h = 50$ and $L/R = 20$. Note as in the case of angle-ply plates a different critical buckling load is obtained for positive and negative shear.

The effect of boundary conditions in conjunction with cylinder length on critical buckling load is discussed in reference 2.4.8.

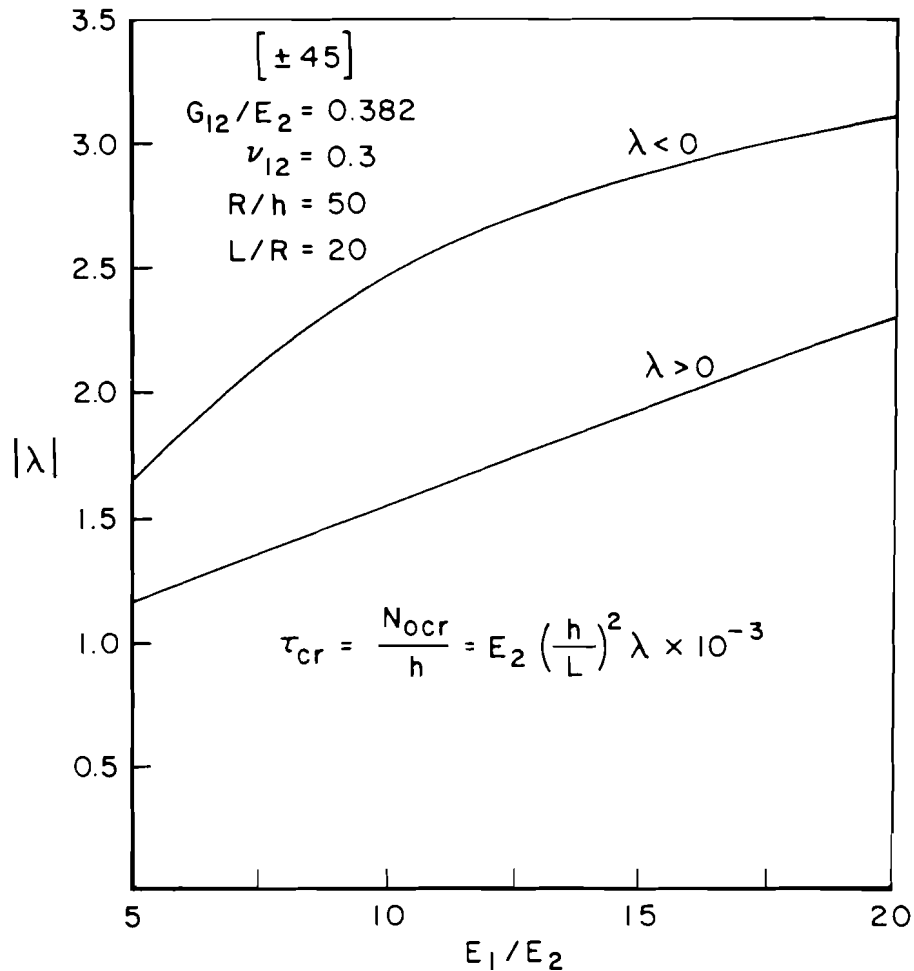


FIGURE 2.4.10. EFFECT OF ANISOTROPY RATIO ON THE CRITICAL BUCKLING LOAD OF AN ANGLE-PLY CYLINDER UNDER SHEAR LOADING

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CHAPTER III. JOINTS

3.0 INTRODUCTION.

The design of efficient joints in fibrous composite structures is, without doubt, the most critical task in ensuring the cost effectiveness of such structures. Joints, more than any other consideration, pervade all aspects of manufacturing and assembly. Joints, be they bonded or bolted, have a far stronger influence on the specification of the laminate stacking sequence than any considerations of optimization in the "clean" structure elsewhere. Even the allowable strain levels in the composite laminates are set by joints which can occur either in the original design or for which provision must be made for repairs in service. The design of optimized fibrous composite structures is achieved by first designing the joints and then filling in the gaps in between, by omitting some plies locally. The reverse procedure, of optimizing the basic structure first and then discovering that those fiber patterns or stiffener shapes compromise the design of efficient joints, is to be avoided.

3.1 TYPES OF JOINTS.

Joints in fibrous composite structures come in two basic forms; adhesive bonded joints which are intended to be permanent and which should always be designed to be stronger than the surrounding structure, and mechanically fastened joints which are used at manufacturing breaks and which can be disassembled if necessary. The mechanically fastened joints - bolts or rivets - are invariably weaker than the basic surrounding structure. The choice of selection between these two kinds of joining depends on the load intensity and the possible need for disassembly.

Thin, lightly-loaded composite structures are almost invariably adhesively bonded, unless there is some overriding need to provide for disassembly. The repair of such structures is easily accomplished by simple bonded patches and such a repaired structure need be no weaker than the original one.

Most highly-loaded composite structures are joined mechanically. The reasons for this are that such structures are frequently so large as to need manufacturing breaks to aid in assembly. They are often so expensive that it is out of the question to consider unrepairable throw-away structures with no provision for repair. The execution of adhesive-bonded repairs on large thick composite structures is extremely difficult and, if not done properly, quite unreliable. Consequently, if repairs must be done by mechanical fasteners, there is no point in not using mechanical fasteners for the original construction also.

If there is no need of provision for repairability, extremely efficient highly-loaded composite structures can be built by eliminating all of the major stress concentrations either by using bonded joints exclusively instead of bolts, or by confining all bolts within special low modulus softening strips to protect the basic laminate. Another approach to achieve this goal is to cure the entire assembly in a single operation. It should be noted, however, that this last approach often destroys any damage tolerance of the structure at the same time. Indeed, all thick composite structures on transport aircraft need to be stitched together by bolts to ensure the confinement of any foreign object impact damage, even if there were no other need for those fasteners. This technique to suppress

the spread of delaminations in thick laminates is the direct counterpart of the use of fiber stitching for the same purpose in those thin laminates that are just a little too thick for the brittle resins used today.

The elimination of highly-loaded joints, where possible, can be just as important as their efficient design. For example, the wing of the Lear Fan all-composite aircraft has one-piece full-span spars and skins because the alternatives would have been twelve major spar cap splices at the sides of the fuselage or six for a centerline splice. Indeed, besides the substantial weight saving of the one-piece approach adopted, a half wing (with its complex splices) would have cost more to build than a whole wing (without them). The AV-8B Harrier composite wing derives its weight saving from the same philosophy.

The location of splices in areas of low loads can be very important also. Attempts to build one-piece fuselage frames for the fuselage of the Lear Fan had been fraught with problems due to thermal distortion of the plastic tooling that was used to save development money. When the skin splices were relocated to improve producibility, the opportunity was taken to replace the one-piece J-section frames on the prototype by the combination of a continuous L-section inner ring and four T-section segments for the outer ring. The splices in the outer ring were located near the skin splices, at $+45^\circ$ to the horizontal, so that the tees could be prebonded to the skin panels. Now, because the Lear Fan fuselage is out of round, those frame cap splices were thus located in an area in which the hoop loads due to cabin pressure were in the opposite sense to the frame-bending loads induced by the pressure. The net effect was that these outer cap splices were located in the areas of least load transfer. The frames were not made as four J-section pieces, one for each skin panel (top, bottom, and sides) for a number of reasons. The first is that, at the optimum locations for the outer cap splices, the two load increments would be additive on the inner cap, meaning that would be the worst possible position for a cut in the inner frame cap. Several other reasons for the choice made pertained to ease of fabrication at the subassembly and final assembly stages, these reasons were just as compelling as the one cited above. Effectively, it was the consideration of the joints that designed those frames.

The material in this chapter is divided essentially into a section on adhesive-bonded joints followed by one on mechanically-fastened joints. That leaves any possible combination of the two uncovered. That subject is treated in depth in some of the references cited here. As a general rule, bonding and bolting do not mix when both load paths are intact, because of the very much greater stiffness of the former load path. However, the combination can be very beneficial with regard to damage confinement and to repairs.

This chapter is not prepared as a compendium of material data. Rather, the prime object is to explain the underlying mechanisms behind the behaviors observed so that reliable designs can be made, using analysis methods that cover all of the important effects.

3.2 ADHESIVELY BONDED JOINTS.

3.2.1 Basic Principles.

The prime function of adhesively bonded joints is to transfer load in shear. Consequently, most of the pioneering analyses have considered shear transfer alone. The exception is that of Goland and Reissner (reference 3.2.1) for the

unsupported single-lap joint, in which the eccentricity in load path causes bending of the adherends which cannot be equilibrated without the presence of adhesive peel stresses which peak sharply at the ends of the overlap. It has since become apparent (reference 3.2.2) that all bonded joints have induced peel stresses which, depending on the geometry, can be significant or insignificant. These peel stresses are automatically associated with interlaminar tension stresses within fibrous composite adherends and these stresses can combine with the interlaminar shear stresses associated with the adhesive shear stresses. It is possible, therefore, that in the vicinity of a bonded joint, failure could occur in an adherend outside the joint or within the bonded overlap, in the adhesive under a combination of shear and peel stresses, or by a delamination within the adherend.

Each of these possible failure modes is governed by a distinct set of differential equations and most do not develop at the same rate in terms of the applied load. Consequently, the designers and analysts can select joint configurations to minimize the undesirable effects for each application and they can incorporate local details to improve the performance of the basic designs. Obviously, the objective is to attain the maximum strength possible by failing in an adherend outside the joint. To achieve this goal, each joint must have what seems at first to be an excess of shear strength but what is really a provision for creep resistance, an allowance for flaws and damage tolerance, and a considerable increase in the statistically allowable joint strength on the basis of having excluded the possibility of failure in the most variable link. In addition, it is imperative (and usually quite simple to accomplish) that the joint proportions are such as to restrict any induced peel stresses or tendency for the adherend to delaminate to a level of insignificance in comparison with the shear stresses. Otherwise, parasitic effects could prevent the bond from developing its full potential shear strength.

This situation above, of failure modes possibly being influenced by multiple variables, has led to two widely different approaches to the analysis of adhesive/bonded joints. On the one hand, those methods developed purely for the creation of structural hardware have tended to be simplified by excluding minor effects from consideration. This has permitted a straightforward inclusion of nonlinear adhesive shear behavior, without which accurate failure predictions would be impossible, even for brittle adhesives. On the other hand, many recent analyses have focused attention on one or more effects which have been excluded from the design-oriented analyses, in order to explain phenomena observed in laboratory test coupons. Unfortunately, the added mathematical complexity has meant that such analyses have been restricted to perfectly linear elastic behavior. Both approaches have their place since, without the knowledge gained from the latter, it would be more difficult to make the appropriate modifications to the design so as to ensure that all those effects omitted from the former analyses really were minor. It should be noted that, in order to obtain experimental data with which to verify the latter kinds of analyses, it has sometimes been necessary to create an artificially brittle adhesive, totally unlike the commercially available structural adhesives.

An awareness of these two approaches to the design and analysis of adhesive-bonded joints is necessary to interpret the many contributions to this art that can be found in the literature. Most of the material presented here falls into the former category of being design-oriented, but those minor effects which could

easily become major had they not been attended to are also explained. An outgrowth of these two approaches to the analysis of adhesive-bonded joints is that, quite recently, there has arisen a considerable interest in effects which are known from experiment to occur only in laboratory test coupons. The application of fracture mechanics methods falls into this category. Such research contributes to the overall understanding of the subject, but there is a very real danger that the unsuitability of such analyses for structurally configured joints may be overlooked. Indeed, the theme of reference 3.2.3 is an exposition of the pitfalls of trying to correlate behavior of adhesive-bonded joints in test coupons and structural hardware, respectively.

The origin of the unsuitability of most test coupons to generate specific design data is quite simple to understand. The objective of test coupons is always to fail the adhesive since, otherwise the adhesive capabilities in that joint cannot be evaluated. The objective of the design of efficient bonded structure joints is to ensure that the adhesive never fails. The purpose of at least some of the data generated on test coupons must be the rational design of structurally configured joints, without any expectation of a one-for-one transfer of apparent adhesive failure stresses. Some of the data generated by short-overlap test coupons, such as the well-known 0.5 inch overlap with aluminum alloy adherends 0.064 inch thick (ASTM D 1002), is very confusing because of the lack of a means to trade off the joint strength due to the added strain to failure in a ductile adhesive against the increment due to a higher peak shear stress in another more brittle adhesive. For several years now, there has been a growing tendency to perform design analyses on the basis of a continuous stress-strain curve to failure to characterize the adhesive under shear loading. Such curves are generated on either a napkin-ring torsion test or on a thick-adherend shear test, as described in the body of the text.

Another of the ground rules that needs to be established before proceeding to specific joint analyses, concerns the mode of failure and, more specifically, the exclusion of failures due to the inability of an adhesive to adhere to an improperly prepared surface. The related problems with metal bonding are well known and understood (see, for example, reference 3.2.4) but, since the most prevalent form involves what seems to be the corrosion of the cladding on 7075 aluminum alloys (which could never occur to a resin matrix), the need for equal concern about the surface preparation for fibrous composites is not as well recognized. The difficulty with interfacial failures is that they do not fail under conditions that can be characterized in terms of external loads and cycles. Furthermore, they cannot usually be detected ahead of time by nondestructive testing after the bonding has been completed.

Therefore, it will be assumed herein that such problems will be excluded from mathematical consideration on the basis that the correct and improper techniques have been identified (see references 3.2.5 and 3.2.6). Interfacial failures would always occur at a lower load level than would be needed to fail the adhesive layer cohesively, which occurs under conditions which can be characterized mathematically.

The polymers from which adhesives are formulated are known to exhibit time-dependent properties. Such effects can be accentuated by the selection of particular joint configurations. Indeed, the difference between the behavior of short-overlap bonded joints when tested slowly and at high frequency is particularly dramatic, as explained in reference 3.2.7. Nevertheless, for

reasons that are also explained in this work, it was possible to show by the artificially severe testing of realistic joints that such behavior can be controlled to the extent that it has no effect on structural joint strengths or lives. For that reason, this subject will not be fully discussed in this section. The adhesive will be modeled as an elastic-plastic material in shear and an elastic material in peel. The latter characterization is adequate because the adhesive does not fail in peel; the resin matrix fails instead and that material is invariably quite brittle for the 350° F cured epoxies used today.

Within the limitations described above, joint analysis methods have been developed in references 3.2.2 and 3.2.8 to 3.2.12 that have permitted the use of highly-loaded adhesive bonds on the primary structure of modern military aircraft. Neither have any failures been experienced when such joints have been used in past or current civil and/or military R&D programs. The basic success of this approach has permitted its extension to variable adhesive properties, to account for such things as voids, porosity and nonuniform moisture absorption as well as variable thickness glue lines.

3.2.2 Adhesive Stress-Strain Curves in Shear.

The analyses in this section of the shear strength of adhesive-bonded joints are performed on the basis of elastic-plastic nonlinear stress-strain curves for an adhesive layer roughly the same thickness as used in the structural joints (nominally 0.005 inches thick). Such a model is shown in figure 3.2.1 for a typical ductile adhesive suitable for use on subsonic transport aircraft. The very considerable non-linear behavior is quite evident. Comparative curves for a ductile adhesive and a brittle adhesive of the type used in hotter applications as on supersonic fighters or in the vicinity of power plants, are shown in figure 3.2.2. While the relative degrees of nonlinear behavior in figure 3.2.2 are very different, a single-straight line elastic model would seem to be inappropriate for either kind of adhesive. The inclusion of the nonlinear behavior is important even for brittle adhesives, because the joint strength is shown to be defined by the total area under the stress-strain curve.

Besides the elastic-plastic model for ultimate failure in figure 3.2.1 which was derived by equating the hatched areas shown, there is also a different model for a lesser load which would not induce such a high maximum strain in the adhesive. The utilization of the bilinear characteristic shown would permit the use of only a single characteristic regardless of the particular load level applied, but the added mathematical complexity has precluded its use to date for all but the balanced double-lap joint. It was shown explicitly in closed form in reference 3.2.9 that any two-straight line adhesive shear characteristic having the same failure stress and strain as well as strain energy would develop precisely the same strength in such a joint. Consequently, the use of the bilinear model would be for convenience only; it would not be significantly more accurate than the most appropriate elastic-plastic model.

There have been attempts, such as in reference 3.2.13, to use mathematically more precise adhesive models, such as the Ramberg-Osgood formulation. Unfortunately the added mathematical complexities have so far prevented significant progress in that direction.

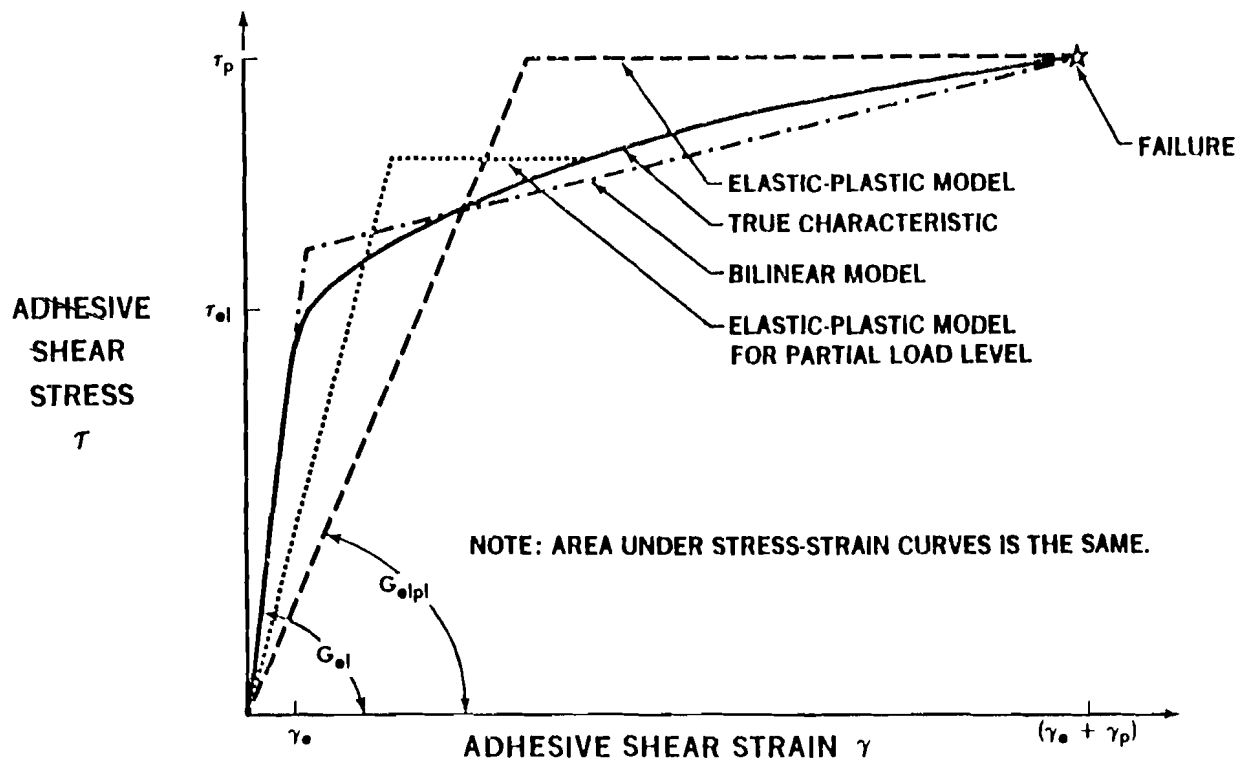


FIGURE 3.2.1 ADHESIVE SHEAR STRESS-STRAIN CURVE AND MATHEMATICAL MODELS

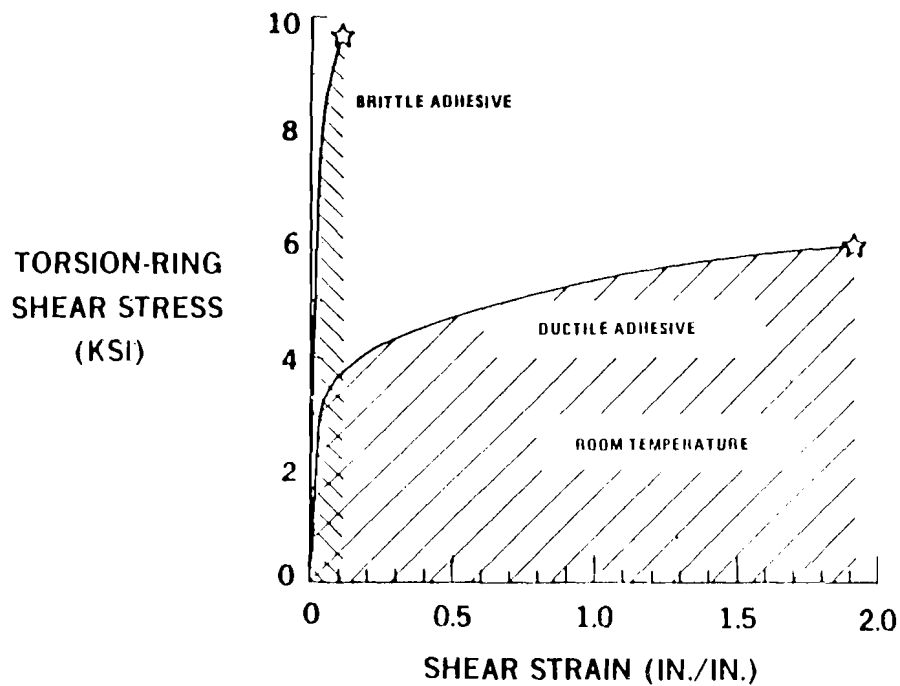


FIGURE 3.2.2 ADHESIVE STRESS-STRAIN CURVES IN SHEAR

These adhesive stress-strain curves vary substantially with the environment as shown in figure 3.2.3, for a typical 250° F cured modified epoxy adhesive. (The ductile adhesive curve in figure 3.2.2 is for a 350° F cured modified epoxy adhesive tested at room temperature.) What is of particular significance about the curves in figure 3.2.3 is that, while the individual characteristics initial modulus, stress at knee (kink), failure stress, and failure strain differ widely, the area under each of the curves is quite similar, so the strength of properly proportioned structural joints would be insensitive to the precise temperature within that range. Yet, on the basis of very different lap-shear test results, many would be surprised by such a close similarity of the structural joint strengths with much longer overlaps. The joint proportions would be sized on the basis of that environment (usually the hot/wet one) that requires the maximum overlaps. The insensitivity of the joint strengths to variations within the operating environment is limited by another effect that is not shown in figure 3.2.3, i.e., at a sufficiently high temperature, known as the glass transition temperature T_g , adhesives undergo a change of state from a glassy to a rubbery substance.

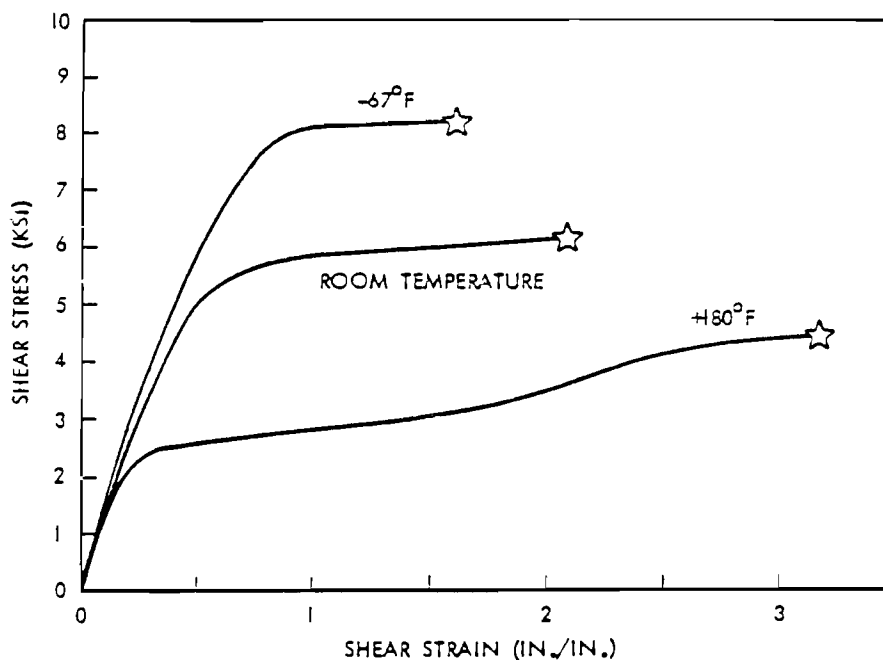


FIGURE 3.2.3 STRESS-STRAIN CHARACTERISTICS OF ADHESIVE FILM IN SHEAR, SHOWING TEMPERATURE DEPENDENCE

Above that temperature there is such a pronounced drop-off in shear strength, as shown by the lap-shear results in figure 3.2.4, that each adhesive becomes unsuitable for use. The particular temperature at which this drop-off occurs is a function of the chemical formulation of the adhesive. Figure 3.2.4 shows that, whereas the brittle adhesive has less strength than the ductile one, it is able to retain it up to a higher service temperature. Actually, even the brittle adhesives become quite ductile at the upper limits of their operating regime. There are not yet any adhesives which can remain ductile over a very considerable

temperature range. That is why polyurethane are used for the sub-zero applications in space, with the plasticized epoxies or phenolics used for moderate temperatures (-67°F to 160°F max.) and aluminum powder filled epoxies and phenolics used for temperatures as high as 300°F but at a substantial sacrifice in strength at the lower temperatures. There is no universal adhesive available today.

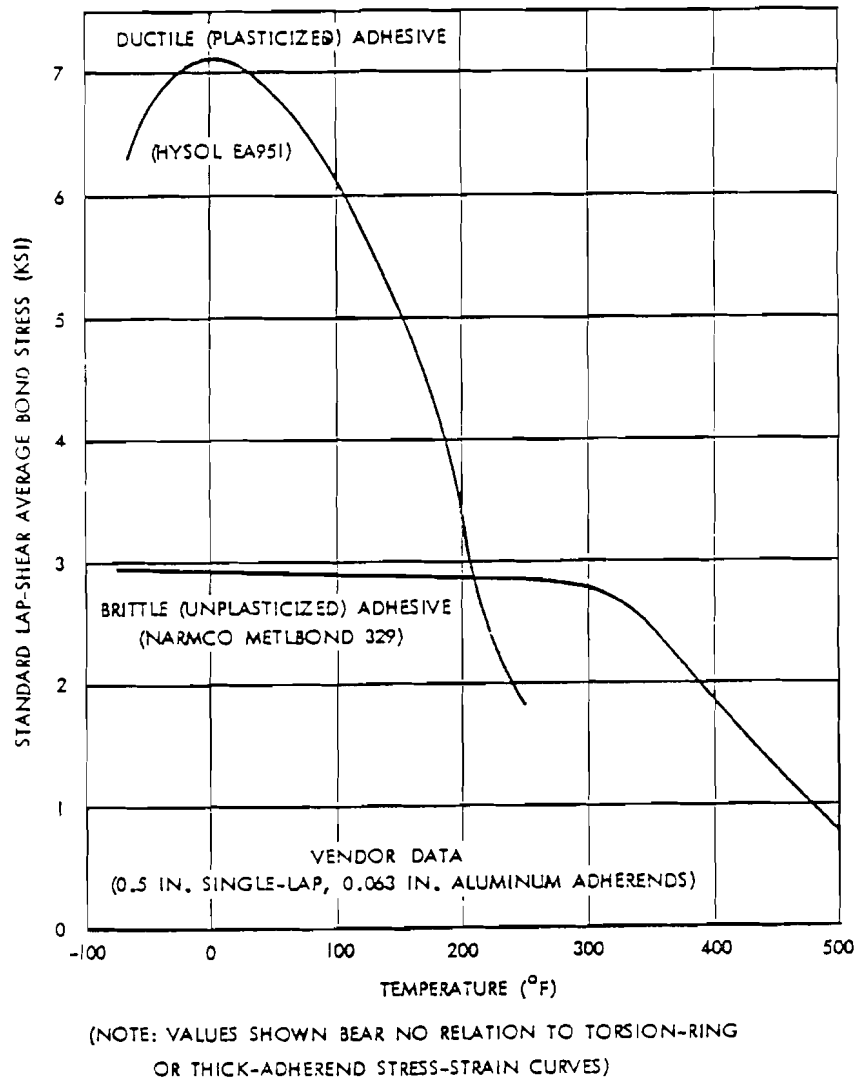


FIGURE 3.2.4 COMPARISON OF DUCTILE AND BRITTLE ADHESIVE STRENGTHS AT VARIOUS TEMPERATURES

Adhesive characteristics can also be affected by porosity, which occurs as a result of the gap between the adherends during the cure of the adhesive being slightly greater than nominal. Complete local flaws result from the capillary action of the adhesive as it melts, should that gap be slightly greater still. Porous bonds occur for single layers of adhesive film somewhere within the range of 0.010 inch to 0.015 inch, with the exact gap varying with each particular adhesive and the amount of edge damming to restrict the flow. Gaps of 0.020 inch or more would invariably be completely disbonded with a single layer of adhesive,

but could be made solid by the combination of multiple layers of adhesive film and very careful edge damming. Reference 3.2.14 contains several stress-strain curves for porous bonded joints. While porous bonds would appear to be very weak on the basis of test coupon assessments, their effect on the strength of structural joints is usually slight, if detectable at all, because the porosity tends to occur in areas of naturally low load transfer. Furthermore, the porous bonds are thicker and not strained as much as the adjacent sound bonds which pick up some of the load that would have otherwise been transmitted through the porous areas had they not been such. This subject is explained fully in reference 3.2.14.

The considerable straining of the ductile adhesives in figures 3.2.2 and 3.3.3 beyond the knees (kinks just above the proportional limit) might suggest that the whole area could be used for nominal designs. That might be possible in rare situations subject only to one applicable load during the life of the structure. More normal design practice is to reserve that nonlinear strength for load redistribution around flaws, for overloads, and for damage tolerance. The frequently occurring loads, such as 1-P cabin pressures or 1-G maneuver loads should not strain the adhesive beyond the knee in the curve. If such a philosophy is not followed, and the adhesive is ever allowed to become the weak link in the load chain, there is a possibility that any load redistribution, around what might initially be quite a small flaw or area of damage, could spread catastrophically across the total bond area. That could never happen if, for every inch of bond, the shear strength of a properly manufactured bond is always greater than the cross-sectional strength of the adjacent structure; the adherends would then always fail outside the bonded area.

Adhesive shear stress-strain curves of the type shown in figures 3.2.2 and 3.2.3 are customarily acquired from one of two test techniques. These are described in figures 3.2.5 and 3.2.6. The napkin ring specimen (figure 3.2.5 is the more difficult to make and only Rutherford (reference 3.2.15) and his colleagues at Singer General Precision have consistently made a success of this specimen. Kreiger (reference 3.2.16), of the Bloomingdale Division of American Cyanamid, has made very substantial contributions to the rational design of adhesive-bonded structural joints by the development of the KGR-I and KGR-II extensometers of which the former is used with the thick-adherend specimen in figure 3.2.6.

The thick-adherend specimen in figure 3.2.6 is made by first bonding together large thick aluminum plates which are then slit into one-inch wide-strips. Finally, each specimen is carefully slotted to create an effective half-inch overlap as shown. The slitting and slotting operations must be performed with extreme care for brittle adhesives. The thick-adherend specimen is inherently conservative in the sense that the abrupt cut-off the adhesive layer at each end of the test specimen is more severe than the natural fillet that occurs in practical construction. That point is explained in figure 3.2.7, which shows also how the refined elastic analyses which account for the variation of shear stress through the adhesive layer thickness are more representative of test coupons, while the simpler analyses upon which they try to improve are actually more representative of practical bonded structures construction because of the benign effects of the fillets.

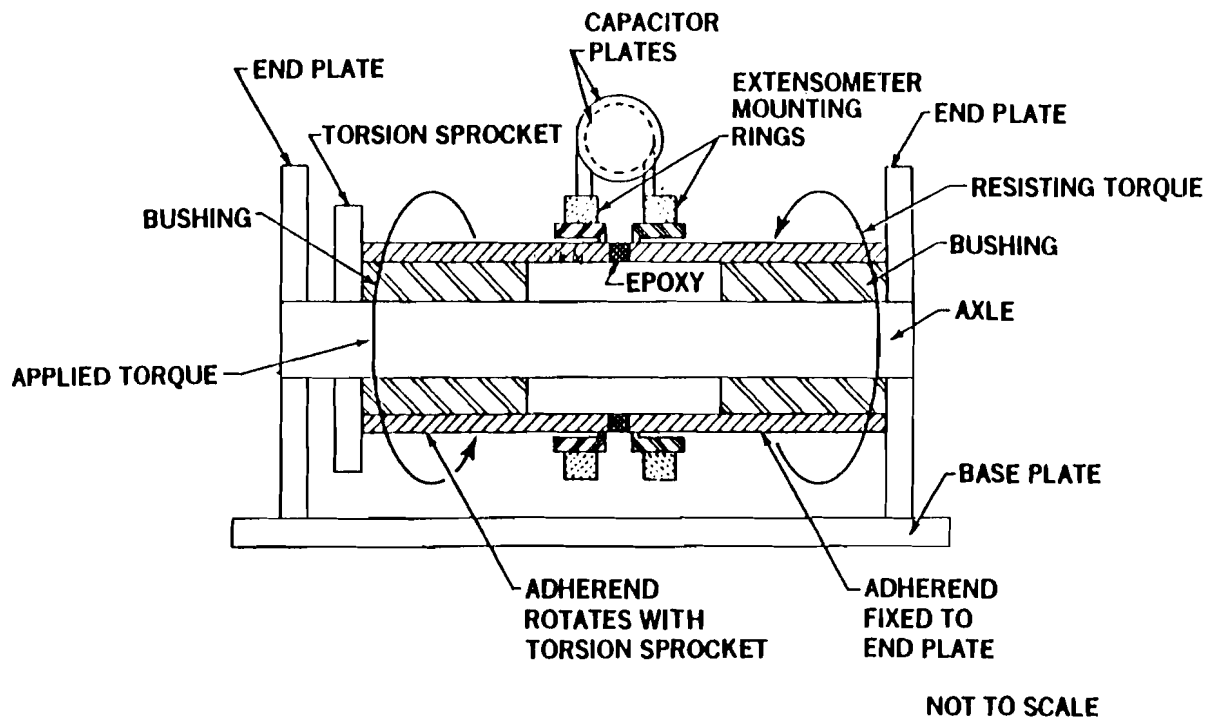


FIGURE 3.2.5. SCHEMATIC OF NAPKIN-RING ADHESIVE SHEAR TEST APPARATUS

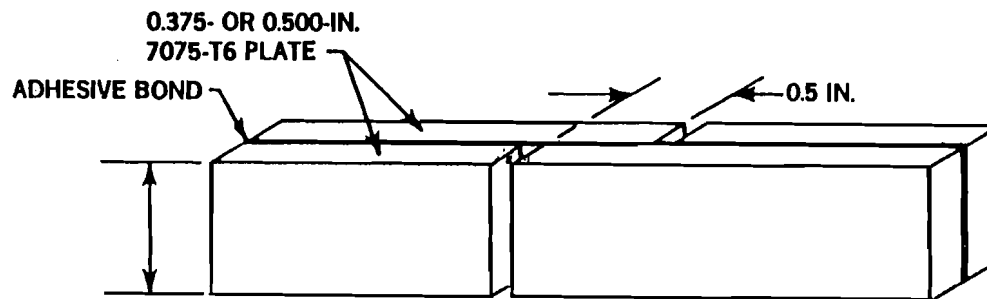


FIGURE 3.2.6. THICK-ADHEREND TEST SPECIMEN

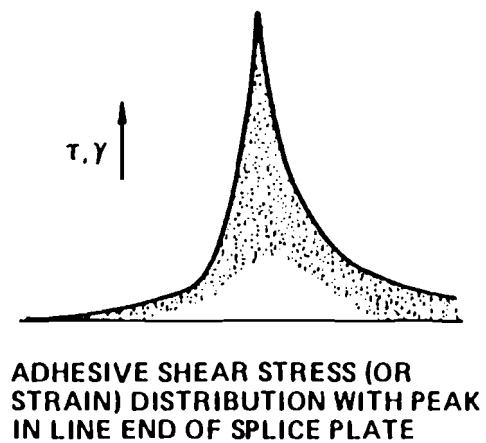
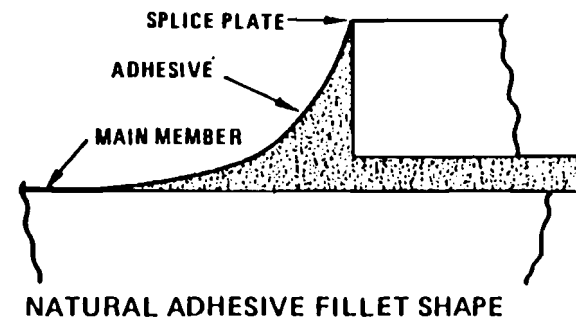
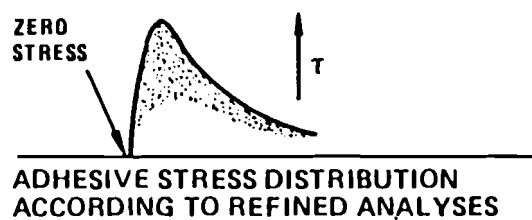
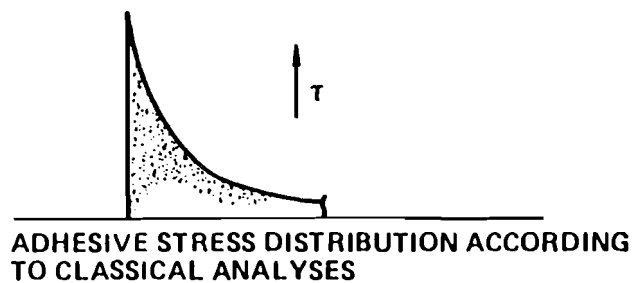
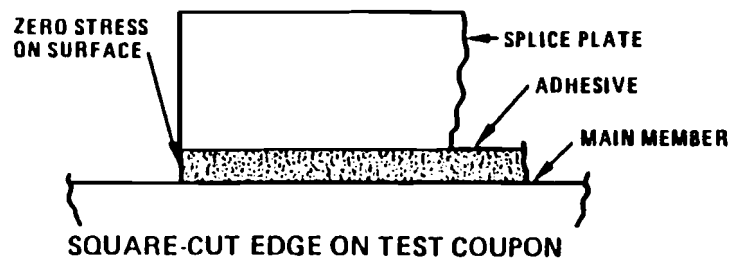


FIGURE 3.2.7 ADHESIVE STRESSES AT ENDS OF OVERLAPS

3.2.3 Effect of Adherend Flexibility on Adhesive Shear Stress Distribution.

There are three basic sources of nonuniform adhesive shear load transfer in bonded joints. These are (1) the adherend extensibility in a balanced joint, (2) the influence of adherend stiffness imbalances, from one end of the joint to the other and (3) adherend thermal mismatch, as between dissimilar materials. The latter two effects are shown below to be true inefficiencies which should be minimized or eliminated wherever possible. The first effect, however, is a necessity for long-life joints. The adhesive shear strain concentration due to adherend extensibility is shown in figure 3.2.8 for identical adherends. It can be seen that the stiffer adherends promote a more uniformly loaded joint while flexible adherends have little bond load transfer in the middle of the overlap. The joint strength is defined by the load at which the adhesive shear strain attains the maximum allowable value, at both ends of the overlap simultaneously. The great majority of the load transfer is accomplished through the plastic zones at the ends of the overlap. The importance of the width and depth of the elastic troughs is explained fully below.

The middle diagram in figure 3.2.9 explains how the uniform adherend stress, in a scarf joint between identical adherends, is associated with an essentially uniform bond strain and stress.

The same local areas of high bond stresses at the ends of the overlap and large lightly-loaded troughs apply equally to the case of in-plane shear loading, in (figure 3.2.10). There is, however, a slight difference between the exponents governing the adhesive stress and strain distributions because tensile and compressive lap shear loads are governed by the Young's modulus E of the adherends while the distribution for the in-plane shear loads is governed by the adherend in-plane shear modulus G .

3.2.4 Influence of Adherend Stiffness Imbalance.

A further source of nonuniformity in the transfer of load through an adhesive bond is a stiffness imbalance between the adherends being joined. This is explained in figure 3.2.11. There is an aggravated stress and strain concentration in the adhesive at that end of the joint from which the more extensible adherend extends. The same end of the joint is critical for both tensile and compressive lap shear loads and also for in-plane shear. In comparison to a stiffness-balanced joint, this imbalance reduces the joint strength by unloading the less critical end and developing a smaller plastic adhesive zone there. If the imbalance were sufficiently severe, there would be virtually no load transfer at one end of the joint while in a balanced joint, there would be an equal load transfer at each end of the joint, as shown in figure 3.2.8.

This reduction in joint strength due to adherend stiffness imbalance applies not only to uniformly thick adherends, but also to scarf joints as shown in the bottom illustration in figure 3.2.9. This problem is of particular importance in regard to stepped-up joints, as shown in figure 3.2.12. Titanium to composite joints are used only in instances of very high load transfer, so it is particularly important not to lose nearly half the joint strength just to save on tooling and layup cost by maintaining flat outer surfaces as shown at the top of figure 3.2.12. The titanium step plate usually should be significantly thinner than the fibrous composite member because the Young's modulus of the former is typically twice as high as that for most practical carbon epoxy laminates.

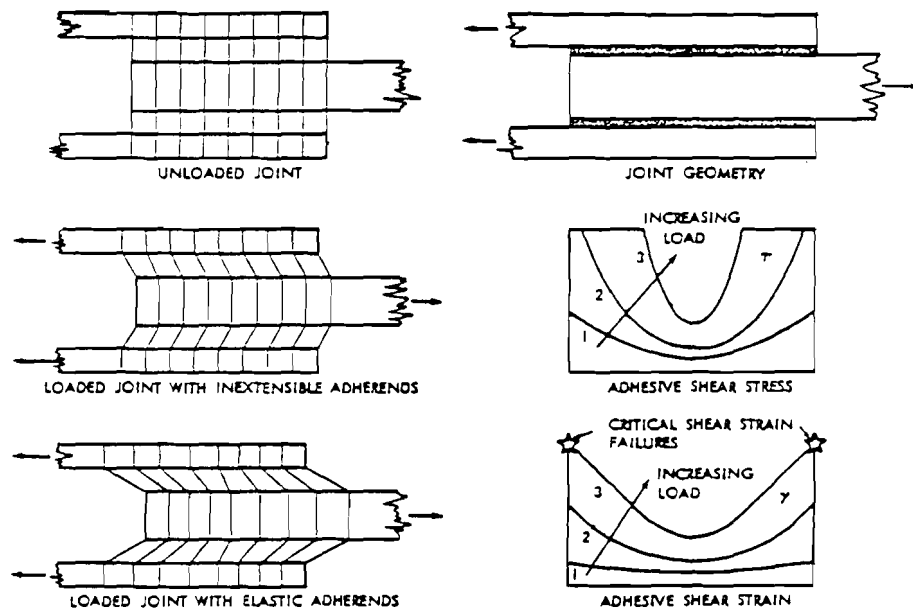


FIGURE 3.2.8 SHEARING OF ADHESIVE IN BALANCED DOUBLE-LAP JOINTS

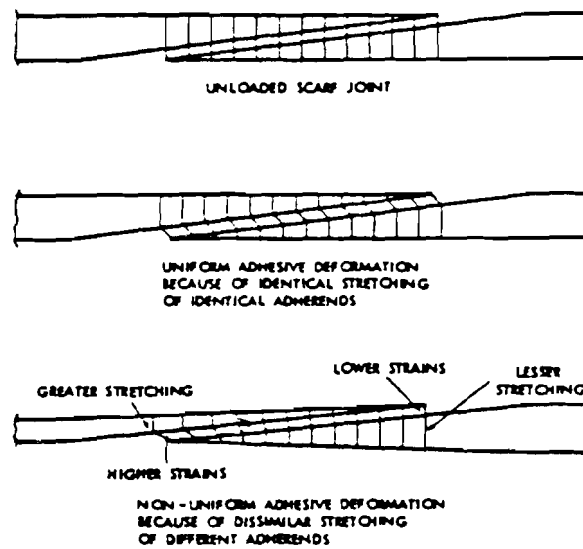


FIGURE 3.2.9 SHEARING OF ADHESIVE IN BALANCED AND UNBALANCED SCARF JOINTS

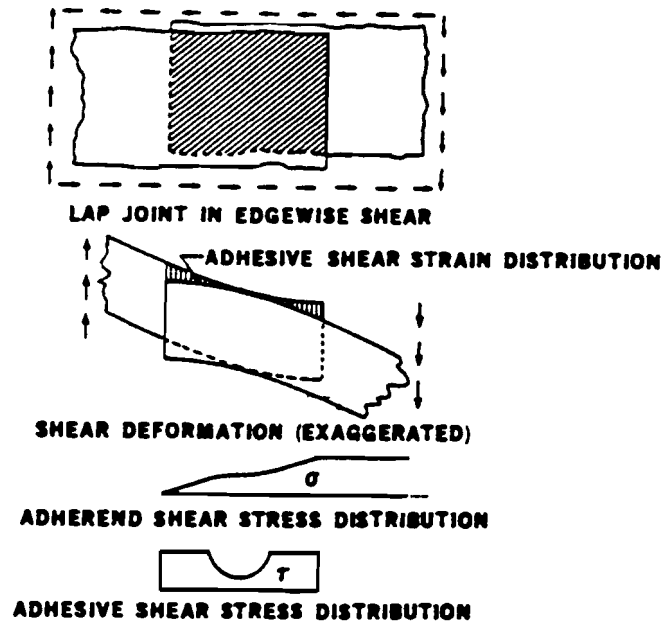


FIGURE 3.2.10 ADHESIVE-BONDED JOINT LOADED BY IN-PLANE SHEAR

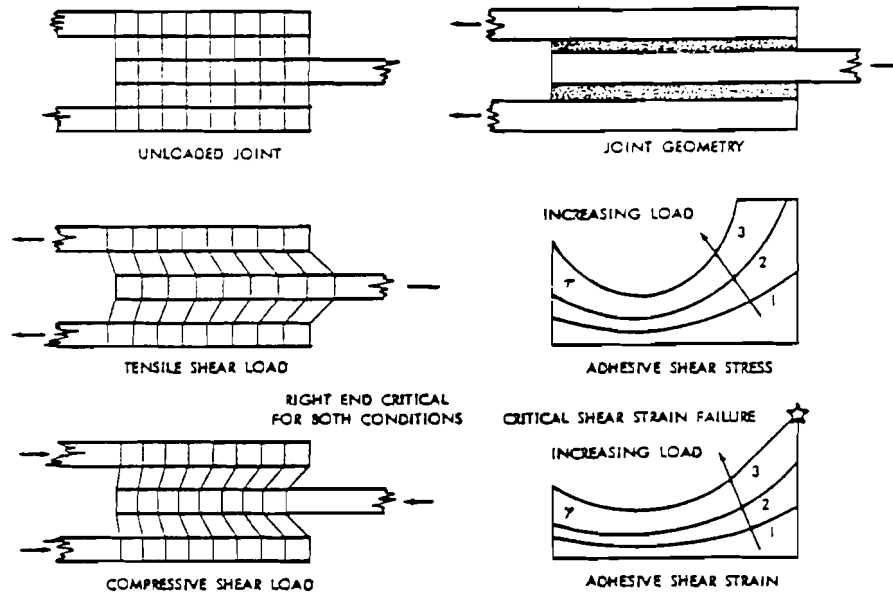


FIGURE 3.2.11 DEFORMATIONS AND ADHESIVE SHEAR STRAINS IN STIFFNESS-UNBALANCED BONDED JOINTS

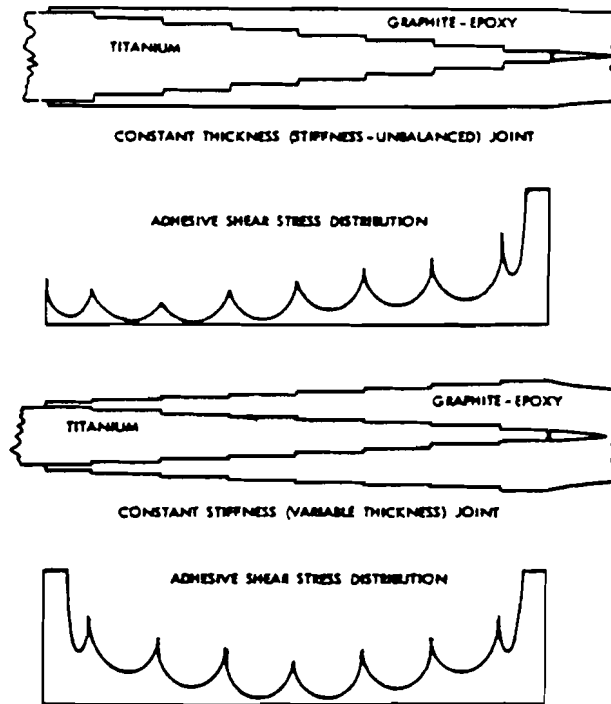


FIGURE 3.2.12 STIFFNESS BALANCED AND UNBALANCED STEPPED-LAP JOINTS

3.2.5 Effect of Adherend Thermal Mismatch.

A further reduction in bond strength is caused by adherend thermal mismatch, which is acute for some of the metal-to-composite combinations and is still significant for aluminum-to-titanium joints. The problem develops because high-strength adhesives are customarily cured at a temperature far above their operating temperatures. This situation arises from many sources, only some of which can be avoided. For instance, there is no way to build and store on the earth a component for a space flight without a substantially different temperature for the assembly. Likewise, when a fibrous composite skin for a high-speed fighter wing is co-cured and bonded (with an adhesive layer) to a titanium step plate, to ensure a good fit of the details, there is inherently a high temperature differential between the cure temperature and the sub-zero air temperature at high altitude. Furthermore, some assemblies are so large as to preclude the use of room-temperature curing adhesives because of an inadequate working life for the adhesive before it cures. Nevertheless, wherever possible, every effort should be made to avoid inducing these residual stresses. For example, a 250° F cured ductile adhesive can be superior to an otherwise better 350° F cured ductile adhesive because so much more of the strength of the latter is dissipated reacting to internal stresses and no longer available to transfer external mechanical loads.

The mechanics of this adherend thermal imbalance are shown in figure 3.2.13 with the metal, which has a higher coefficient of thermal expansion, sandwiched between composite outer adherends. The metal tends to shrink more than the composite during the cooling down to operating temperature. This shrinkage is partially resisted by the composite adherends, which are themselves compressed, thereby setting up residual bond stresses which detract from the joints capability to resist mechanically applied loads.

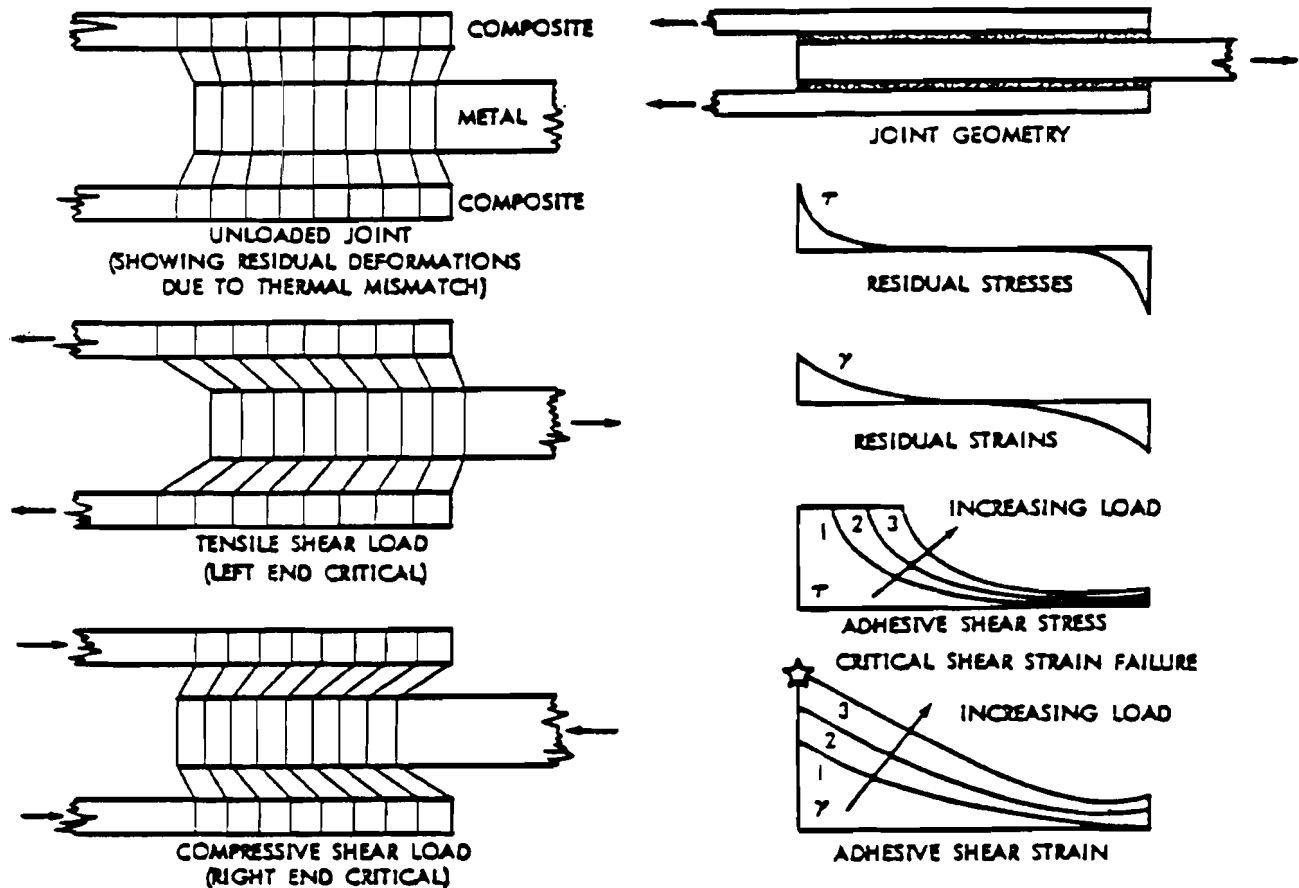


FIGURE 3.2.13 DEFORMATIONS AND ADHESIVE SHEAR STRAINS IN THERMALLY-MISMATCHED BONDED JOINTS

If the overlap between the bonded dissimilar material is short enough - e.g., up to about an inch, adhesive creep can eliminate, or at least minimize, this problem provided that the temperature excursion of the bonded structure in service is not too great. However, for very long overlaps, the thermally-induced residual stresses must remain because the adhesive cannot possibly creep far enough to relieve such stresses. For this reason, the experimental evidence for the existence of such thermal stresses is based on the unambiguous observations on the curvature of long bonded "bi-metallic" strips (reference 3.2.17). The strength reduction, one would predict for short-overlap joint coupons, is rarely, if ever, seen.

An important characteristic of the so-called thermal stress imbalance is that the critical end of the joint changes with the direction of the applied load, as shown in figure 3.2.13. The consequence of simultaneous adherend stiffness imbalance and thermal mismatch is that the joint strength, at the more critical end, can change between tensile and compressive loading. An associated quirk of this behavior is shown in figure 3.2.14 whereby, for a sufficiently large temperature differential between cure and operation, both the maximum and minimum joint strengths have the same sign, to the left of point T_1 . The physical meaning of this condition is that such a joint would not fail, provided the load remained applied, but the complete removal of the load would cause the joint to break apart. For a temperature differential greater than at point T_2 , not even trickery can keep the joint together. The straight lines shown in figure 3.2.14 would be true only if the various coefficients of thermal expansion remained constant throughout the entire temperature range. Actually, they do not and the straight lines shown would actually be curved.

In the case of in-plane shear loading, the thermal mismatch effect become most severe in the corners of the bonded area instead of all along one or more edges. It then becomes necessary to consider orthogonal shear strains in the adhesive instead of just those along a single axis.

Whenever there is an adherend thermal mismatch, there is also a strong possibility of potential galvanic corrosion problems between the dissimilar materials. That calls for the use of a dielectric separator, such as a surface layer of glass epoxy on a carbon-epoxy laminate or a carrier within the adhesive layer.

3.2.6 Peel-Stress Problems and Alleviation in Adhesive Bonded Joints .

In addition to the shear stresses in the adhesive, the other dominant characteristic of adhesive-bonded joints is peel stresses induced by those shear stresses. Figure 3.2.15 shows how the peel stresses are needed to balance the shift in axis of the shear load in the bond to the stretching load in the adherend. The failure sequence shown, when applied to filamentary composite adherends, is dramatic. The adhesive is usually not damaged in such a case; the interlaminar tensile strength of the adherends is even less than the peel strength of most adhesives. With a thick metal adherend, the failure will be in the adhesive, with a peel failure preventing the attainment of the potential shear strength of the bond.

In most cases, the peel-stress problem is not insurmountable. Some of the unrealized shear strength can be traded off for additional peel strength, as shown in figure 3.2.16. There may be a loss of shear strength at the ends of the outer adherends shown, because of the decrease in adherend stiffness there. The outer adherends stretch more there with the inner adherend, minimizing the adhesive shear load. At the other end of the joint, on the other hand, the untapered adherends are associated with a more abrupt shear load transfer and that end of the joint now becomes critical first. It should be noted that figure 3.2.16 is not drawn to scale. More realistic detailing is shown in figure 3.2.17, which was prepared for bonded aluminum alloy structure. A nominal tip thickness of 0.020 inch is to be preferred for fibrous composite adherends. The local thickening of the glue layer shown at the top right of figure 3.2.16 is beneficial for two reasons. It increases the flexibility of the tip of the adherends and thereby decreases the severity of the peel stresses induced.

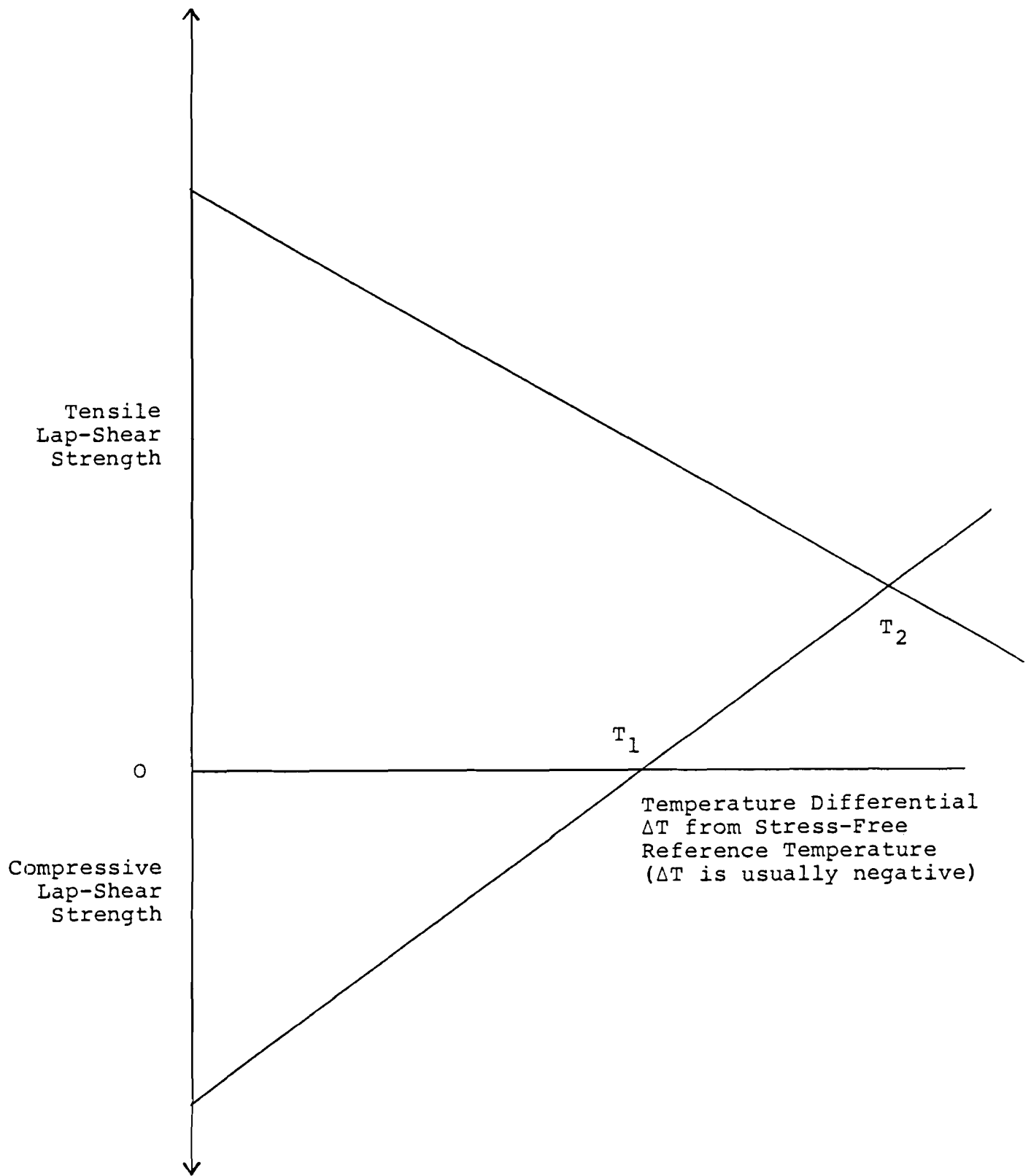


FIGURE 3.2.14 EFFECT OF TEMPERATURE DIFFERENTIAL

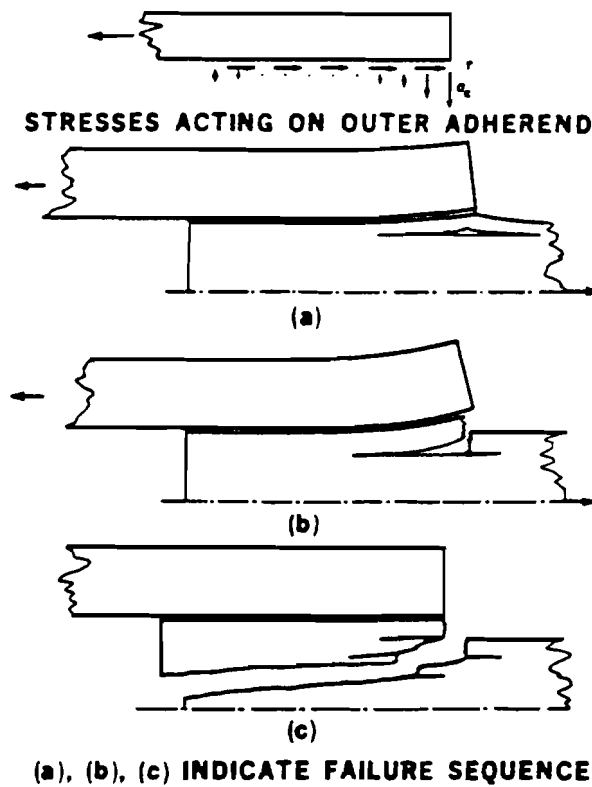


FIGURE 3.2.15 PEEL STRESS FAILURE OF THICK COMPOSITE JOINTS

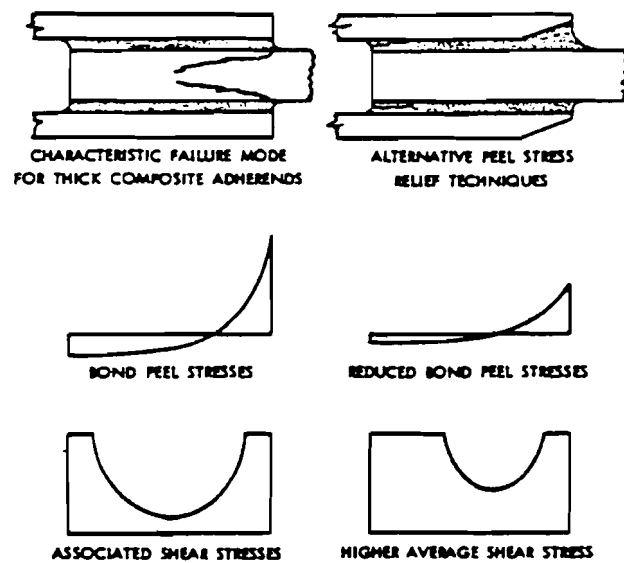


FIGURE 3.2.16 ALLEVIATION OF ADHESIVE PEEL STRESS

However, and more importantly, it increases the gage length of the glue layer and thereby minimizes the strains and stresses.

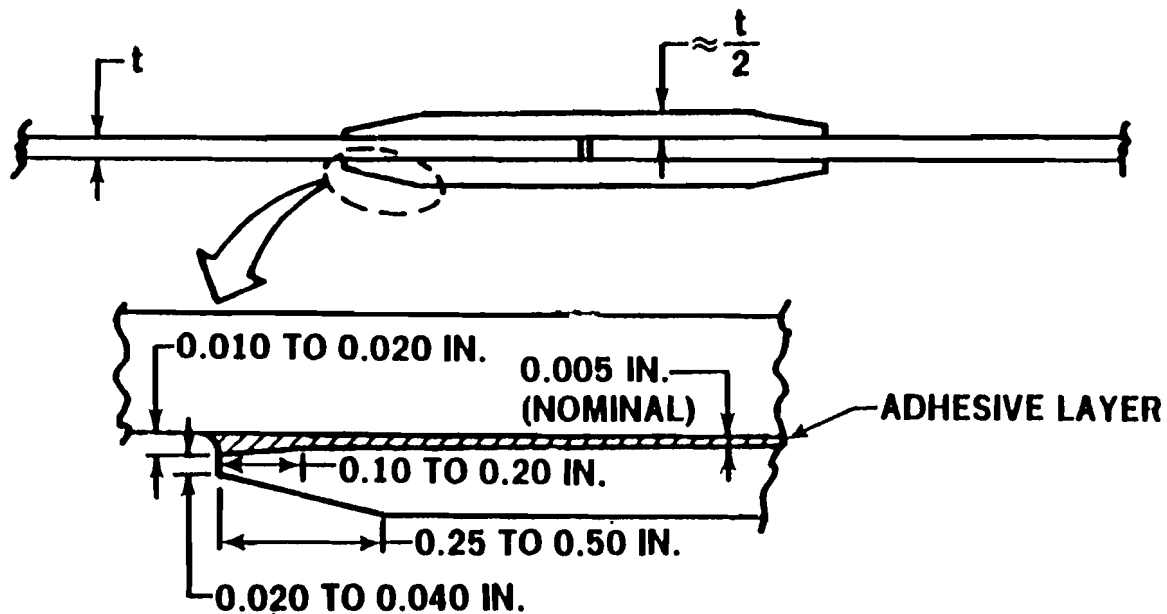


FIGURE 3.2.17. TAPERING OF EDGES OF SPLICE PLATES TO RELIEVE ADHESIVE PEEL STRESSES

There is no need for such modifications at the other end of the joints shown in figure 3.2.16 because the normal stress there is compressive, for tensile lap-shear load, as shown in figure 3.2.15. Compressive lap-shear loading would not require any special treatment at either end of the joint; the high normal stresses at the right ends of the joints in figures 3.2.15 and 3.2.16 would then be compressive--in which state the adherends and adhesives are typically three times stronger than in interlaminar tension or peel. There would also be a large area of small tensile normal stresses at the left end of those joints. The slight tapering of the ends of the inner adherend, in the manner shown to thicken the glue layer in figure 3.2.17, would be beneficial but probably not necessary.

The detailed modifications in figures 3.2.16 and 3.2.17 increase the joint strength by overcoming the limitations of a peel-stress cut-off. With a sufficiently long overlap, that peel stress relief can be accomplished without any loss of shear strength. This is explained in figure 3.2.18 which shows how the distribution of the load transfer at the right end of those joints is rearranged by the local adherend tapering, but that the joint strength is unchanged for the entire range of adherend tapering because the critical condition is always at the left end, which remains constant. The clear implication of figure 3.2.18 is that there is considerable tolerance to perform the refinements shown in figure 3.2.7.

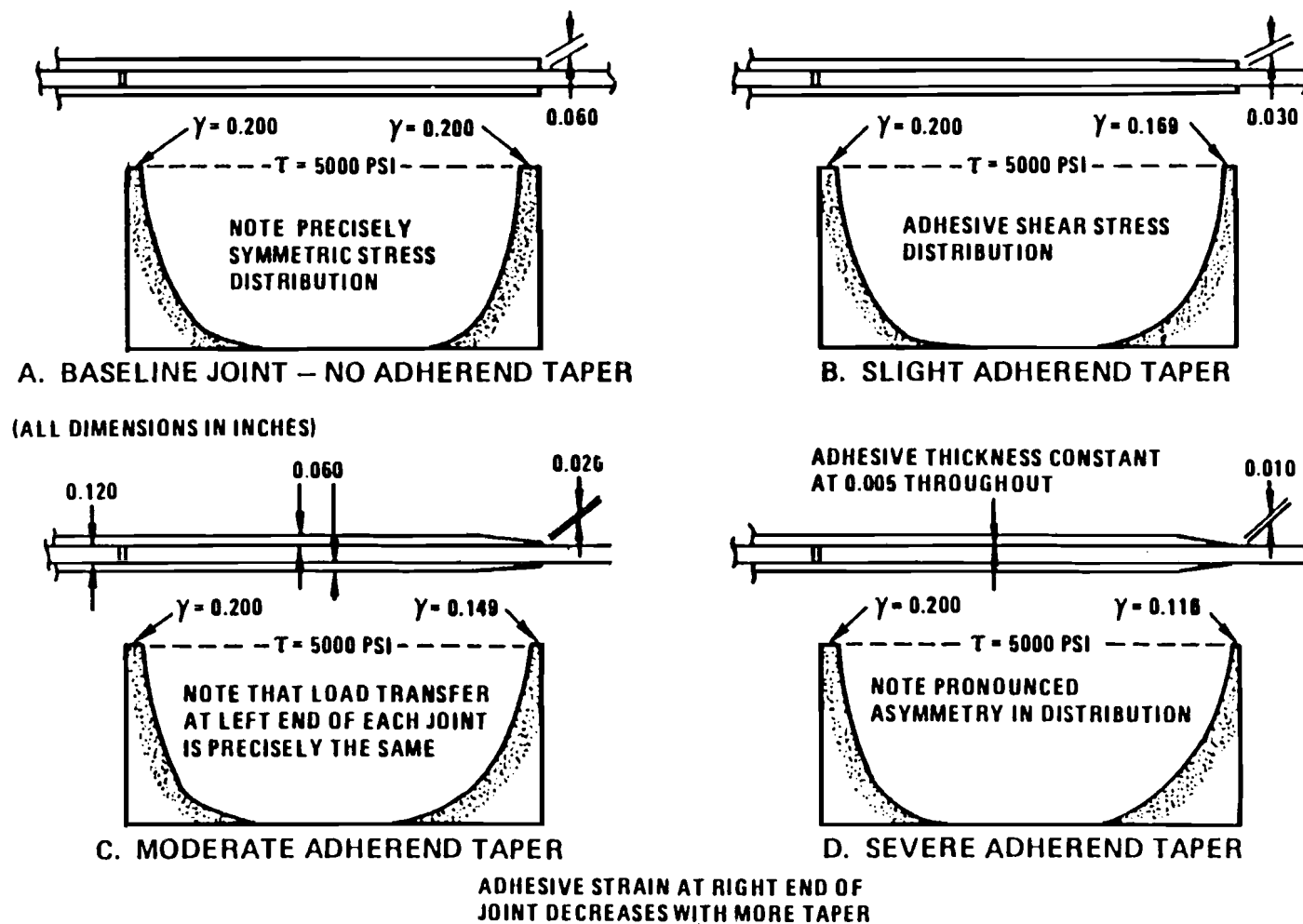


FIGURE 3.2.18 INSENSITIVITY OF ADHESIVE-BONDED JOINT STRENGTH TO MODIFICATIONS AT ONE END OF JOINT ONLY

With progressively thinning of the outer adherend tips in figure 3.2.18, it is seen that the peak adhesive shear strain on the right decreases steadily in relation to the constant peak shear strain at the left end of the joint. That unused adhesive shear strength can be recovered, by deliberately thickening the tapered outer adherends as shown in figure 3.2.19, to actually improve upon the joint shear strength that has previously been limited to adhesive peel stresses. The 24 percent shear strength gain shown in figure 3.2.19 was derived on the basis of a plastic adhesive model and led to the optimum reinforcement of some 32.5 percent. Those tapered splices in figure 3.2.19 are barely more difficult to make than uniform ones, yet the joint retains all of the ease of assembly and generous positioning tolerance associated with the double-strap joints.

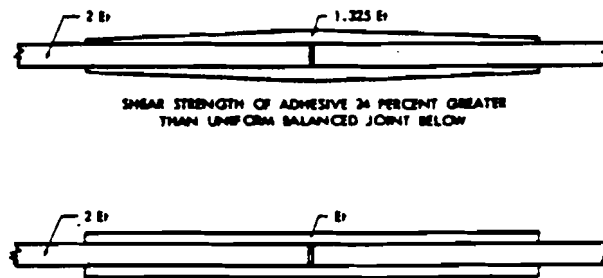


FIGURE 3.2.19 TAPERED-STRAP BONDED JOINTS TO IMPROVE STRENGTH

Reference 3.2.2 contains thorough analyses of the peel stresses included in several configurations of adhesive-bonded joints by eccentricities in load path that are obvious, in addition to the more obscure one discussed here. In every case, some practical scheme was found whereby the peel stresses could be minimized to the point of insignificance. That is not to imply that they should be ignored. They can be crucial if not minimized by appropriate detail design, as is discussed later in this article. However, it is not appropriate to accept a set of circumstances whereby induced peel stresses cause a significant loss of joint shear strength which would demand the derivation of a theory which accounts for shear stresses and peel stresses in the adhesive simultaneously. The only need for such a theory known to the author is to explain the results from test coupons that had been designed to fail nominally in pure shear, but which failed prematurely because of induced peel stresses.

3.2.7 Adhesive Bonded Joint Configurations .

There are several different configurations for adhesive-bonded joints and the most common are shown in figure 3.2.20. Among them, these different joints can be used throughout a wide range of laminate thicknesses and load intensities. Their relative uses are placed in perspective in figure 3.2.21, from which it is evident that each is most suitable for a particular range of thicknesses. There is no point in building an unnecessarily complex joint to transfer the load in and out of thin laminates. Likewise, it would be foolish to expect the more simple configurations to suffice for thick laminates.

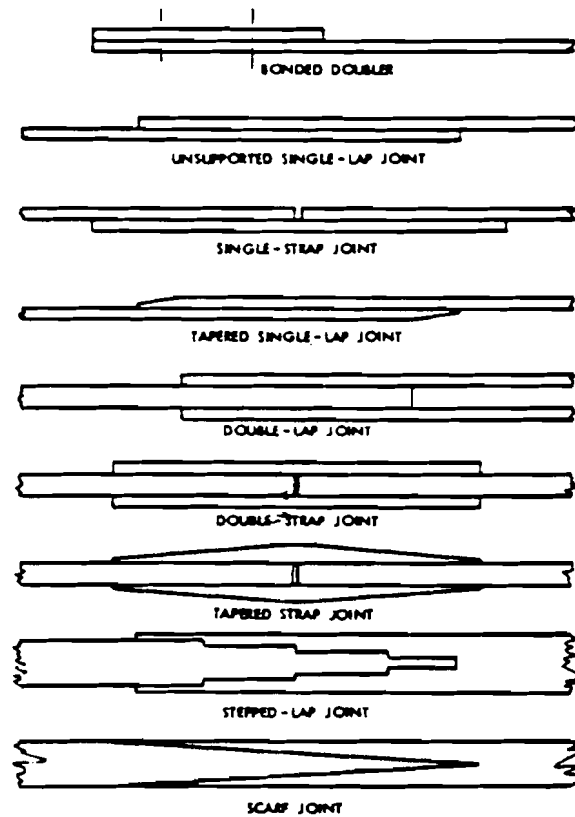


FIGURE 3.2.20 IDENTIFICATION OF BONDED JOINT TYPES

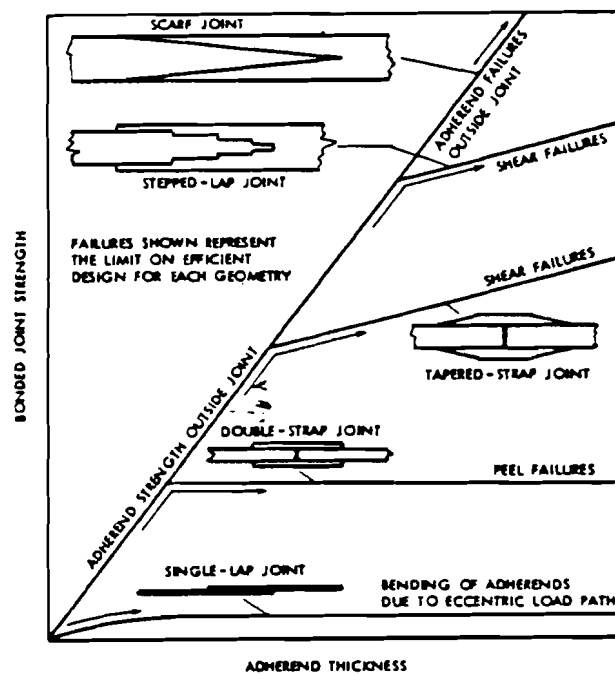


FIGURE 3.2.21 RELATIVE USES OF DIFFERENT BONDED JOINT TYPES

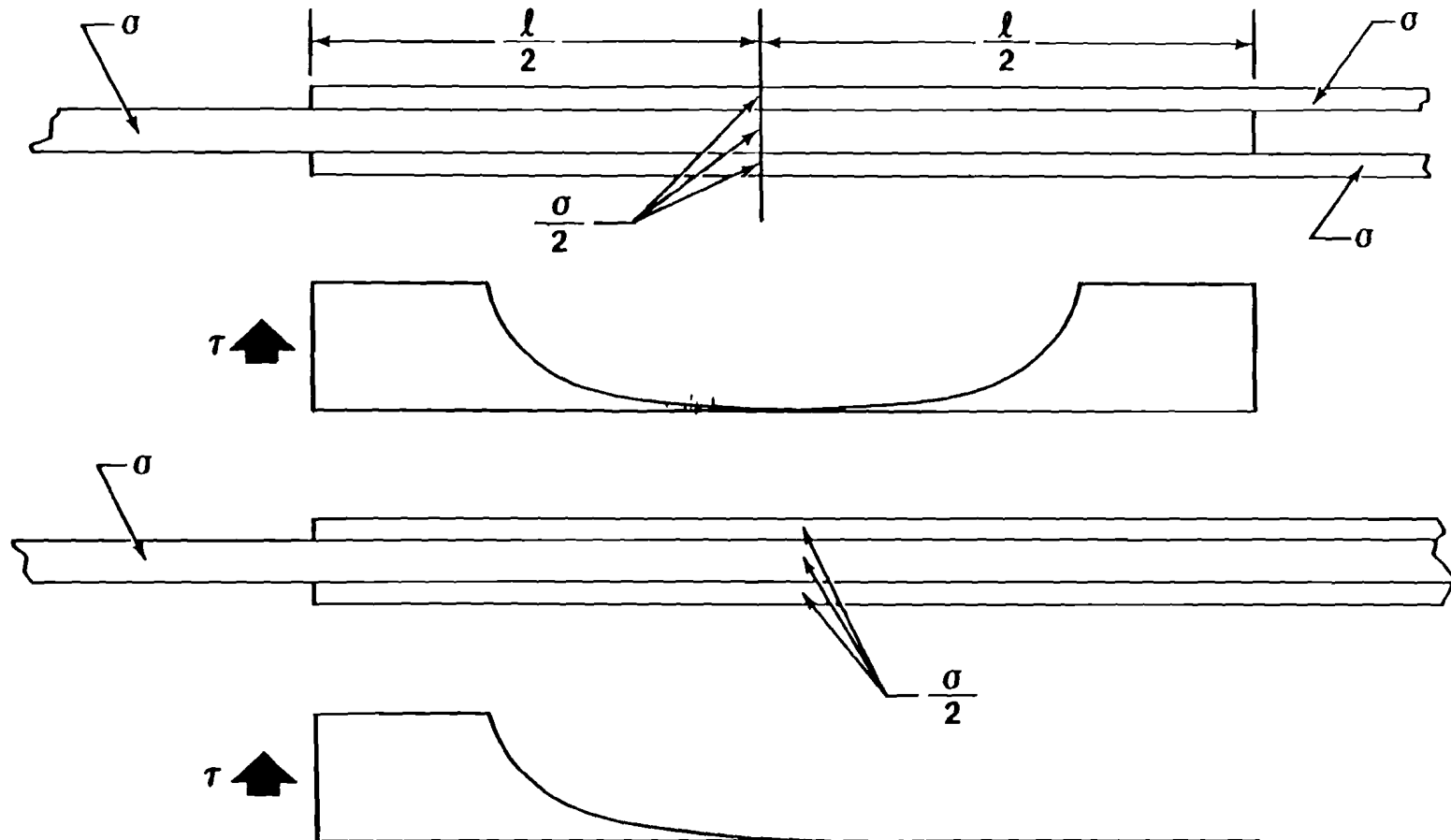
The simplest and most common bonded joint is the bonded (or integral) doubler and it is often not even considered to be a structural joint at all. Yet, as is shown in figure 3.2.22, the load transfer through the adhesive in load-sharing bonded doublers is just as intense as in the case of a full-transfer bonded joint. This fact has often been overlooked in the bonding of doublers to reinforce bolt holes where the load is introduced in test coupons reference 3.2.18. A tendency for such doublers to disbond during the test program reported in reference 3.2.19 (albeit mainly due to poor surface preparation) led to a thorough assessment of the load transfer at the tip of bonded doublers. It was found in reference 3.2.18 that, except for thin doublers (less than about 0.030 inch) it is necessary to taper the tips of the doublers very gently. A slope of one-in-ten is appropriate for near to quasi-isotropic laminates while a slope of one-in-twenty is more suitable for the higher loads associated with all-unidirectional material. These same slopes would hold for integral skin buildups within the basic laminates. There appears to then be no upper limit on the total thickness of any such transitions or reinforcements.

The unsupported single-lap joint is probably the best known adhesive-bonded joint between metal adherends, on the basis of its extensive use for quality control and in military specifications. However, its low structural efficiency unless it is used with what seems to be an unnecessarily generous overlap-to-thickness ratio (between 50 and 100 to 1), has led to its infrequent use with the advanced fibrous composites. The need for such long overlaps is due not to any weakness of the adhesive in shear but to a need to permit very gradual normal bending deflections to minimize the bending and interlaminar tension stresses in the laminated adherend. This can be seen from the top curve in figure 3.2.23. Such unsupported joints should never be used for compressive loads because, then, the initial eccentricity in load path would be amplified by the action of any load instead of being alleviated as it is under tensile loading. The efficiency of such joints can be improved in either case by providing normal supports with which to react the kick loads at the ends of the bonded overlap. Such a supported single-lap joint would then be analyzed as one side of a double-lap joint which was twice as thick. That same analysis method is also appropriate whenever the load transferred is in-plane shear, because there is then no effective eccentricity in load path tending to make the joint deflect (unless the entire panel were to buckle). The transfer of in-plane shear loads, rather than of tensile or compressive lap shear loads, is governed by differential equations of precisely the same form but with different exponents in the various stress distributions.

The single-strap joint shown in figure 3.2.20 barely qualifies as a joint at all; a better description would be as a severe built-in stress concentration. The abrupt eccentricity where the skins built together is necessarily associated with a high bending moment in the middle of the splice plate. In addition, the same area is subject to high adhesive peel stresses and an associated tendency for fibrous composite adherends to delaminate there. The analysis of the peel stresses and adherend bending stresses in this joint is given in reference 3.2.2, which explains also two techniques for alleviating the problems.

Thicker single-lap joints need the ends of the adherends tapered as in figure 3.2.20 primarily to prevent a premature failure by induced peel stresses (as explained above) before the full potential shear strength of the adhesive could be developed. A secondary benefit is that, provided both ends are tapered, there is an actual increase in that shear strength because of a reduction in the severity of the peaks in the shear strain distribution.

- SAME ADHESIVE STRESSES IN EACH CASE



- SAME MAXIMUM ADHESIVE SHEAR STRAIN FOR SAME ADHERENDS AND METAL STRESSES

FIGURE 3.2.22 DOUBLERS VERSUS JOINTS

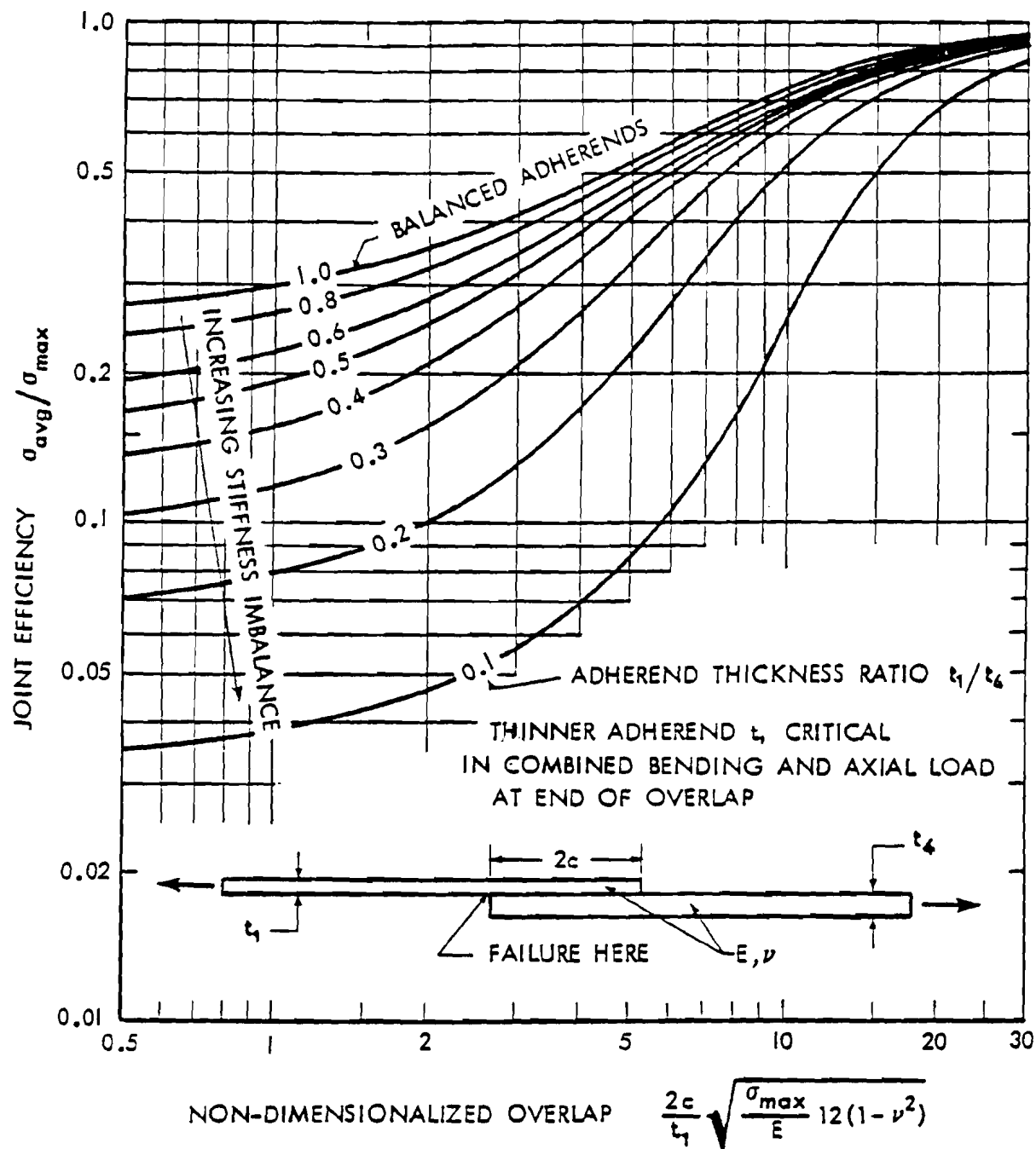


FIGURE 3.2.23 EFFECT OF ADHEREND STIFFNESS IMBALANCE ON ADHEREND BENDING STRENGTH OF SINGLE-LAP BONDED JOINTS

It is appropriate at this point to explain the inter-relation between the three distinct failure modes defining the strength envelope of adhesive-bonded joints. The strongest joints do not fail in the adhesive at all; they fail outside the joint area at a load equal to 100 percent of the strength of the adherend(s). The next higher strength occurs when the load that can be developed is limited by the shear strength of the adhesive. The weakest failure mode is associated with failure of the adhesive under peel loads, or with the even more premature failure by delamination of fibrous composite adherends, as described above. Now, the strength of the joint is proportional to the laminate thickness (to the first power) if it fails outside the joint under direct tension, compression, or in-plane shear, in the absence of bending. The shear strength of the adhesive-bonded joint, however, is proportional to only the square root of the laminate thickness. The peel strength drops off even faster, being proportional to the quarter power of the laminate thickness. These effects are explained in figure 3.2.24. Because of these different dependencies on the adherend thicknesses, each possible failure mode is not equally severe in any particular joint; usually one will dominate over the others. Consequently, seemingly similar adhesive-bonded joints will fail by different modes at quite different structural efficiencies, primarily as a function of the thickness of the members making up the joint. These phenomena are characterized in figure 3.2.25 for carbon-epoxy laminates. It is also evident that the joint strengths and failure modes are sensitive to the adhesive properties. The dominance of peel-stress failures in figure 3.2.25, for the thicker adherends, makes clear the importance of using the peel-stress alleviation techniques in figure 3.2.17. Figure 3.2.25 does not take into account the benefits from such relief.

Laminates of somewhat greater thickness than those for which single-lap joints are adequate, require two bond surfaces to transfer the load. The transition in thickness is not exact, being a function of the adhesive as well as of the thermal environment, but is typically of the order of 0.06 to 0.07 inches for aluminum alloy adherends or for near quasi-isotropic carbon-epoxy laminates. Lower thicknesses would apply for stronger steels or unidirectional composite members. The optimum overlap-to-thickness ratio for double-lap or double-strap joints (see figure 3.2.20) is of the order of 30 to 1 for ductile adhesives used with structure subsonic transport aircraft structures. Despite the lack of any obvious gross eccentricity in load path, as with single-lap or single-strap joints, double-lap joints are also subject to peel-strength cut-offs, as explained above and illustrated in figure 3.2.15.

However, just tapering the outside of the splice plates (while leaving the basic members at a uniform thickness) will not create any additional shear strength in the joint. (Neither will it usually create any decrease in shear strength, but it could if the overlap were too short.) The tapered-strap joint shown in figure 3.2.20 employs deliberate thickening of the splice plate at its middle to achieve, simultaneously, the alleviation of peel stresses and an increase in shear strength, (figure 3.2.19). The uniform double-strap joint is limited to the splicing of about 1/8 to 3/16 inch of aluminum alloy, or equivalent. The use of the reinforced and tapered splice straps might raise that limit to as high as 1/4 inch without a significant increase in the difficulty of manufacturing the joint.

Only the stepped-lap or scarf joint should be considered for the joining of members more than 1/4 inch thick by adhesive bonding. The suggestion in figure 3.2.21 that the theoretical scarf joint can always develop the entire strength of

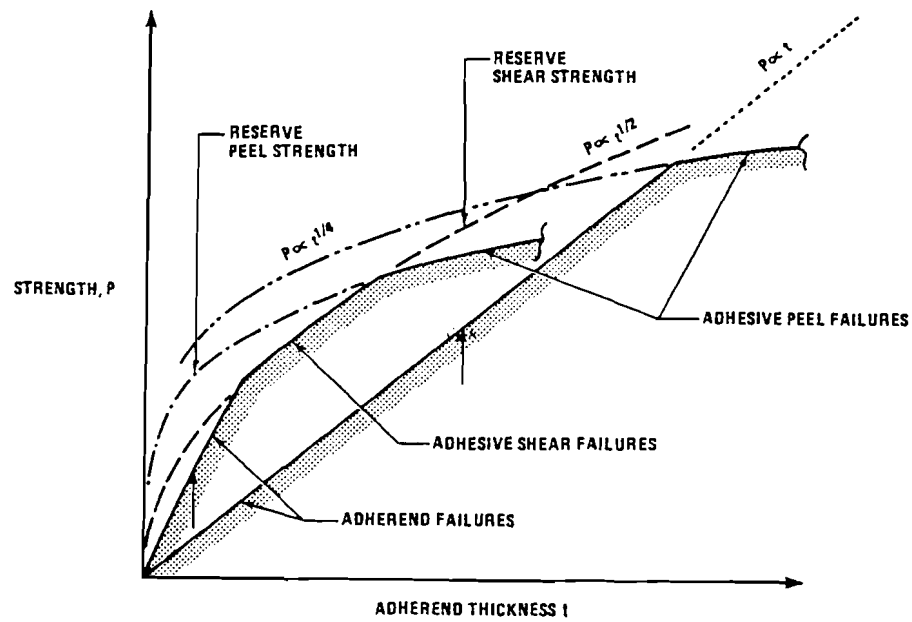


FIGURE 3.3.24 RELATIVE SEVERITY OF ADHESIVE SHEAR AND PEEL STRESSES

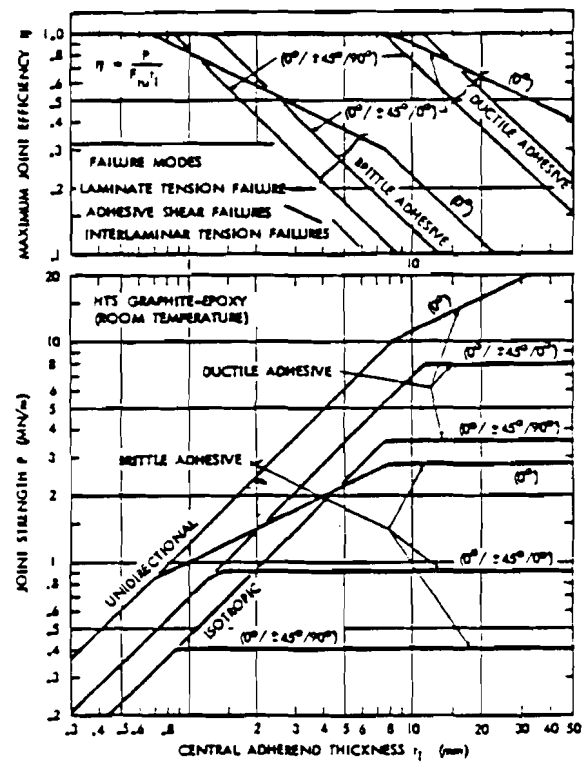


FIGURE 3.2.25 MAXIMUM EFFICIENCY AND JOINT STRENGTHS FOR GRAPHITE-EPOXY DOUBLE-LAP JOINTS

the laminate, outside the joint, is usually violated by the necessity of a finite thickness at the tip of the taper. One researcher has even calculated that a tip thickness of only 0.001 inch is equivalent to a stress concentration of 25 percent in the perfectly-elastic adhesive shear stress distribution. For this reason, the analysis of scarfed joints is often better performed by a stepped-lap joint program rather than one for scarf joints. This point is discussed in reference 3.2.20, which explains those conditions when a stepped-lap joint should not be treated as an approximation to a scarf joint. The key difference is that the shear strength of a true scarf joint continues to increase as the scarf angle is decreased, while the strength of a stepped-lap joint with a fixed number of steps does not continue to increase indefinitely as the total overlap is increased. This characteristic is confirmed by the analyses shown in figure 3.2.26 of various five-step bonded joints in which only the step lengths were varied. A four-fold increase in overlap was predicted necessary to increase the joint strength by barely ten percent. The prime factor in determining the strength of a stepped-lap joint is the number of steps, provided that there is not some gross error in the dimensioning of some critical step. Figure 3.2.27 shows that the number of steps is also very important in the use of a stepped-lap joint analysis when modeling the load transfer through a scarf joint. With the 32-step approximation shown, the thin end of the scarf is only 0.01 inch thick, yet there is an appreciable stress concentration in the adhesive shear stress. That calculation indicates why the stepped lap joint model is more appropriate than the scarf joint analysis for real scarf joints; the finite tips thickness is too significant to ignore.

3.2.8 Double-Lap Bonded Joints.

The first analysis of adhesive-bonded joints to merit special attention is that of Volkersen (reference 3.2.21) who, in 1938, computed the elastic adhesive shear stress distributions along the lap length in terms of the differential stretching of the adherends. This analysis pertains to symmetric (double-lap) joints, even though it was presented for single-lap joints. This analysis lacks the out-of-plane bending effects that Goland and Reissner later included in their analysis of unsupported single-lap joints. In 1973, Hart-Smith (reference 3.2.9) included nonlinear adhesive behavior, in the form of the elastic-plastic model discussed earlier, in this analysis. That approach forms the basis of the treatment presented here. That simple specification of the adhesive behavior permitted closed-form explicit predictions of the joint strengths. The inclusion of adhesive plasticity has a major effect on the results. The analysis is presented in Appendix III-A.

Figure 3.2.28 is a nondimensionalized expression of the solution for the special case of stiffness-balanced joints in which there is no adherend thermal mismatch. Those excluded effects move the curves in figure 3.2.28 around but do not change the form of the basic behavior. It is evident that, for "short" overlaps (with the extent of "short" increasing with greater adhesive plasticity), the joint strength is proportional to the bonded overlap. However, for "long" overlaps, the joint strength is essentially independent of the overlap and asymptotic towards a constant value for each amount of adhesive plasticity. There is, therefore, no point in increasing the bonded overlaps indefinitely. The "optimum" line shown has an origin which is explained in full below. (It suffices to say here that overlaps to the right of that line can increase neither the joint strength nor its environmental durability. When this figure was published in references 3.2.8, 3.2.9, 3.2.22, and 3.2.23, the optimum line was

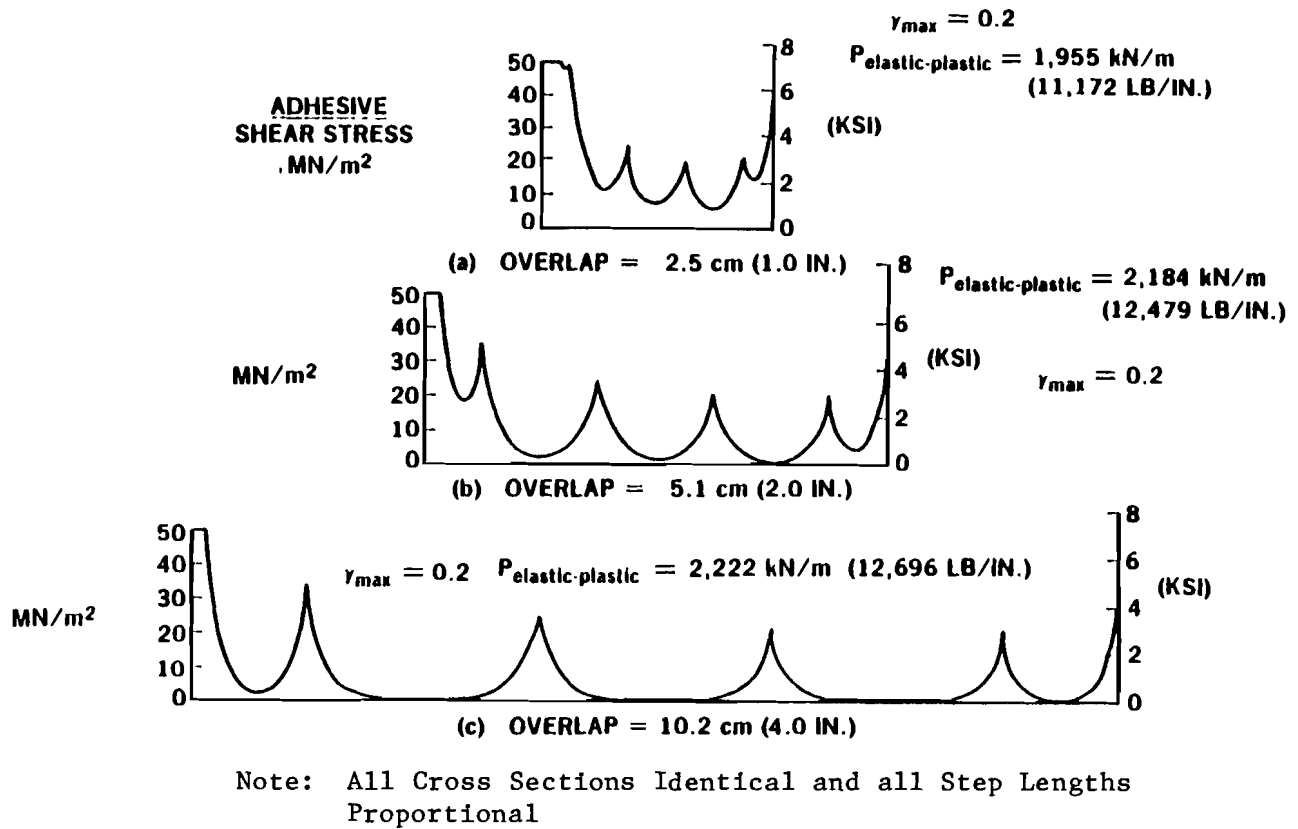


FIGURE 3.2.26 LIMITED EFFECT OF OVERLAP ON STRENGTH OF STEPPED-LAP BONDED JOINTS

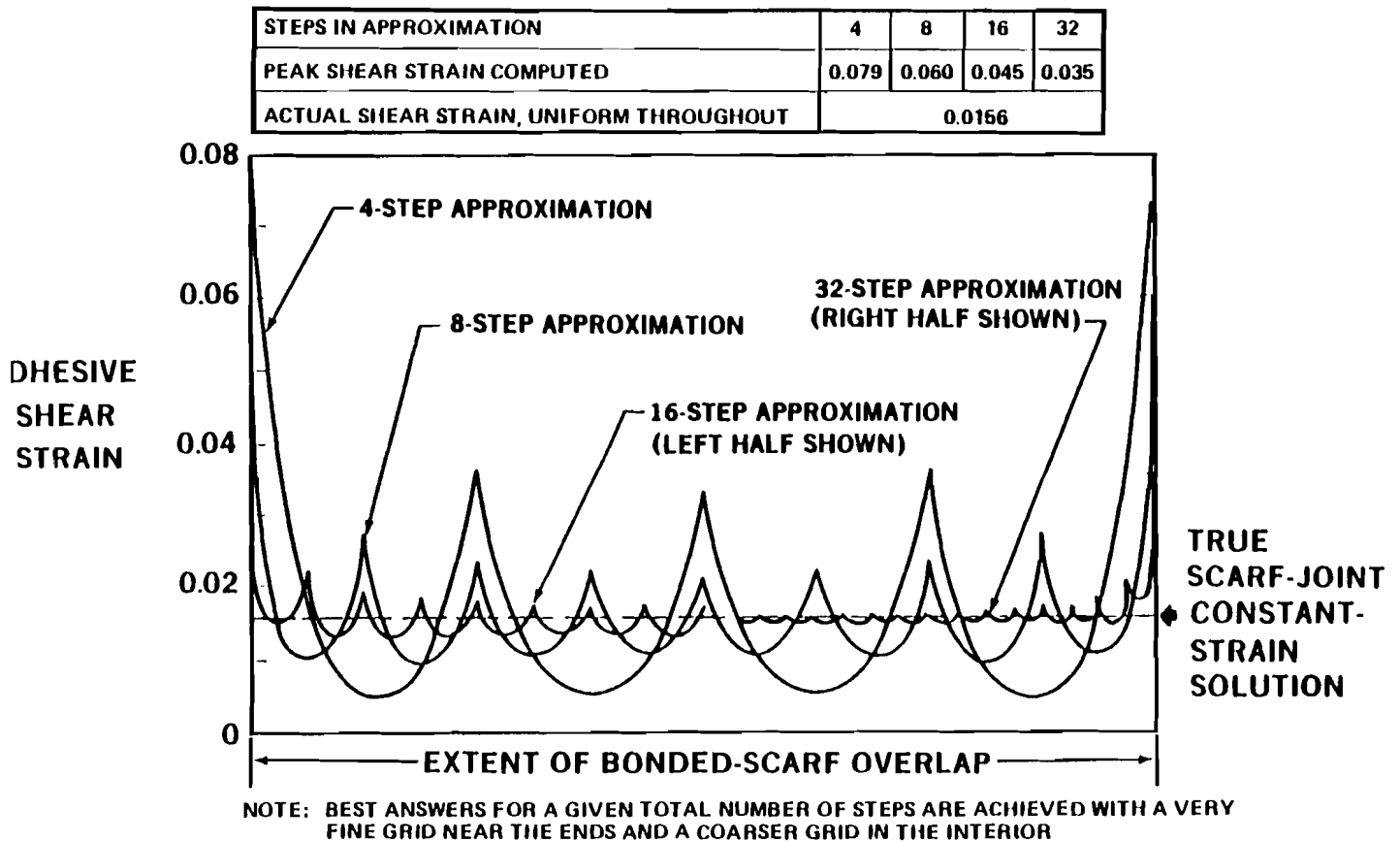


FIGURE 3.2.27 INFLUENCE OF PRECISION OF MODELING ON ANALYSIS OF SCARF JOINT

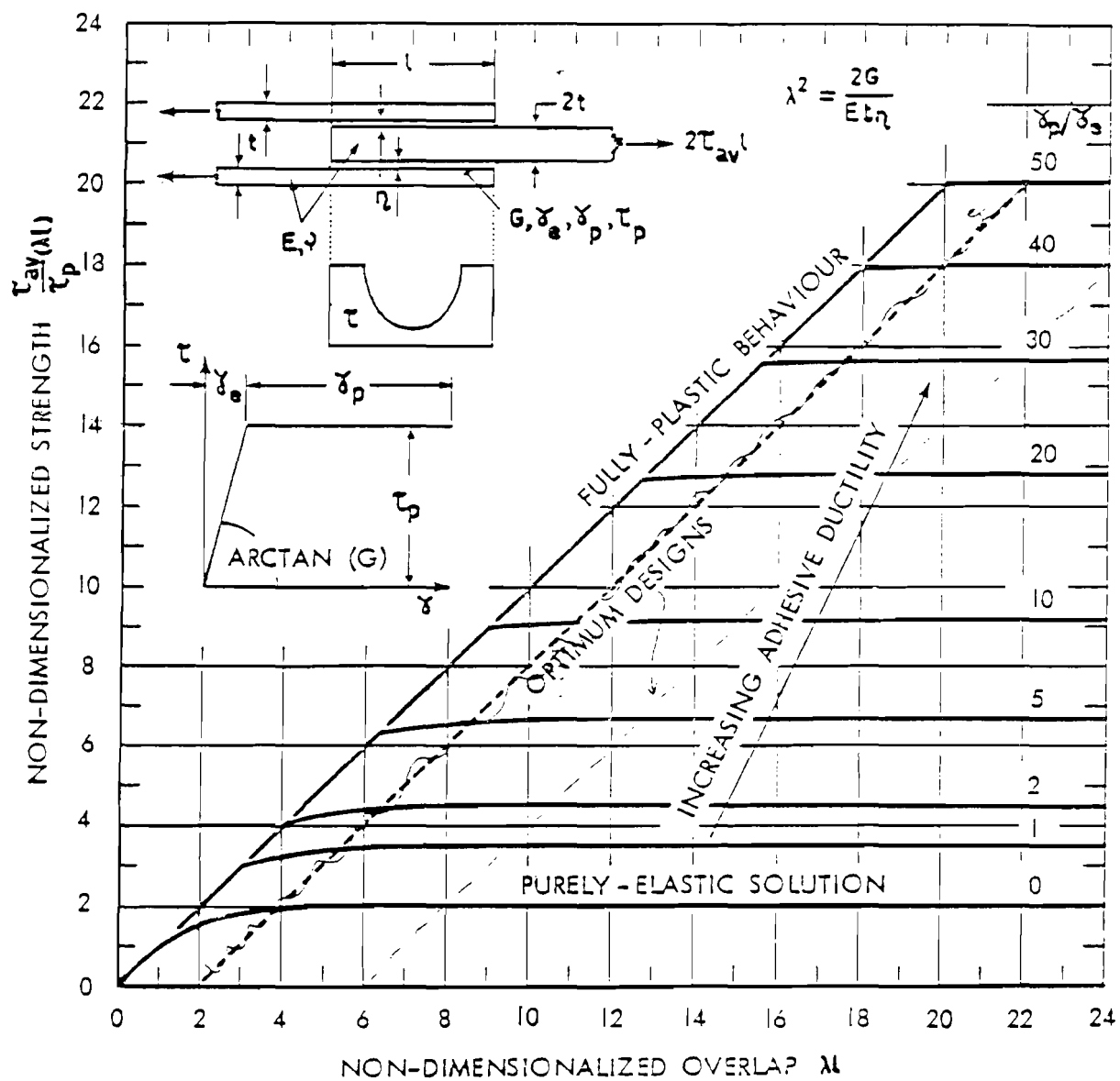


FIGURE 3.2.28 SHEAR STRENGTH OF DOUBLE-LAP BONDED JOINTS

shown further to the left, because its deviation then was based on consideration of static strengths only, without an awareness of the need for still longer overlaps to resist creep rupture failures in prolonged service.) The physical reason for the independence of the joint strength from the bonded overlap is that the load transfer is always confined to one or both ends of the overlap, with a deep elastic trough in between.

3.2.9 Differences Between Behavior of Test Coupons and Structural Joints.

The analysis above for double-lap and double-strap joints indicates substantially different behavior of the adhesive in "short" overlap test coupons, on the one hand, and "long" overlap structural joints, on the other. This mixed influence of lap length on the shear strength of such bonded joints is explained more fully in figure 3.2.29, of which the top curve is easily seen to have the same form as a typical curve from figure 3.2.28. The joint strength is initially proportional to the short overlap, as at A. Then, as the overlap is increased still further, the strength increases only very slightly as the elastic trough is developing at B. Finally, no matter how much longer the overlap is made, as at C, the joint strength remains constant and the maximum adhesive shear stress decreases asymptotically towards zero, while the load transfer zones keep the same width but move steadily further apart.

The design of such joints is thus reduced to the simple task of identifying the overlap for which the minimum stress has become low enough. Any further increase in overlap cannot increase the joint strength. This process is explained in figure 3.2.30.

The formula from which the overlap is calculated, as the sum of plastic and elastic zones, is

$$l = \frac{t_i \sigma_{ult}}{2\tau_p} + \frac{6}{\lambda}, \quad (3.2.1)$$

in which σ_{ult} is the ultimate strength of the central adherend. The results can be summarized by the simple approximation that each bonded overlap is roughly 30 times as long as the thickness of the inner adherend. The same ratio would also apply to typical cross-ply carbon epoxy laminates. This length of 30t for structural joints is in marked contrast to the typical 8t used for test coupons.

Figures 3.2.31, 3.2.32, and 3.2.33 depict the maximum and minimum adhesive shear strains, for each of three environments, as a function of the bonded overlap and the thickness of aluminum alloy adherends. In curves A and B in each figure, the joint strength is limited by the adherend strength outside the joint, rather than by the adhesive shear strength as in the case of curve C. Those adherend strength limits cause an abrupt change in the character of curves A and B in each of these figures. For very short overlaps, the joint strength is limited by the plastic adhesive shear strength and the minimum adhesive shear strain is almost as high as the maximum. For longer overlaps, to the right of the vertical discontinuities, there is a definite upper limit on the maximum adhesive shear strain at a value less than sufficient to fail the adhesive. The margin is greater for the thinner adherends, in curves A. The minimum shear strain in the adhesive decrease asymptotically to zero for long overlaps in all of the curves.

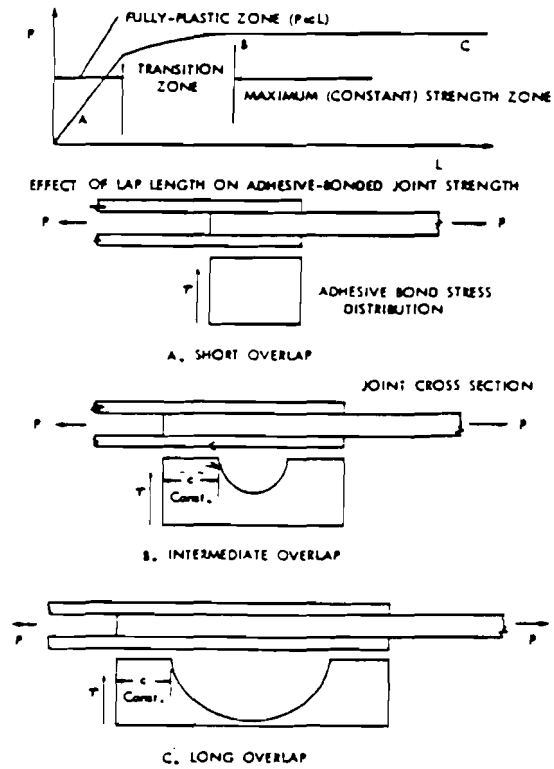


FIGURE 3.2.29 INFLUENCE OF LAP LENGTH ON BOND STRESS DISTRIBUTION

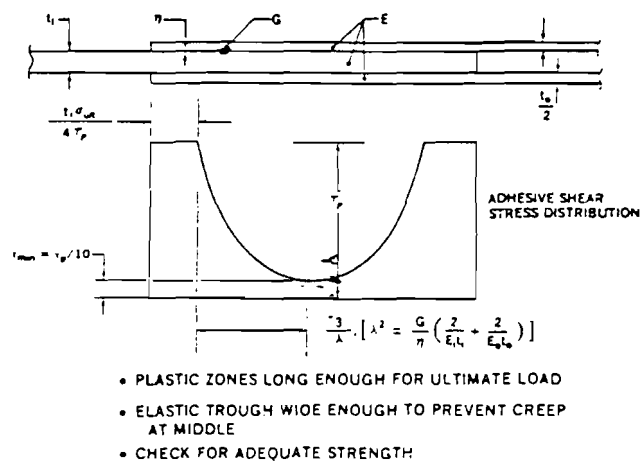


FIGURE 3.2.30 DESIGN OF DOUBLE-LAP BONDED JOINTS

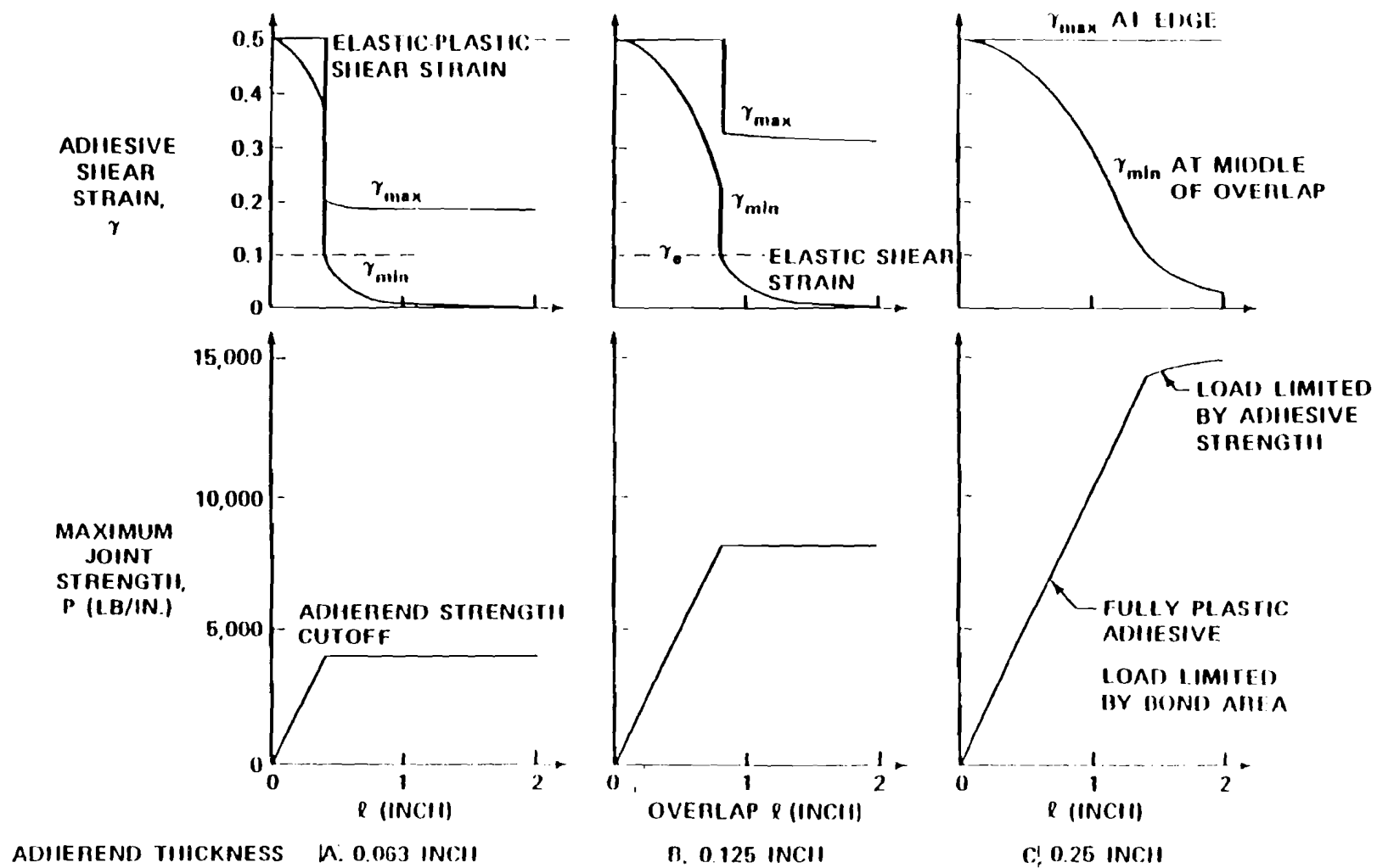


FIGURE 3.2.31 EFFECTS OF OVERLAP AND THICKNESS ON MAXIMUM AND MINIMUM ADHESIVE-BOND SHEAR STRAINS (ROOM TEMPERATURE ENVIRONMENT)

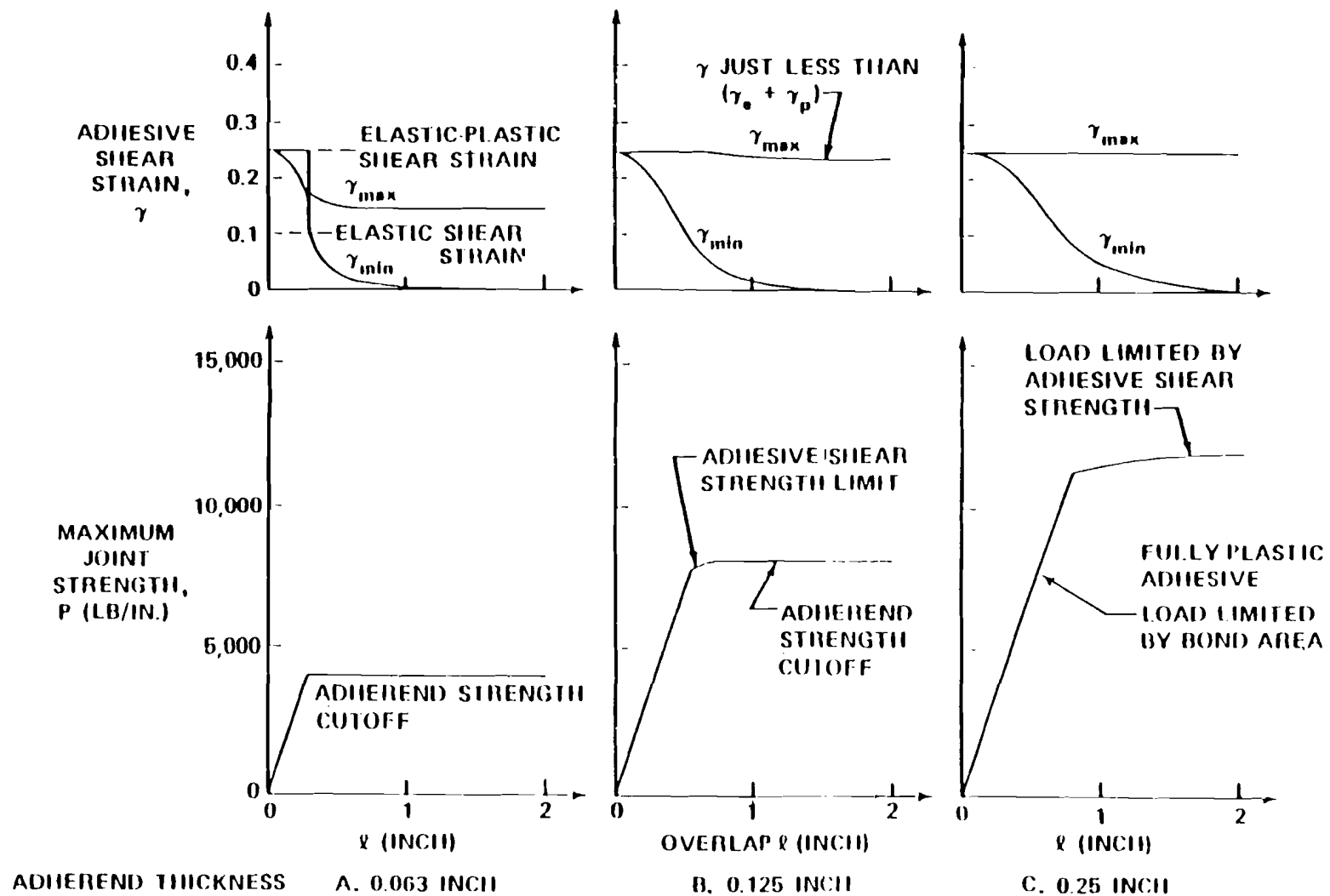


FIGURE 3.2.32 EFFECTS OF OVERLAP AND THICKNESS ON MAXIMUM AND MINIMUM ADHESIVE-BOND SHEAR STRAINS (COLD ENVIRONMENT)

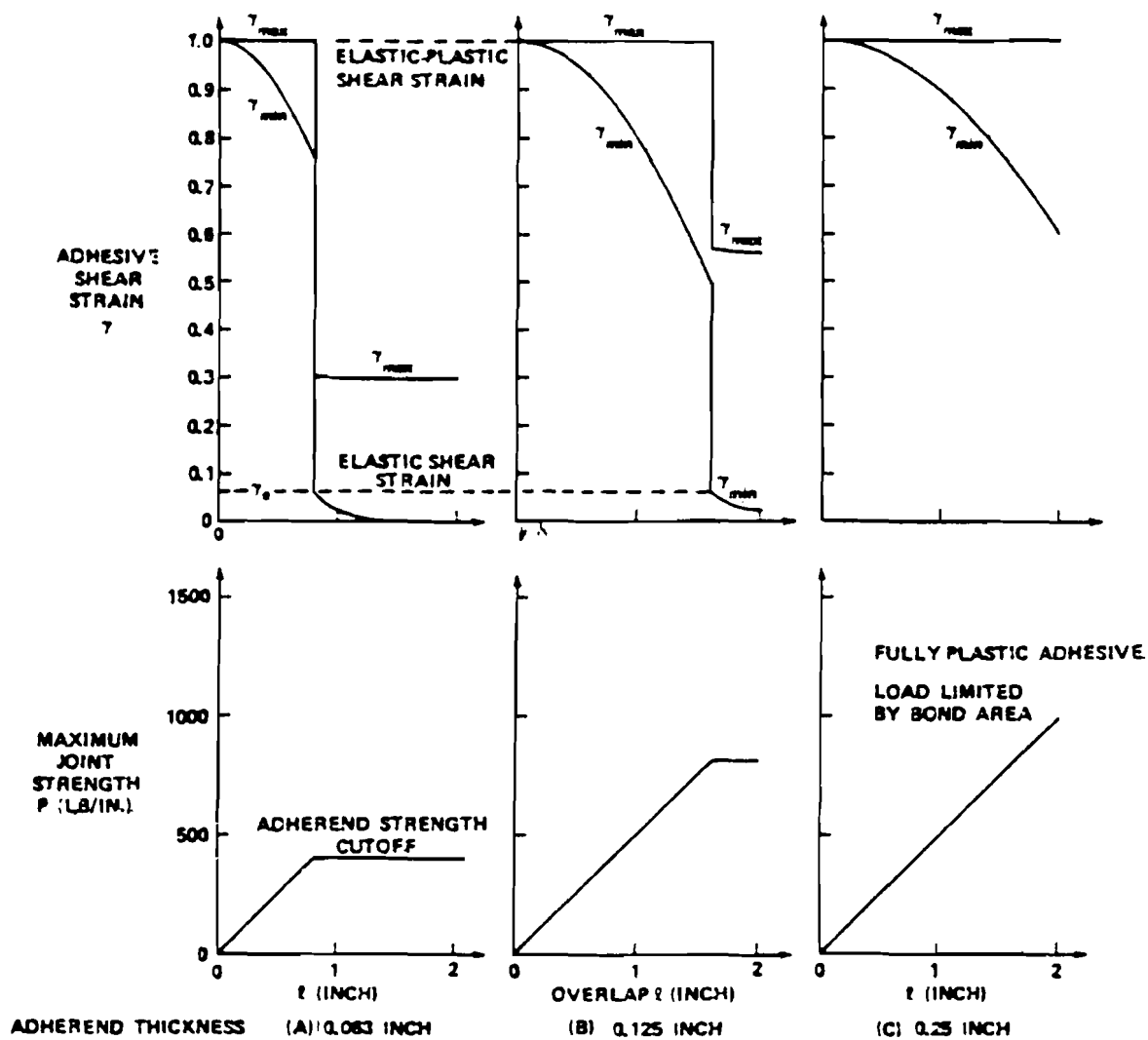


FIGURE 3.2.33 EFFECTS OF OVERLAP AND THICKNESS ON MAXIMUM AND MINIMUM ADHESIVE-BOND SHEAR STRAINS (HOT, WET ENVIRONMENT)

Thus, only for short overlap test coupons could the adhesive fail by creep rupture. Provided that this simple joint geometry is not used for adherends that are so thick as to require a more complex joint, as in curves C, the adhesive can never be the weak link when the overlap is long enough, even in the case of load redistribution around bond flaws.

The vertical precipices in curves A and B occur at longer overlaps for the hot/wet environment, in figure 3.2.33, so that becomes the design condition. In such a case, τ_p has the smallest value in equation (3.2.1) and so does the exponent λ . It is usual for the upper temperature limit to set the overlaps. However, as can be seen from curves B in figure 3.2.32, the maximum adherend thickness suitable for bonding with the simple double-lap configuration may be set by the lowest temperature, at which the adhesive becomes more brittle.

Figures 3.2.31 and 3.2.33 explain how short-overlap joints would inevitably fail by creep rupture under sustained or repeated loads. The adhesive in long-overlap joints also undergoes creep under sustained loads; the difference is that this creep does not accumulate. This is explained in figure 3.2.34. In the lower diagram (b), the minimum adhesive shear strain is almost as much creep as at the edges. In the upper diagram (a), on the other hand, there can be no creep under any circumstances between points D and E because the adhesive shear strain at A is too low. Thus the creep which occurs between points F and G due to the high strain at C grows at a diminishing rate. Furthermore, since there is also no creep in the adherends, between points F and D and G and E respectively, the removal of the load induces residual stresses in the adherends in such a manner as to push the adhesive back to its original state during periods of load reversal or load removal.

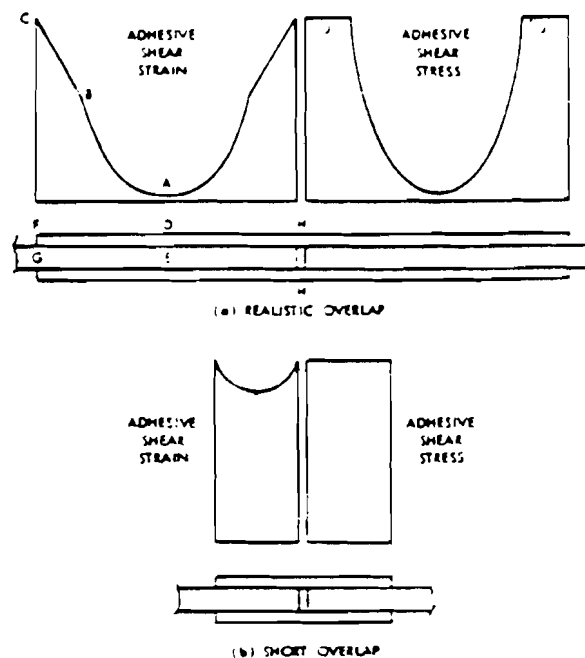
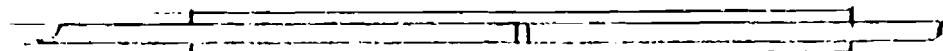
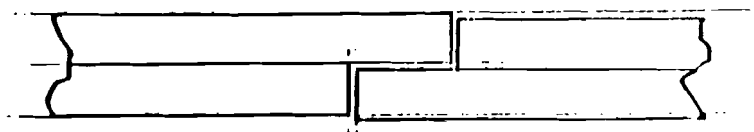


FIGURE 3.2.34 NONUNIFORM STRESSES AND STRAINS IN BONDED JOINTS

These two distinct behaviors for short- and long-overlap joints (figure 3.2.34) make it clear that the behavior of each under load is very different. This is the reason it is important to interpret very carefully the results of any adhesive test coupons, i.e., to make sure that they either do apply to structurally configured joints, or that the data is capable of being reinterpreted meaningfully. There is frequently no correlation between the two situations, as explained in figure 3.2.35. That is not to imply that all coupon testing is bad; just that it must be assessed with extreme caution, as explained in reference 3.2.3.

3.2.10 Single-Lap Bonded Joints.

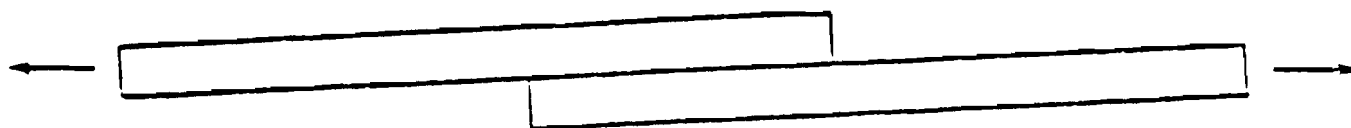
Double-lap and double-strap joints have been shown in aforementioned sections above to have a strength that is essentially independent of the overlap for structurally configured joints. Unsupported single-lap and single-strap joints (figure 3.2.20) differ from all the other bonded joints in that these eccentric joints exhibit a marked dependence on the overlap-to-thickness (ℓ/t) ratio of the adherends. The distinctive failure phenomenon are depicted in figure 3.2.36 for both brittle (composite) and ductile (metal) adherends. The analysis of these joints is presented in Appendix III-B.



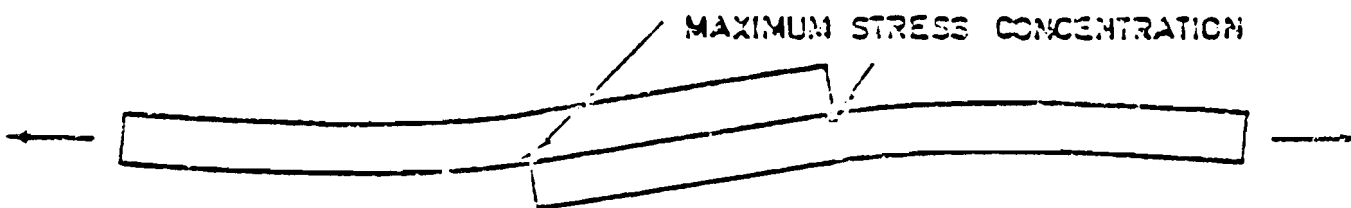
- UNIFORM SHEAR STRAIN IN ADHESIVE
- ADHESIVE CREEP OCCURS EVERYWHERE
- ONLY WEAK ADHESIVE LOCKED-IN STRESSES TO RESTORE TO ORIGINAL STATE
- JOINT CANNOT RECOVER DURING UNLOAD PERIODS, SHORT OR LONG
- ENVIRONMENTALLY DEGRADED
- USEFUL TEST TO INVESTIGATE ADHESIVES
- RESULTS NOT NECESSARILY INDICATIVE OF CONFIGURED JOINTS

- NONUNIFORM ADHESIVE SHEAR STRAIN
- ADHESIVE CREEP OCCURS ONLY AT ENDS OF OVERLAP – NONE IN ELASTIC TROUGH
- HIGH LOCKED-IN METAL STRESSES TO RESTORE TO ORIGINAL CONFIGURATION
- JOINT RECOVERS DURING LONG UNLOAD PERIODS
- NO EVIDENCE OF IN-SERVICE DEGRADATION, EXCEPT AT INTERFACE WITH POOR SURFACE TREATMENT
- MOST FAILURES IN METAL
- NOT SUITABLE FOR COMPARING ADHESIVES
- NECESSARY TO VERIFY ENVIRONMENTAL DURABILITY

FIGURE 3.2.35 THICK ADHEREND VERSUS AIRCRAFT CONFIGURATION BONDED JOINTS



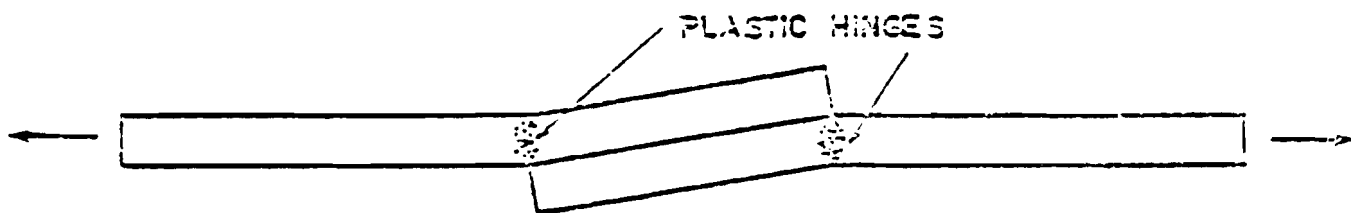
LOW LOAD LEVEL



DEFORMATION OF ADHERENDS UNDER MODERATE LOAD



"INTERLAMINAR" FAILURE OF FILAMENTARY COMPOSITE ADHERENDS



HIGH LOAD LEVEL (METAL ADHERENDS)



PERMANENTLY DEFORMED (METAL) ADHERENDS
AFTER FAILURE OF ADHESIVE

FIGURE 3.2.36 FAILURE OF SINGLE-LAP BONDED JOINTS WITH
BRITTLE AND WITH YIELDING ADHERENDS

The relation defined by equation 3.2.B21 is plotted in figure 3.2.37 for aluminum alloy adherends of various overlap-to-thickness ratios for a range of nominal adherend stresses outside the joint areas. Increases in the bonded overlap (2c) are shown to be a very powerful means of increasing the joint efficiency because the entire adherend outside the joint can then be operated safely at a higher stress.

The primary consequence of the bending moment $\hat{M}_e$, due to the eccentricity in the load path, is a reduction in the remote adherend stress for a given maximum allowable stress. The ratio of this average to maximum stress represents the structural efficiency of the panel - not just of the joint - and for isotropic adherends is expressible in the form

$$y = \frac{\sigma_{avg}}{\sigma_{max}} = \frac{\sigma_{avg}}{\sigma_{avg} + 6M/t^2} = \frac{1}{1 + 3/[1 + \frac{1}{6}\epsilon^2 c^2]} \quad (3.2.2)$$

as plotted in the lower portion of figure 3.2.38. The upper portion is the inverse of this quantity and represents the severity of the stress concentration. Those upper curves in figure 3.2.38 are directly comparable with the lower curves in figure 3.2.39 and give some indication of the conversation inherent in the Goland and Reissner solution.

The bending moment induced in the adherends is thus seen to be greatly dependent on the ℓ/t ratio because, for long overlaps, the adherends can easily deflect to towards the load axis, under a tensile load, to relieve the bending moment. Out-of-plane (normal) deflections associated with short overlaps would necessarily involve higher curvatures and bending stresses. Out-of-plane deflections associated with compressive bending would increase the bending moment beyond that which linear static analysis would predict, instead of relieving it. However, there would no longer be a peel-stress problem of the nature described below. In-plane shear loading would induce out-of-plane displacements only at the corners of the bonded area and neither of the analyses given here would cover that case; it would be appropriate to support the corners and use the double lap analysis in such a case, treating such a single-lap joint as one side of a double-lap joint of twice the total thickness.

In order to minimize these bending stresses associated with the eccentricity in load paths, single-lap and single-strap joints should have very large overlaps, typically 50 to 100 times the adherend thickness - not to decrease the adhesive shear stresses because of a large bond area, but to decrease the bending stresses in the adherends. As a rule of thumb, ℓ/t ratios of about 10:1 are associated with joint strengths of only about 1/3 of the remote strength away from the joint, while an ℓ/t ratio of 50 corresponds with about 75 percent efficiency and an ℓ/t of 100 with better than 90 percent. The derivations below explain how the same reduction in bending moment has a proportionate reduction in the peak adhesive peel stress also.

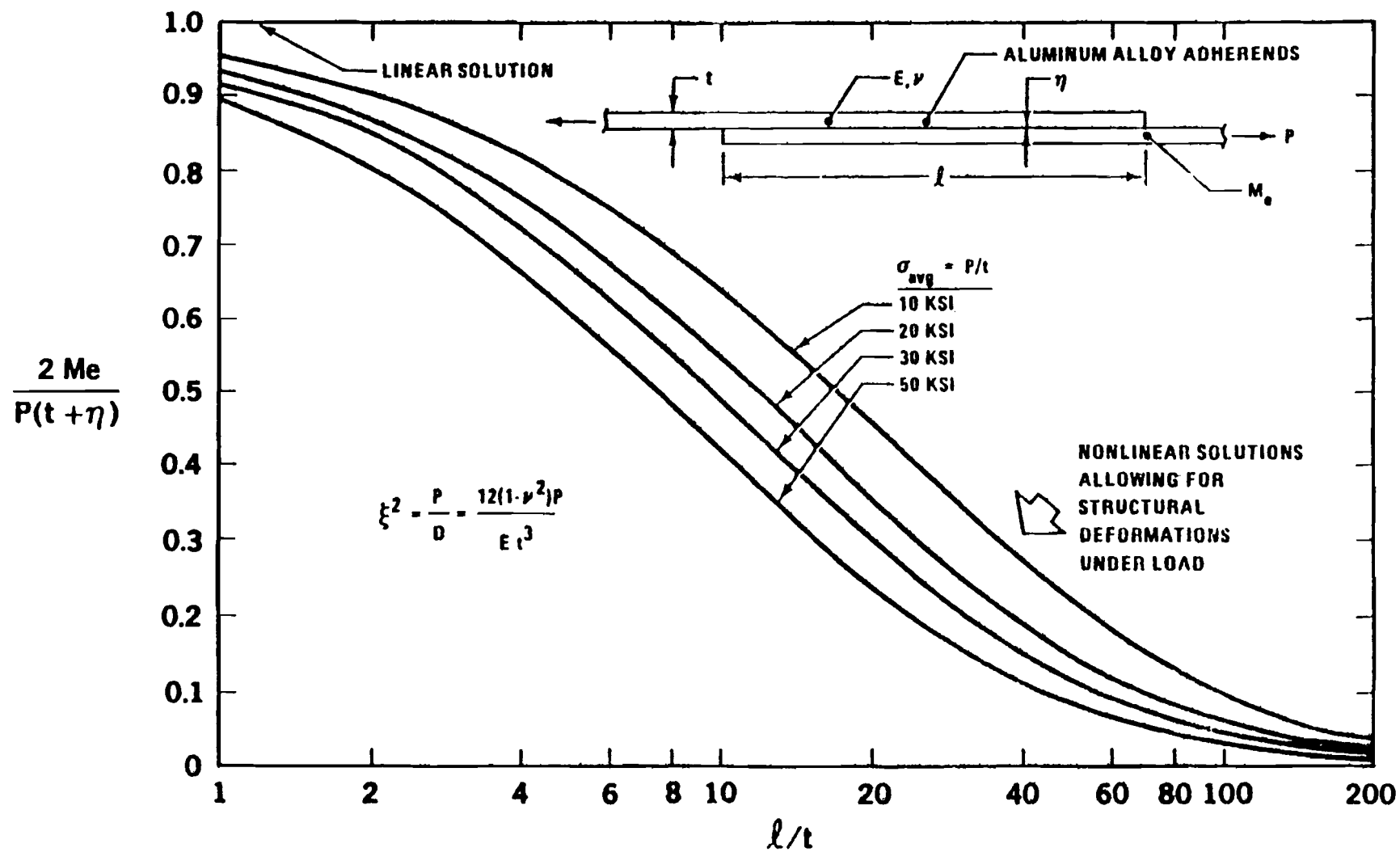


FIGURE 3.2.37 NONDIMENSIONAL ADHEREND BENDING MOMENT IN SINGLE-LAP JOINTS

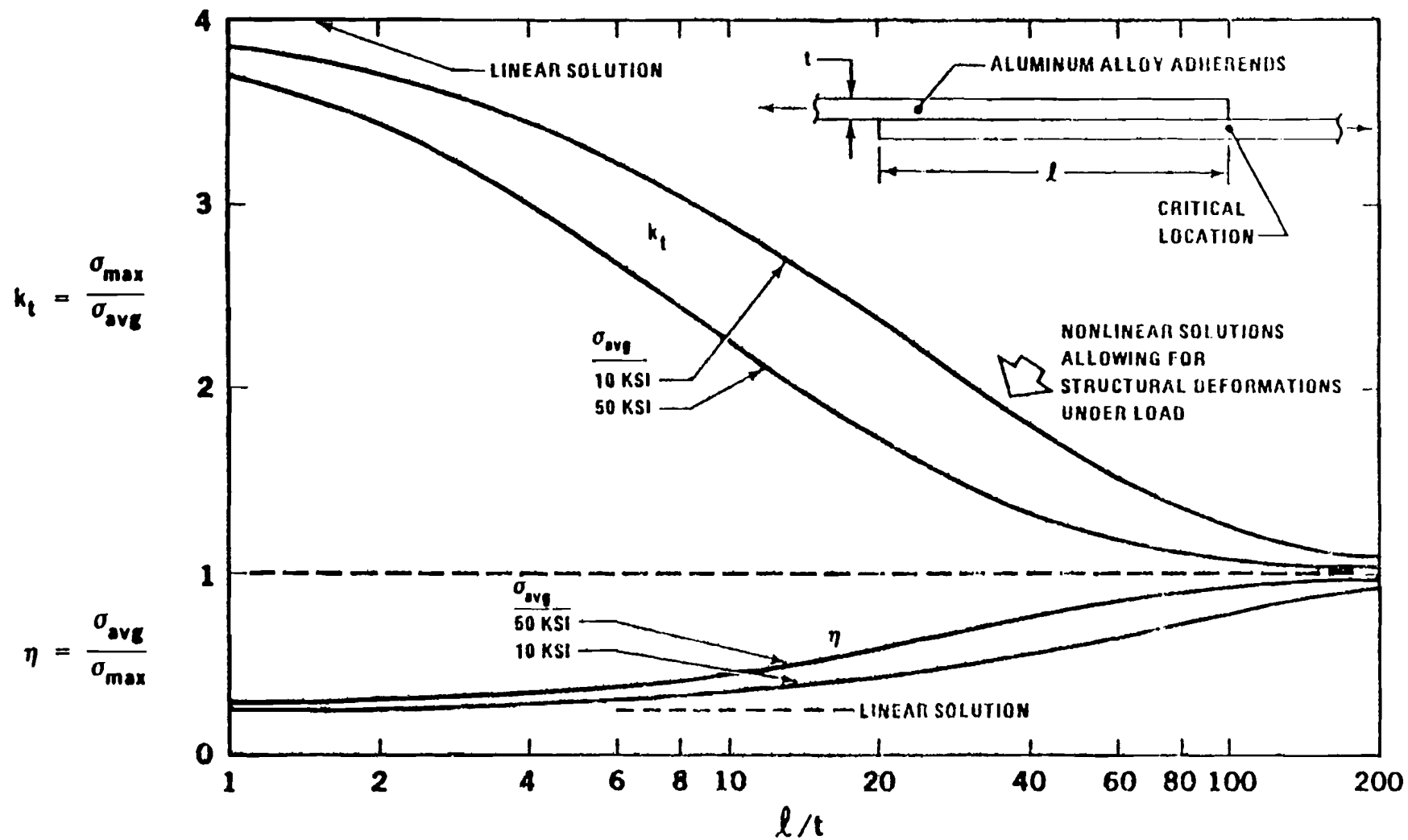


FIGURE 3.2.38 ADHEREND STRESS CONCENTRATION AND JOINT EFFICIENCY OF SINGLE-LAP JOINTS

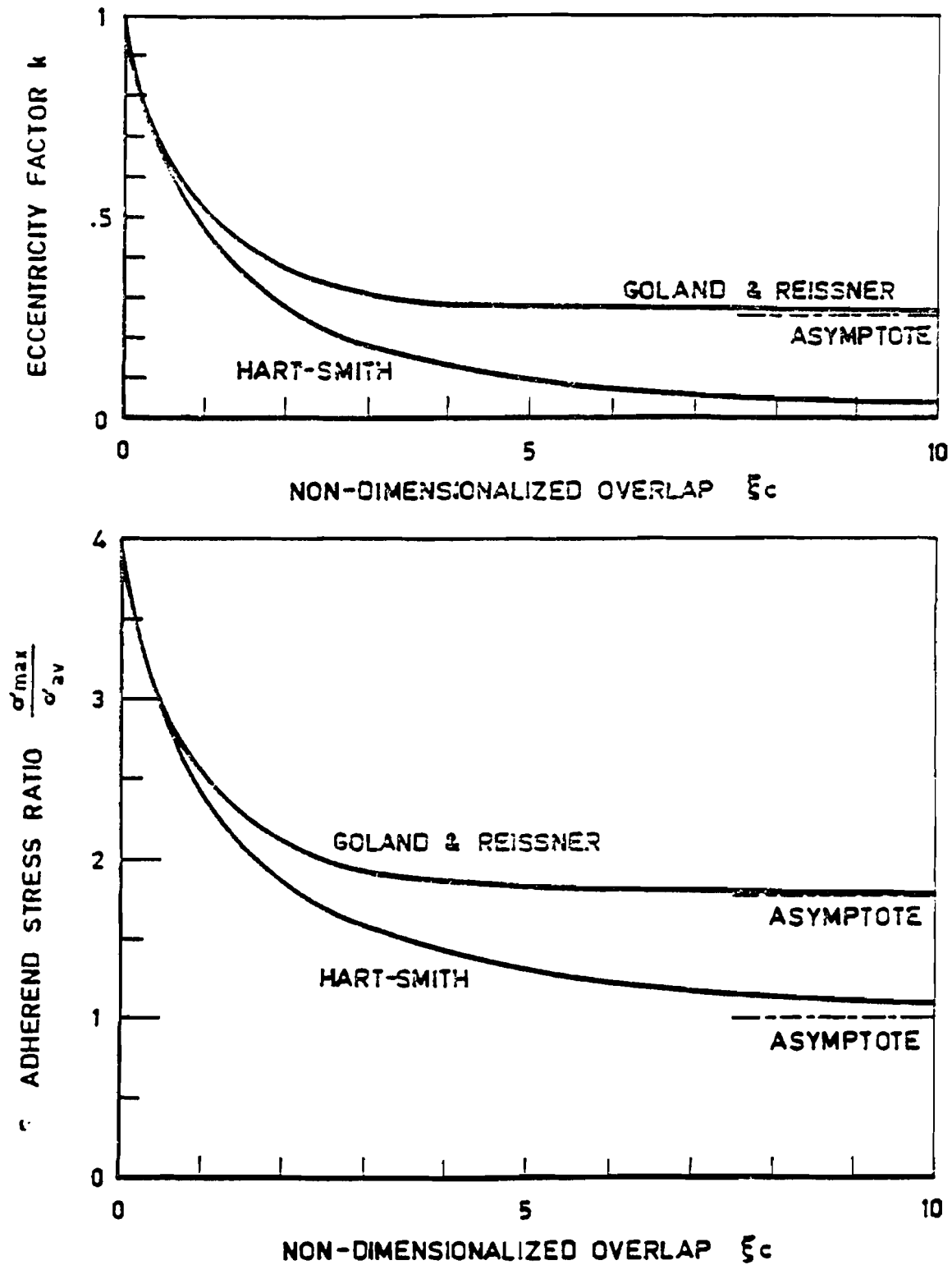


FIGURE 3.2.39 COMPARISON BETWEEN PRESENT SOLUTION FOR ADHEREND STRESSES AND THAT OF GOLAND AND REISSNER

Results of the analysis are presented in figure 3.2.40. They depict the nondimensional shear strengths of adhesive-bonded single-lap joints and, while of generally similar form to figure 3.2.28 for double-lap joints, contains certain distinct differences. Note the factor of two differences between the ordinate scales, to reflect the number of layers of adhesive. For long overlaps and very small eccentricities (see the effect of that parameter in figure 3.2.41), the adhesive in an unsupported single-lap joint is twice as effective as in a double-lap joint - because of the contribution from the equation illustrated in figure 3.2.40. On the other hand, for short overlaps with no eccentricity relief, the shear strength of the adhesive in single-lap joints is only half as high as for the equivalent geometry in a double lap joint.

Figures 3.2.40 and 3.2.41 do not lend themselves to any of the simplified rules deduced for figure 3.2.28 but, since the strength of single-lap joints is rarely, if ever, limited by the adhesive shear strength (as shown in figures 3.2.42 and 3.2.43), that complexity is of little consequence. Those figures (taken from reference 3.2.10) show how, for thin adherends or a large ℓ/t ratio, the joint efficiency η [defined in Equation (3.2.B23) and not to be confused with the adhesive layer thickness y] is limited by the bending moment in the adherend, just outside the joint. For thick adherends or small ℓ/t ratios, much weaker joints are associated with induced peel failures. There are no adhesive shear failures depicted at all in figure 3.2.42 and only a few in the upper right corner of figure 3.2.43, and those are at thicknesses beyond the normal upper limit of about 0.07 inch adherends.

The derivations above have referred to joints between identical adherends since, otherwise, the adhesive shear and peel stresses would not be governed by uncoupled equations. A more general analysis is given in reference 3.2.10 and relies on the adhesive plasticity to uncouple those equations. Figure 3.2.44 shows the effects of the adherend stiffness imbalance on the adherend bending strength of single-lap bonded joints. The upper curve in this diagram characterizes the efficiency of single-lap bonded joints between identical adherends, showing the powerful influence of the overlap-to-thickness ($2c/t$) term. The other curves show how adherend dissimilarities aggravate the effect of the eccentricity in load path by concentrating the adherend bending at that end of the overlap from which the thinner adherend extends.

Reference 3.2.10 also contains expended derivations for layered adherends rather than for the isotropic materials discussed here. There are no new effects not evident from the present results, but the simple $6/t^2$ relation between the bending and stretching stiffnesses no longer applies. These effects tend to be small because there must always be fibers parallel to the load direction on each side of the adhesive layer in order to avoid the weakness associated with the rolling of lateral fibers in such a position. Then, in order that the laminate does not warp during cool-down after cure, the outermost fibers must also be parallel to the load direction. Those specified layers of fibers then dominate the adherend bending stiffnesses. Nevertheless, this effect should not be ignored completely, since single-lap joints are suitable only for the bonding of thin adherends and they suffer most from the effects of non-homogeneity.

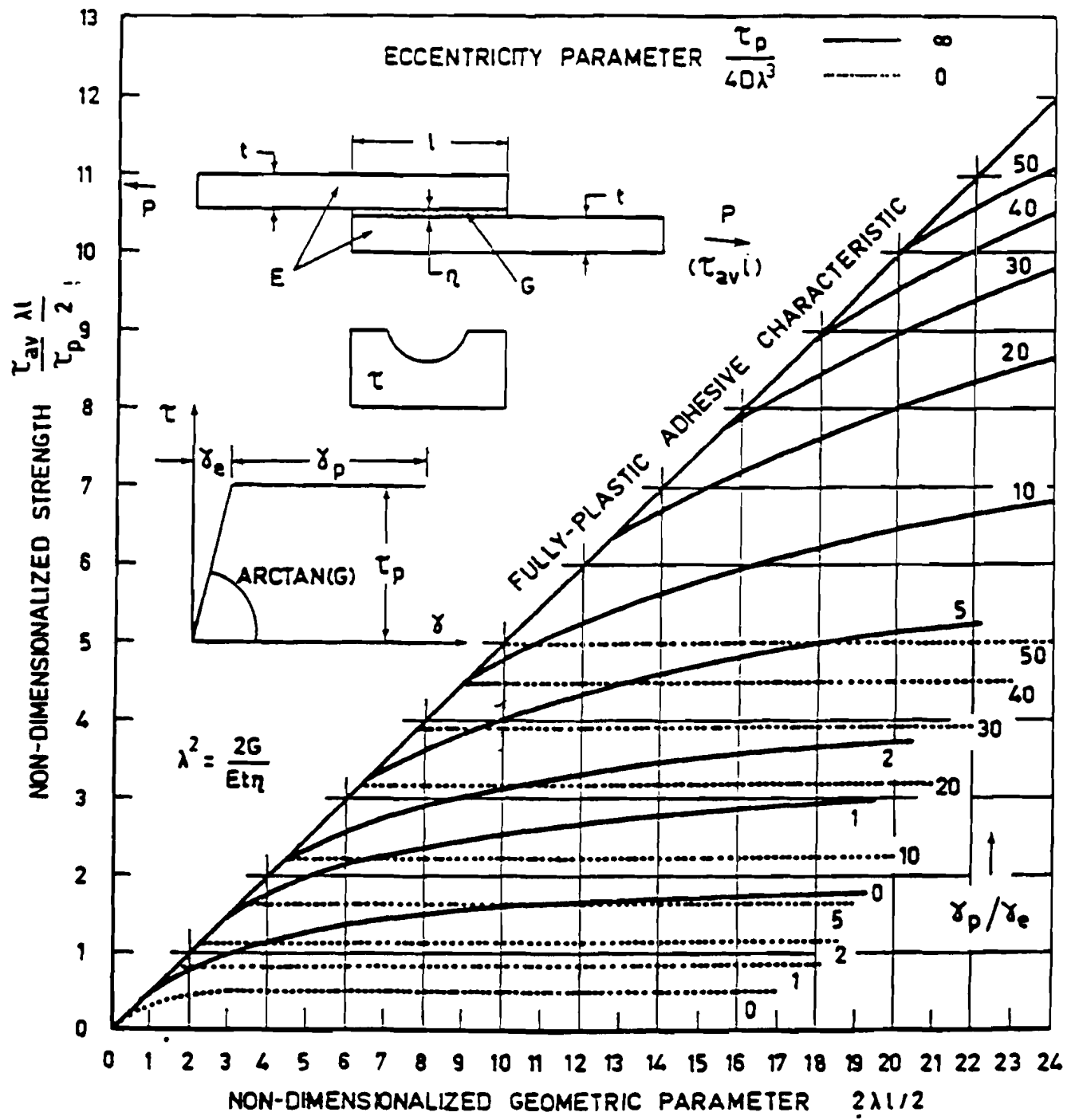


FIGURE 3.2.40 INFLUENCE OF ADHESIVE PROPERTIES ON SHEAR STRENGTH OF SINGLE-LAP BONDED JOINTS

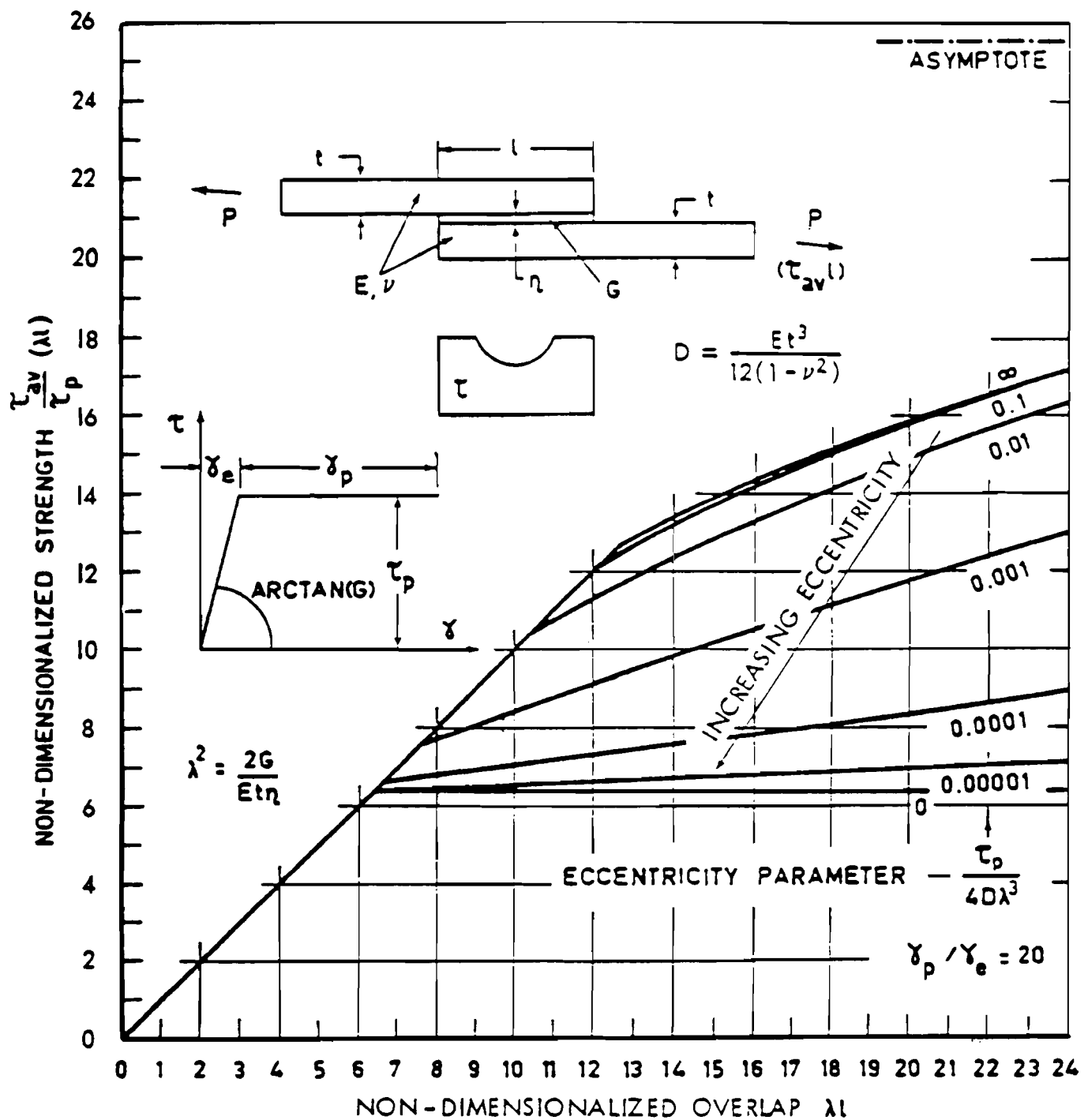


FIGURE 3.2.41 INFLUENCE OF ADHEREND PROPERTIES ON SHEAR STRENGTH OF SINGLE-LAP BONDED JOINTS

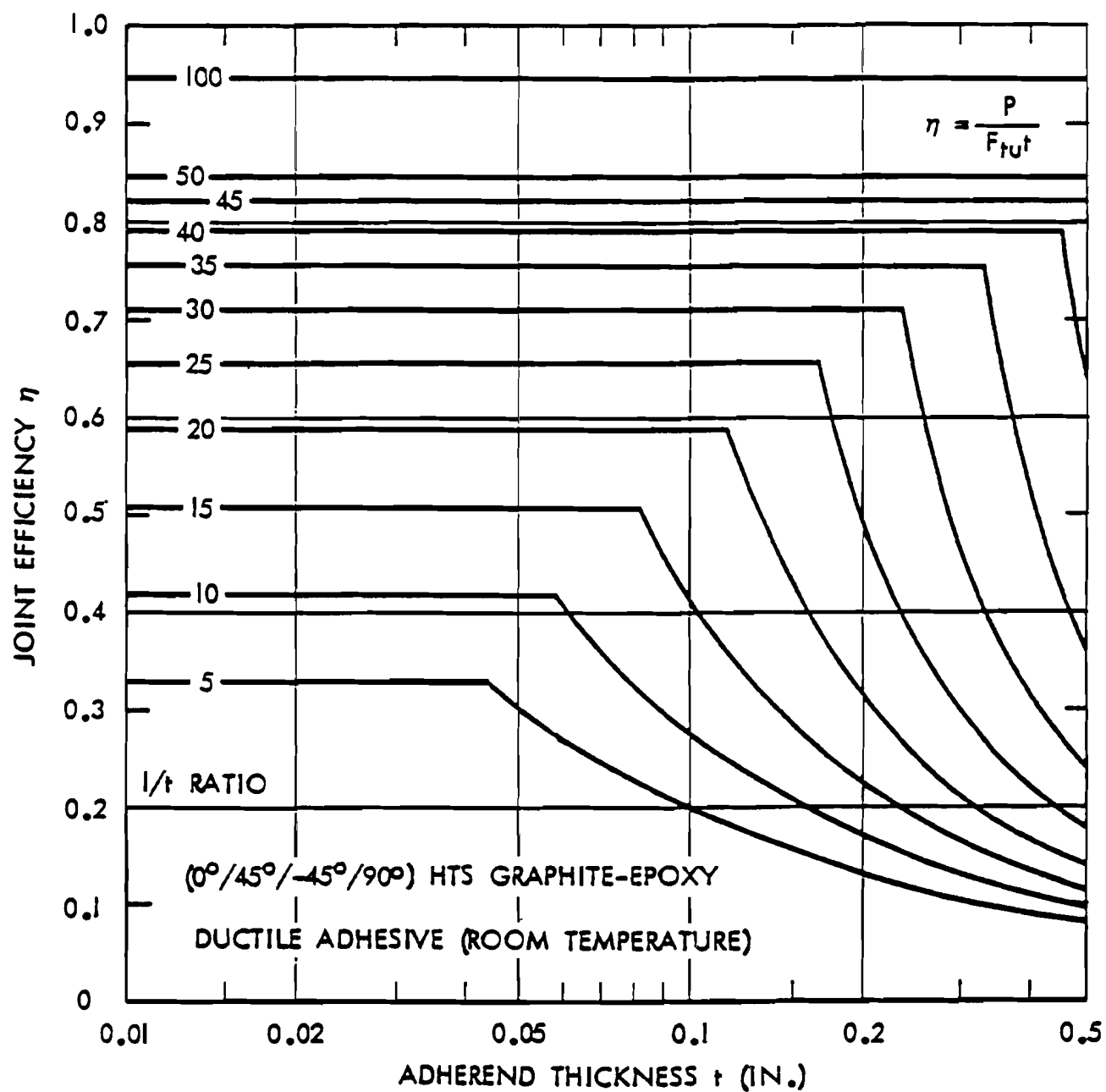


FIGURE 3.2.42 JOINT EFFICIENCY CHART FOR SINGLE-LAP COMPOSITE JOINTS

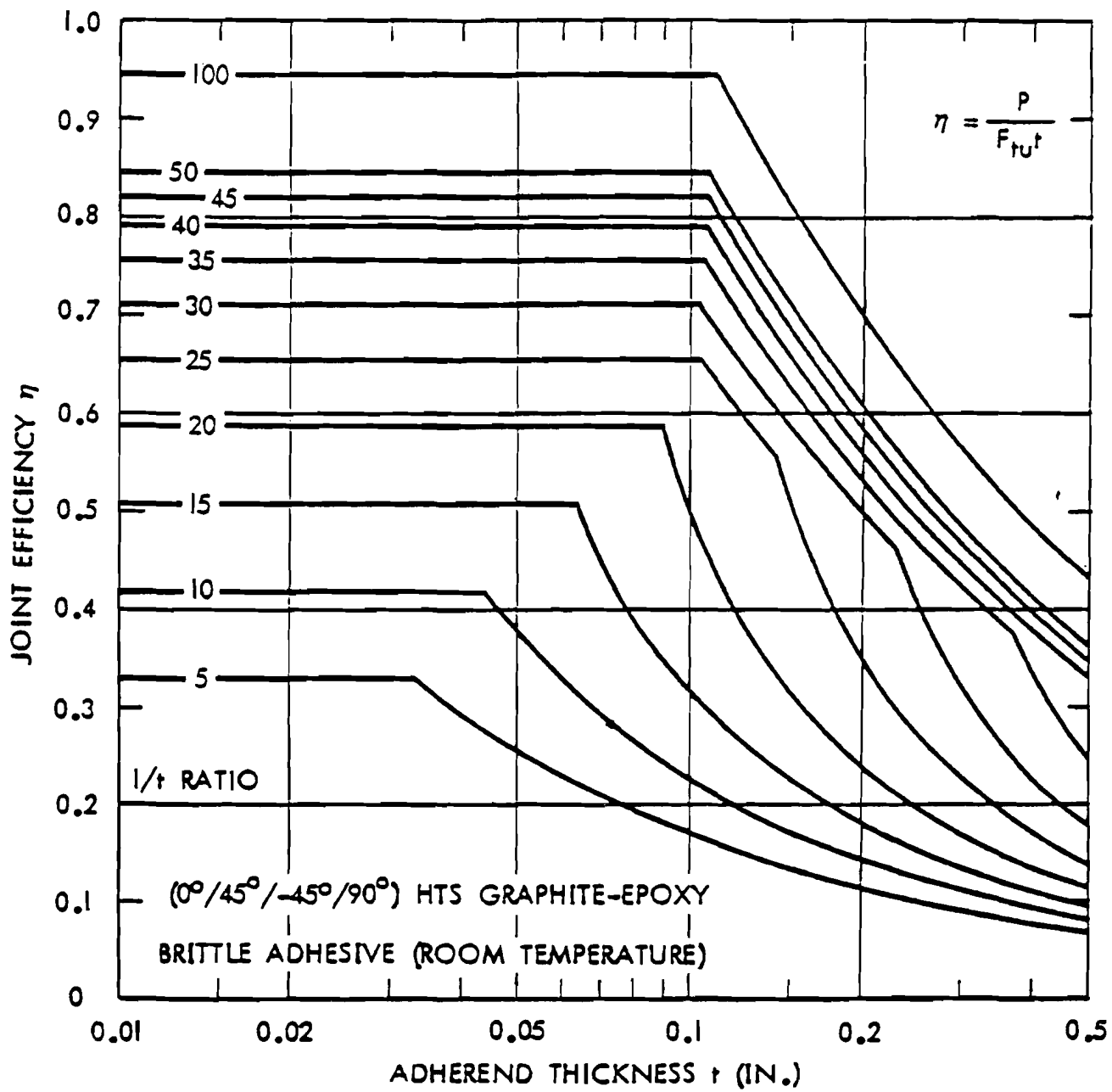


FIGURE 3.2.43 JOINT EFFICIENCY CHART FOR SINGLE-LAP COMPOSITE JOINTS

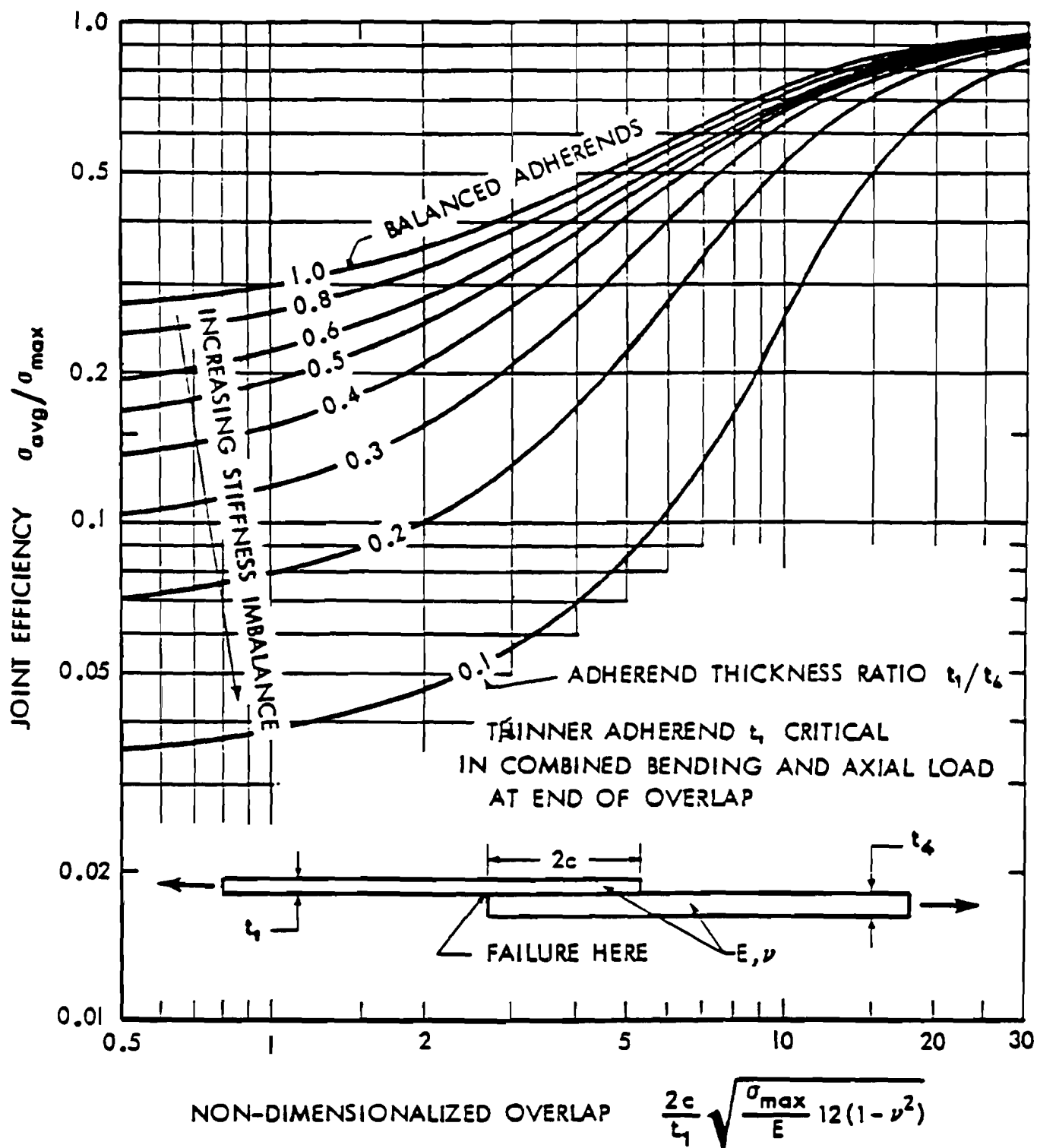


FIGURE 3.2.44 EFFECT OF ADHEREND STIFFNESS IMBALANCE ON ADHEREND BENDING STRENGTH OF SINGLE-LAP BONDED JOINTS

The single-strap joints are also sensitive to the ℓ/t ratio in establishing the bending moments and peel stresses at the ends of each bonded overlap. The values are no longer identical as shown in reference 3.2.2, and the need for very long overlaps is even greater because of the lack of length with which to provide flexibility at the point where the two skins butt together.

3.2.11 Adhesive-Bonded Scarf Joints.

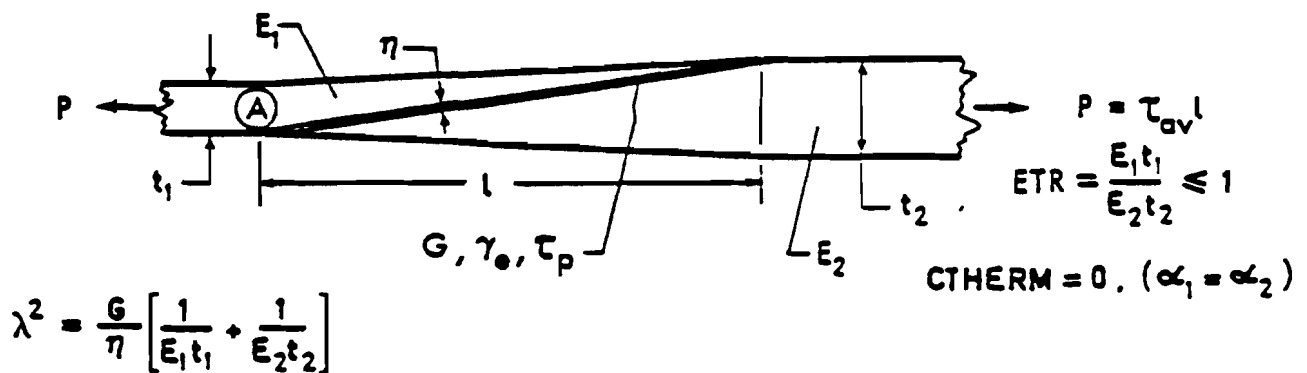
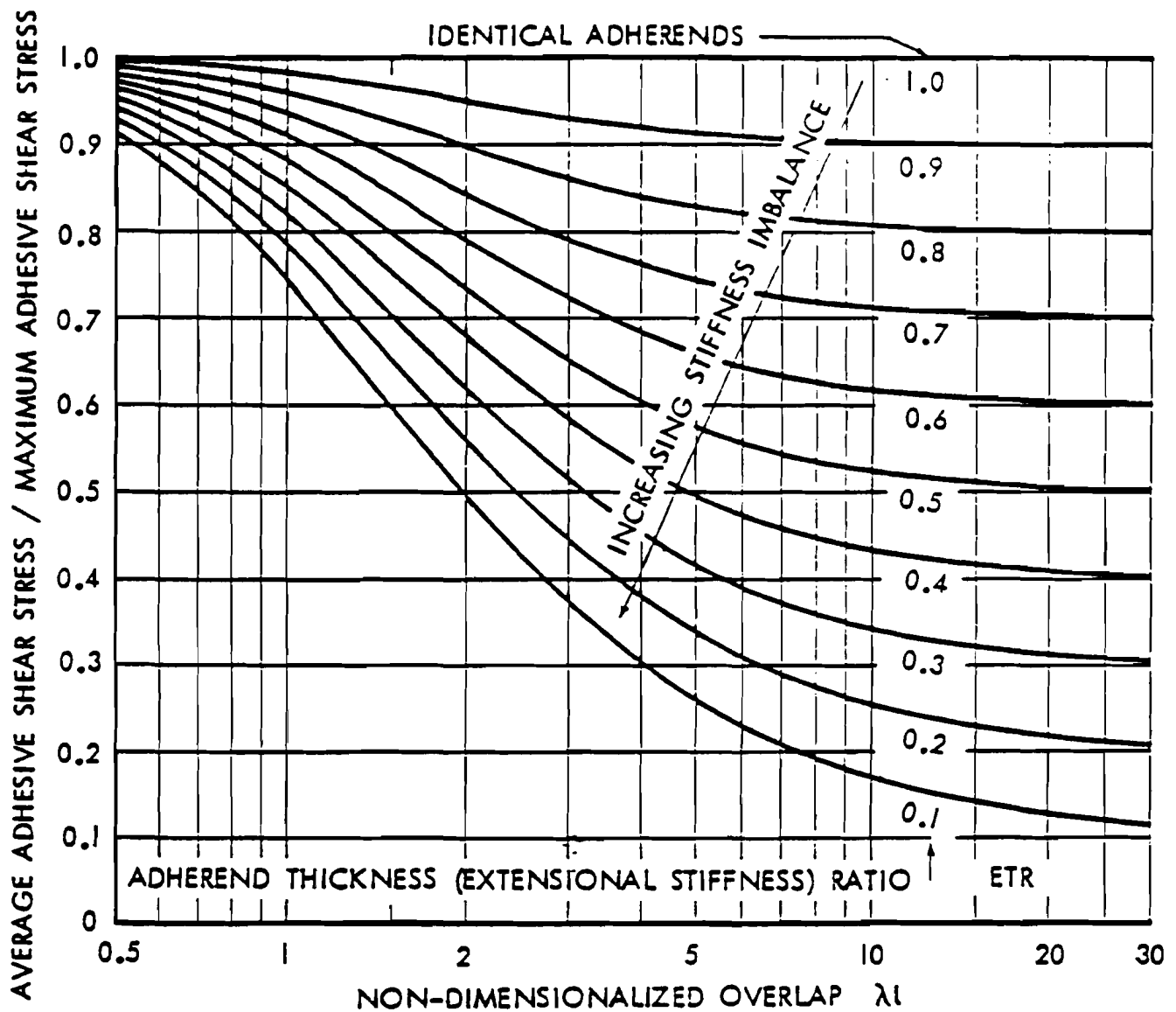
An elastic-plastic analysis for the strength of adhesive-bonded scarf joints was derived in reference 3.2.11 for both identical adherends and those with a stiffness imbalance. The solution is an idealized case, in the sense that the adherends are assumed to be tapered to feathered edges. The added stress concentrations associated with finite tip thicknesses have led to the use of the stepped-lap joint programs to analyze scarf joint also. Consequently, the real interest in the analysis of scarf joints is not as an end in itself, but as a means of establishing the differences between the behavior of scarf and stepped-lap joints in order to model each properly.

Despite the mathematical complexity of the analysis for tapered adherends, one surprisingly simple result is deduced, regardless of plasticity or adherend thermal mismatch. If the overlap is great enough, for a scarf joint,

$$\frac{\tau_{av}}{\tau_p} \quad \text{the lesser of} \quad \frac{E_1 t_1}{E_2 t_2} \quad \text{and} \quad \frac{E_2 t_2}{E_1 t_1}$$

This is consistent with the well known characteristic of scarf joints between identical adherends for which the shear stress is nominally uniform. The significance of this result is that a scarf joint, balanced or not, can always be designed to carry and required load by making the scarf small enough and thereby increasing the bond area. This is in clear contrast with the characteristics of stepped-lap joints, as is explained in the next section, but can require such impractical geometries as scarf angles of less than one degree to achieve this end, possibly requiring a joint longer than the members being joined. Figure 3.2.9 identifies a further problem associated with very small scarf angles in combination with adherend stiffness imbalance. With reference to the bottom diagram in figure 3.2.9, the load in the thicker member builds up more rapidly than the thickness because so much of the load transfer is confined to the left end of the joint. There is, therefore, a serious risk that the tip of the stronger member might fail prematurely, under static loads or by fatigue, if the scarf angle is less than a balance between the adhesive shear strength τ_p and adherend strength F_{tu} or F_{cu} would indicate.

The most significant characterization of the strength of adhesive-bonded scarf joints is shown in figure 3.2.45. The right hand end of figure 3.2.45 matches the asymptotes defined in Equation 3.2.2.



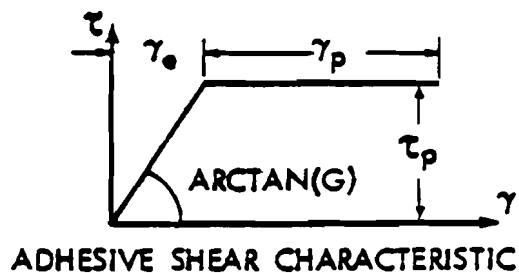
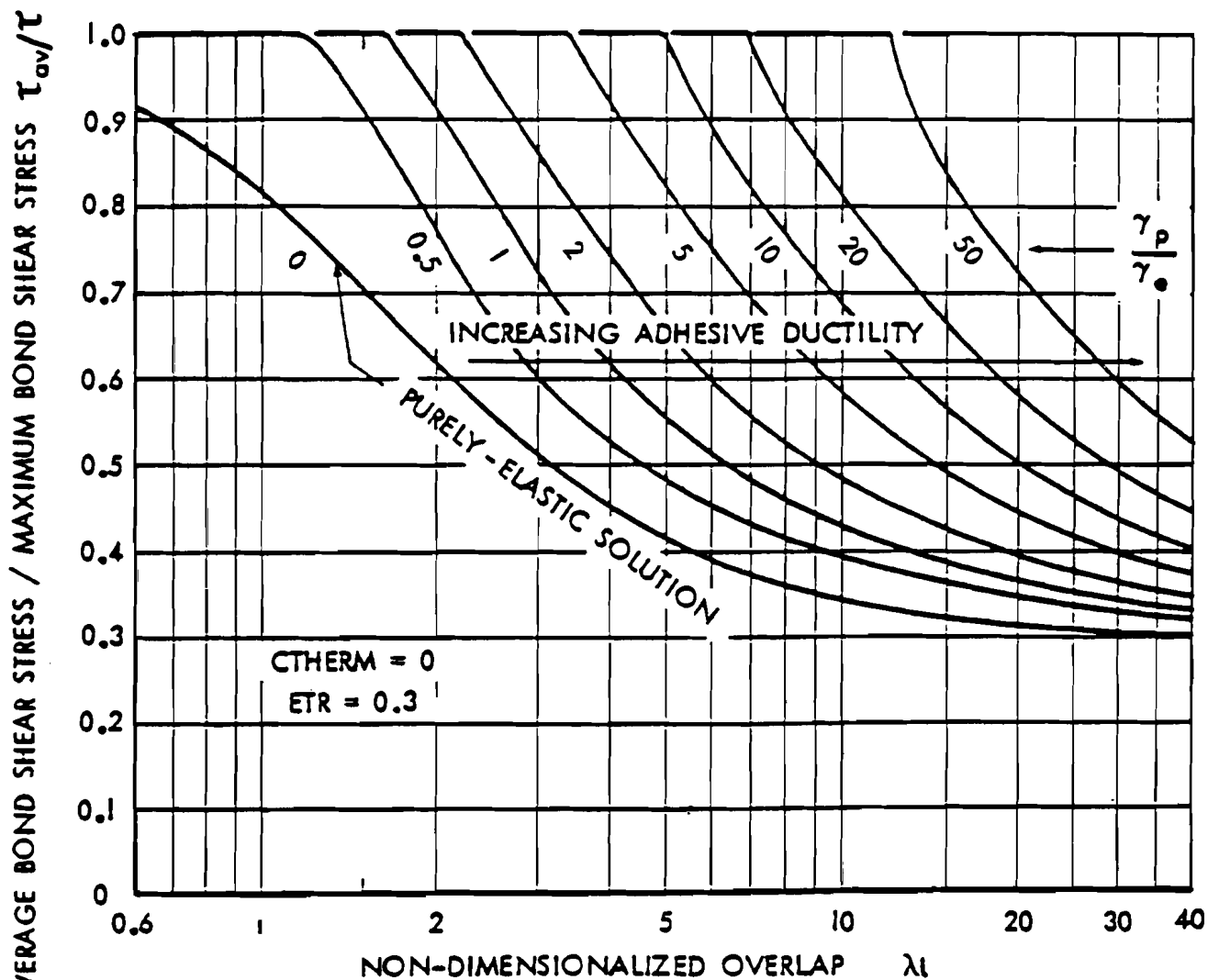
LOCATION A IS CRITICAL FOR BOTH POSITIVE (TENSILE LAP-SHEAR) AND NEGATIVE (COMPRESSIVE LAP-SHEAR) VALUES OF LOAD P

FIGURE 3.2.45 EFFECT OF ADHEREND STIFFNESS IMBALANCE ON ELASTIC STRENGTH OF BONDED SCARF JOINTS

An example of how adhesive plasticity modifies each of the curves in figure 3.2.45 is shown in figure 3.2.46, for the curve $ETR = 0.3$. The adherend stiffness ratio ETR as defined in figure 3.2.45, is the lesser of the E_t ratios in Equation (3.2.3). It is evident from figure 3.2.46 that adhesive plasticity effects substantial strength increases for short to moderate overlaps but that eventually, the elastic behavior dominates for long overlaps. The effect of adherend thermal mismatch is shown in figures 3.2.47 and 3.2.48, from which it is evident for scarf joints that the thermal mismatch effect can be minimized by sufficiently increasing the overlaps. This characteristic is in marked contrast with that for uniform adherends which behave essentially independently of the precise value of any long overlap.

3.2.12 Stepped-Lap Bonded Joints.

The most widely used of the analyses for adhesive-bonded joints are those for the stepped-lap joints, shown near the bottom of figure 3.2.20. There are two reasons for this. The first reason is the need, which surfaced late in the 1960s, for a means of analyzing the boron epoxy to titanium stepped-lap joints used on the tails of modern fighter aircraft. The first such analysis seems to be the elastic derivation by Corvelli (reference 25), which was followed soon after by the present author's nonlinear (elastic-plastic) program A4EG (reference 11). The second reason for the popularity of such computer programs is that they, alone, can be used to model all of the other joints, with the exception of those with a dominant eccentricity in load path. There is no need to provide for eccentricities in well-designed stepped-lap joints. The versatility of this joint analysis program has led to its further development (references 3.2.12 and 3.2.20) into the A4EI version which accounts also for variations in the adhesive. Such variations include thickness, porosity, flaws, and softening or drying out locally due to different moisture contents. With the passage of time, these joints have been used on the tails of the F-14, F-15, F-16 and F-18 as well as on the wings of the F-18. The load intensities of these successful primary structural applications are so high (up to about 30,000 pounds per inch) as to have inspired considerable confidence in the use of adhesive bonding, even though their cost of fabrication has been so great as to make people wish that fibrous composites yielded more around bolt holes so that the weaker, but cheaper, mechanical joints could be used without such a weight penalty. There are also numerous applications of such joints in research and development programs. The analysis for the internal stresses and strains in stepped-lap adhesive-bonded joints has evolved in stages, as recorded in references 3.2.11, 3.2.20, and 3.2.12. Within each mathematical step, all properties must be constant. However, within each physical step of the typical stepped-lap joint shown in figure 3.2.55, it is possible to add artificial breaks to permit changes in such quantities as adhesive porosity or layer thickness. Also, any step that is unnecessarily long must be subdivided to permit the numerical solution of the problem; this is done automatically within the coding for the A4EI program in reference 3.2.12. Figure 3.2.55 shows the sign convention and nomenclature necessary for the analysis.



(See Figures 2, 3 and 4 for notation)

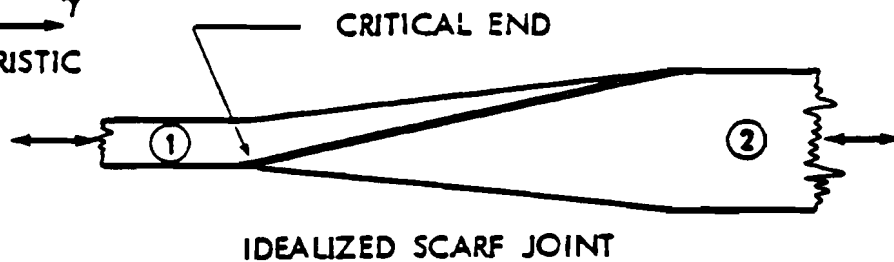
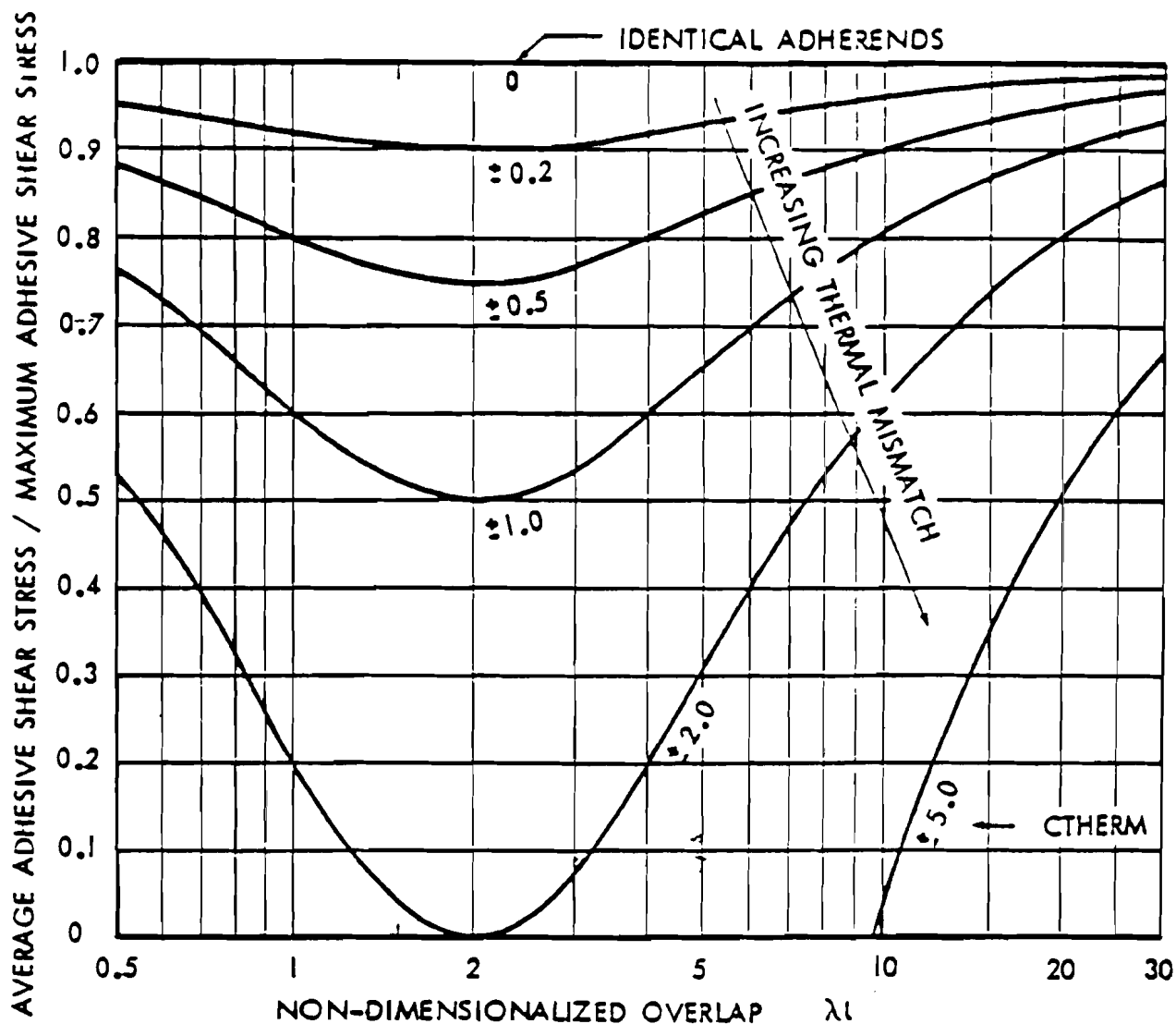


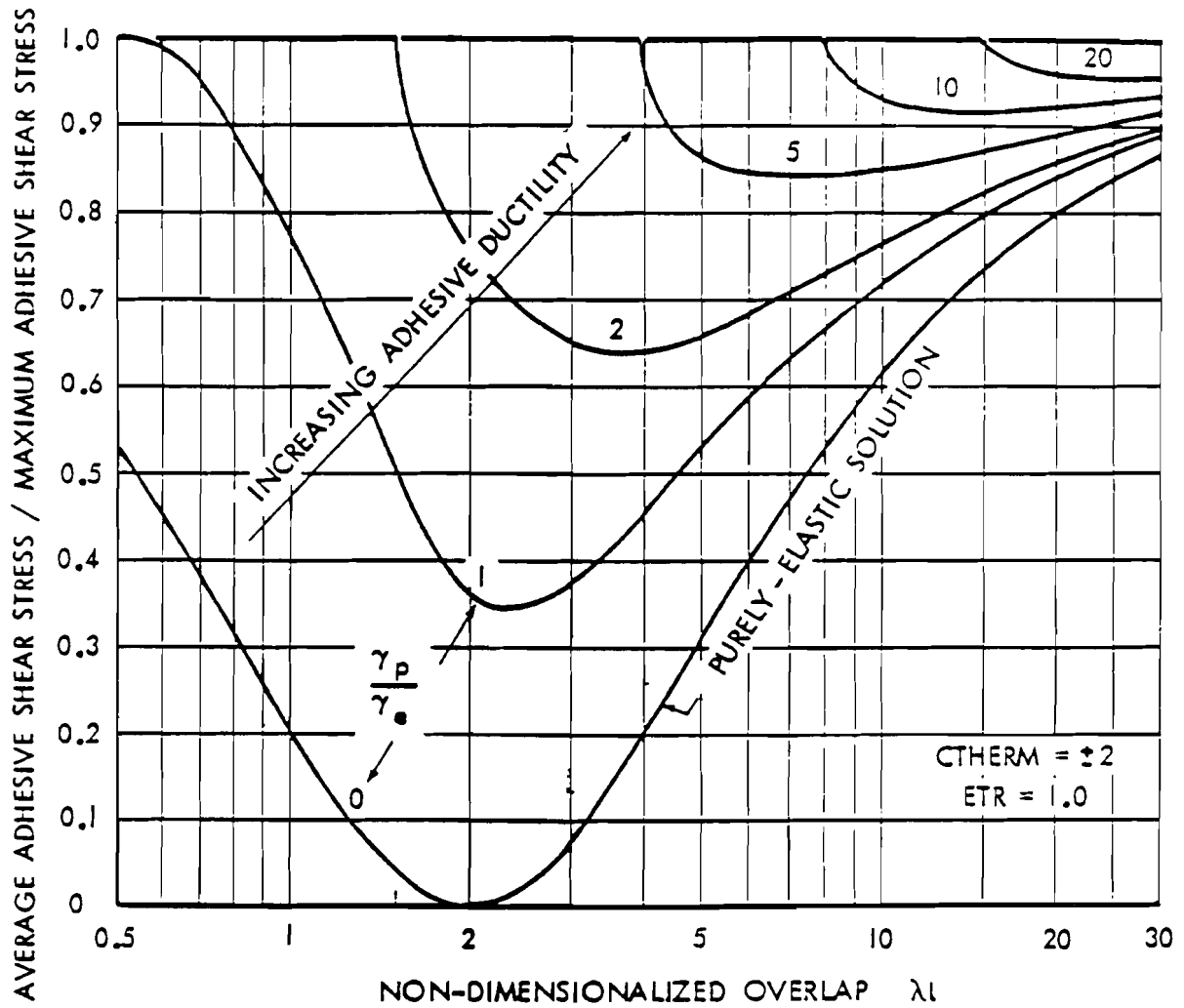
FIGURE 3.2.46 EFFECT OF ADHESIVE PLASITCITY IN REDUCING STRENGTH LOSS DUE TO ADHEREND STIFFNESS IMBALANCE FOR BONDED SCRAF JOINTS



$$\lambda^2 = \frac{G}{\eta} \left[\frac{1}{E_1 t_1} + \frac{1}{E_2 t_2} \right] \quad C_{THERM} = \frac{(\alpha_2 - \alpha_1) \Delta T \lambda}{\tau_p \left(\frac{1}{E_1 t_1} + \frac{1}{E_2 t_2} \right)}, \quad \Delta T = T_{\text{operating}} - T_{\text{stress free}}$$

LOCATION A CRITICAL FOR $C_{THERM} < 0$ AND $P > 0$
 LOCATION A CRITICAL FOR $C_{THERM} > 0$ AND $P < 0$
 LOCATION B CRITICAL FOR $C_{THERM} < 0$ AND $P < 0$
 LOCATION B CRITICAL FOR $C_{THERM} > 0$ AND $P > 0$

FIGURE 3.2.47 EFFECT OF ADHEREND THERMAL MISMATCH ON ELASTIC STRENGTH OF BONDED SCARF JOINTS



(See Figures 2, 3 and 4 for notation)

FIGURE 3.2.48. EFFECT OF ADHESIVE PLASTICITY IN REDUCING STRENGTH LOSS DUE TO ADHEREND THERMAL MISMATCH FOR BONDED SCARF JOINTS

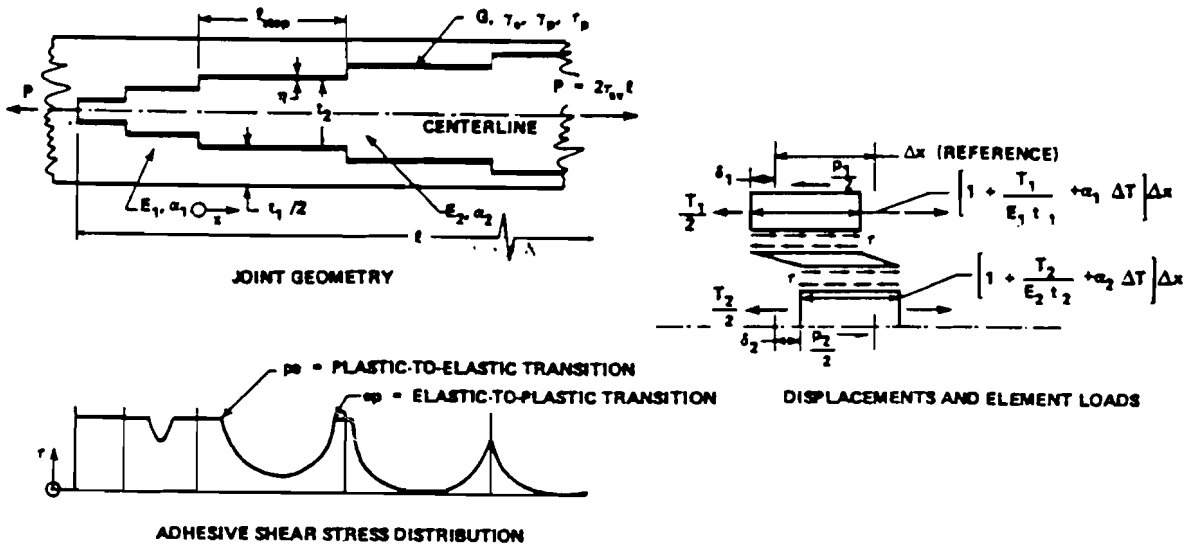


FIGURE 3.2.49. NOTATION AND GEOMETRY FOR ADHESIVE-BONDED STEPPED-LAP JOINT ANALYSIS

Numerous worked examples of the analysis of stepped-lap bonded joints can be found in references 3.2.11, 3.2.26, 3.2.20 and 3.2.12. Some of the highlights of the lessons learned from such analyses and from actual experience with hardware are discussed in the following paragraphs.

The strength of these bonded joints is known to be particularly sensitive to a number of factors. Perhaps the most powerful factor is the need for adherend stiffness balance from one end of the joint to the other, as explained in figure 3.2.12. The lower joint has almost twice the strength of the upper one, even though that strength increase has been obtained by removing material rather than by adding reinforcement.

The remaining design variables are the length and number of steps. Of these two, the dominant effect is the number of steps. The length effect, for a given number of steps, is extremely slight, as can be seen from the three joint analysis in figure 3.2.26. Each joint had five steps and all thicknesses were the same at the corresponding stations. An increase in overlap by a factor of four is seen to increase the strength by only 14 percent. This highlights the large difference between the mechanism of load transfer in scarf joints (for which such an increase in bond area would have had a substantial effect on the joint strength) and stepped-lap joints. The reason why the stepped-lap joints are insensitive to the overall joint length, as the sum of a series of uniform steps, is that each of those steps is governed mathematically by precisely the same differential equation as applies to double-lap joints. The independence of the strength of these joints to precise overlaps in the usual design range is shown in figure 3.2.29.

The strength increases associated with maximizing the number of steps in the joint are substantial, as shown by a comparison between figures 3.2.9, 3.2.8 and 3.2.6 of reference 3.2.12. The strength is maximized by having a separate step for each ply of composite tape or cloth and such a procedure may be appropriate for the highest load transfer between thick members. As a good starting point in

design for typical composite laminates that are either a quasi-isotropic or only slightly orthotropic, each step should be about 0.5 inch long and the thickness increment should be in the range 0.02 to 0.03 inch on each side of what is usually a titanium step plate. The exact increment should, of course, be a precise match for the corresponding increment of laminate thickness - for example, four layers of unidirectional tape at 0.005 inch each would require 0.020 inch steps, while two layers of satin-weave cloth at 0.013 inch each would require 0.026 inch steps.

Just as with double-lap joints, it is vital that at least one of the steps near the middle of the total bonded overlap contain a very lightly-stressed deep elastic trough to prevent creep rupture of the joint. That step should probably be in the range 0.75 to 1.0 inch long and its adequacy should be checked as part of the analysis. That is the reason program A4EI prints out not only the high adhesive stresses at the end of each overlap but also the low stress at the middle of each step.

It should be noted that the suggestions above are intended only to provide the starting point of the design process using the A4EG or A4EI computer programs. There is still usually room for improvement for a minor tweaking of the design once the internal stress distribution has been characterized. It is still desirable to establish that the potential shear strength of the adhesive would exceed by at least fifty percent the strength of the weaker adherend outside the joint. Otherwise the joint could become the weak link in the chain and suffer catastrophic disbonding due to only a minor local flaw. This issue is discussed more fully below in the content of load redistribution around flaws and of the usual insensitivity of well-designed joints to quite large flaws.

Practical experience with bonded stepped-lap joints has taught that the layer of fibers or filaments immediately adjacent to the metal plate should never be perpendicular to the direction of the applied load. Otherwise, a much weaker joint results because of the tendency of such layers to split transversely and roll within themselves. This is easily understood by considering an axially loaded test coupon with bonded load-introduction tabs. The preferred fiber direction adjacent to the adhesive layer would obviously be 0° , aligned along the specimen. A 90° ply would be quite unsuitable, with an intermediate performance coming from a $+45^\circ$ or -45° layer. Even with a large panel loaded primarily in shear, the same 0° direction is to be preferred, because whichever 45° layer would be optimal for shear in one direction would be the worst possible for shear in the opposite sense. This consideration defines where the 0° plies are to be located within the stack-up, and therefore influences the basic laminate outside the joint as well.

As the loads transferred increase to the level of tens of thousands of pounds per inch, the joint proportions are sufficiently critical to be of concern from the point of view of manufacturing tolerances. Experience in fabrication has established that it is important not to overlap the end of the composite layers on to the next step or to stop them significantly short on their own steps. Otherwise, the laminate will be wrinkled within the joint area and will probably suffer internal delaminations during cyclic loading. Those, in turn, can so redistribute the internal load transfer as to cause a reduction in joint strength. Such problems become more acute as the incremental step thicknesses become larger. Indeed, for step thicknesses significantly greater than those recommended here, the author's analyses will not suffice. It would then be

necessary to use an analysis which included a check on the possibility of delaminations within the composite adherends, as is done in reference 3.2.27. A further important use for the analysis of stepped-lap bonded joints is the establishment of the sensitivity of the joint strength to minor deviations from the nominal design. This task becomes progressively more important after the strength has been raised by repeated optimization of the detailed dimensions.

3.2.13 Insensitivity of Adhesive Bonded Joints to Flaws .

To understand the effects of flaws and porosity on the strength of adhesive-bonded joints, it is necessary first to characterize the behavior of such joints having no imperfections. Relatively thin, lightly loaded members are incapable of overloading the adhesive - the adherends break outside the joint, as explained in curves A in figures 3.2.36, 3.2.37 and 3.2.38.. Much thicker members can transmit far greater loads and therefore, are capable of failing any adhesive bonds between them, as shown in curves C of the same three figures. It follows then that, for each adhesive, there is a certain thickness of adherends (for a given material) for which the adherend strength precisely balances the shear strength of the adhesive (at least for a particular environment). One should infer, therefore, that relatively thin bonded structures should have considerable tolerances to bond flaws and any other imperfections or damage. Conversely, for the greater adherend thicknesses for which the adhesive is not strong enough to fail the adherends outside the joint, one would expect the bond to be very sensitive to any flaws or imperfections. This topic is discussed fully in reference 3.2.14.

Acceptance criteria for adhesive bond flaws and porosity must account for the thickness and strength of the members being joined as well as adhesive properties and the service environments. It is not possible to prescribe meaningful acceptance criteria for bond flaws based merely on their location and severity as assessed by nondestructive testing inspections.

It should be evident from the discussion above that the prime effect of bond flaws is a reduction in the thickness of members that can be bonded satisfactorily. For thin adherends, most suitable for bonding, even the flawed bonds are strong enough. Indeed most flawed bonded structures have their strengths or lives reduced by attempts to repair them.

This insensitivity of adhesive bonds to flaws, for thin and only moderately thick structures, was explained during Douglas' PABST (Primary Adhesively Bonded Structure Technology) program work. Figures 3.2.50 to 3.2.53 show precise calculations for three different flaws in bonded aluminum skin splices, using a 250 F cured modified epoxy ductile adhesive. The load shown corresponds with 1.3 times the maximum cabin pressure in service. The large areas of lightly loaded adhesive in the unflawed joint in figure 3.2.61 provided ample reserve to rearrange the load transfer across the flaws. It is noteworthy that none of the flaws caused any increase in the peak bond stress induced by the constant. The major effect detected is that the joint flexibilities in figures 3.2.51 to 3.2.53 are slightly greater than shown in figure 3.2.50 because more of the aluminum is highly stressed. Consequently, in a real structure in which random bond flaws are surrounded by nominally perfect bonds, any flawed bonds divert some of their share of the loads to the adjacent strong bonds. This two-dimensional load redistribution adds to the insensitivity shown by the one-dimensional analyses in figures 3.2.50 to 3.2.53.

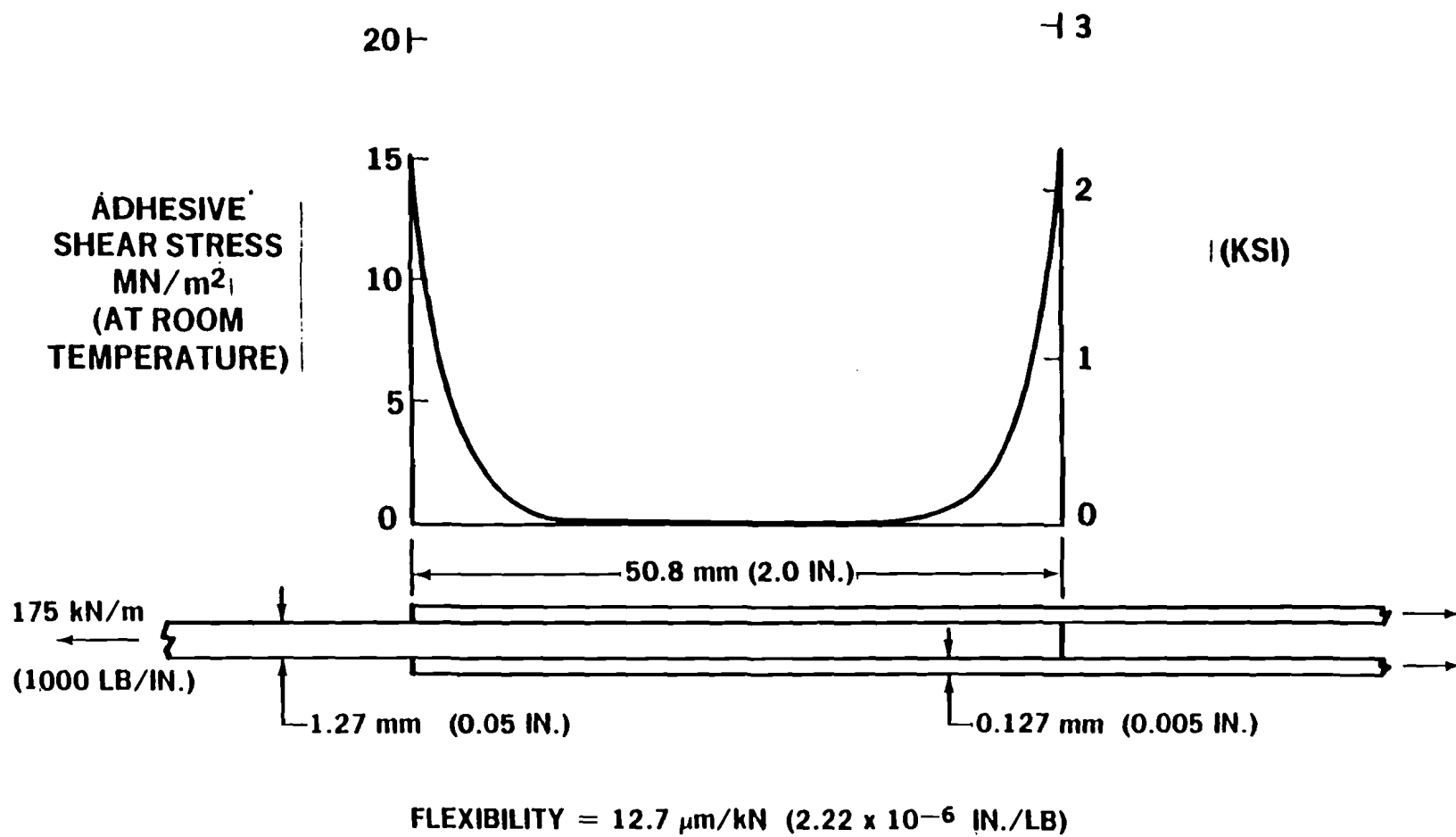


FIGURE 3.2.50. ADHESIVE SHEAR STRESSES IN BONDED JOINTS

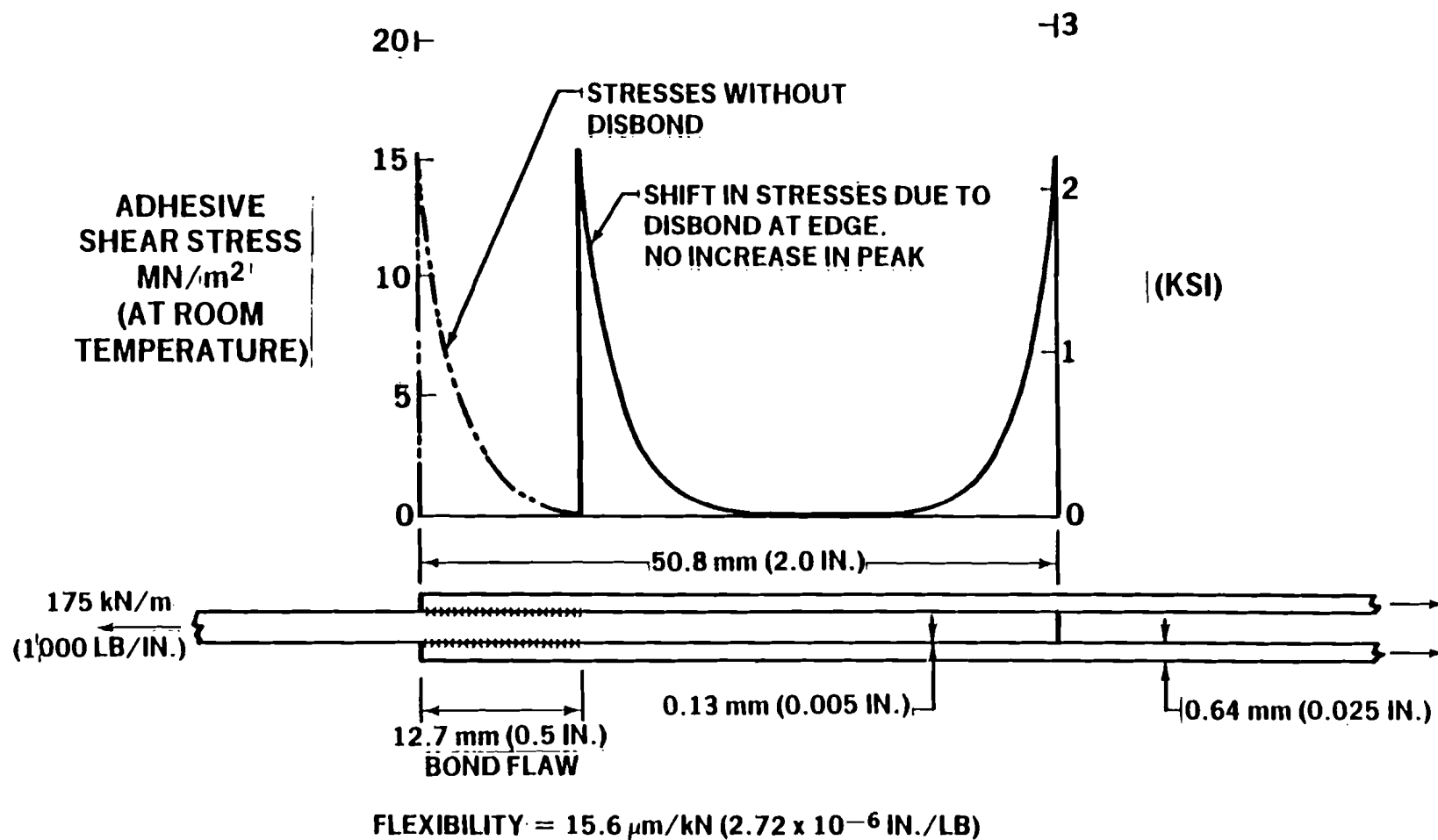


FIGURE 3.2.51. ADHESIVE STRESSES IN FLAWED BONDED JOINTS (WITHOUT DISBOND)

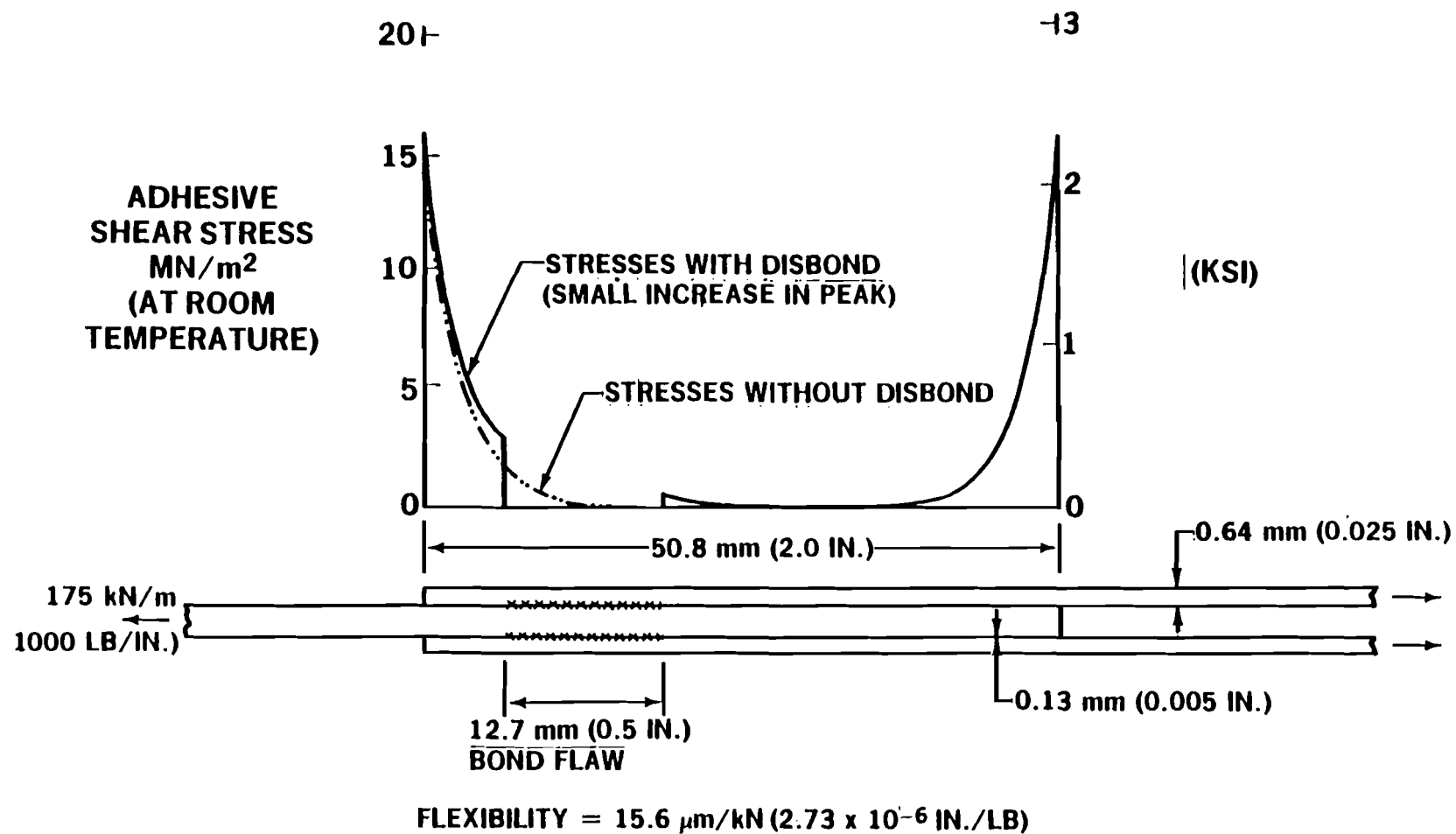


FIGURE 3.2.52. ADHESIVE STRESSES IN FLAWED BONDED JOINTS (WITH DISBOND)

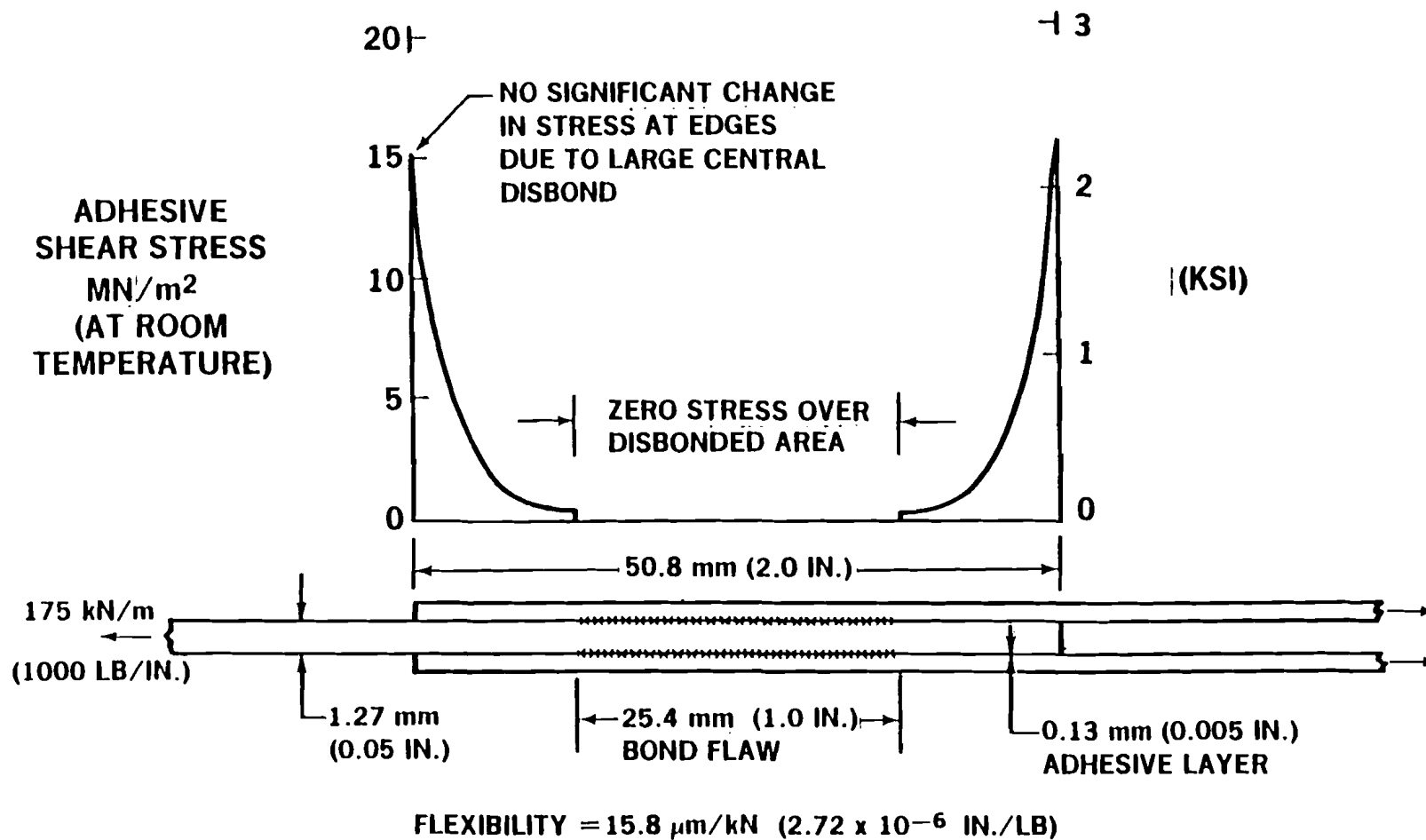


FIGURE 3.2.53. ADHESIVE STRESSES IN FLAWED BONDED JOINTS (LARGE CENTRAL DISBOND)

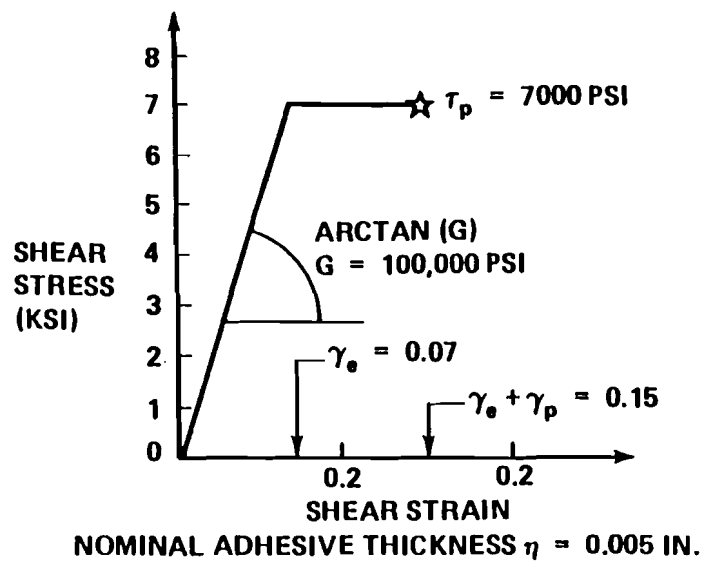
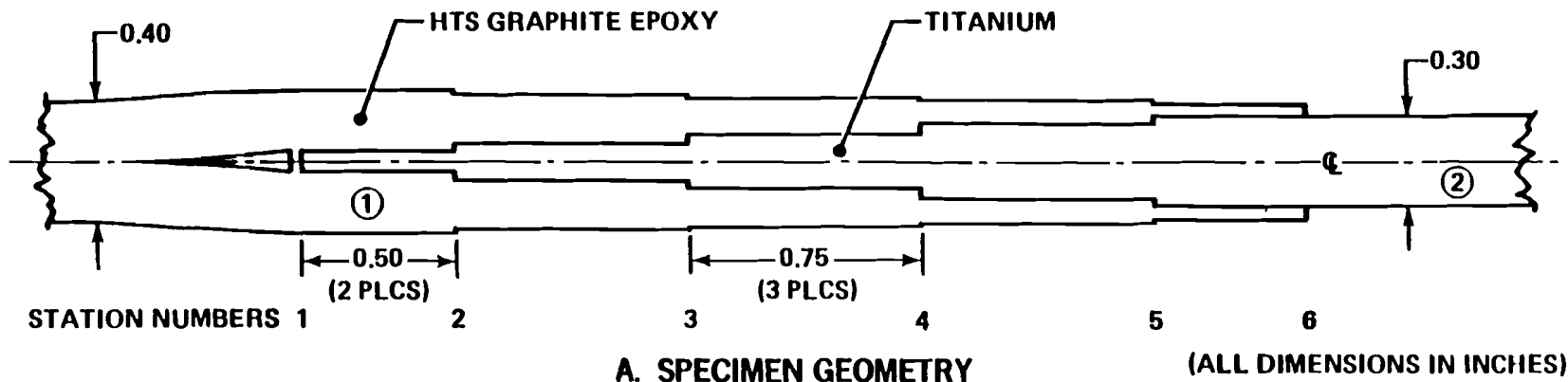
Porosity also tends to have much less effect on the strength of adhesive-bonded joints than one would at first expect, for two reasons. The first is that porosity normally occurs only in the interior of a joint (as with the flaw in figure 3.2.53) in an area in which there would be virtually no load transferred anyway. Porosity is caused by a gap between the adherends that is greater than normal yet not quite large enough for capillary action to create a complete void as would occur with still greater gaps. Hence, porosity would be rare in areas of high load transfer at the edges of overlaps; there the problem is more one of pinching off the glue layer and making it too thin, as explained in reference 3.2.28. The added thickness associated with porous bond layers, inevitably decreases the strains in the adhesive. Again, the stress in the area of the weak bond is reduced and the load is transferred elsewhere in the bond region.

The discussion above has concentrated on adhesively bonded joints between relatively thin adherends of essentially uniform thickness. However, porosity and flaws can also occur in thick stepped-lap joints, particularly if made by secondary bonding of pre-cured details instead of the more usual co-cure and bond operation.

Figure 3.2.54 describes a typical stepped-lap joint. The adherends are much stronger than the adhesive in this case, so that effects of any imperfections anywhere in the bond should be noticeable. Figure 3.2.55 shows the adhesive shear stress distribution for this joint under tensile shear loads for the case of a nominally perfect bond with a uniform thickness of 0.005 inch. [For compressive shear loads, the stress and strain distributions would be mirror images (end-for-end) and of the opposite sign.] The lesser adherend strength is 48,000 pounds per inch while the adhesive shear-strength is only 13,749 pounds per inch under both tensile and compressive shear loads, because of the precise adherend stiffness balance.

Figure 3.2.56 shows what happens as the result of a misfit whereby the central step has been given a complete void because the adhesive is set there at 0.015 inch. The intermediate steps are modeled as having a 50 percent loss of shear strength due to a less severe thickening of the bond layer, to 0.010 inch. The adhesive bond shear strength is predicted to be reduced only to 13,350 pounds per inch, which is clearly far less severe than the proportional effective loss of bond area. That is because the critical location remains in the outermost steps. The added adhesive strains in the porous bond steps, with respect to the same steps in figure 3.2.55, are caused by the loss of load transferred by the central step.

Figure 3.2.57 depicts the further losses of strength due to a complete void throughout all of the bond except the outermost steps. The strength would be further reduced to 8,350 pounds per inch. That represents a strength retention of 61 percent even though the effective bond area had been reduced to only 31 percent of the original extent. It is significant that this large loss of strength is associated with a shift of the critical adhesive location from the left end of the overlap to the left side of the complete bond flaw. It is precisely because such a shift did not occur in Figure 3.2.56 that there was so much less strength loss associated with the less severe bond flaw.



ADHERENDS "1"
MATERIAL: TITANIUM
 $E = 12 \times 10^6$ PSI
 $F_{tu} = 120$ KSI
 $\alpha = 0$

ADHERENDS "2"
MATERIAL: TITANIUM
 $E = 16 \times 10^6$ PSI
 $F_{tu} = 130$ KSI
 $\alpha = 0.000006/^{\circ}\text{F}$

C. ADHEREND PROPERTIES

FIGURE 3.2.54. TYPICAL STEPPED-LAP ADHESIVE-BONDED JOINT

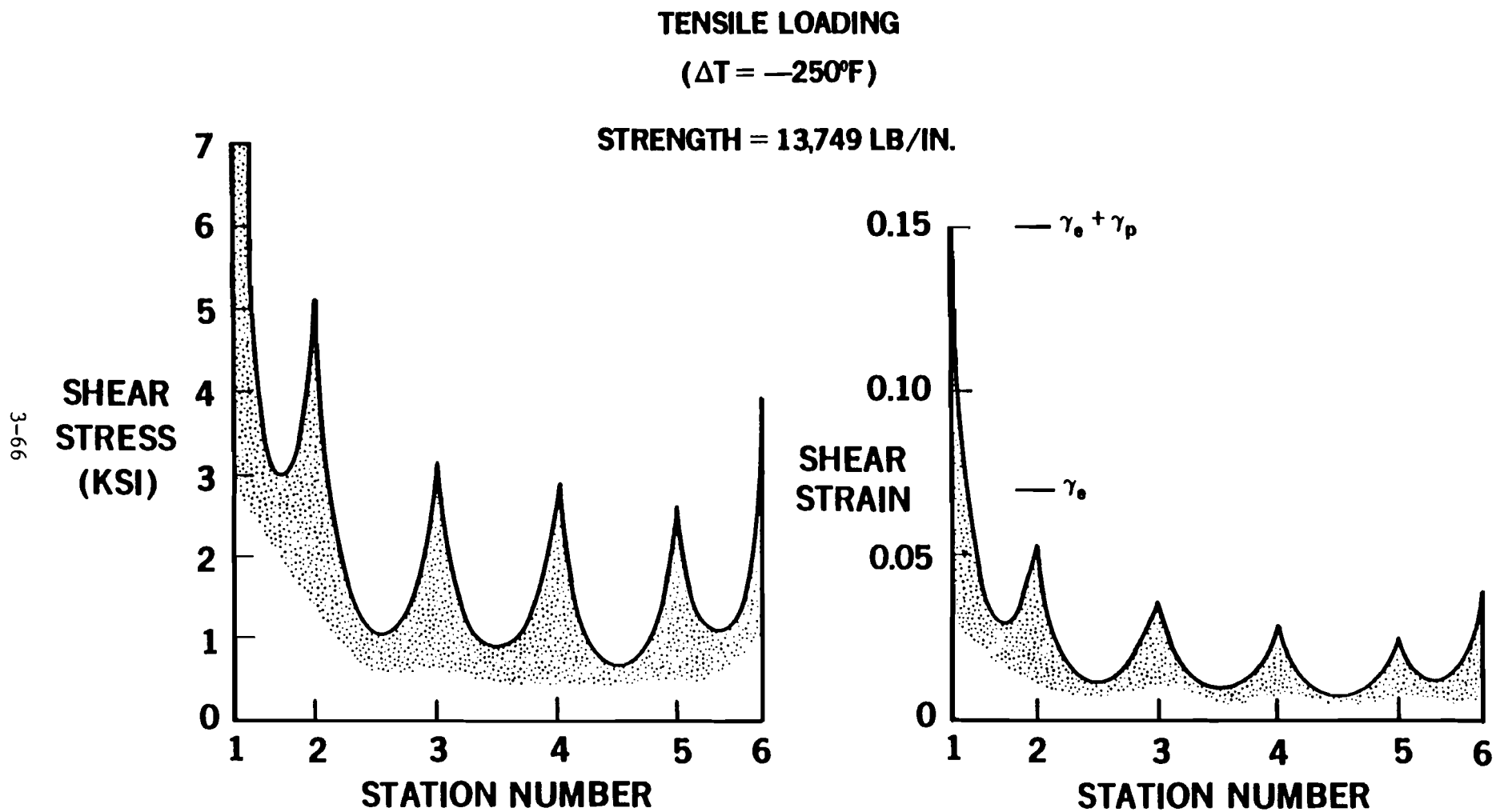


FIGURE 3.2.55. ADHESIVE SHEAR STRESSES AND STRAINS IN UNUNFLAWED JOINT

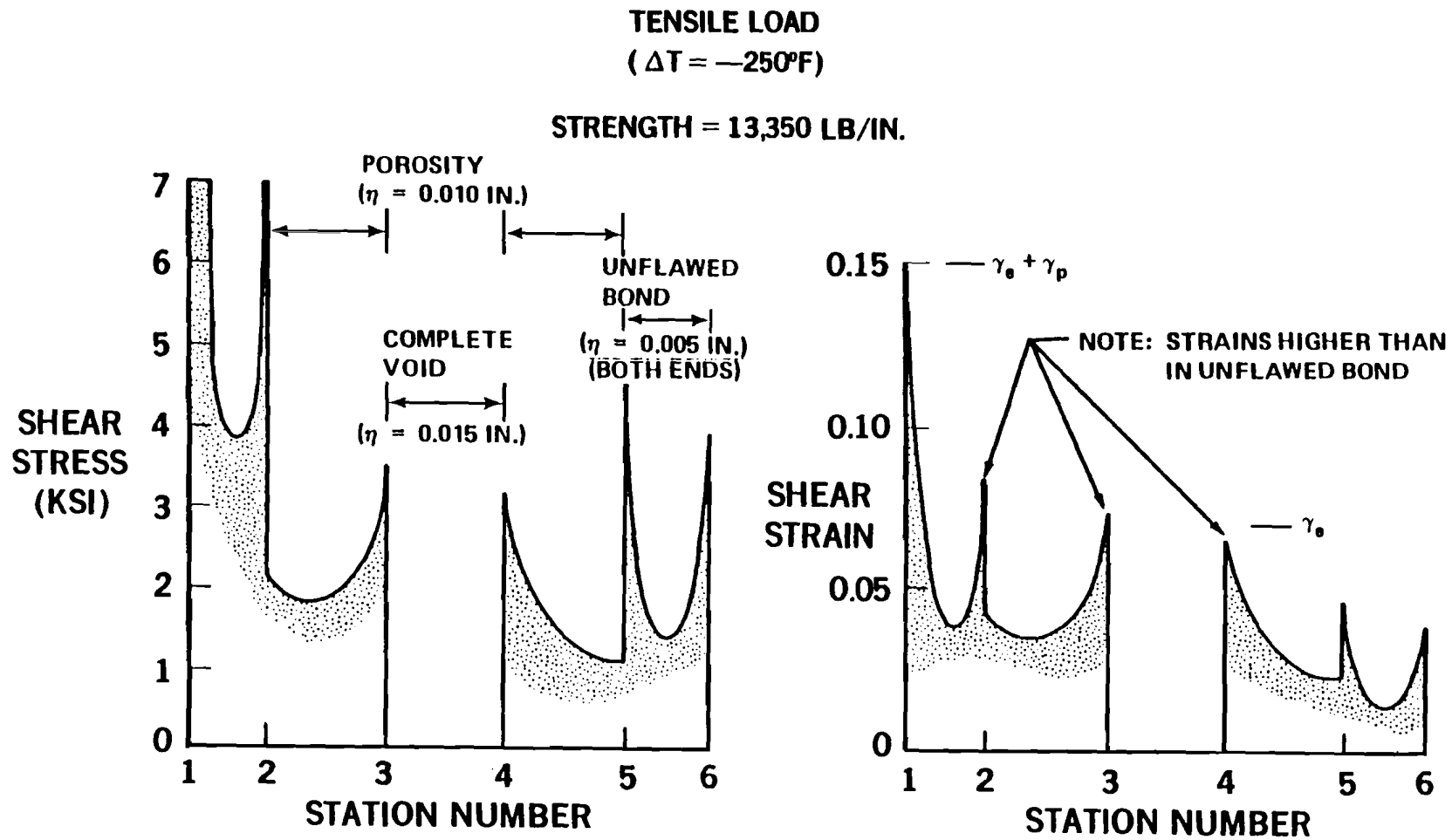


FIGURE 3.2.56. SHEAR STRESSES AND STRAINS IN FLAWED, POROUS BONDED JOINT

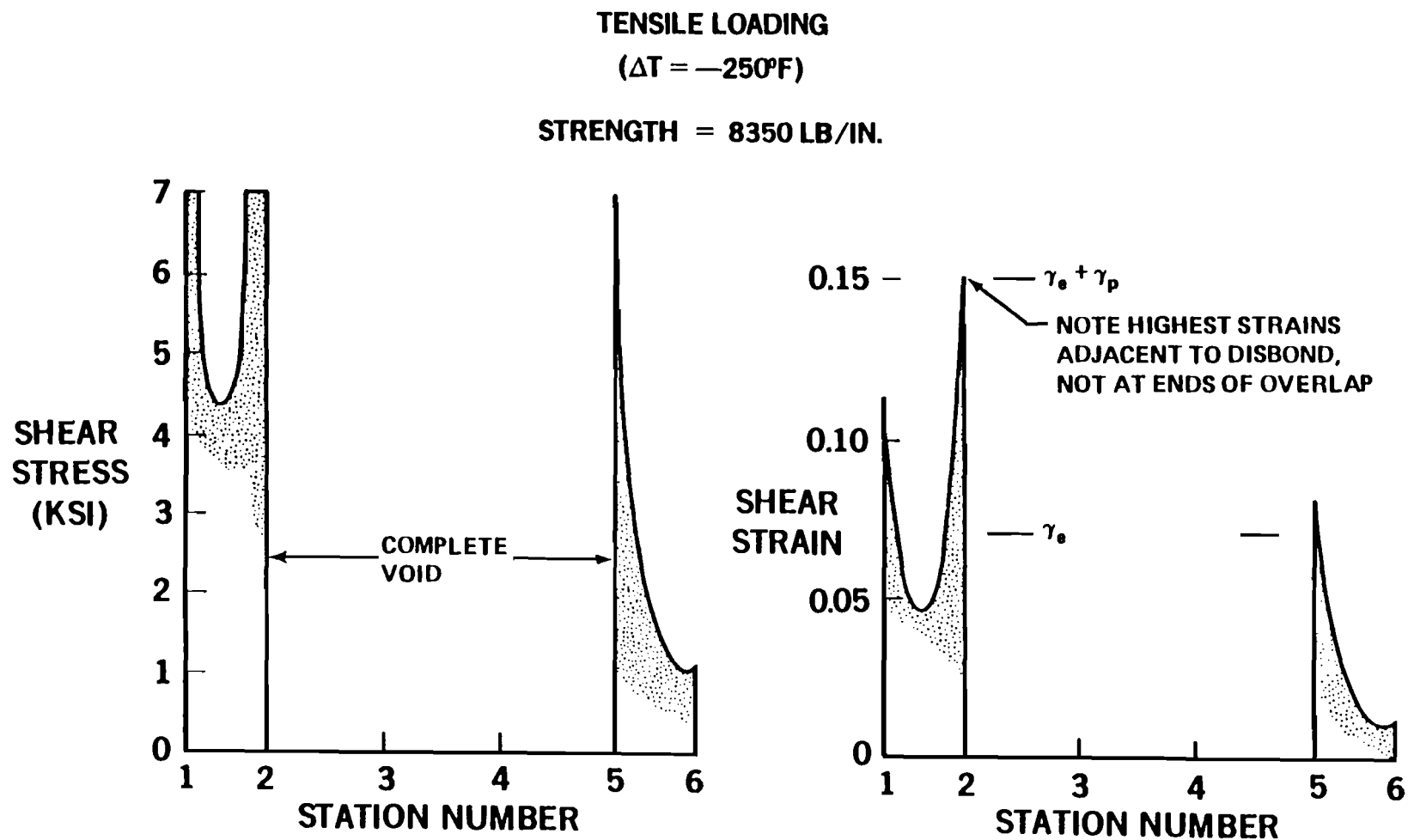


FIGURE 3.2.57. SHEAR STRESSES AND STRAINS IN FLAWED ADHESIVE-BONDED JOINT

The examples above teaches us that adhesive-bonded joints exhibit far more tolerance of bond flaws than nondestructive inspection would lead one to believe. The first effect of a flaw is to cause a local load redistribution around the periphery of that flaw. Such a redistribution need not have any effect on the critical location in the adhesive, particularly if that critical location is remote from the flaw as is usually the case. Only if the flaw is so large that the local redistribution does cause a shift in the critical bond location will there be a substantial loss of bond strength.

The potential significant consequence of bond flaws is not evident from the one-dimensional analysis described above. A two-dimensional analysis is needed to explain this phenomenon. Consider the well-designed bonded joint shown in figure 3.2.58. Despite the large size of the bond flaw there is no possibility of propagating a disbond because the adherends limit the amount of potential load redistribution. Those adherend are simply incapable of imparting sufficient load to fail the bond, even just along side of the flaw. Consequently there could be no possible justification for fail-safe (or "chicken") rivets in such a splice; the pressurized fuselage splice in figures 3.2.50 and 3.2.53 belongs in such a category.

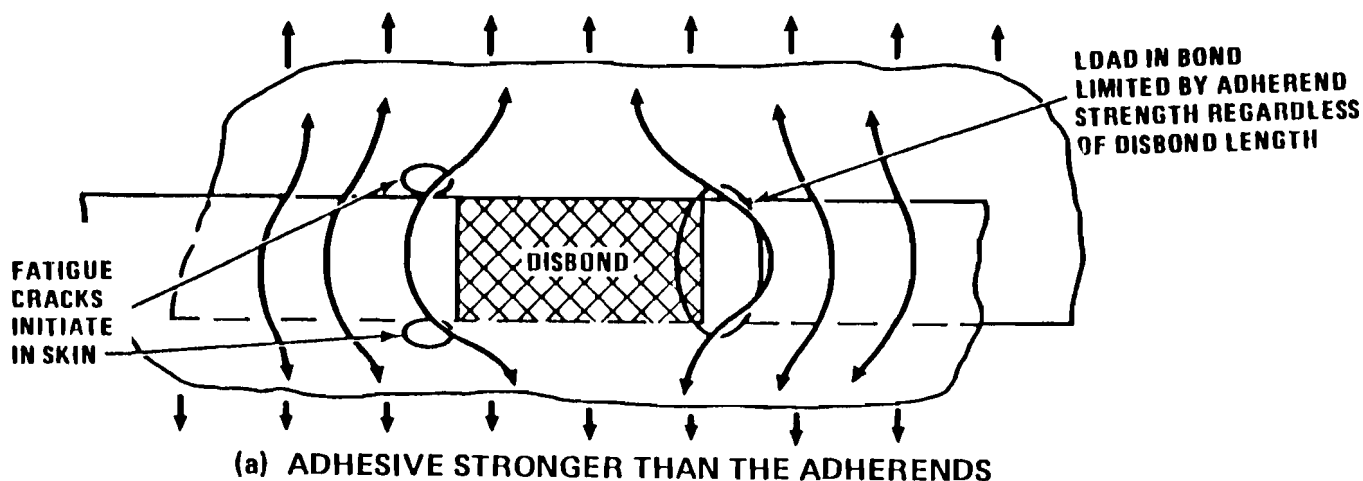


FIGURE 3.2.58 LOAD REDISTRIBUTION AROUND TWO-DIMENSIONAL ADHESIVE-BOND FLAWS

However, if the bond were to be weaker than the adherends, as in the example shown in figure 3.2.59, a bond flaw would now act like a finite crack in a metallic plate. The remote stress would need to be restricted if the initial disbond were not to be propagated catastrophically. In this case, the overloaded areas of the adherends immediately adjacent to the flaw could initiate the progressive failure of the bond. Such a joint, even without the flaw, must be considered improperly designed. Only in such a case would fail-safe rivets be beneficial. This is explained in figure 3.2.60. The adhesive is stiffer and loaded preferentially, but the rivets in the disbanded area would eventually pick up load when the relative motion between the adherends became sufficient. These rivet loads would then decrease the load being diverted around the disbond and arrest its propagation. Reference 3.2.12 contains a more indepth discussion of the combination of bonding and bolting, with particular emphasis on when the combination is no better than either joining method alone. There is also an explanation of how the two work together in the separate parts of a damaged joint, along with an argument for the stitching of all thick fibrous composite structures with fasteners in order to confine the spread of any damage either at a bond surface or within the laminate.

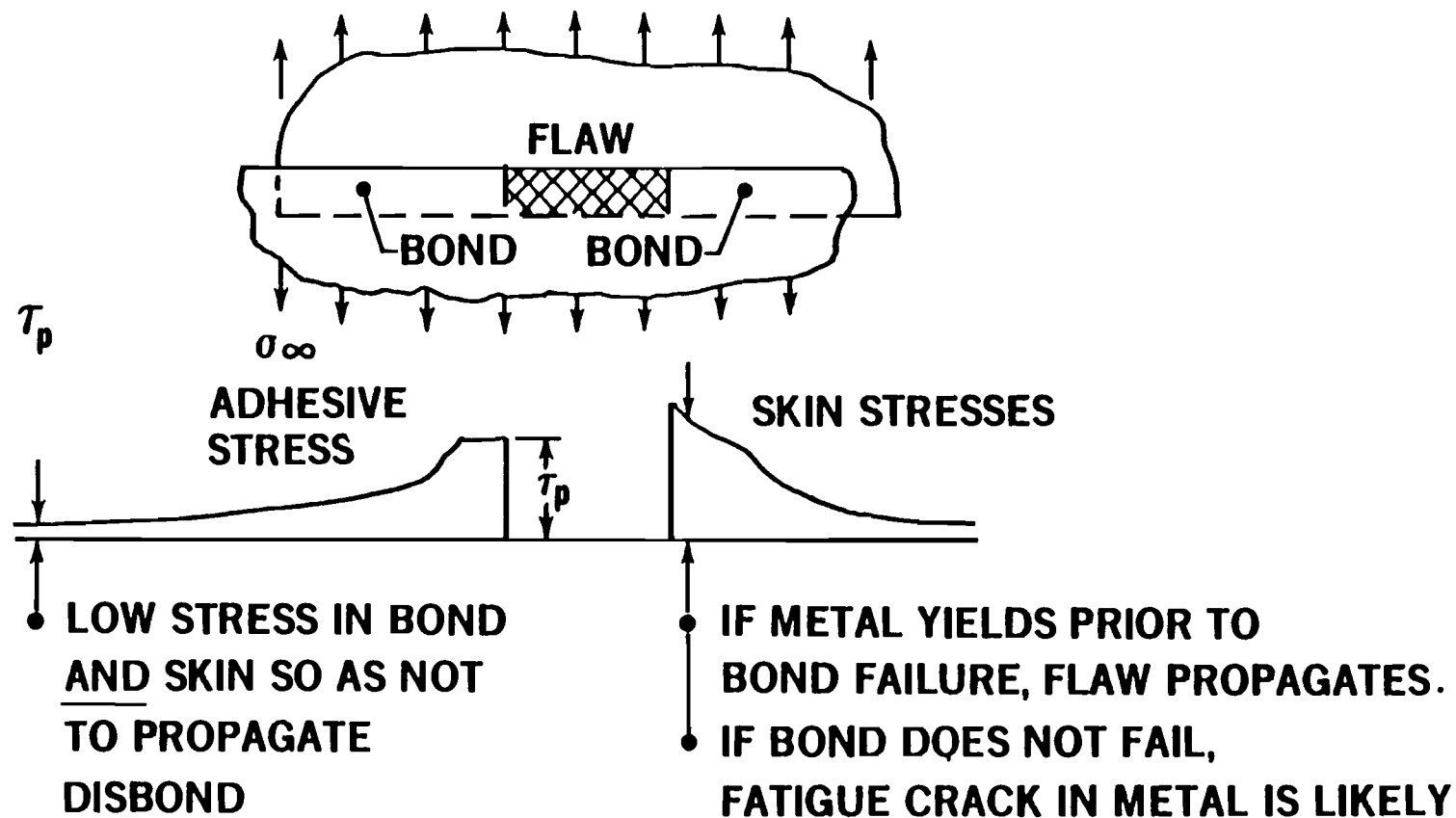


FIGURE 3.2.59 REDISTRIBUTION OF LOAD AT FLAWS IN BOND

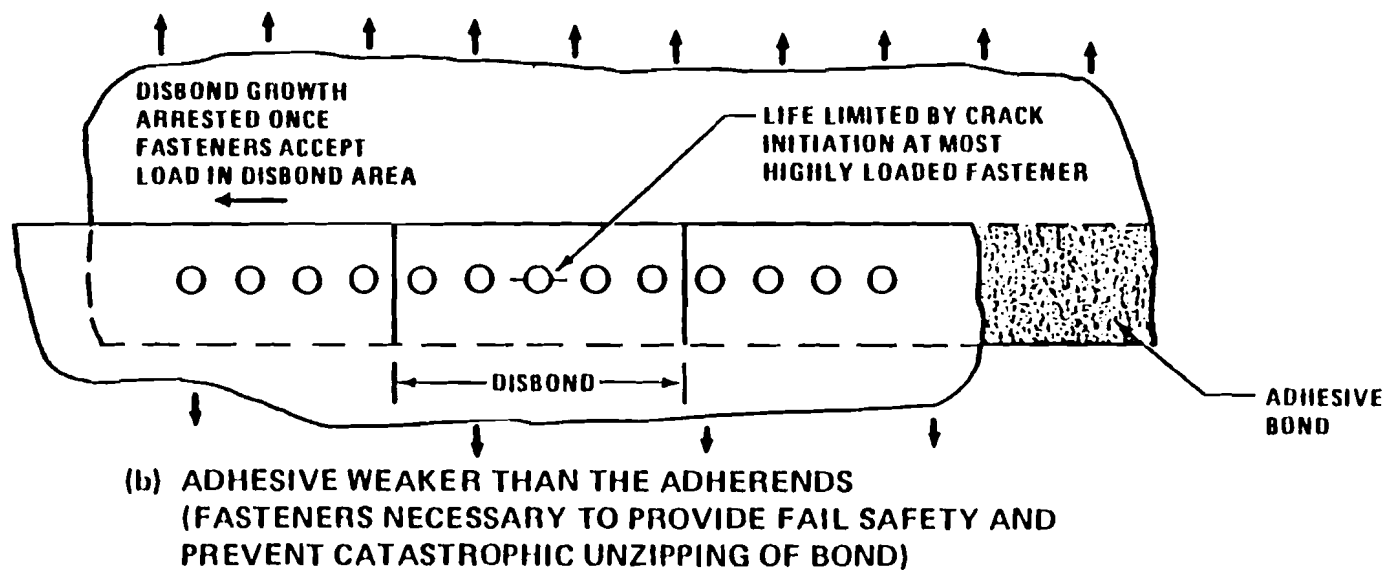


FIGURE 3.2.60 LOAD REDISTRIBUTION AROUND TWO-DIMENSIONAL ADHESIVE-BOND FLAWS

3.2.14 Miscellaneous Considerations.

The results summarized above have been confirmed to direct tensile or compressive lap-shear loads. (Derivations are presented in Appendices III A, B, and C). There are minor differences for the treatment of in-plane shear loads. (The full derivations are contained in reference 3.2.9, 3.2.10 and 3.2.11.) The key difference is that the adherend extensional stiffnesses E_t are replaced by the shear stiffnesses G_t , where G is not to be confused with the adhesive shear modulus. The exponents λ are thus increased slightly, so that the stress concentrations in the adhesive are more severe. Consequently, overlaps sized for tensile or compressive adherend loads will be more than adequate for in-plane shear also. For near isotropic materials, the strength is also usually more critical in tension or compression than shear since the latter is essentially biaxial tension and compression. However, for a composite laminate near an all +45° pattern, the critical check on strength would be made for in-plane shear loads. The governing differential equations all have precisely the same form, so the same computer analysis programs can be used with just a modification to some of the coefficients. The coding makes such provisions.

It is always possible to tailor the load transfer across a bonded joint by tapering, thinning or stepping the adherends. However, damage in service can create abrupt load transfers in bonded structures at locations where originally there had been only trivial running shear loads passing through a bond. Such a situation can arise in bonded stiffened metal structures and has been investigated in reference 3.2.29. Consider the case of a fatigue crack growing through a stiffener somewhere in the middle of a bonded panel, with the crack originating at an improperly bucked fastener where the panel is secured to the substructure. When that crack has finally propagated right through the stiffener, there is developed a very abrupt load transfer out of the stiffener into the skin. A number of secondary failures are possible. If the glue is strong enough, a second crack will grow in the skin. Alternatively, if the stiffener is too massive for the glue there would be a disbond, which may or may not be self arresting. The exact counterparts of these situations in fibrous composite structures have yet to be analyzed and it is likely that the additional failure mode of delaminations within the composite will complicate the picture. Nevertheless, it is probable that the technique for minimizing such problems in bonded metal structures will still be effective in bonded or co-cured composite structures. That technique is to use many light stiffeners to minimize the magnitude of the possible unplanned load transfers to minimize the secondary spreading of any initial damage. If other considerations dictate the use of a few large stiffeners in certain areas, as around cutouts, those highly-loaded members should be mechanically fastened or both bonded and bolted.

3.2.15 Inspection Requirements.

Load transfer through adhesive bonds is not effected uniformly, but is concentrated in narrow bands immediately adjacent to the discontinuities in or termination of the members bonded together. Therefore, it is appropriate that the inspection techniques and acceptable flaw sizes reflect these different intensities of load levels by appropriate zoning within each bonded joint rather than general zoning across the structure. The former approach was used successfully at Douglas during the PABST program (reference 3.2.31) and, while the acceptance limits were shown to be too stringent from the structural viewpoint, the underlying philosophy was justified.

Figures 3.2.61 and 3.2.62 serve to illustrate this two-level inspection procedure. Briefly, only flaws no larger than 1/4 inch in diameter were accepted within half an inch of any discontinuity in the bonded structure, while flaws up to four times as large would be tolerated in the lightly loaded elastic troughs, provided they were not too close together. Consequently, the large-area bonded doublers that sometimes trapped air bubbles in their interior were not damaged by futile attempts to perform unnecessary repairs in areas of zero load transfer.

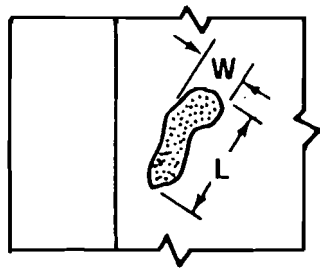
Figure 3.2.27 explains the use of visual inspection of adhesive bonds, which is actually a far more important indicator of the integrity of a bond than are ultrasonic methods. The visual inspection is effective only at the edges of a bonded overlap, but that is precisely where the load is transferred and where ultrasonic inspection is ineffective. Ultrasonic inspection is also incapable of distinguishing between good bonds and an intimate fit (reduced strength bond) caused by lack of flow, of the adhesive during cure. However, visual inspection works well in that regard and, in combination with proper surface preparations prior to bonding and monitoring of the cure cycle, can assure the strength and durability of an adhesively bonded structure.

Actually, the subject of acceptable flaw sizes in adhesive bonds should be addressed from two viewpoints -- before they are created and afterwards. The most economical bonded structures, both to build and maintain, are those containing the least flaws. While it may seem to be paradoxical that the highest quality structures cost less to build than those of inferior quality, that is usually the case. The reason is that recurrent bond flaws are associated with a misfit between the details and that, in turn, causes considerable hand reworking which would not have been necessary had the design and tooling been made compatible in the first place. This point is explained in detail in reference 3.2.29. Overly generous acceptable flaw sizes tend to encourage the use of inadequate, or unsuitable, tooling and result in high recurring costs for inspection and salvage, not to mention the cost of recording the existence of each flaw so that its rediscovery twenty years hence will not be cause for unwarranted alarm. The minimization of overall cost of bonded structures is thus associated with the minimization of the size and number of flaws. Conversely if, in spite of all efforts, there are occasional flaws in the finished structure, the maximization of structural durability is usually associated with the maximization of the size of flaw that is accepted without "repair". There are obviously individual exceptions that would require repairing to meet strength requirements but, nevertheless, there are far more instances of repairs to bonded structures being counterproductive.

3.3.16 Concluding Remarks.

This chapter contains outlines of the features included in the elastic-plastic analyses of various adhesive-bonded joints of different configurations; double-lap, single-lap, scarf and stepped lap.

There is also a discussion of the characteristics that govern the behavior of adhesive-bonded joints. Of particular importance is the frequently fundamentally different behavior of adhesive layers in test coupons, when compared with the corresponding behavior of structurally configured joints.



LONG
DIMENSION
(cm)

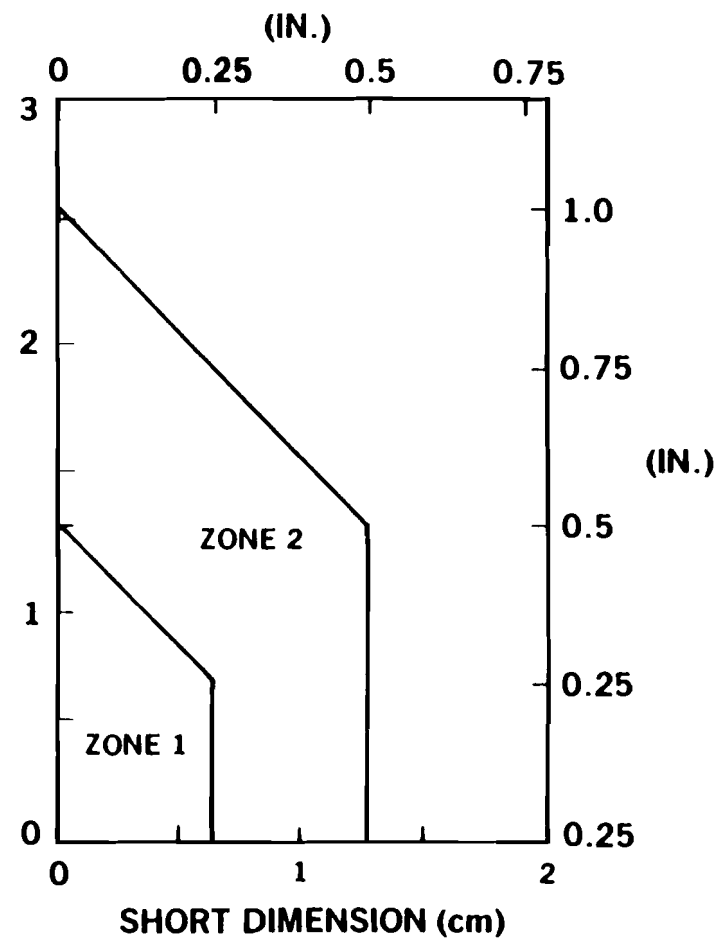
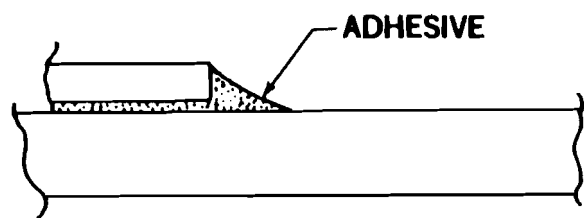
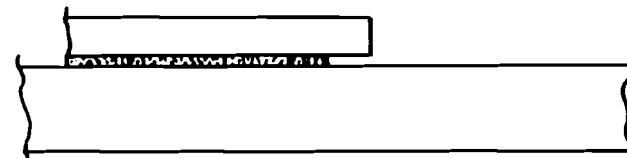


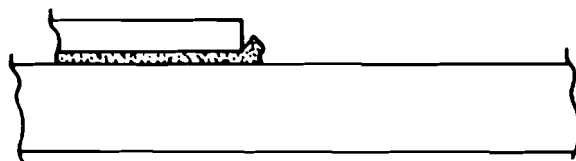
FIGURE 3.2.61. ACCEPTABLE BOND SIZES FOR PABST PROGRAM



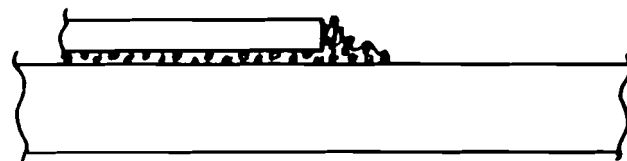
GOOD FLOW AND WETTING



**RECESSED FILLET, PROBABLY
DUE TO POOR FIT OR LACK
OF PRESSURE**



**POOR FLOW, PROBABLY DUE
TO SLOW HEAT-UP RATE**



**POROUS SPEW, PROBABLY DUE
TO MOISTURE IN ADHESIVE OR
EXCESSIVE HEAT-UP RATE**

FIGURE 3.2.62. GOOD AND BAD ADHESIVE FILLETS

The prime purpose of this material is to convey and understand the causes of strengths and weaknesses in adhesively bonded joints in fibrous composite structures. With that knowledge, it is possible to design strong, reliable joints, and to analyze them far more simply than would have been possible had not known potential deficiencies been excluded.

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APPENDIX 3.2A - ANALYSIS OF DOUBLE-LAP BONDED JOINTS

The differential equation governing the load transfer through the adhesive in a symmetric double-lap (or double-strap) joint can be derived on the basis of the coordinate system, elemental forces and displacements shown in 3.2.A1.

The analysis begins with the conditions of longitudinal equilibrium for the adherends

$$\frac{dT_o}{dx} + \tau = 0, \quad \frac{dT_i}{dx} - 2\tau = 0, \quad (3.2.A1)$$

in which τ is the adhesive shear stress and T_i is the total load, at a particular station, in all of the inner adherend and T_o is the load in only one of the two outer adherends. The thermo-elastic relations for the adherends are

$$\frac{d\delta_i}{dx} = \frac{T_i}{E_i t_i} + \alpha_i \Delta T, \quad \frac{d\delta_o}{dx} = \frac{T_o}{E_o t_o} + \alpha_o \Delta T \quad (3.2.A2)$$

in which

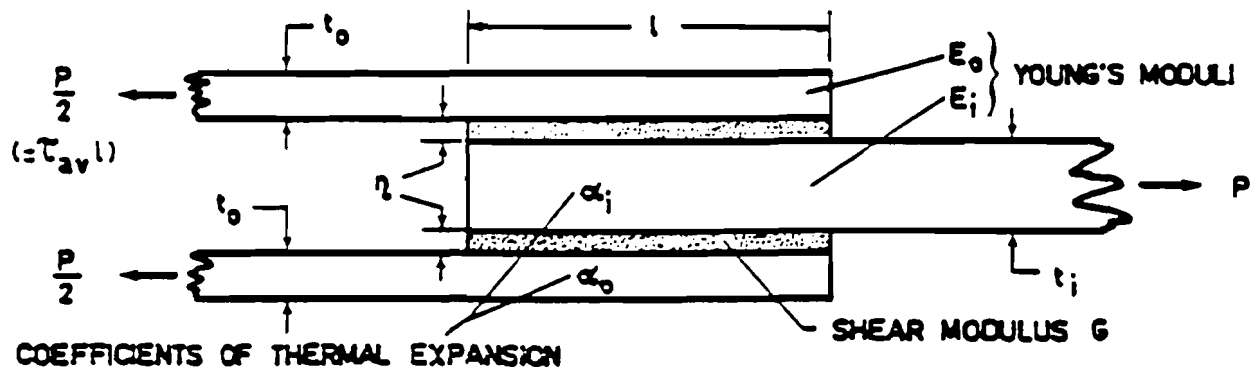
$$\Delta T = T_{\text{operating}} - T_{\text{stress-free}} \approx T_{\text{operating}} - T_{\text{cure}} \quad (3.2.A3)$$

represents the temperature differential from the reference state. The adherend extensional moduli are E_i and E_o respectively. The adherend displacements δ_i and δ_o are identified in figure 3.2.A1.

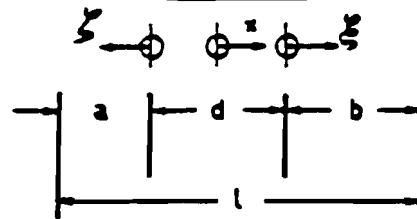
Any elastic adhesive behavior would be governed by the relation

$$\tau = G\gamma = \frac{G}{\eta}(\delta_i - \delta_o), \quad (3.2.A4)$$

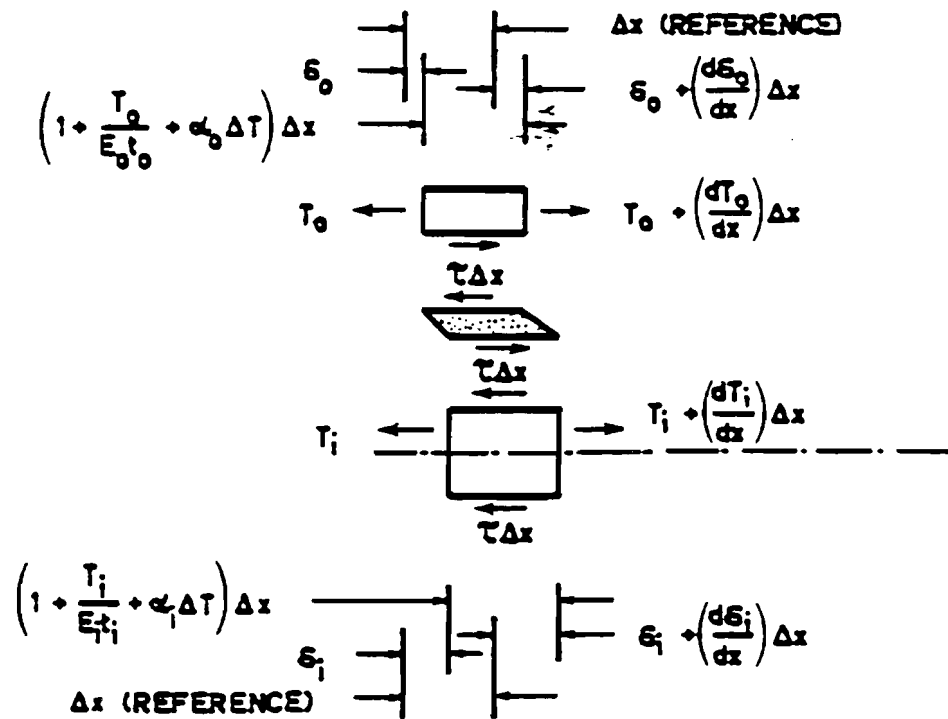
in which G is the adhesive layer shear modulus and η its thickness. This simple approximation of a uniform shear stress across the entire thickness of the adhesive layer has been questioned by a number of authors because it clearly violates the condition of freedom from stress on the exposed surfaces at the ends of the overlap. Elastic analyses which have accounted for this effect have reduced the predicted peak elastic shear stress substantially and moved it minutely, away from the end of the overlap, as shown in figure 3.2.7. However, this effect becomes insignificant if the adhesive is loaded sufficiently to strain it into the plastic zone beyond the base in the stress-strain curve in figure 3.2.1. Furthermore, the right side of figure 3.2.7 raises serious questions about whether the refinement of the elastic analysis is appropriate for structural adhesive-bonded joints in which the fillet is not scraped off because doing so is known to weaken the joint. Now, if the applied load on the joint



GEOMETRY



ADHESIVE BEHAVIOUR



DISPLACEMENTS AND FORCES

FIGURE 3.2.A1. CO-ORDINATE SYSTEM AND DEFORMATIONS IN BONDED JOINTS (DISSIMILAR ADHERENDS)

was sufficiently high, equation 3.2.A4 would apply only within the interior of the joint. At one or both ends there would be plastic behavior governed by the relations

$$\tau = \tau_p, \quad \gamma = (\delta_i - \delta_o) / \eta, \quad (3.2.A5)$$

with τ_p being the plastic adhesive shear stress cut-off.

In either of the possible adhesive behaviors, the analysis starts from the differentiation of the second of equation (3.2.A5). Thus

$$\frac{d\gamma}{dx} = \frac{1}{\eta} \left(\frac{d\delta_i}{dx} - \frac{d\delta_o}{dx} \right) = \frac{1}{\eta} \left(\frac{T_i}{E_i t_i} - \frac{T_o}{E_o t_o} \right) + \frac{\Delta T}{\eta} (\alpha_i - \alpha_o). \quad (3.2.A6)$$

A further differentiation yields

$$\frac{d^2\gamma}{dx^2} = \frac{\tau}{\eta} \left(\frac{2}{E_i t_i} + \frac{1}{E_o t_o} \right), \quad (3.2.A7)$$

which is valid throughout both the elastic and plastic zones.

Throughout the adhesive elastic trough,

$$\frac{d^2\gamma}{dx^2} = \frac{G}{\eta} \left(\frac{2}{E_i t_i} + \frac{1}{E_o t_o} \right) \gamma \quad (3.2.A8)$$

and the solution is of the form

$$\gamma = A \sinh(\lambda x) + B \cosh(\lambda x). \quad (3.2.A9)$$

Here, the exponent λ defines the steepness of the adhesive shear stress gradients. The coefficients A and B are determined by the boundary conditions. Depending on the load level, there are many possible solutions, as discussed in reference 3.2.9. These conditions follow from equation (3.2.A6) in which the right-hand side is known, as well as directly from equation (3.2.A9). Thus, the relation

$$\frac{d\gamma}{dx} = A\lambda \cosh(\lambda x) + B\lambda \sinh(\lambda x) = \frac{1}{\eta} \left(\frac{T_i}{E_i t_i} - \frac{T_o}{E_o t_o} \right) + \frac{\Delta T}{\eta} (\alpha_i - \alpha_o) \quad (3.2.A10)$$

is used in the determination of the coefficients A and B.

The plastic adhesive behavior is governed by a different equation. From equation (3.2.A7.)

$$\frac{d^2\gamma}{dx^2} = \frac{1}{\eta} \left(\frac{2}{E_i t_i} + \frac{1}{E_o t_o} \right) \tau_p = \text{constant} , \quad (3.2.A11)$$

whence

$$\gamma = \left(\frac{\lambda^2 \tau_p}{2G} \right) x^2 + Cx + F . \quad (3.2.A12)$$

The transition between elastic and plastic behavior occurs at some location, which is initially unknown, but at which

$$\tau = \tau_p \quad \text{and} \quad \gamma = \gamma_e = \tau_p / G \quad (3.2.A13)$$

and at which there is continuity of the gradient $d\gamma/dx$ on the basis of Equation (3.2.A10).

The precise algebraic manipulations to complete the solution are beyond the scope of this work, but are recorded in full in reference 3.2.9. Figure 3.2.28 is a nondimensionalized expression of this solution for the special case of stiffness-balanced joints in which there is no adherend thermal mismatch.

The asymptotic strengths shown in figure 3.2.28 are given by the relation:

$$\left(\frac{\tau_{av}}{\tau_p} \right) \left(\frac{\lambda \ell}{2} \right) \rightarrow \sqrt{1 + 2 \frac{\gamma_p}{\gamma_e}} . \quad (3.2.A14)$$

which can be re-expressed in the form

$$P = 2\tau_{av}\ell = 4\sqrt{\eta\tau_p \left(\frac{1}{2}\gamma_e + \gamma_p \right)} \sqrt{Et} . \quad (3.2.A15)$$

Now, the quantity under the first radical is the adhesive strain energy to failure, per unit bond area. That quantity, alone, is seen to characterize the entire influence of the adhesive behavior on the joint strength. The second radical, in which, t is the thickness of a single outer adherend as shown in figure 3.2.28, defines the entire contribution of the adherends to the shear

strength of the adhesive layer. That is the origin of the relation shown in figure 3.2.24. The transition between "short and "long overlap behavior occurs approximately when

$$l_{\text{transitional}} \approx \frac{P}{2\tau_p} \quad (3.2.A16)$$

In the more general case, in which either or both adherend stiffness imbalance or thermal mismatch may occur, equation (3.2.A14) is replaced by the lesser value of two expressions. Depending on which end of the joint is more critical. There are

$$\frac{\tau_{av}}{\tau_p} \left(\frac{\lambda l}{2} \right) \rightarrow \left[\sqrt{1 + 2 \frac{\gamma_p}{\gamma_e}} - \frac{(\alpha_i - \alpha_o) \Delta T \lambda}{\tau_p \left(\frac{2}{E_i t_i} + \frac{1}{E_o t_o} \right)} \right] \left[\frac{1 + E_i t_i / 2 E_o t_o}{2} \right] \quad (3.2.A17)$$

for $E_i t_i < 2 E_o t_o$ and

$$\frac{\tau_{av}}{\tau_p} \left(\frac{\lambda l}{2} \right) \rightarrow \left[\sqrt{1 + 2 \frac{\gamma_p}{\gamma_e}} - \frac{(\alpha_o - \alpha_i) \Delta T \lambda}{\tau_p \left(\frac{1}{E_o t_o} + \frac{2}{E_i t_i} \right)} \right] \left[\frac{1 + 2 E_o t_o / E_i t_i}{2} \right] \quad (3.2.A18)$$

for $2 E_o t_o < E_i t_i$.

These two expressions are cyclic with respect to the adherent contributions. Again, the joint strengths for "long" overlap joints are seen to be independent of that overlap.

The analyses above for the shear strength of double-lap bonded joints does not take into account any adhesive peel stresses. Yet, there must be some, as explained in figure 3.2.A2. The complete derivation of the adhesive peel stress distribution is given in references 3.2.2 and 3.2.9. A simple explicit solution can be obtained if it is assumed that the weakness in peel strength (or in interlaminar tension) is not so great as to prevent the generation of any plastic adhesive shear behavior. The differential equilibrium eqs. for the element of the outer adherent are:

$$\frac{dM}{dx} = V - \tau_p \frac{t_o}{2} \quad (3.2.A19)$$

for moment equilibrium and

$$\frac{dV}{dx} = \sigma_c \quad (3.2.A20)$$

for normal equilibrium. The characteristic equation of the outer adherent undergoing elastic bending is

$$\frac{d^2 w_o}{dx^2} = - \frac{M}{D_o}, \quad (3.2.A21)$$

while the assumed elastic adhesive peel stress is defined as

$$\sigma_c = E_c \frac{w_o}{\eta}, \quad (3.2.A22)$$

in which E_c is the normal modulus of the adhesive in peel. Strictly, for composite adherents, some allowance should be made for the additional flexibility of the adherents under such a stress state, as in reference 3.2.9, by introducing an effective (and lower) peel modulus given by

$$\frac{1}{E_c'} = \frac{1}{E_c} + \frac{k_i}{E_{in}} + \frac{k_o}{E_{on}}. \quad (3.2.A23)$$

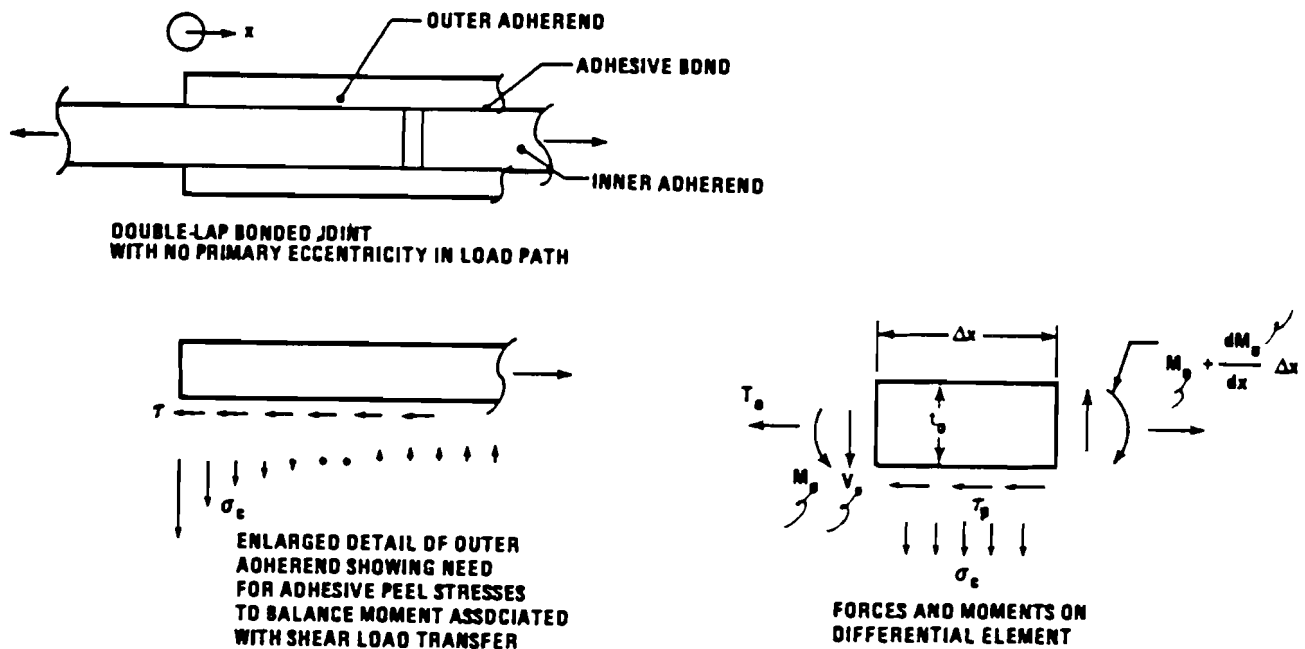


FIGURE 3.2.A2. INDUCED PEEL STRESSES IN DOUBLE-LAP ADHESIVE-BONDED JOINTS

Here, E_{in} and E_{on} are the transverse tensile moduli of the inner and outer adherents, respectively and k_i and k_o refer to the thickness of laminate affected as a multiple of the adhesive layer thickness. The solution of equations (3.2.AI9) to (3.2.A23) is then governed by the classical differential equation for a beam on an elastic foundation.

That is

$$\frac{d^4 w_o}{dx^4} - \frac{E_c'}{D_o \eta} w_o = 0 \quad (3.2.A24)$$

and, on introducing the notation

$$\chi^4 = E_c' / (4D_o \eta) , \quad (3.2.A25)$$

the relevant solution is

$$w_o = e^{-\chi x} [A \cos(\chi x) + B \sin(\chi x)] , \quad (3.2.A26)$$

assuming that the overlap is sufficiently long not to need the other two terms with a positive exponent. Since there can be no bending moment at the free edge of the splice or doubler;

$$\frac{d^2 w_o}{dx^2} = 0 \quad \text{at} \quad x = 0 , \quad (3.2.A27)$$

The final solution is shown in reference 3.2.2 to be

$$\sigma_c = \frac{\tau_p t_o E_c'}{4\chi^3 D_o \eta} e^{-\chi x} \cos(\chi x) = \tau_p t_o e^{-\chi x} \cos(\chi x) , \quad (3.2.A28)$$

which has a peak value, at the very end of the bonded overlap, given by

$$\sigma_{c \max} = \tau_p t_o \chi = \tau_p \left[\frac{3(1 - \nu^2) E_c'}{E_o} \right]^{\frac{1}{4}} \left(\frac{t_o}{\eta} \right)^{\frac{1}{4}} . \quad (3.2.A29)$$

Peak peel stress is seen to be aggravated by high peel moduli, by thicker outer adherents, and by thin glue pinched off at the ends of the overlaps.

Equation (3.2.A29) contains no explicit mention of the applied load, because of the assumption made about the local uniformity of the adhesive shear stress p at the end of the overlap. What equation (3.2.A29) does indicate is that the load necessary to cause a peel-stress failure -- or delamination of an adherent, usually in the middle one as shown in figure 3.2.A18 -- varies as the quarter power of the tip thickness of the outer adherents. This relation is shown in figure 3.2.24 in the explanation of how the shear stress failures and peel-stress failures do not occur simultaneously.

The importance of this peel-stress analysis is not some precise characterization of the peel-stress distribution. Rather, it is the proof that the peel-stress problem can be removed by restricting the tip thickness t_0 of the outer adherent, as shown in figure 3.2.17. The locally thickened glue layer is also seen to be beneficial. It should be noted that the only minor effect of the inner adherent is on the effective peel modulus E_c^1 , even though the inner adherent is where the failure would be observed. The dominant effect of the adherents in this analysis is the tip thickness of the outer adherents.

APPENDIX 3.2B - ANALYSIS OF SINGLE-LAP BONDED JOINTS

The Volkersen approach to the analysis of shear transfer through bonded joints cannot be used to characterize the behavior of unsupported single-lap joints. The modifications introduced by Goland and Reissner (reference 3.2.1) are needed to account for the effects of the out-of-plane deflections due to eccentricities in load path. The free-body diagram used to set up the equilibrium and compatibility equations, for such a joint with identical adherents, is shown in figure 3.2.B1. The corresponding analysis for the bonding of dissimilar adherents is given in reference 3.2.10.

The first part of the analysis is the determination of the bending moments in the adherents at each end of the bonded overlap. It is in this detail that the analysis by Goland and Reissner can be improved significantly (see reference 3.2.10 and 3.2.24). Their subsequent analysis for the internal shear and peel stresses in the adhesive layer is algebraically correct but is rendered quantitatively inaccurate by not having the appropriate boundary conditions. Incidentally, this failure to determine correct boundary conditions pervades almost all published analyses of the stresses in single-lap joints.

For purposes of analysis, the joint is divided into the four sections shown in figure 3.2.B1. Throughout region 1, the longitudinal stress resultant (force per unit lateral width) is uniform, at the value P of the applied load. The stress couple M_1 (inch pound per unit width) is then given by the equilibrium equation.

$$M_1 = P \left[\left(\frac{t + \eta}{\ell + c} \right) \frac{x}{2} - w_1 \right] \quad \text{for} \quad 0 \leq x \leq \ell . \quad (3.2.B1)$$

According to the classical theory for the infinitesimal bending of thin cylindrically deformed plates,

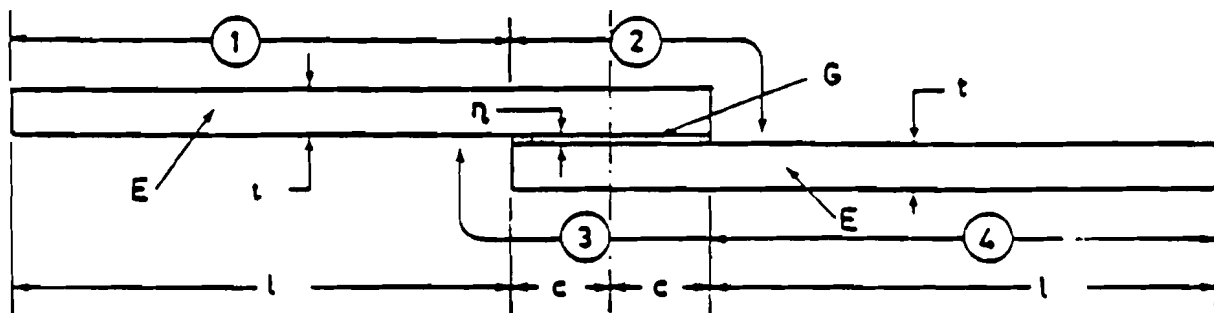
$$\frac{d^2 w_1}{dx^2} = - \frac{M_1}{D} = - \frac{P}{D} \left[\left(\frac{t + \eta}{\ell + c} \right) \frac{x}{2} - w_1 \right] , \quad (3.2.B2)$$

the solution of which is

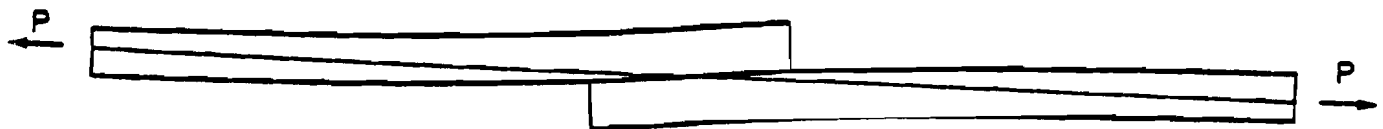
$$w_1 = A_1 \cosh(\xi x) + C_1 \sinh(\xi x) + \left(\frac{t + \eta}{\ell + c} \right) \frac{x}{2} , \quad (3.2.B3)$$

where

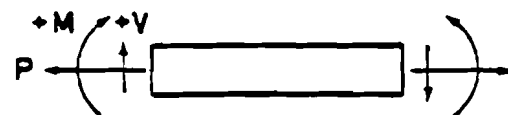
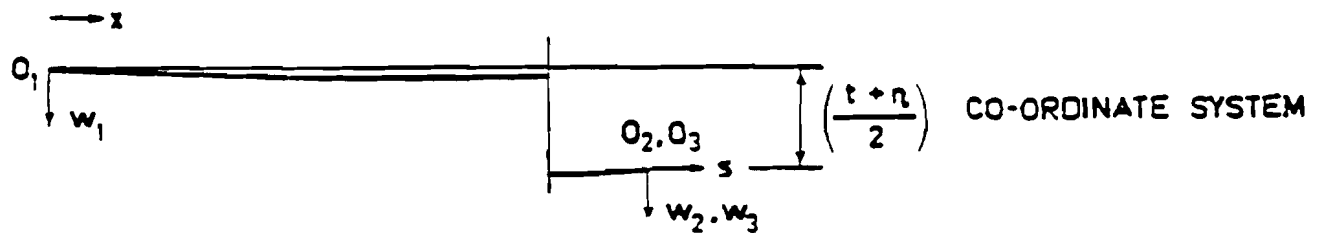
$$\xi^2 = P/D . \quad (3.2.B4)$$



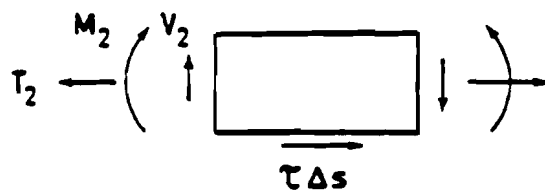
UNLOADED LAP JOINT



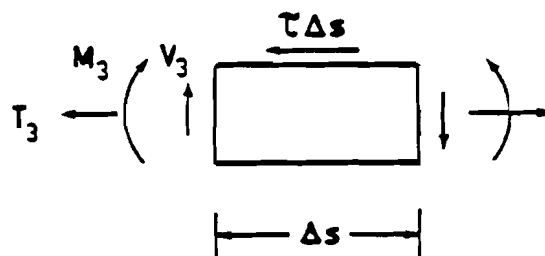
LAP JOINT DEFLECTING UNDER LOAD



SIGN CONVENTION



LOADS ON ELEMENT OF REGION 2



LOADS ON ELEMENT OF REGION 3

FIGURE 3.2.B1. CO-ORDINATE SYSTEM AND DEFORMATIONS IN BALANCED SINGLE-LAP BONDED JOINTS

In the normal case of the structural application of single-lap bonded joints, the support at $x = 0$ is flexible and far removed from the overlap area ($\ell \gg c$) so that, regardless of whether the adherent is simply supported there or built-in,

$$w_1 \approx B_1 e^{\xi x} + \left(\frac{t + \eta}{\ell + c} \right) \frac{x}{2} . \quad (3.2.B5)$$

This point is explained fully in reference 3.2.2.

Turning attention now to regions 2 and 3 in figure 3.2.B1, moment equilibrium requires that

$$\frac{dM_2}{ds} - V_2 + \tau \left(\frac{t + \eta}{2} \right) = 0 , \quad \frac{dM_3}{ds} - V_3 + \tau \left(\frac{t + \eta}{2} \right) = 0 , \quad (3.2.B6)$$

while longitudinal force equilibrium requires that

$$\frac{dT_2}{ds} + \tau = 0 , \quad \frac{dT_3}{ds} - \tau = 0 , \quad (3.2.B7)$$

and transverse force equilibrium requires that

$$\frac{dV_2}{ds} + \sigma_c = 0 \quad \text{and} \quad \frac{dV_3}{ds} - \sigma_c = 0 . \quad (3.2.B8)$$

From plate theory, with the size convention depicted in figure 3.2.B1.

$$\frac{d^2 w_2}{ds^2} = - \frac{M_2}{D} \quad \text{and} \quad \frac{d^2 w_3}{ds^2} = - \frac{M_3}{D} . \quad (3.2.B9)$$

Assuming that the adhesive properties are linearly elastic in transverse (normal) tension,

$$\frac{\sigma_c}{E_c} = \frac{(w_3 - w_2)}{\eta} . \quad (3.2.B10)$$

For fibrous composite adherents, the adhesive peel modulus E_c should really be replaced by an effective modulus E_c' in the same manner as in eq. 3.2.A23.

Based on the analyses developed in reference 3.2.10, it is possible to proceed directly with an approximate solution of these equations. The precisely antisymmetric average adherent displacement throughout the bonded area is taken

to be

$$\frac{1}{2}(w_2 + w_3) \approx A_{23}s^3 + C_{23}s \approx w_2 \approx w_3 . \quad (3.2.B11)$$

The similarity of the slopes dw_2/ds and dw_3/ds is also assumed, but the distinction between the higher derivatives must be maintained since, at the ends of the overlaps ($s = \pm c$),

$$\frac{d^2 w_3}{ds^2} \approx \frac{d^3 w_3}{ds^3} \equiv 0 \quad \text{at } s = -c \quad \text{and} \quad \frac{d^2 w_2}{ds^2} = \frac{d^3 w_2}{ds^3} \equiv 0 \quad \text{at } s = +c . \quad (3.2.B12)$$

Therefore, it is sometimes necessary to set

$$w_2 = [(w_2 + w_3) + (w_2 - w_3)] / 2 \quad (3.2.B13)$$

to complete the solution.

The displacement at the edge of the overlap ($x = \ell$, $s = -c$) is given by

$$w_1 \Big|_{x=\ell} = \left(\frac{t+\eta}{\ell+c} \right) \frac{\ell}{2} + B_1 e^{\xi \ell} \approx \frac{1}{2}(w_2 + w_3) \Big|_{s=-c} = -A_{23}c^3 - C_{23}c . \quad (3.2.B14)$$

Matching the slopes at the same location leads to the results

$$\frac{dw_1}{dx} \Big|_{x=\ell} \approx \frac{1}{2} \left(\frac{t+\eta}{\ell+c} \right) + B_1 e^{\xi \ell} \approx \frac{1}{2} \frac{d(w_2 + w_3)}{ds} \Big|_{s=-c} = 3A_{23}c^2 + C_{23} . \quad (3.2.B15)$$

The continuity of the bending moment at that location requires that

$$\frac{d^2 w_1}{dx^2} \Big|_{x=\ell} \approx B_1 \xi^2 e^{\xi \ell} = -\frac{M_e}{D} \equiv \frac{1}{2} \left[\frac{d^2(w_2 + w_3)}{ds^2} - \frac{d^2(w_2 - w_3)}{ds^2} \right]_{s=-c} = -6A_{23}c - \frac{M_e}{2D} , \quad (3.2.B16)$$

in which M_e is the bending moment in the continuous adherent at the end of the overlap. From equation (3.2.B16),

$$\frac{M_e}{D} = -B_1^2 \xi^2 e^{\xi \ell} = 12A_{23}c . \quad (3.2.B17)$$

The equation above can be solved simultaneously. Thus,

$$\left. \begin{aligned} \left(\frac{t+\eta}{\ell+c}\right)\frac{\ell}{2} + B_1 e^{\xi\ell} &= -A_{23}c^3 - C_{23}c, \\ \frac{1}{2}\left(\frac{t+\eta}{\ell+c}\right) + B_1 \xi e^{\xi\ell} &= 3A_{23}c^2 + C_{23}, \\ B_1 \xi^2 e^{\xi\ell} &= -12A_{23}c. \end{aligned} \right\} \quad (3.2.B18)$$

These are easily solved to yield

$$\frac{1}{2}\left(\frac{t+\eta}{\ell+c}\right)(\ell+c) + B_1 e^{\xi\ell}(1+\xi c) = 2A_{23}c^3 = -B_1 e^{\xi\ell}\left(\frac{\xi^2 c^2}{6}\right), \quad (3.2.B19)$$

whence the bending moment M_e at the ends of the overlap ($x = \ell$, $s = -c$) is given by

$$M_e = \frac{P(t+\eta)/2}{[1 + \xi c + \frac{1}{6}\xi^2 c^2]} = k P(t+\eta)/2, \quad (3.2.B20)$$

as given in equation (3.2.39) of reference 3.2.10.

This result differs significantly from the equivalent result of Goland and Reissner (reference 3.2.1) that

$$M_e = \frac{P(t+\eta)/2}{[1 + 2\sqrt{2} \tanh(\xi c/2\sqrt{2})]}. \quad (3.2.B21)$$

The significant difference is that, according to equation 3.2.B20

$$M_e \rightarrow 0 \quad (3.2.B22)$$

for large overlaps and large distances between the support and bonded joints, whereas M_e remains finite according to equation 3.2.B21, no matter how long the overlap becomes. Thus, the Goland and Reissner solution overestimates the bending moment at the ends of the overlap and, in turn, the peak adhesive shear and peel stresses. The reason for this is that Goland and Reissner lumped together the elements 2 and 3 in figures 3.2.B1 for determining the boundary conditions of the ends of the overlap. The differences between the two solutions -- equations 3.2.B20 and 3.2.B21 is shown in figure 3.2.B2.

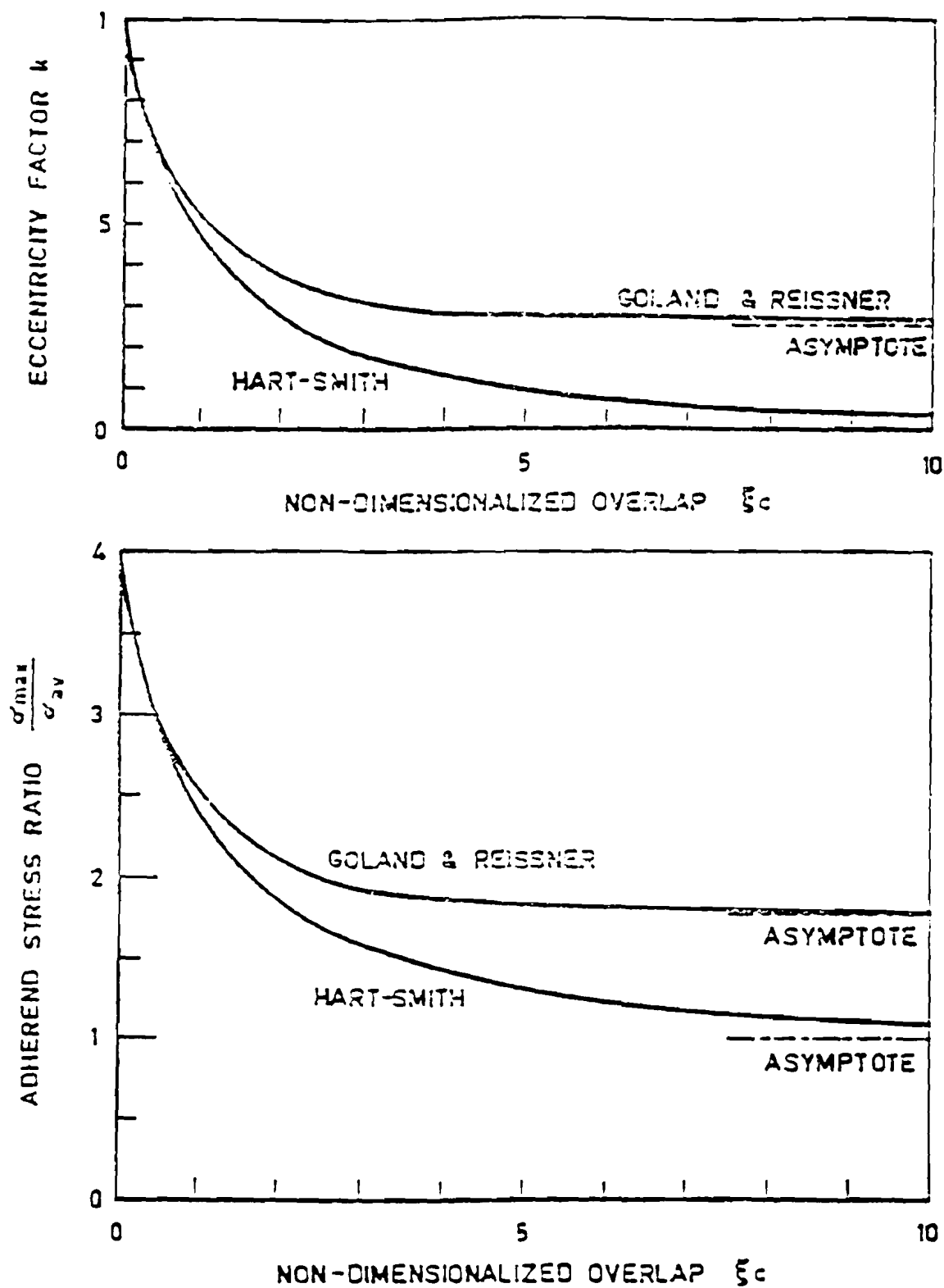


FIGURE 3.2.B2. COMPARISON BETWEEN PRESENT SOLUTION FOR ADHERENT STRESSES AND THAT OF GOLAND AND REISSNER

The ratio of average to maximum stress represents the structural efficiency of the panel -- not just of the joint -- and for isotropic adherents is expressible in the form

$$\eta = \frac{\sigma_{avg}}{\sigma_{max}} = \frac{\sigma_{avg}}{\sigma_{avg} + 6M/t^2} = \frac{1}{1 + 3/\left[1 + \xi_c + \frac{1}{6}\xi^2 c^2\right]}, \quad (3.2.B23)$$

The inverse of this quantity represents the severity of the stress concentration.

Now that the bending moment M_e at the ends of the overlap has been established, it is possible to solve for the distribution of the adhesive peel stresses (or interlaminar tension stresses). From equations (3.2.B9) and (3.2.B10),

$$\frac{d^2(w_3 - w_2)}{ds^2} = -\frac{1}{D}(M_3 - M_2) = \left[\frac{\eta}{E_c}\right] \frac{d^2\sigma_c}{ds^2}. \quad (3.2.B24)$$

Differentiating twice, and substituting from equations (3.2.B6) and (3.2.B8), it follows that

$$\left(\frac{\eta}{E_c}\right) \frac{d^3\sigma_c}{ds^3} = -\frac{1}{D}(V_3 - V_2) \quad (3.2.B25)$$

and

$$\left(\frac{\eta}{E_c}\right) \frac{d^4\sigma_c}{ds^4} = -\frac{2\sigma_c}{D}, \quad (3.2.B26)$$

or

$$\frac{d^4\sigma_c}{ds^4} + 4\chi^4\sigma_c = 0, \quad (3.2.B27)$$

where

$$\chi^4 = \frac{E_c}{2\eta D}. \quad (3.2.B28)$$

Given that the adhesive peel stresses must be symmetric about the middle of the overlap ($s = 0$), the appropriate solution of equation (3.2.B27) is

$$\sigma_c = A \cos(\chi s) \cosh(\chi s) + B \sin(\chi s) \sinh(\chi s). \quad (3.2.B29)$$

The completion of this analysis can be found in reference 3.2.2, from which the results below have been taken.

For moderately long overlaps (structural joints rather than test coupons), the peak adhesive peel stress at $z = \pm c$ is shown there to be given by

$$\sigma_{c \max} = \frac{E_c M_e}{2\eta D \chi^2} = M_e \sqrt{\frac{E_c}{2\eta D}} . \quad (3.2.B30)$$

Thus, the peak peel stress in the adhesive is directly proportional to the adherent bending moment M_e induced by the eccentricity in load path. This explains why it is so important to design in an adequate overlap for single-lap joints, to reduce that bending moment. Equation (3.2.B6) can be expressed in a slightly different form

$$\frac{\sigma_{c \max}}{\sigma_{avg}} \propto k \sqrt{\left(\frac{E_c}{E}\right) \left(\frac{t}{\eta}\right)} \quad (3.2.B31)$$

showing that, for a given adherent operating stress σ_{avg} , the maximum induced adhesive peel stress $\sigma_{c \max}$ can be minimized by the use of low-modulus (E_c) adhesives, by thick adhesive layers (η) and by thin ends of the adherents (t), as well as by increasing the overlap to decrease the eccentricity coefficient (k) in equation (3.2.20).

The adhesive shear stress distribution can be determined as follows. Defining by u_2 and u_3 the longitudinal displacements of the adherents on each side of the adhesive layer in figure 3.2.B1, the stress-strain relations for plane strain of the assumed elastic adherents give

$$\frac{du_2}{ds} = \frac{1}{Et} \left[T_2 + \frac{6(1-\nu^2)M}{t} \right] , \quad \frac{du_3}{ds} = \frac{1}{Et} \left[T_3 - \frac{6(1-\nu^2)M}{t} \right] . \quad (3.2.B32)$$

The adhesive properties are assumed to be perfectly elastic-plastic in shear between the adherents and are given by

$$\left. \begin{aligned} \gamma &= \frac{\tau}{G} = \left(\frac{u_3 - u_2}{\eta} \right) & \text{for } \gamma \leq \gamma_e \\ \tau &= \tau_p & \text{for } \gamma \geq \gamma_e \end{aligned} \right\} . \quad (3.2.B33)$$

The equation governing the adhesive shear strain distribution can be derived as follows

$$\frac{d\gamma}{ds} = \frac{1}{\eta} \left(\frac{du_3}{ds} - \frac{du_2}{ds} \right) = \frac{1}{Et\eta} \left[T_3 - T_2 - 6(1 - \nu^2)(M_3 + M_2)/t \right], \quad (3.2.B34)$$

whence

$$\frac{d^2\gamma}{ds^2} = \frac{1}{Et\eta} \left[2\tau + 6(1 - \nu^2)\tau - 6(1 - \nu^2)(V_3 + V_2)/t \right] \quad (3.2.B35)$$

and

$$\frac{d^3\gamma}{ds^3} = [2 + 6(1 - \nu^2)] \left(\frac{1}{Et\eta} \right) \frac{d\tau}{ds}. \quad (3.2.B36)$$

For the elastic behavior,

$$\frac{d^3\tau}{ds^3} = [1 + 3(1 - \nu^2)] \frac{2G}{Et\eta} \frac{d\tau}{ds} = [1 + 3(1 - \nu^2)] \lambda^2 \frac{d\tau}{ds} \quad (3.2.B37)$$

and the solution is

$$\tau = A \cosh(2\lambda's) + B \sinh(2\lambda's) + C, \quad (3.2.B38)$$

in which the exponent $(2\lambda')$ for single-lap joints is related to that (λ) for double-lap joints given in equation (3.2.B8) by

$$(\lambda')^2 = \frac{[1 + 3(1 - \nu^2)]}{4} (\lambda)^2. \quad (3.2.B39)$$

It is evident that the elastic adhesive shear stress gradients are nearly twice as high for single-lap joints as for double-lap ones. However, the bigger difference between these two families of joints derives from the term C in equation (3.2.B38) that is completely lacking from the corresponding equation (3.2.B9). It is shown in reference 3.2.10 that, for long overlaps,

$$C \rightarrow \frac{3}{4} \tau_{avg}. \quad (3.2.B40)$$

For the zones of plastic adhesive behavior, equations (3.2.B36) and (3.2.B33) yield

$$\gamma = As^2 + Bs + C \quad (3.2.B41)$$

since, in those zones, $d\tau/ds \approx 0$. The complete solution, including the width of the plastic zone which changes with the load intensity, is assembled by matching the various boundary conditions as in reference 3.2.10.

APPENDIX 3.2C - ANALYSIS OF ADHESIVE-BONDED SCARF JOINTS

The joint geometry and notation for the analysis of a bonded scarf joint with a single bond interface are shown in figure 3.2.C1. The major difference with respect to the analysis of joints having uniform adherents is that the adherent stiffnesses in equations (3.2.A2) and (3.2.A6) are no longer uniform. There is also a change in notation from inner and outer adherents to adherent numbers one and two. The stiffness distributions are now given by

$$(Et)_1 = E_1 t_1 \left(1 - \frac{x}{\ell} \right) \quad \text{and} \quad (Et)_2 = E_2 t_2 \left(\frac{x}{\ell} \right). \quad (3.2.C1)$$

The entire analysis is given in reference 3.2.11 and summarized here. The analysis is restricted to joints without eccentricities because it would be unusual to go to the trouble of scarfing a joint to improve its strength while not simultaneously eliminating any eccentricities in the load path that could cause bending of the adherents and require additional analysis of the type given above for unsupported single-lap joints.

The conditions for horizontal equilibrium of a differential element with the joint are

$$\frac{dT_1}{dx} + \tau = 0, \quad \frac{dT_2}{dx} - \tau = 0. \quad (3.2.C2)$$

The stress strain relations for the adherents materials, accounting for thermo-elastic effects yield

$$\frac{d\delta_1}{dx} = \frac{T_1}{(Et)_1} + \alpha_1 \Delta T, \quad \frac{d\delta_2}{dx} = \frac{T_2}{(Et)_2} + \alpha_2 \Delta T. \quad (3.2.C3)$$

The adhesive shear strain is again taken to be uniform across the thickness of the bond. That is

$$\gamma = (\delta_2 - \delta_1) / \eta. \quad (3.2.C4)$$

The elastic adhesive shear stress is still given by

$$\tau = G\gamma = G(\delta_2 - \delta_1) / \eta. \quad (3.2.C5)$$

Differentiation of equation (3.2.C4) yields

$$\frac{d\gamma}{dx} = \frac{1}{\eta} \left(\frac{d\delta_2}{dx} - \frac{d\delta_1}{dx} \right) = \frac{1}{\eta} \left[\frac{T_2}{(Et)_2} - \frac{T_1}{(Et)_1} \right] + \frac{\Delta T}{\eta} (\alpha_2 - \alpha_1). \quad (3.2.C6)$$

The variation in the adherent stiffnesses prevent the derivation of an explicit closed form solution. Therefore a power series solution is sought and, for the elastic adhesive area,

$$\frac{\tau}{\tau_p} = \sum_{n=1}^{\infty} A_n \phi^{n-1}, \quad (3.2.C7)$$

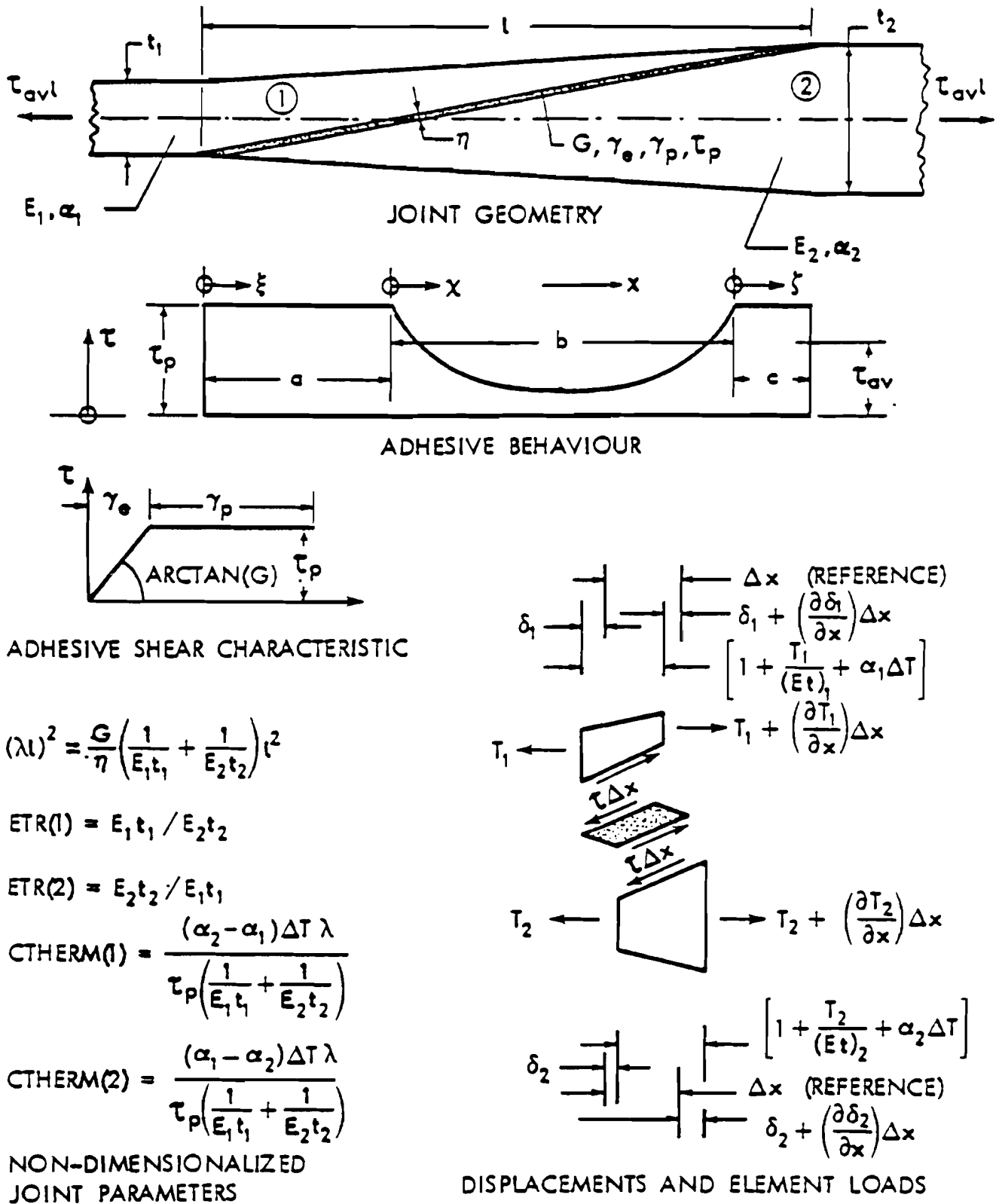


FIGURE 3.2.C1. NOTATION AND GEOMETRY FOR ADHESIVE-BONDED SCRAF JOINT ANALYSIS

where

$$\phi = x / \ell . \quad (3.2.C8)$$

Substituting equation (3.2.C7) into the integrated form of equation (3.2.C2),

$$T_1 = \tau_{av} \ell - \tau_p \ell \sum_1^{\infty} \frac{A_n}{n} \phi^n , \quad T_2 = \tau_p \ell \sum_1^{\infty} \frac{A_n}{n} \phi^n . \quad (3.2.C9)$$

The substitution of equations (3.2.C9) and (3.2.C1) into equation (3.2.C6) then yields

$$\begin{aligned} \frac{d(\tau/\tau_p)}{d\phi} &= \sum_1^{\infty} (n-1) A_n \phi^{(n-2)} \\ &= \frac{G\ell^2}{\tau_p \eta} \left[\frac{\tau_p \sum_1^{\infty} \frac{A_n}{n} \phi^{n-1}}{E_2 t_2} - \frac{\tau_{av} - \tau_p \sum_1^{\infty} \frac{A_n}{n} \phi^n}{E_1 t_1 (1 - \phi)} \right] + \frac{G\ell}{\tau_p \eta} (\alpha_1 - \alpha_2) \Delta T . \end{aligned} \quad (3.2.C10)$$

Equation (3.2.C10) is then multiplied throughout by (1) to convert the equation directly into a form suitable for solution by recurrence relations between the coefficients A_n for each power of ϕ . However, the direct solution in terms of the boundary conditions was found to be numerically divergent in reference 3.2.11 in the sense that successive terms in the series in equation (3.2.C7) do not decrease steadily with increasing values of n . Yet it was found that, even so, the successive contributions of the average shear stress

$$\frac{\tau_{av}}{\tau_p} = \sum_1^{\infty} \frac{A_n}{n} \quad (3.2.C11)$$

did converge rapidly, so valid solutions were obtained for the joint strength even though the adhesive shear stress distribution themselves could not be determined. This incomplete definition of the stress field is not normally a problem since those undefined stresses are everywhere else less severe than at the more critical end of the overlap. However, the inability to estimate the minimum adhesive shear stress, in order to assess the creep resistance of the joint can be of concern if the joint is such as to have an inherently low stress concentration.

Within any plastic zones in the adhesive, a different solution is required for tapered adherents. Equation (3.2.C9) is then modified to read

$$T_1 = \tau_{av} \ell - \tau_p x, \quad T_2 = \tau_p x \quad (3.2.C12)$$

Then, from Equations (3.2.C4) and (3.2.C3),

$$\gamma = \frac{(\delta_2 - \delta_1)}{\eta} = \frac{1}{\eta}(\alpha_2 - \alpha_1)\Delta T + \int_0^x \frac{T_2}{(Et)_2} dx - \int_0^x \frac{T_1}{(Et)_1} dx. \quad (3.2.C13)$$

This equation can be integrated once the variation of the extension stiffnesses is specified, as in equation (3.2.C1). That solution can then be matched with the elastic solution at the transition from elastic to plastic behavior as well as to the boundary conditions. Equation (3.2.C7) is modified for this purpose to read

$$\frac{\gamma}{\gamma_e} = \sum_{n=1}^{\infty} A_n \phi^{n-1} \quad (3.2.C14)$$

since, for elastic behavior.

$$\tau / \tau_p = \gamma / \gamma_e \quad (3.2.C15)$$

and the left side of equation 3.2.C10 is modified accordingly.

APPENDIX 3.2D - ANALYSIS OF STEPPED-LAP BONDED JOINTS

Figure 3.2.49 shows the sign convention and nomenclature necessary for the analysis. The analysis begins with the equilibrium equations for a differential element of one of the steps.

$$\frac{dT_1}{dx} - p_1 + F(1-D)\tau = 0, \quad \frac{dT_2}{dx} + p_2 - F(1-D)\tau = 0, \quad (3.2.D1)$$

in which

$$\left. \begin{array}{l} F = 1 \text{ for one layer of adhesive} \\ F = 2 \text{ for two layers of adhesive} \end{array} \right\} . \quad (3.2.D2)$$

The term D represents the effective disbond or porosity at that step, and the shear stress τ characterizes the adhesive shear stress on the reduced area that is fully effective. The factor $1-D$ is needed in order to relate the shear stress τ to the differential motion between the adherents in terms of adhesive stress-strain characteristics measured on unflawed adhesive bonds. The adherent forces per unit width, T_1 and T_2 , refer to the total load which may be in one or two physical members, depending on the number of adhesive-bonded surfaces.

The thermo-elastic relationships for the adherents are

$$\frac{d\delta_1}{dx} = \frac{T_1}{E_1 t_1} + \alpha_1 \Delta T, \quad \frac{d\delta_2}{dx} = \frac{T_2}{E_2 t_2} + \alpha_2 \Delta T, \quad (3.2.D3)$$

in which

$$\Delta T = T_{\text{operating}} - T_{\text{stress-free}} \approx T_{\text{operating}} - T_{\text{cure}} \quad (3.2.D4)$$

and is usually negative. At a first approximation, the adhesive shear strain is represented as

$$\gamma = (\delta_2 - \delta_1) / \eta \quad (3.2.D5)$$

and is assumed to be constant across the thickness of the adhesive.

Differentiation of eq. (3.2.D5) and the elimination of the displacements σ_1 and σ_2 via equation (3.2.D3) leads to the result

$$\frac{d\gamma}{dx} = \frac{1}{\eta} \left(\frac{d\delta_2}{dx} - \frac{d\delta_1}{dx} \right) = \frac{1}{\eta} \left[\frac{T_2}{E_2 t_2} - \frac{T_1}{E_1 t_1} + (\alpha_2 - \alpha_1) \Delta T \right]. \quad (3.2.D6)$$

A further differentiation of eq. (3.2.D6), in conjunction with equation (3.2.D1), leads to the expression

$$\frac{d^2\gamma}{dx^2} - \frac{F(1-D)}{\eta} \left(\frac{1}{E_1 t_1} + \frac{1}{E_2 t_2} \right) \tau = - \frac{1}{\eta} \left(\frac{p_1}{E_1 t_1} + \frac{p_2}{E_2 t_2} \right), \quad (3.2.D7)$$

In which the quantities p_1 and p_2 represent externally applied running loads along each step.

Within the elastic region of adhesive behavior, as with light applied loads or near the middle of the bonded overlap, the adhesive shear stress is related to the shear strain by the equation.

$$\tau = G\gamma = G(\delta_2 - \delta_1) / \eta \quad (3.2.D8)$$

while, throughout any plastic adhesive areas,

$$\tau = \tau_p = \text{constant} . \quad (3.2.D9)$$

Adhesive peel stresses are customarily omitted from the analysis because the proportions in figure 3.2.49, that yield the highest shear strength, have very thin outermost (end) steps, rendering any peel stresses insignificant.

The combination of equations (3.2.D7) and (3.2.D8) yields, for the governing differential equation for the elastic behavior,

$$\frac{d^2\gamma}{dx^2} - F(1-D)\frac{G}{\eta} \left(\frac{1}{E_1 t_1} + \frac{1}{E_2 t_2} \right) = - \frac{1}{\eta} \left(\frac{p_1}{E_1 t_1} + \frac{p_2}{E_2 t_2} \right) \quad (3.2.D10)$$

or, on introducing the exponent

$$\lambda^2 = F(1-D)\frac{G}{\eta} \left(\frac{1}{E_1 t_1} + \frac{1}{E_2 t_2} \right) , \quad (3.2.D11)$$

$$\frac{d^2\gamma}{dx^2} - \lambda^2 \gamma = \text{constant} . \quad (3.2.D12)$$

The elastic adhesive shear stresses are then expressed as

$$\gamma = A \cosh(\lambda x) + B \sinh(\lambda x) + C , \quad (3.2.D13)$$

in which

$$C = \frac{1}{\lambda^2 \eta} \left(\frac{p_1}{E_1 t_1} + \frac{p_2}{E_2 t_2} \right) . \quad (3.2.D14)$$

Integration of equations (3.2.D1) in turn, making use of eq. (3.2.D8), leads to the expressions.

$$T_1 = T_{1\text{ref}} + p_1 x - F(1-D)G \left[\frac{A}{\lambda} \sinh(\lambda x) + \frac{B}{\lambda} \left(\cosh(\lambda x) - 1 \right) + Cx \right] \quad (3.2.D15)$$

and

$$T_2 = T_{2\text{ref}} - p_2x + F(1-D)G \left[\frac{A}{\lambda} \sinh(\lambda x) + \frac{B}{\lambda} (\cosh(\lambda x) - 1) + Cx \right] . \quad (3.2.D16)$$

The values of the integration constants $T_{1\text{ref}}$ and $T_{2\text{ref}}$ depend on the origin of x adopted. In the coding of the computer programs A4EG, ...H, and ...I, it has been convenient to adopt the start of each step as the origin of x for that step. If there is some adhesive plasticity within the step, that origin is then moved to the start of the elastic adhesive behavior. Integrating eqs. (3.2.D15) and (3.2.D16) again and substituting the integrals into eq. (3.2.D3) yields the following,

$$\begin{aligned} \delta_1 = \delta_{1\text{ref}} + \alpha_1(\Delta T)x + \frac{1}{E_1 t_1} \left\{ T_{1\text{ref}}x + \frac{p_1 x^2}{2} \right. \\ \left. - F(1-D)G \left[\frac{A}{\lambda^2} (\cosh(\lambda x) - 1) + \frac{B}{\lambda^2} (\sinh(\lambda x) - \lambda x) + \frac{Cx^2}{2} \right] \right\} \end{aligned} \quad (3.2.D17)$$

and

$$\begin{aligned} \delta_2 = \delta_{2\text{ref}} + \alpha_2(\Delta T)x + \frac{1}{E_2 t_2} \left\{ T_{2\text{ref}}x - \frac{p_2 x^2}{2} \right. \\ \left. + F(1-D)G \left[\frac{A}{\lambda^2} (\cosh(\lambda x) - 1) + \frac{B}{\lambda^2} (\sinh(\lambda x) - \lambda x) + \frac{Cx^2}{2} \right] \right\} . \end{aligned} \quad (3.2.D18)$$

The subroutine in the FORTRAN IV digital computer program A4EI used to solve the equations above for the elastic adhesive behavior employs the following analysis technique to determine the integration constants A and B . At the start of the bonded joint, it is assumed that there is no load in member(s) 2. That is

$$T_2 \Big|_{x=0} \equiv 0 \quad (3.2.D19)$$

and the entire load there is carried in member(s) 1:

$$T_1 \Big|_{x=l} \equiv P . \quad (3.2.D20)$$

Program A4EI makes an attempt to reinput the joint data if that convention is violated, but such a mistake with the input to A4EG just wastes the run and requires a manual redefinition of the problem. It is also necessary to assume, or prescribe, the state of adhesive stress and strain at the start of the overlap.

$$\gamma \Big|_{x=0} = A + C \quad (3.2.D21)$$

from equation (3.2.D13), with the constant C defined in equation (3.2.D14). Of course, one cannot prescribe all the boundary conditions, because that would preclude satisfying the boundary conditions at the other end of the overlap. In the case of an adhesively bonded joint, the conditions would dictate that

$$T_1 \Big|_{x=\ell} = 0 \quad \text{and} \quad T_2 \Big|_{x=\ell} = P + \int_0^{\ell} (p_1 - p_2) dx \quad (3.2.D22)$$

while, for an adhesively bonded doubler, the corresponding condition would be

$$\gamma \Big|_{x=\ell} = 0 . \quad (3.2.D23)$$

By assuming a complete set of boundary conditions at $x=0$ and adjusting the unknown iteratively, one is eventually able to match the boundary conditions at the other end of the bonded overlap, as explained in references (3.2.11) and (3.2.26).

The integration constant A is evaluated from the specified (or subsequently computed) adhesive shear strain at the start of the step under consideration. The other constant B is derived from equation (3.2.D6), also evaluated at the start of that step. That is

$$\frac{d\gamma}{dx} = A\lambda \sinh(\lambda x) + B\lambda \cosh(\lambda x) = \frac{1}{\eta} \left(\frac{T_2}{E_2 t_2} - \frac{T_1}{E_1 t_1} + (\alpha_2 - \alpha_1) \Delta T \right) \quad (3.2.D24)$$

so that, at $x = 0$,

$$B = \frac{1}{\lambda \eta} \left(\frac{T_2}{E_2 t_2} - \frac{T_1}{E_1 t_1} + (\alpha_2 - \alpha_1) \Delta T \right) \Big|_{x=0} . \quad (3.2.D25)$$

The values of γ , τ , T_1 , T_2 , σ , and σ , at the end of that step then follow from equations (3.2.D13), (3.2.D8), (3.2.D15), (3.2.D16), (3.2.D17) and (3.2.D18), respectively. These in turn, specify the initial conditions for the succeeding step.

The inclusion of adhesive nonlinear (plastic) behavior in the analysis is similar, with one of the major complications being the unknown location of the transition between elastic and plastic behavior. The plastic adhesive shear stress is constant throughout as given in eq. (3.2.D9). Therefore, for the plastic zones, eq. (3.2.D7) becomes

$$\frac{d^2\gamma}{dx^2} = \frac{\lambda^2 \tau_p}{G} - \frac{1}{\eta} \left(\frac{p_1}{E_1 t_1} + \frac{p_2}{E_2 t_2} \right) . \quad (3.2.D26)$$

Thus, throughout plastic adhesive areas,

$$\frac{d^2\gamma}{dx^2} = H = \text{constant} , \quad (3.2.D27)$$

so that

$$\gamma = \frac{1}{2}Hx^2 + Jx + K . \quad (3.2.D28)$$

It then follows from Equations (3.2.D1) that

$$T_1 = T_{1\text{ref}} + p_1x - F(1-D)\tau_p x \quad (3.2.D29)$$

and

$$T_2 = T_{2\text{ref}} - p_2x + F(1-D)\tau_p x . \quad (3.2.D30)$$

Equations (3.2.D3) may then be integrated to yield

$$\delta_1 = \delta_{1\text{ref}} + \alpha_1(\Delta T)x + \frac{1}{E_1 t_1} \left(T_{1\text{ref}}x + \frac{p_1 x^2}{2} - F(1-D)\tau_p \frac{x^2}{2} \right) \quad (3.2.D31)$$

and

$$\delta_2 = \delta_{2\text{ref}} + \alpha_2(\Delta T)x + \frac{1}{E_2 t_2} \left(T_{2\text{ref}}x - \frac{p_2 x^2}{2} + F(1-D)\tau_p \frac{x^2}{2} \right) . \quad (3.2.D32)$$

In Equation (3.2.D28), K is set equal to the value of γ at the start of that step, where $x = 0$;

$$K = \gamma_{\text{ref}} = \gamma \Big|_{x=0} , \quad (3.2.D33)$$

while the other constant J is evaluated via Equation (3.2.D6):

$$J = \frac{d\gamma}{dx} \Big|_{x=0} = \frac{1}{\eta} \left(\frac{T_2}{E_2 t_2} - \frac{T_1}{E_1 t_1} + (\alpha_2 - \alpha_1)\Delta T \right) \Big|_{x=0} . \quad (3.2.D34)$$

Very few individual steps of stepped-lap joints have fully plastic behavior throughout. Any plastic behavior is frequently confined to the ends of the outermost steps, as shown in figure 3.2.49, because in order to achieve the excellent fatigue behavior for which bonded structures are renowned, it is desirable not to operate the adhesive beyond its proportional limit for frequently recurring loads. (The plastic strength is best reserved for overloads and for local load redistribution around flaws, as explained in the next section.) Figure 3.2.D1 shows typical adhesive shear stress distributions in stepped-lap bonded joints. Therefore, in performing elastic-plastic analyses of adhesively bonded stepped-lap joints, it is necessary to compute the extent of any plastic zones. The result is then compared with the actual extent of the step under consideration to see whether there is a transition in behavior. Starting from eqs. (3.2.D28) and (3.2.D33)

$$\gamma = \frac{1}{2}Hx^2 + Jx + \gamma_{\text{ref}} , \quad (3.2.D35)$$

in which the constant J is given by Equation (3.2.D34). It is necessary to find the lesser value of x for which

$$\gamma = \gamma_e . \quad (3.2.D36)$$

Equation (3.2.D35) is first rearranged to read

$$\frac{1}{2}Hx_p^2 + Jx_p + (\gamma_{\text{ref}} - \gamma_e) = 0 , \quad (3.2.D37)$$

so that maximum extent of plastic adhesive zone is given by

$$x_p = -\frac{J}{H} \pm \sqrt{\left(\frac{J}{H}\right)^2 - \frac{2(\gamma_{\text{ref}} - \gamma_e)}{H}} . \quad (3.2.D38)$$

Now, since $J = d\gamma/dx < 0$ at $x = 0$ if γ is positive, the minus sign in front of the radical holds there. The positive sign covers the far end of the joint and negative loads. If x_p is less than the actual step length, step, there is a transition to elastic behavior for some or all of the remainder of that step. If x_p is greater than the step length, that step is fully plastic and eqs. (3.2.D35), (3.2.D29), (3.2.D30), (3.2.D31) and (3.2.D32) are used to determine the plastic boundary conditions at the start of the next step. In solving for the possible length of the elastic adhesive zone, one must provide both for the case of shear strains of the same sign (positive or negative) at each end of the step (or portion thereof) and for the case of a reversal of sign in the shear stress. The procedure to determine the length of the elastic adhesive zone is thus reduced to an iterative evaluation of the distance required for the adhesive shear strain to become

$$\gamma = \gamma_e \quad \text{or} \quad -\gamma_e . \quad (3.2.D39)$$

If the elastic zone does not extend to the far end of the step being analyzed, it is necessary to complete the analysis using the plastic equations above. In effect, each step might be broken up mathematically into as many as three sub steps to handle the changes in behavior.

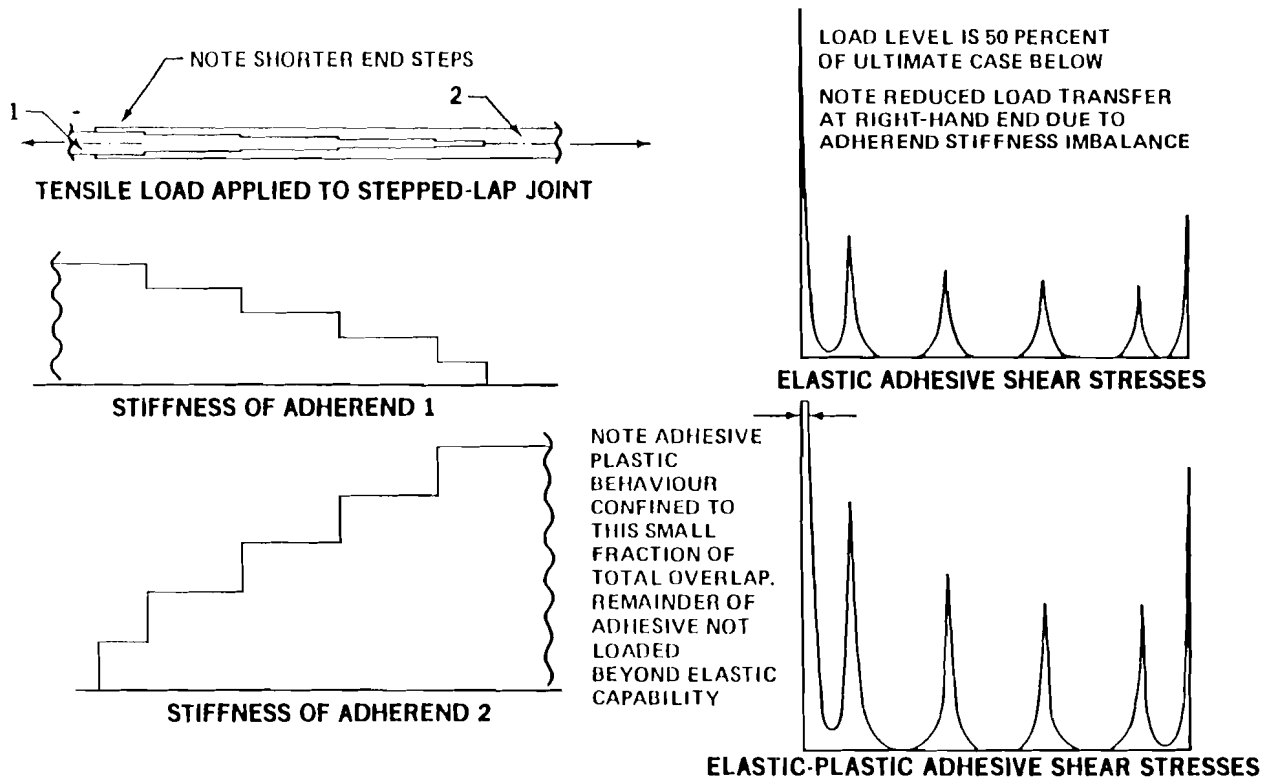


FIGURE 3.2.D1. TYPICAL ELASTIC-PLASTIC ANALYSIS OF STEPPED-LAP BONDED JOINT

3.3 MECHANICALLY FASTENED JOINTS.

3.3.1 Basic Principles.

The transfer of load via bolted joints in advanced composites is the one function that, more than anything else, limits the possible structural efficiency of large primary structures. The stress concentrations around bolt holes in such materials, at failure, can be effectively eliminated only at the expense of providing for the repairability of thick composite structures. Consequently, bonded and bolted joints in composite structures are very different; the former should always be stronger than the basic structure and the latter never can be.

The analysis of mechanical joints in advanced fibrous composite structures is more complicated than in conventional ductile metal alloys. This situation arises because bolt holes in composites fail at loads that can be predicted neither by perfectly-elastic analyses nor by fully plastic ones. This is evident from figure 3.3.1, which shows that the structural efficiency of single-bolt joints in composite structures lies roughly half way between those for window-glass, on the one hand, and ductile metal alloys, on the other. More than a minor perturbation from either extreme analysis is needed to predict the failures for composites. Were it possible to perform meaningful analyses from bolted joints in composites on the basis of simply a bearing stress allowable and a net-section tension strength allowable, the middle curve in figure 3.3.1 would have to have the same form as the two-straight-line characteristic for joints in ductile metal alloys. While fiberglass laminates come close to that extreme due to massive delaminations prior to failure, the experimental evidence for such a representation is lacking for carbon-fiber and boron-filament reinforced laminates. Indeed, the evidence points more to a constant gross-section failure stress than to a net-section one.

The mechanism whereby fibrous composite laminates behave differently from conventional metal alloys follows from the nonhomogeneity of the mixture of fibers and resin. Three distinct failure modes are possible in tension, with a fourth exhibited under compressive load. The fiber-to-matrix bond can be broken locally without failing either the fiber or the matrix; this is the real key to load redistribution and relief of stress concentrations. Next, there can be a delamination between layers of fibers or a splitting of the resin between the fibers within a single lamina; again no fiber need be broken. Finally, the load level may be high enough to actually break fibers and cause a loss of measured strength; the previous failures are usually followed by an increase in laminate strength. The failure mode that is peculiar to compressive loading is that of microbuckling of the fibers because of limited support from the matrix. Even though fibrous composite laminates are often analyzed as homogeneous non-isotropic materials, and such modelling usually suffices for predicting stiffnesses, it is important not to think of them as such when predicting failure strengths and explaining experimental observations.

Figure 3.3.2 explains how this behavior as a distinctly two phase material causes composite laminates to develop much higher bolted joint strengths than simple analysis as a one-phase homogenous material would suggest. With reference to the left side of figure 3.3.2, it is apparent that a very high local stress concentration develops in the fiber aligned with the load immediately outside the bolt. On the right side of the bolt, however, a much lower peak stress exists over a finite length. Therefore, considering the fibrous composite as the

heterogeneous mixture it really is, it becomes evident that, if the initial failure is by intra-ply splitting or inter-ply delamination, rather than of fiber breaking, a much higher load can be transmitted than elastic orthotropic theory would predict. This mechanism is responsible for roughly a factor of two differences between the load levels at the initial (noncatastrophic and often undetectable) damage and at the final failure for carbon epoxy laminates. Much the same increase is evident because the resin turns from clear to milky.

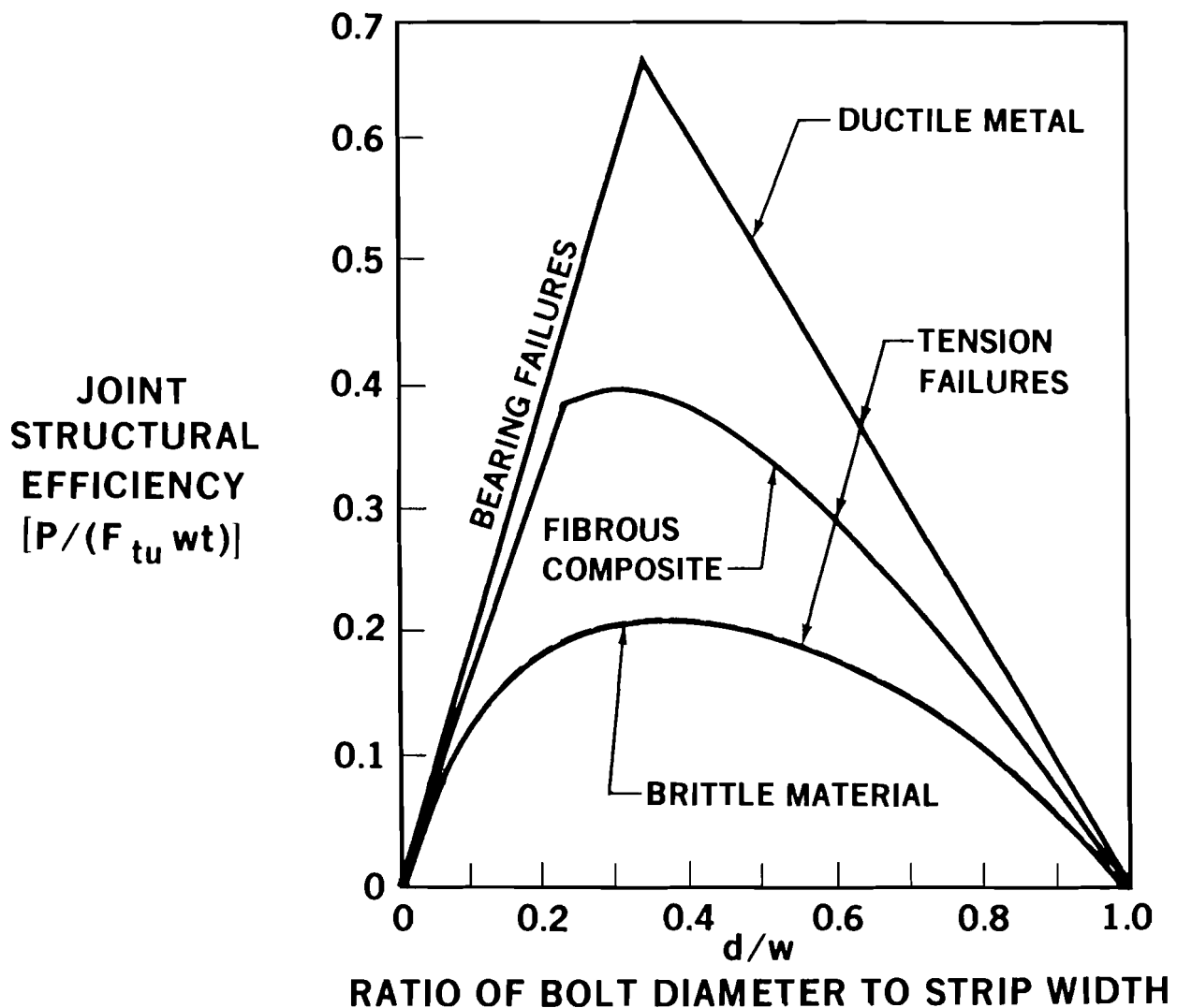


FIGURE 3.3.1. RELATION BETWEEN STRENGTHS OF BOLTED JOINTS IN DUCTILE, FIBROUS COMPOSITE AND BRITTLE MATERIALS

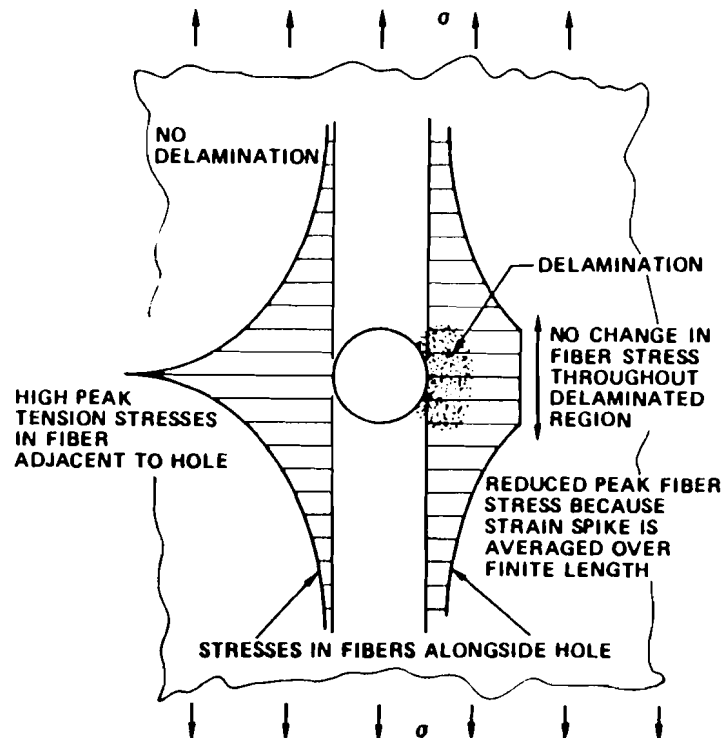


FIGURE 3.3.2. STRESS CONCENTRATION RELIEF IN FIBROUS COMPOSITES BY DELAMINATIONS

A related phenomenon is associated with interference-fit fasteners or with gentle fatigue cycling. The delaminations around the bolt hole can be enlarged by the careful installation of interference-fit fasteners, or they can be grown at a steadily decreasing rate, with the effect that the stress concentration factor has apparently disappeared. In such a case, the laminate strength has been shown by tests to approach the unnotched strength, albeit for a cross-sectional area reduced by the presence of the hole.

The analysis and design methods presented here were developed in references 3.3.1 and 3.3.2 and are based on theoretical elastic-isotropic stress concentration factors, modified substantially by empirically established correlation factors. These techniques have been shown in reference 3.3.3 to be capable of handling highly-loaded, efficient, multi-row bolted joints at load intensities in the order of 50,000 pounds per inch at strain levels of 0.005 in the basic carbon epoxy laminates.

The importance of the structural efficiency of bolted composite joints can be understood by a simple comparison between the weights of equivalent metal and composite structures. Most claims for the superiority of composites over metals are based on unnotched specimens; some are even deduced from monolayer properties before cross-plying to make real structures. Now, for a common load P , per unit width on aluminum and carbon-epoxy structures,

$$P = \sigma_a t_a = \sigma_c t_c$$

The relative weights are then given by

$$\frac{wt_c}{wt_a} = \frac{\rho_c t_c}{\rho_a t_a} = \frac{\rho_c \sigma_a}{\rho_a \sigma_c} = \frac{\rho_c \sigma_a}{\rho_a E_c \epsilon_c}$$

Therefore, composite structures are to be preferred whenever the aluminum stress σ_a is low -- be it so for minimum gage reasons, pure conservatism, or crude design practice -- or when the operating strain ϵ_c in the composite can be kept high by the absence of bolt holes. This combination favoring composites arises frequently in fairings and minimum-gage control surfaces. Conversely, the combination of a high operating stress in large efficiently-designed aluminum structure with a restriction on the composite strain allowable caused by large bolt holes has inhibited the use of large primary composite structures on transport aircraft.

This trade-off is illustrated in figures 3.3.3 and 3.3.4, for a quasi-isotropic laminate and for a fiber pattern more suitable for a wing skin, respectively. It can be seen from figure 3.3.4 that a composite strain level of 0.003 is needed to just break even with the weight of aluminum structure designed to a gross-section stress of 50 ksi, which is typical for wing skins. Yet the same 0.003 strain has, for many years, also been the upper limit of composites at loaded bolt holes; it corresponds with an efficiency of 0.33 in figure 3.3.1, which refers to average ultimate conditions at room temperature. Obviously, more efficient bolted joints are needed for primary composite structures and that was the purpose of the investigation reported in reference 3.3.3. On the other hand, figure 3.3.3 indicates that the same 0.003 composite strain could yield a weight saving of 50 percent if the aluminum stress were as low as 20 ksi, as is often the case in secondary structure. These two sample comparisons between the relative weights of aluminum alloy and carbon-epoxy laminates suggest that there are appropriate places for both materials in the aircraft of the future. Of more importance in the present context is the clear need to design reliable bolted composite joints that can operate at sufficiently higher strains (0.004 to 0.004) to permit weight savings on large composite primary structures. That is why multi-row bolted joints and the associated interaction between bearing and bypass loads at each bolt location in such a joint are covered in depth here. The derivation of the analysis covering the load sharing between the bolts in such joints is documented in reference 3.3.4.

The interaction between bearing and bypass loads at the various fasteners in a multi-row joint has the effect that multi-row joints are often barely stronger than the optimum single-row joint. This has been confirmed by tests in reference 3.3.1 showing only a ten percent increase in strength resulting from the addition of a second row of bolts in tandem. However, as explained in reference 3.3.3, multi-row joints are virtually the only way of developing a composite structural efficiency high enough to compete with the weight of well-designed highly-stressed aluminum structures. Test results from reference 3.3.3 showed that optimized four-row joints could be fifty percent stronger than the best single-row joints.

These strength limitations have led to the exploration of softening strips in carbon-epoxy laminates to relieve the stress concentration factors at the bolt holes. These softening strips take the form of strips about four diameters wide in which the 0-degree carbon fibers are replaced by 0-degree glass fibers (so that no fibers are actually cut) or by ± 45 -degree carbon-epoxy. Both have been demonstrated to be effective. However, taking advantage of such relief, and consequently operating the remaining structure at a higher strain level than would be possible, otherwise, implies the impossibility of repairing the basic structure back to its original strength should it be damaged. The same comment applied to the use of local bonded or integral reinforcement in the original

MATERIAL — T300/N5208 GRAPHITE-EPOXY

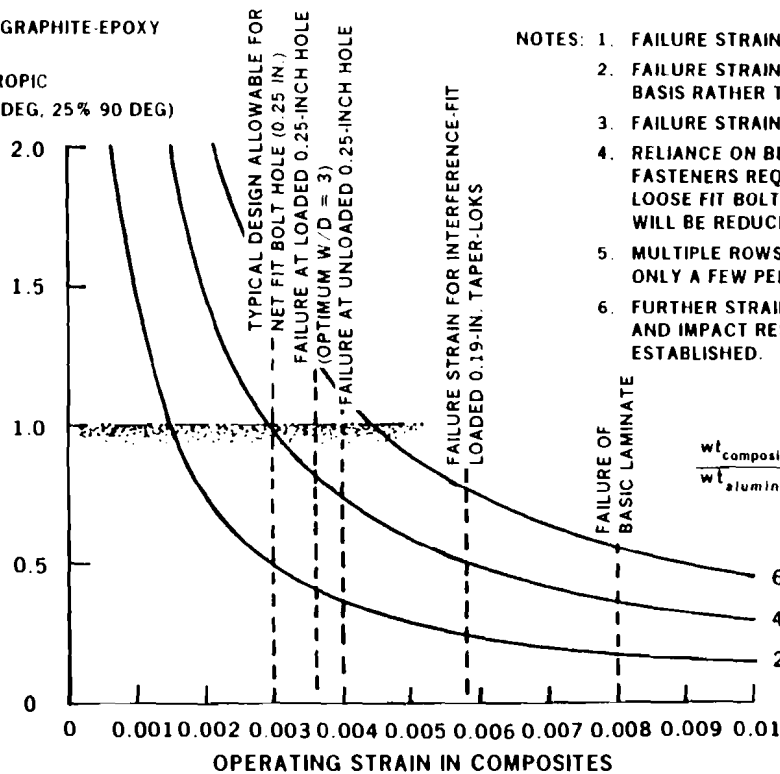
PATTERN — PSEUDO-ISOTROPIC

(25% 0 DEG, 50% ±45 DEG, 25% 90 DEG)

$E = 7.8 \times 10^6$ PSI

$\rho = 0.057$ LB/IN.³

COMPOSITE WEIGHT
ALUMINUM WEIGHT



- NOTES: 1. FAILURE STRAINS ARE LESS FOR LOADED HOLES.
2. FAILURE STRAINS ARE LESS FOR USING STATISTICAL BASIS RATHER THAN AVERAGE.
3. FAILURE STRAINS LESS FOR LARGER HOLES.
4. RELIANCE ON BENEFIT FROM INTERFERENCE FIT FASTENERS REQUIRES ABSOLUTELY NO NET OR LOOSE FIT BOLTS. OTHERWISE, STATIC STRENGTH WILL BE REDUCED BY FACTOR OF TWO.
5. MULTIPLE ROWS OF BOLTS INCREASE STRENGTH BY ONLY A FEW PERCENT
6. FURTHER STRAIN LIMITS FOR DAMAGE TOLERANCE AND IMPACT RESISTANCE ARE NOT YET ESTABLISHED.

$$\frac{wt_{\text{composite}}}{wt_{\text{aluminum}}} = \frac{\rho_c}{\rho_{al}} \times \frac{\sigma_{al}}{E_c E_c}$$

REPAIR POSSIBLE WITH MECHANICAL FASTENERS ALONE

REPAIR REQUIRES ADHESIVE BONDING

FIGURE 3.3.3. RELATIVE WEIGHTS OF ALUMINUM AND GRAPHITE-EPOXY COMPOSITE STRUCTURES

MATERIAL — T300/N5208 GRAPHITE-EPOXY

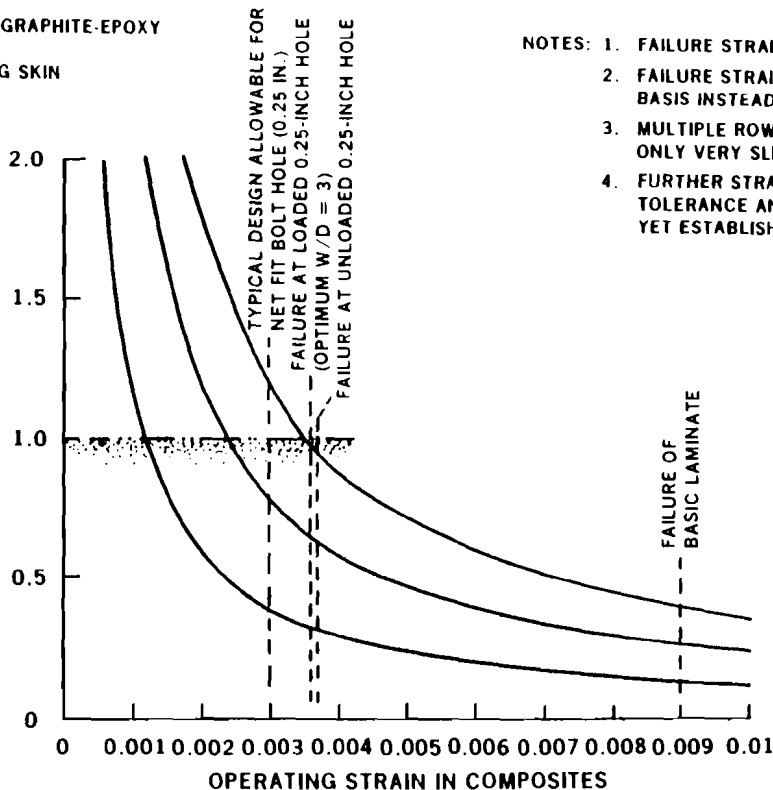
PATTERN — TYPICAL WING SKIN

(37.5% 0 DEG,
50% — 37.5% —45 DEG
12.5% — 25% 90 DEG)

$E = 9.7 \times 10^6$ PSI

$\rho = 0.057$ LB/IN.³

COMPOSITE WEIGHT
ALUMINUM WEIGHT



- NOTES: 1. FAILURE STRAINS ARE LESS FOR LARGER HOLES.
2. FAILURE STRAINS ARE LESS FOR USING STATISTICAL BASIS INSTEAD OF AVERAGE.
3. MULTIPLE ROWS OF BOLTS INCREASE STRENGTH ONLY VERY SLIGHTLY.
4. FURTHER STRAIN LIMITS FOR DAMAGE TOLERANCE AND IMPACT RESISTANCE ARE NOT YET ESTABLISHED.

$$\frac{wt_{\text{composite}}}{wt_{\text{aluminum}}} = \frac{\rho_c}{\rho_{al}} \times \frac{\sigma_{al}}{E_c E_c}$$

REPAIR POSSIBLE WITH MECHANICAL FASTENERS ALONE

REPAIR REQUIRES ADHESIVE BONDING

FIGURE 3.3.4. RELATIVE WEIGHTS OF ALUMINUM AND GRAPHITE-EPOXY COMPOSITE STRUCTURES

structure. Such techniques are most suitable for one-shot vehicles like missiles. However, while the use of softening strips to improve the damage tolerance of all composite structures is related to the same phenomena, it can be justified by entirely different criteria. The purpose then is to confine the spread of damage from foreign object impact, to ensure the survival of the aircraft and the possibility of subsequent repair rather than replacement is of secondary concern.

Not all composite laminates have fiber patterns that lend themselves to efficient bolted or riveted joints. Those patterns which deviate excessively from the quasi-isotropic pattern (25% 0°, 50% 45°, 25% 90°) tend to split badly. The recommended patterns are identified in figure 3.3.5 and are formed by the limit of no more than 3/8 of the fibers in any one of those four directions and no less than 1/8 likewise. Parallel layers of fibers should not be bunched together, but should be thoroughly dispersed to avoid overloading a limited number of effective resin interfaces. Fuller explanations for these design practices are given in the various detailed discussions below.

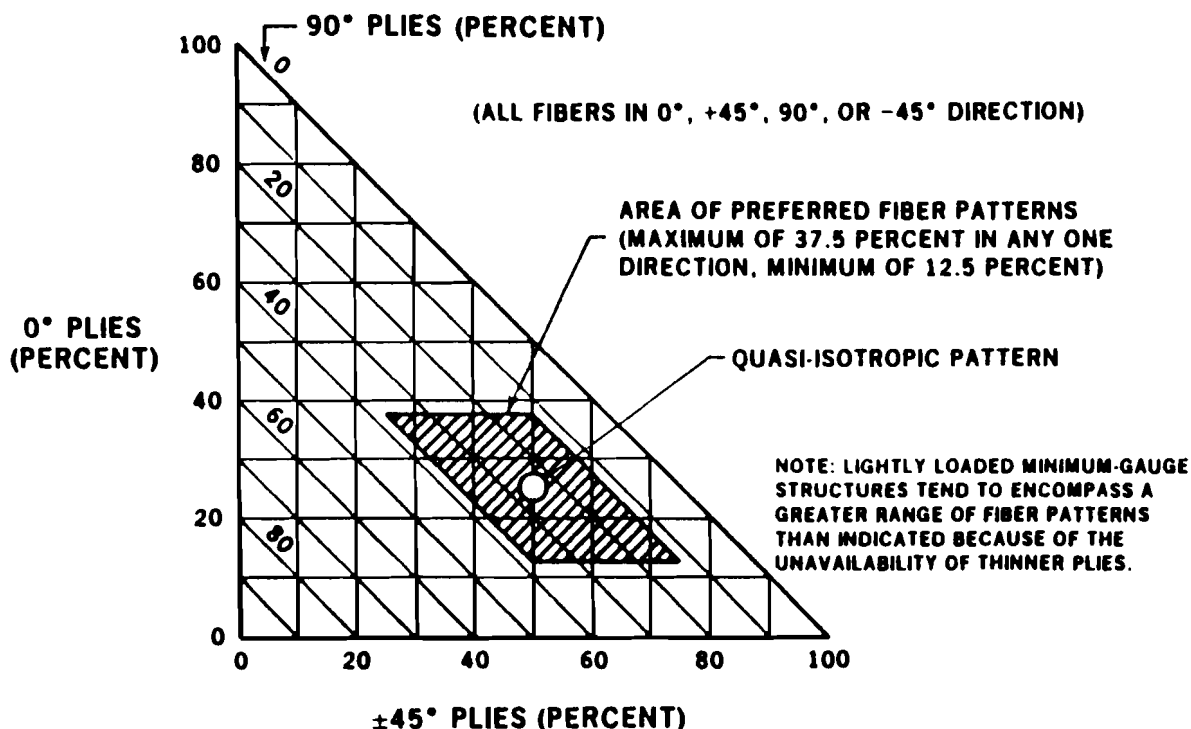


FIGURE 3.3.5. SELECTION OF LAY-UP PATTERN FOR FIBROUS COMPOSITE LAMINATES

3.3.2 Elastic-Isotropic Stress Concentration Factors for Single Holes.

Much of the data on which the design of bolted joints is based have been accumulated from the testing of simple coupons having a single fastener, in double- or single-shear. The variables considered usually include fastener size, laminate thickness, edge distance, width, and sometimes clamping torque. Theories developed to explain these observations and to predict the performance

of more complex structural joints are customarily checked out on such data first. Therefore, these simple joints provide a logical starting point for this work. The separate stress concentration factors for loaded and unloaded holes can then be combined to characterize the behavior at the individual fasteners in multi-row joints.

Reference 3.3.2 contains an in-depth derivation of various expressions for the stress concentrations at bolt holes in elastic-isotropic materials, for a variety of boundary conditions. These expressions were formulated on the basis of a wealth of experimental data and analytical results for specific geometries. The sources for that material are given in reference 3.3.2.

Starting with the single loaded hole in figure 3.3.6, the author has proposed in reference 3.3.1 that the stress concentration factor relating the peak tensile stress at the side of the bolt hole to the average net-section tension stress be expressed as

$$k_{te} = \left(\frac{w}{d} + 1 \right) - 1.5 \frac{(w/d - 1)}{(w/d + 1)} \Theta \quad (3.3.1)$$

in which the parameter Θ is defined as

$$\left. \begin{aligned} \Theta &= 1.5 - 0.5 / (e/w) && \text{for } e/w \leq 1 \\ \Theta &= 1 && \text{for } e/w > 1 \end{aligned} \right\} . \quad (3.3.2)$$

The maximum stress in the plate, adjacent to the bolt hole on the diameter perpendicular to the direction of the load, is given by

$$\sigma_{max} = k_{te} \frac{P}{t(w-d)} . \quad (3.3.3)$$

Eq. (3.3.1) diverges towards infinity as $d/w \rightarrow 0$, as shown in figure 3.3.7, for the case of an isolated hole in a wide plate. Therefore, it is appropriate to re-express the reference stress - which does not decrease asymptotically towards zero - and the average bearing stress is the appropriate one. Consequently,

$$k_{be} = \frac{\sigma_{max}}{P/td} = \frac{k_{te}}{(w/d - 1)} = 1 + \frac{2}{(w/d - 1)} - \frac{1.5\Theta}{(w/d + 1)} . \quad (3.3.4)$$

This expression diverges towards infinity $d/w \rightarrow 1$, as shown in figure 3.3.8. However, it is very well behaved throughout the region of small d/w values, having a limiting value of

$$k_{be} = 1 \quad (3.3.5)$$

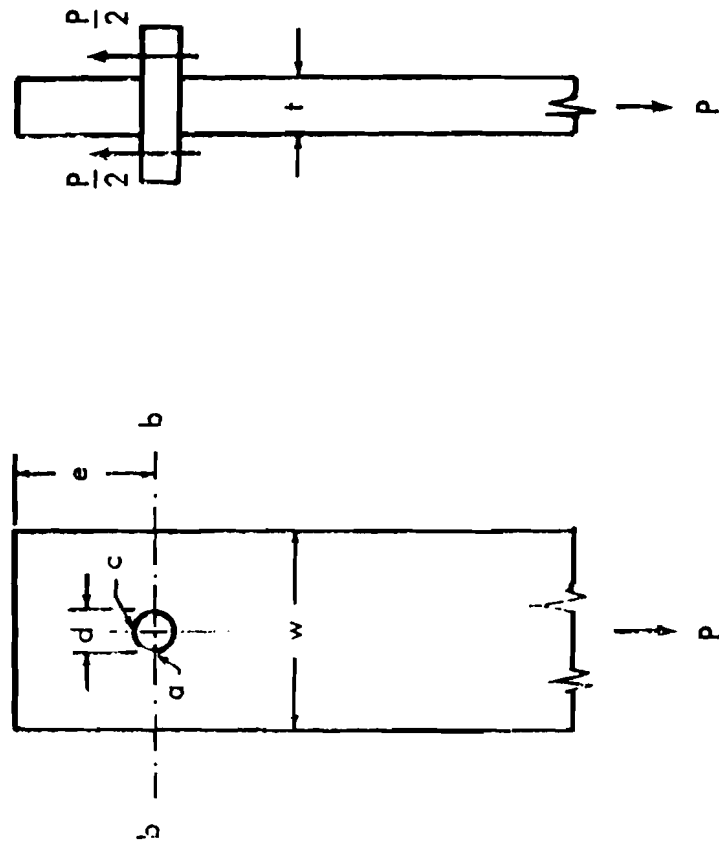


FIGURE 3.3.6. GEOMETRY OF DOUBLE-LAP BOLTED JOINT

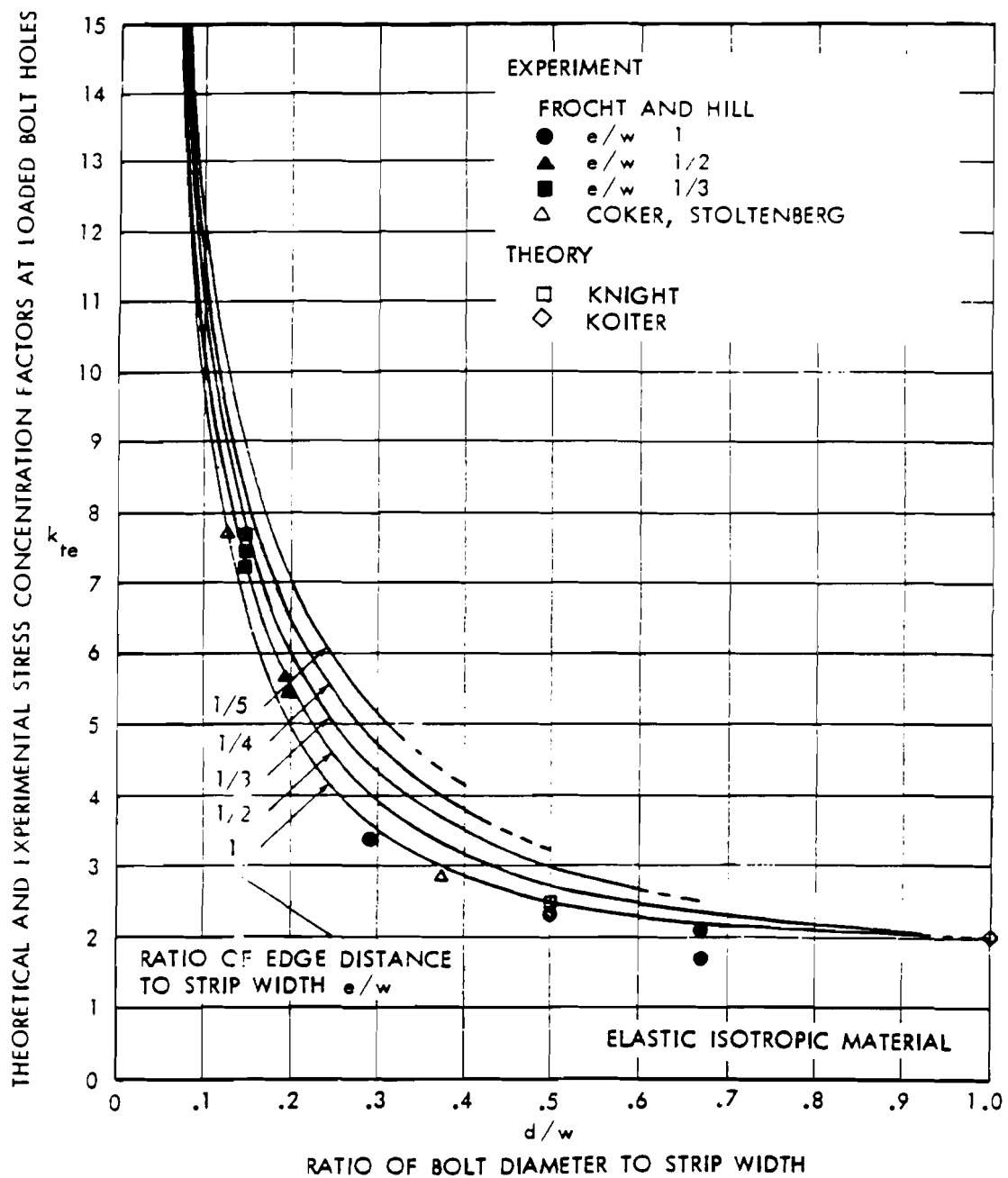


FIGURE 3.3.7. ELASTIC ISOTROPIC STRESS CONCENTRATION FACTORS FOR LOADED BOLT HOLES, WITH REFERENCE TO NET SECTION

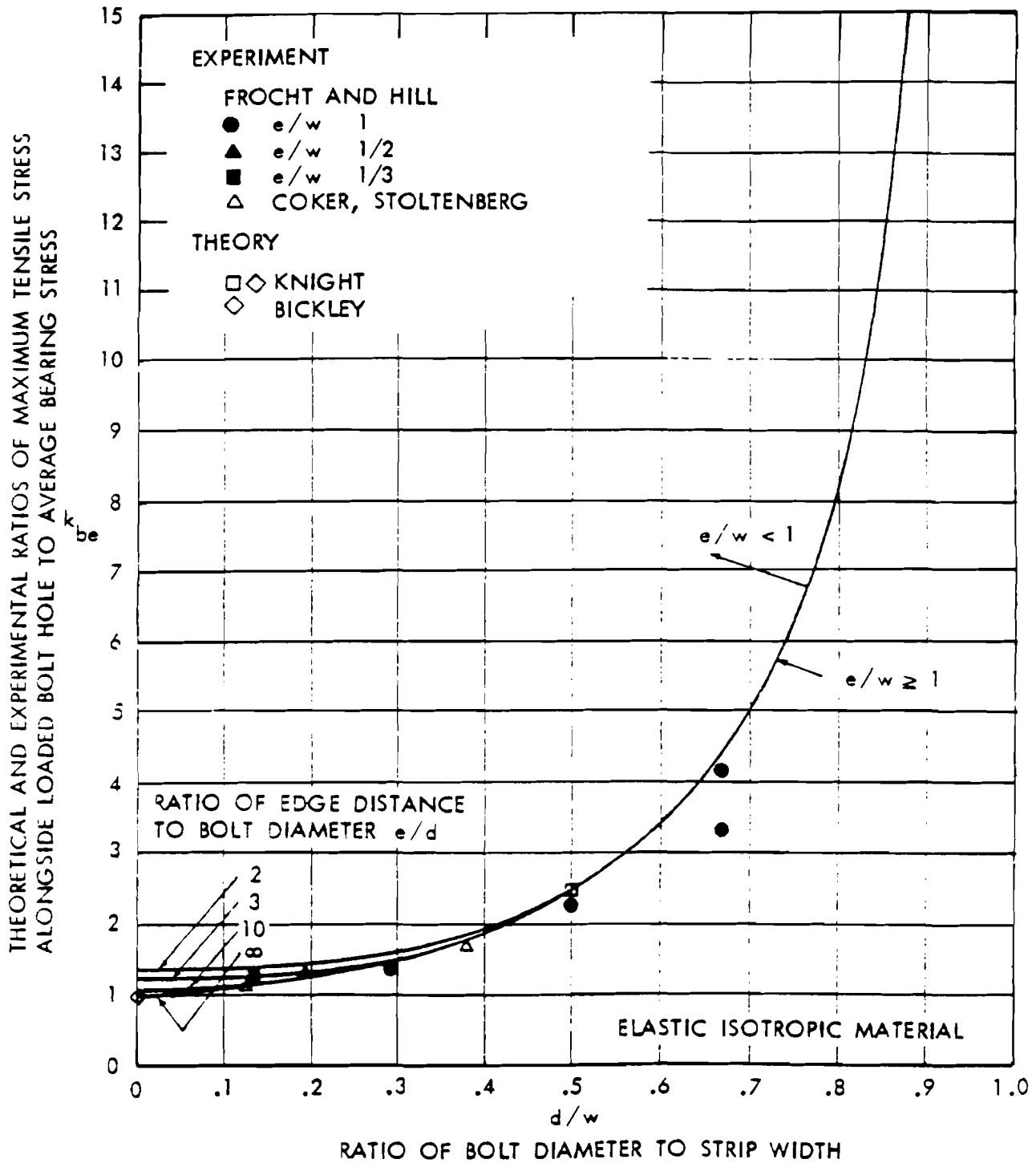


FIGURE 3.3.8. ELASTIC ISOTROPIC STRESS CONCENTRATION FACTORS FOR
LOADED BOLT HOLES, WITH REFERENCE TO BOLT BEARING AREA

for a single small bolt in a large plate with all edges far removed from the bolt hole. The proximity of the end of the test coupon, rather than of the equally-spaced sides, can be shown from eq. (3.3.4) to raise that limit in eq. (3.3.5) to be

$$k_{be} \rightarrow 1 + \frac{3}{4} \left/ \left(\frac{e}{d} \right) \right. \text{ as } d/w \rightarrow 0. \quad (3.3.6)$$

Having too short an edge distance is therefore seen to also increase the stress concentration at the side of the bolt hole by concentrating the stress trajectories closer to the bolt than would have been the case with a more generous edge distance.

Figures 3.3.7 and 3.3.8 contain the theoretical calculations and the test results on which eqs. (3.3.1), (3.3.2) and (3.3.4) were based. The form of those equations established by examining the stress concentration factor from both viewpoints simultaneously. Equation (3.3.4) is clearly a better behaved function with which to visualize the results, but eq. (3.3.1) has the more appropriate reference stress for combining with the bypass loads.

The salient features of the stress field around a loaded bolt hole have been characterized in figure 3.3.9, based on calculations by Bickley (reference 3.3.5). The radial compressive bearing stress peaks at a value some 27 percent higher than the average value. The first printing of reference 3.3.2 contains a miscalculation, showing the peak stress to be twice as high as the average. The location at which the bearing stress drops to zero would vary with hole clearance with interference fit; the position shown corresponds with a nominal net-fit bolt. The peak hoop tension stress occurs at the side of the bolt, having a value equal to the average bearing stress, as indicated by eq. (3.3.5). The hoop tension stress around the back of the bolt is of the same order of magnitude, but distinctly less. Bickley's analyses did not account for any edge effects, so the effect of a short edge distance on the tension stress in figure 3.3.9 is not known. The stresses at both locations would increase in accordance with eq. (3.3.6), but the relative severities have not yet been predicted. Experience suggests that the stress at the back of the bolt would increase more than at the side, however. Therefore, there are three potential failure sites for loaded bolt hole in elastic-isotropic materials. With generous edge distances, failure may occur either by bearing, on the back side of the fastener, or by tension, alongside the bolt hole. The proximity of sides to the hole would favor the net-section tension failure. With a short enough edge distance, a cleavage failure by splitting longitudinally at the back of the hole would be likely.

A clearer indication of the impact of the stress concentration factors in eqs. (3.3.1) and (3.3.4) on the strength of a strip of material is shown by solving for the joint strength;

$$P = (\sigma_{net})_{avg} (w - d)t = \frac{\sigma_{max}}{k_{te}} (w - d)t = \frac{\sigma_{max} t w}{\frac{2}{(1 - d/w)} + \frac{1}{(d/w)} - \frac{1.50}{(1 + d/w)}} \quad (3.3.7)$$

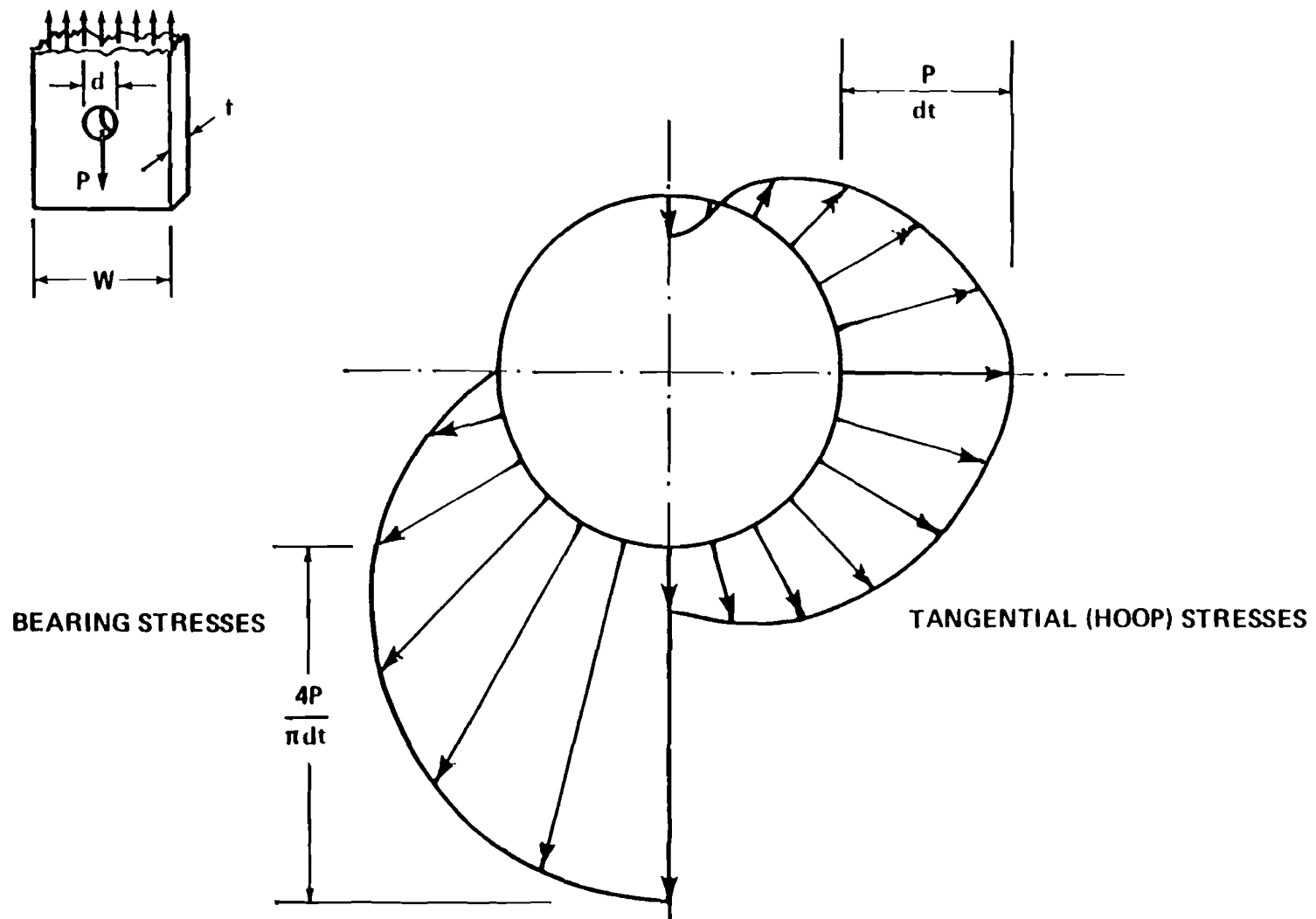


FIGURE 3.3.9. BEARING AND HOOP STRESSES AT BOLT HOLE

Eq. (3.3.7) is plotted nondimensionally in figure 3.3.10. It can be seen that, for a given maximum stress in the (brittle) elastic-isotropic material, the load carried is maximized when

$$d/w = 0.4 \quad (\text{for maximized load}). \quad (3.3.8)$$

This corresponds with a bolt pitch of approximately 2.5 bolt diameters, at which the maximum possible joint efficiency (for this brittle material) would barely exceed 20 percent of the unnotched gross-section strength;

$$P_{\max} \neq 0.21 \sigma_{\max} t w . \quad (3.3.9)$$

Fortunately, even the advanced fibrous composites are not so brittle as to make eq. (3.3.9) a real limit on structural efficiency. This is explained in detail below.

The stress concentration factor at the net-section of a strip containing an unloaded hole is given in reference 3.3.6 as

$$k_{te} = 2 + \left(1 - \frac{d}{w}\right)^3 . \quad (3.3.10)$$

Corresponding with this, one can compute the net-section strength of a strip with a centrally located hole in the form

$$\frac{P}{\sigma_{\max} t w} = \frac{(1 - d/w)}{k_{te}} = \frac{(1 - d/w)}{2 + (1 - d/w)^3} . \quad (3.3.11)$$

Eq. (3.3.10) is plotted in figure 3.3.11, along with various experimental results and theoretical limits. The original sources of those data are identified in reference 3.3.2.

The stress concentration factor of eq. (3.3.10) is applied to a single hole in a single strip. There are some differences between that behavior and that of a strip isolated mathematically out of a wide panel containing a seam of holes. These differences are identified in figure 3.3.12, they are small for $d/w \rightarrow 0$ and substantial as $d/w \rightarrow 1$.

It was suggested in reference 3.3.2 that eq. (3.3.10) be modified to the form

$$k_{te} = 1 + \left(1 - \frac{d}{p}\right)^3 + \left[1 - \left(\frac{d}{p}\right)^2\right]^{2.41} \quad (3.3.12)$$

for the case of a wide panel containing a line of holes. Both expressions are plotted in figure 3.3.13, and it is evident that the difference between the two is significant even for the range of small d/w values ($d/w \leq 0.5$) that are of interest in practical designs. The difference between the two expressions as $d/w \rightarrow 1$ is caused by the ability of the individual narrow strip to bend within the narrow

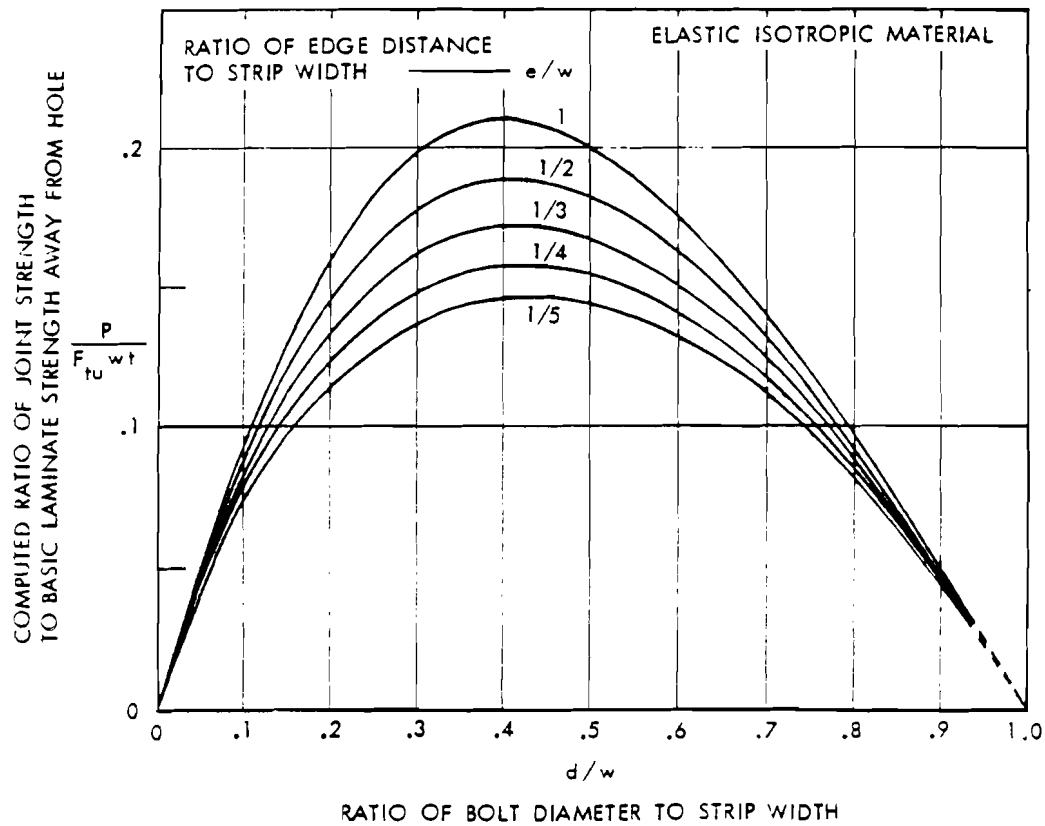


FIGURE 3.3.10. INFLUENCE OF JOINT GEOMETRY ON ELASTIC STRENGTH OF BOLTED JOINTS IN ISOTROPIC MATERIAL

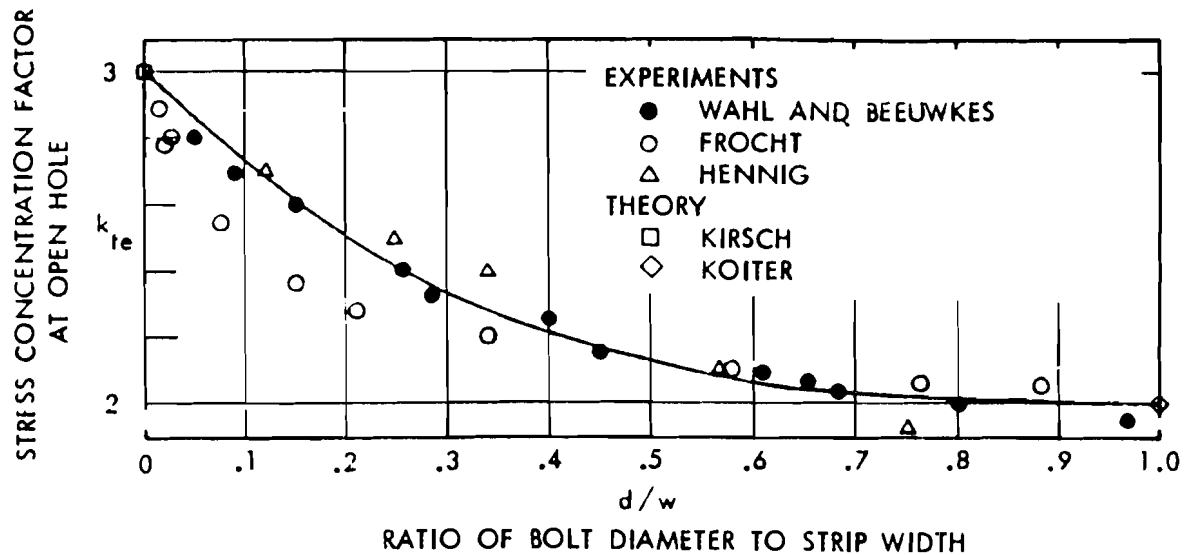


FIGURE 3.3.11. ELASTIC ISOTROPIC STRESS CONCENTRATION FACTORS FOR OPEN HOLES IN STRIPS OF FINITE WIDTH

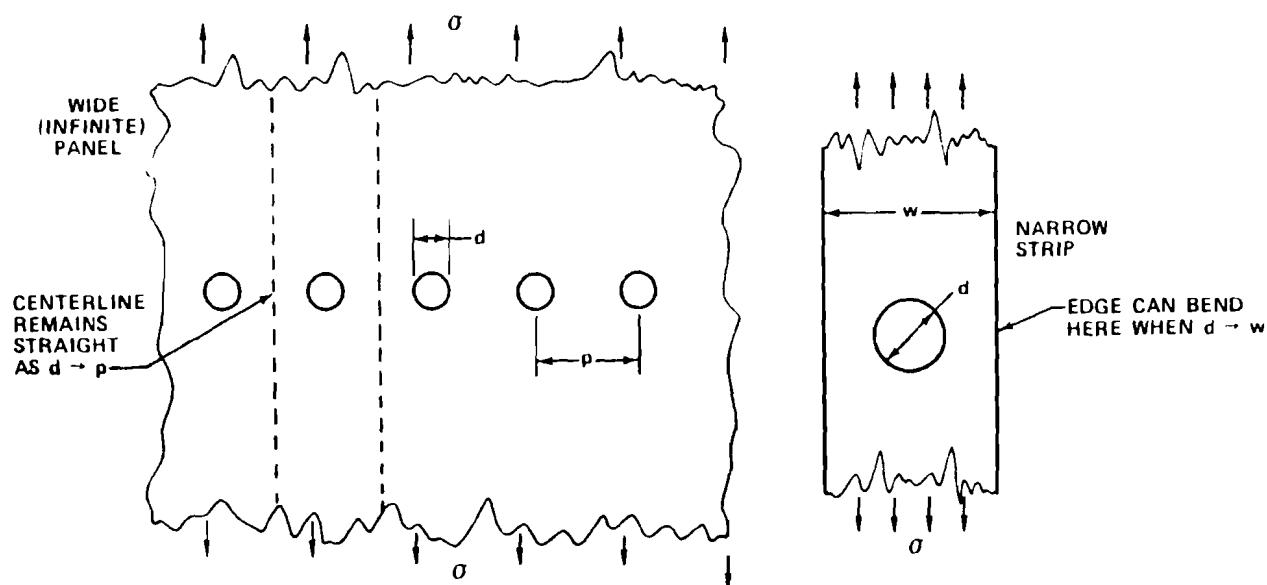


FIGURE 3.3.12. DIFFERENCES BETWEEN BOLTS IN A WIDE PANEL AND A NARROW STRIP

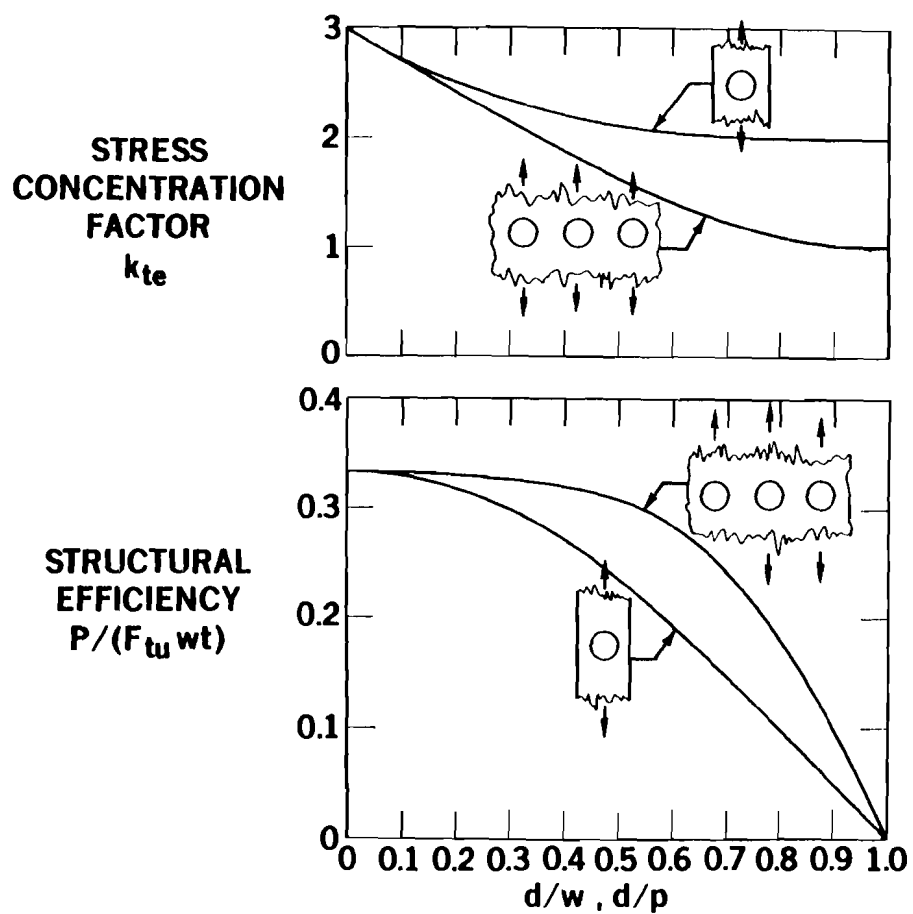


FIGURE 3.3.13. ELASTIC ISOTROPIC STRESS CONCENTRATION FACTORS AT UNLOADED BOLT HOLES

segment between the hole and the edge of the specimen. That bending is prevented by the conditions of symmetry across the dotted lines shown in figure 3.3.12 for the wide panel.

Figure 3.3.13 also shows both expressions equivalent to eq. (3.3.11) for the nondimensionalized net-section strengths of narrow strips and wide panels with many holes. It is seen that the greatest strength is predicted for the smallest holes. Not all published formulae for these stress concentration factors share that necessary characteristic, as explained in reference 3.3.2.

No attempt is made here to account for proximity effects of a seam of such holes aligned along the length, rather than the width, of the specimen. Normal design practice keeps the fasteners far enough apart for that influence not to be of concern.

It is evident that the same sort of modification to distinguish between strips and wide panels should apply also to the stress concentration formulae in eqs. (3.3.1) and (3.3.4) for loaded bolt holes. Consequently, eq. (3.3.1) was modified in reference 3.3.2 to read

$$k_{te} = \frac{p}{d} + \left[1 - \left(\frac{d}{p} \right)^2 \right]^{2.41} - 1.5 \frac{(p/d - 1)}{(p/d + 1)} \Theta . \quad (3.3.13)$$

The differences between the various formulae are depicted in figure 3.3.14, from which it is evident that the distinction is still significant for d/p and d/w ratios of the order of 0.5, even though the major difference is in joint geometries outside normal design practice.

Eq. (3.3.7) can be modified in the corresponding manner and that has been done in the preparation of figure 3.3.15. This shows that the narrow-strip formula would be conservative if applied to the wide panel case. Furthermore, the optimum p/d ratio is closer to 2 than to the 2.5 optimum value for w/d .

Reference 3.3.2 contains several other formulae for unloaded bolt holes; for holes near the edge of a semi-infinite plate and for eccentric holes in individual narrow strips. Although it has not been done yet, equivalent modifications could also be proposed for the loaded bolt hole formulae. However, the need to do so has not yet arisen.

3.3.3 Stress Concentration Factors at Holes in Composite Laminates.

Fortunately, fibrous composite laminates do not behave with the degree of brittleness that their individual fiber and matrix constituents would suggest, so bolted joints in composite structures are stronger than figure 3.3.14 would imply. This increase in strength is indicated in figure 3.3.1, which suggests a substantial relief from the high stress concentration factors in eq. (3.3.1). This relief was accounted for in reference 3.3.1 by the hypothesis of a proportional reduction factor between the theoretical stress concentration factors above and those actually observed at failure for real fibrous composites. Assuming that the joint geometry and fiber patterns are such as always to cause a net tension section failure, it is plausible to suggest that

$$k_{tc} - 1 = C (k_{te} - 1) , \quad (3.3.14)$$

**STRESS
CONCENTRATION
FACTORS
 k_{be} , k_{te}**

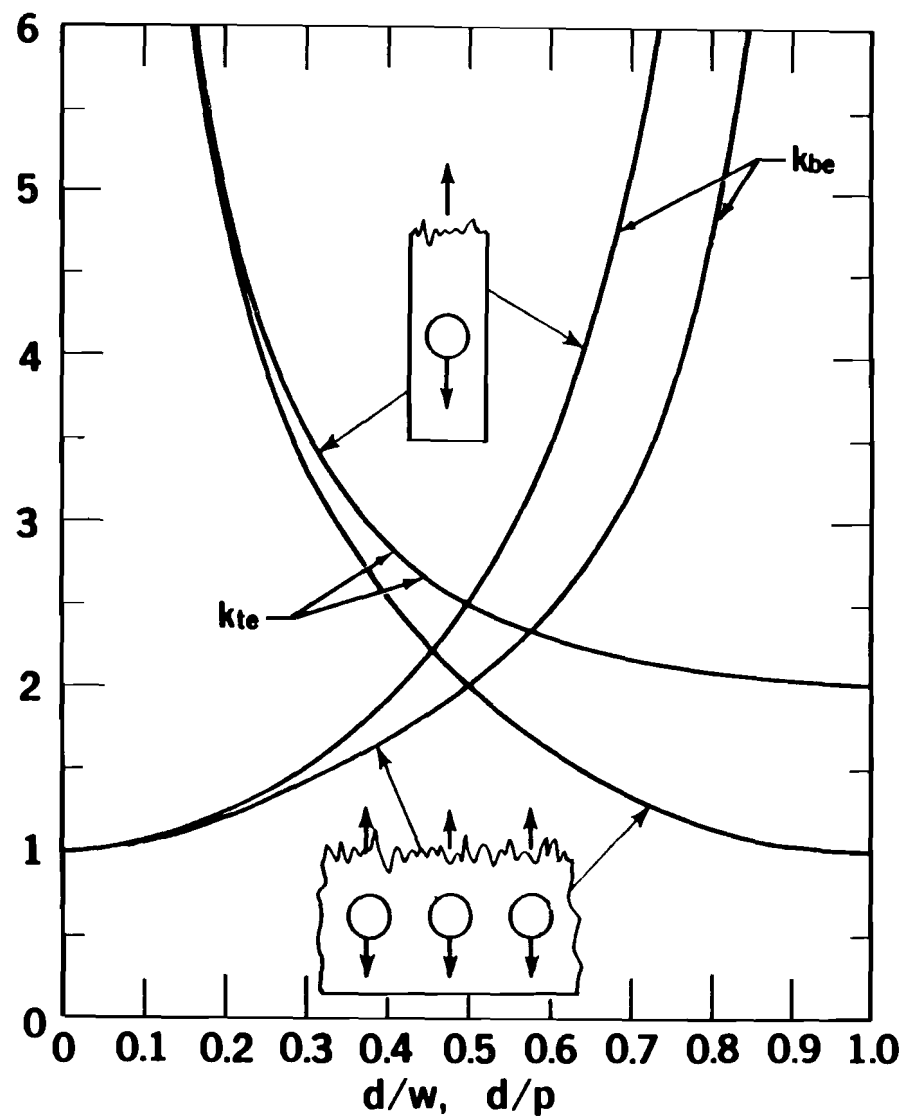


FIGURE 3.3.14. ELASTIC ISOTROPIC STRESS CONCENTRATION FACTORS AT LOADED BOLT HOLES

in which k_{tc} is the apparent stress concentration factor at failure and can be measured experimentally.

$$k_{tc} = F_{tu} t (w - d) / P . \quad (3.3.15)$$

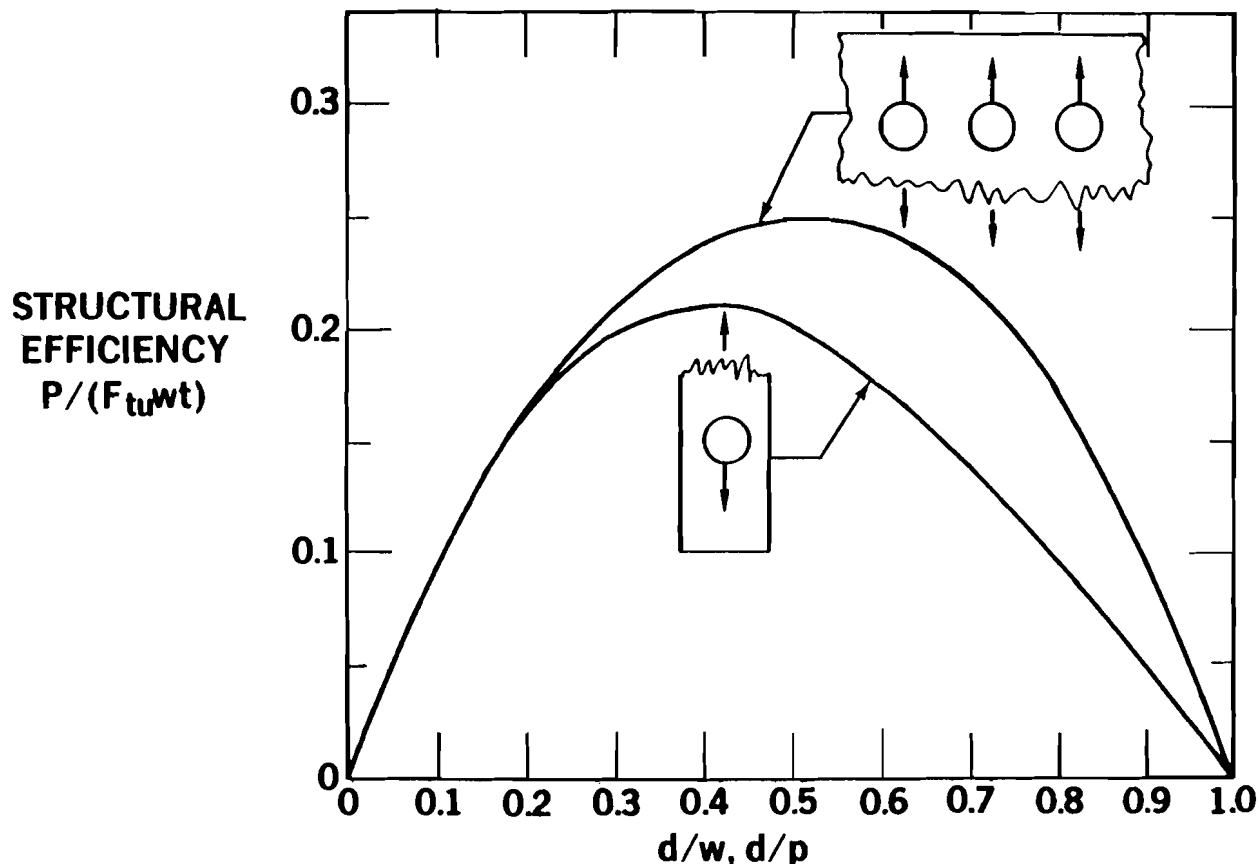


FIGURE 3.3.15. INFLUENCE OF JOINT GEOMETRY ON ELASTIC STRENGTH OF JOINTS IN BRITTLE ISOTROPIC MATERIALS

Such a proportional relief in eq. (3.3.14) would not be expected to hold if there were changes in failure mode associated with the different joints geometries associated with the various values of k_{te} . The proportionality constant, or correlation factor, C could lie in the range from zero (representing complete elimination of the stress concentration factor) to unity, indicating no relief whatever. Figure 3.3.16 shows the predictions, based on the hypothesis (3.3.14), of what would happen for a range of possible values of C that includes the physically probable ones. Those curves in the lower diagram are based on the formula for an individual narrow strip. The lowest curve A in figure 3.3.16 is the same as the uppermost curve in figure 3.3.9. It is apparent from the trend of curves A to G that progressively more relief of the stress concentration factor k_{te} has the predictable effect of increasing the structural efficiency, but that it also has the effect of moving the optimum bolt spacing further apart.

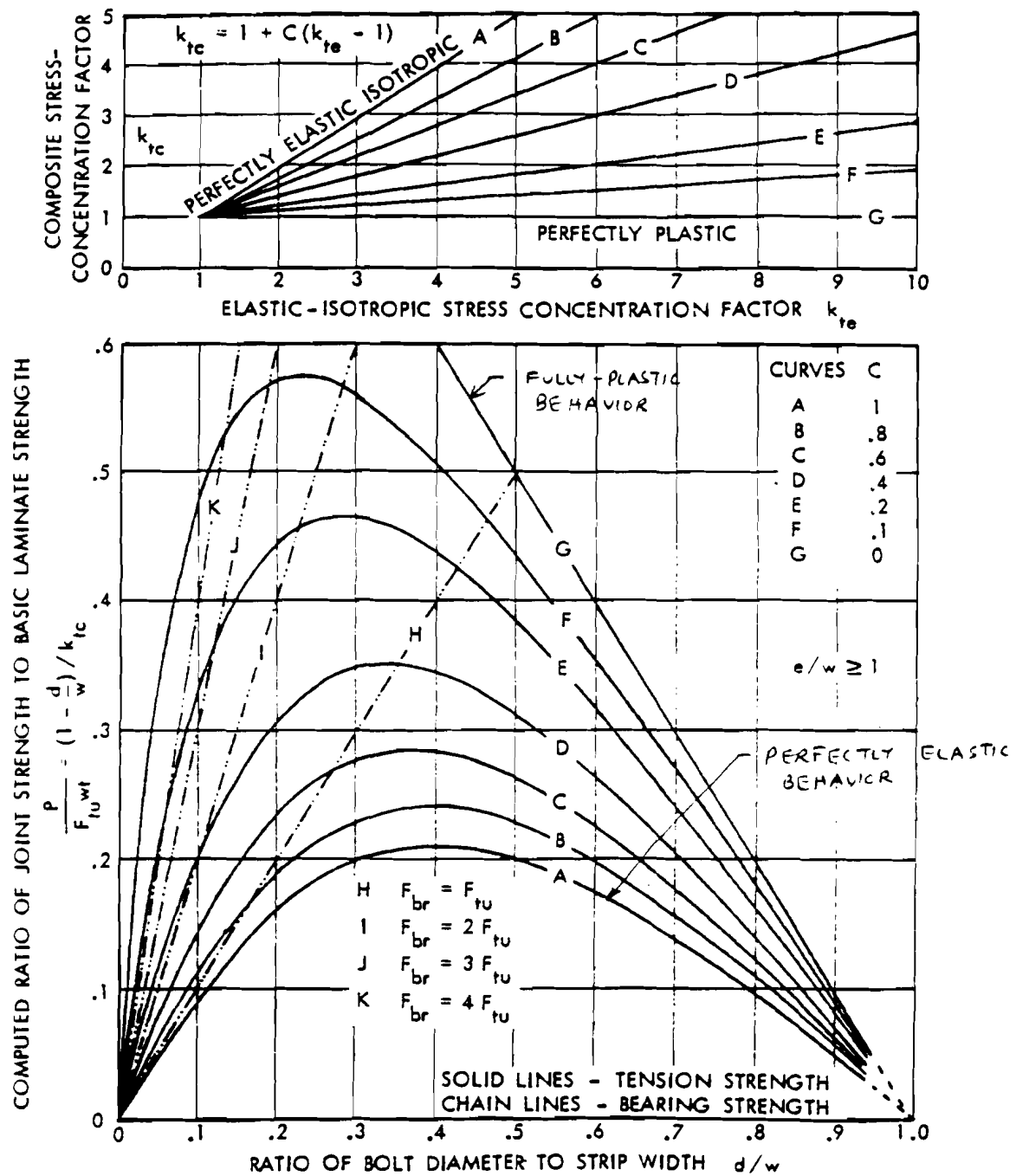


FIGURE 3.3.16. INFLUENCE OF JOINT GEOMETRY ON PREDICTED TENSILE STRENGTHS OF BOLTED JOINTS IN COMPOSITES

Eventually that bolt spacing would become so great that failures would occur by bearing rather than by tension. For that reason, figure 3.3.16 contains a series of radial cut-off lines (H to K) that would preclude the attainment of the extremely high efficiencies in the upper left corner. Since this figure is also capable of accounting for the e/w effects on the bearing and tension-through-the-hole failures, it is a particularly useful form in which to characterize the strength of bolted joints in fibrous composite structures. All that is needed is an experimental determination of the bearing strength F_{br} and of the ultimate laminate strength F_{tu} as well as tests suitable to determine the value of the correlation factor C . These tests are described below, following a discussion of the various possible failure modes.

The principles behind the generation of figure 3.3.16 can equally be applied to the case of unloaded holes, as is done in figure 3.3.17 which shows the corresponding improvements in predicted structural efficiencies. Again, the lower curve in figure 3.3.13 is displaced progressively towards the upper left corner in figure 3.3.17.

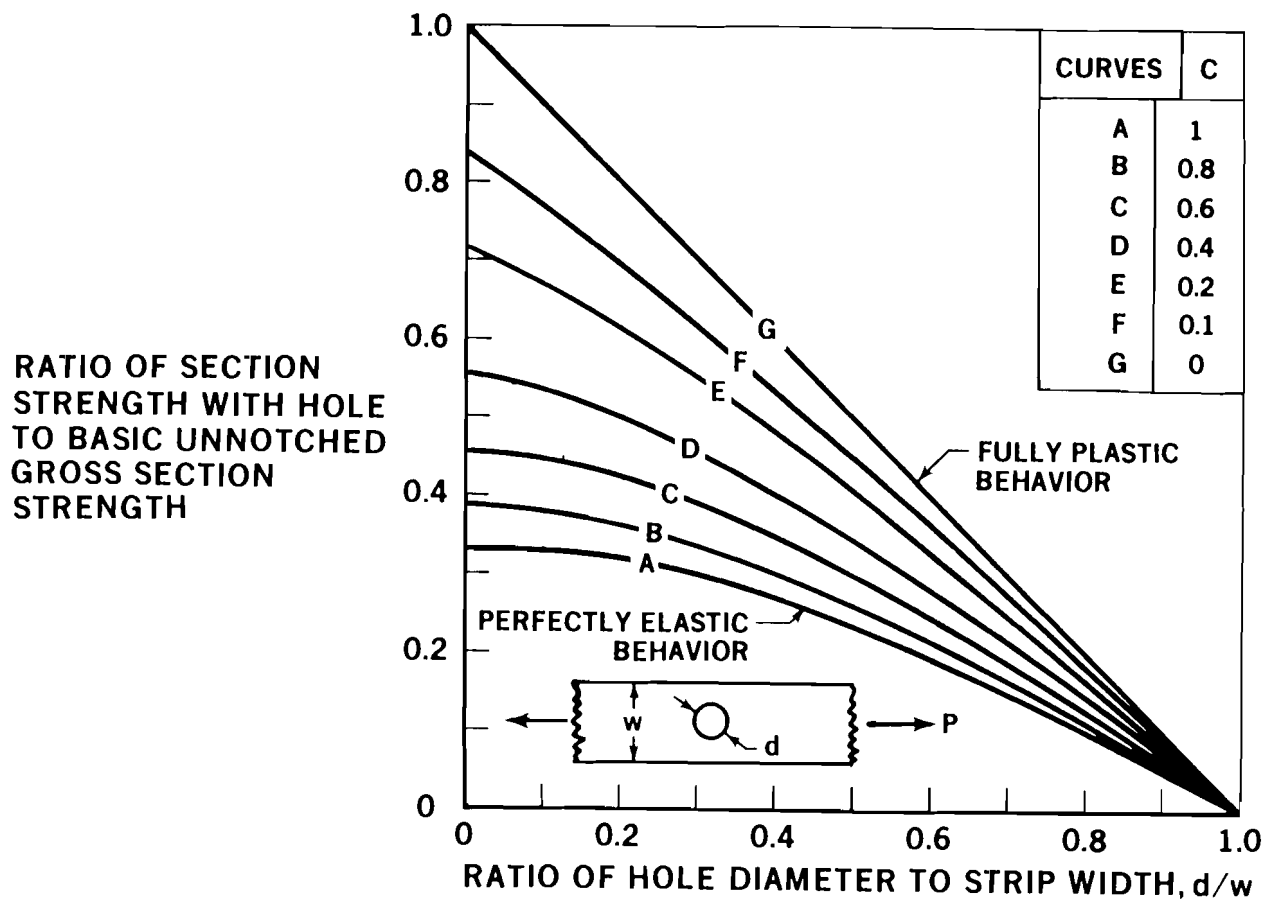


FIGURE 3.3.17. INFLUENCE OF STRESS CONCENTRATION RELIEF ON STRENGTH OF COMPOSITE STRIPS CONTAINING A HOLE

3.3.4 Failure Modes for Fibrous Composite Laminates at Bolt Holes.

Bolt holes in fibrous composite laminates can fail in a number of discrete modes, as shown in figure 3.3.18, or they can fail under combined modes when the joint geometry is near the transition from one mode to another. Of the modes shown in figure 3.3.18, only the tension-through-the-hole and bearing failures should be considered structurally adequate; all others represent weak premature failures and should be eliminated by proper design practices. As a fairly general observation, it should be noted that most designers of composite structures tend to inadvertently limit the efficiency of their structures unnecessarily by using bolts that are too small in diameter and spaced too far apart. There appears to be a reluctance to accept the very large margins on the shear strengths of the fasteners that are associated with those joint proportions needed to maximize the strength of the basic composite elements.

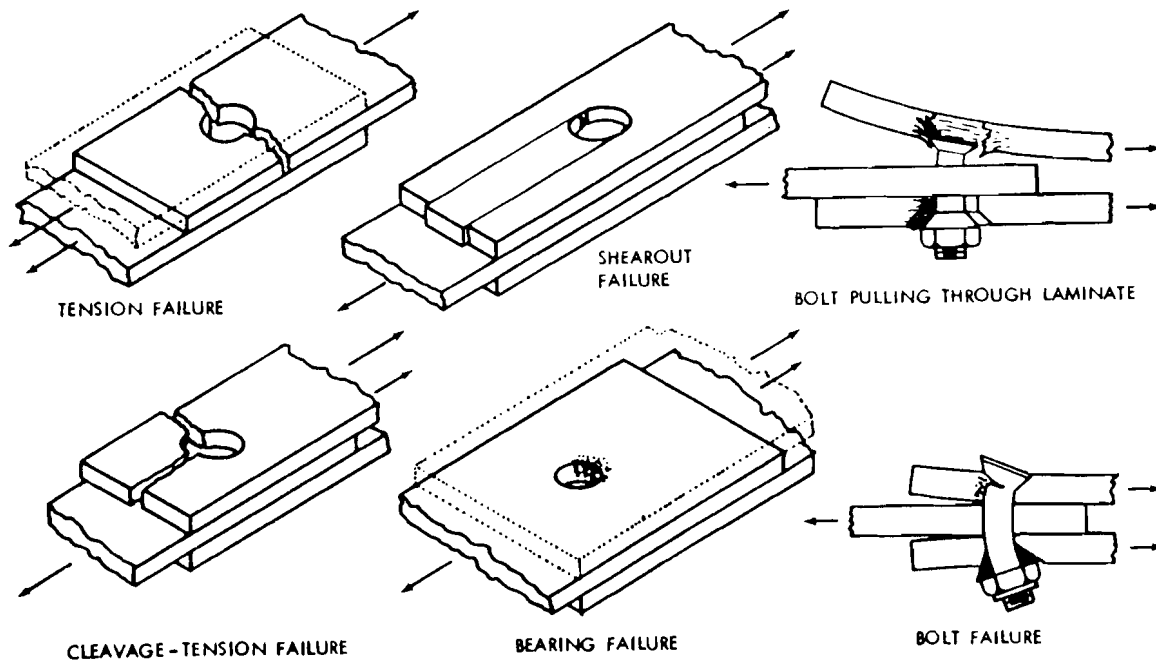


FIGURE 3.3.18. MODES OF FAILURE FOR BOLTED JOINTS IN ADVANCED COMPOSITES

3.3.5 Determination of Correlation Factor.

If the hypothesis implied by eq. (3.3.14) is correct, the value of the correlation factor C could be obtained from a series of tests covering the range of values of k_{te} for which tension-through-the-hole failures are possible. Such tests were run in reference 3.3.1 for Narmco 5208 / T-300 carbon epoxy laminates having three fiber patterns within the shaded area in figure 3.3.5. The results of those tests are recorded in figures 3.3.19 and 3.3.20 and, while the scatter is great, they are consistent with the hypothesis. Values of k_{te} less than 5 are associated with tension-through-the-hole failures. The high stress concentration factors were associated with bearing failures and, if the hypothesis were correct, should have exhibited apparent stress concentration factors higher than those associated with tension failures; the premature bearing failures prevented the development of sufficient load to fail the laminates in tension alongside the bolt holes. The results confirm this and the deviation from the line of best fit through the tension failures grows steadily greater for the larger values of k_{te} .

associated with the maximum values of w/d tested. The correlation factors are 0.25 for the quasi-isotropic pattern (25% 0° , 50% $+45^\circ$, 25% 90°) and 0.41 for the two patterns containing 37-1/2 percent 0-degree plies. The stress concentration factors for the unloaded holes appear to have somewhat less relief than for the loaded holes, but other series of tests indicate the opposite results.

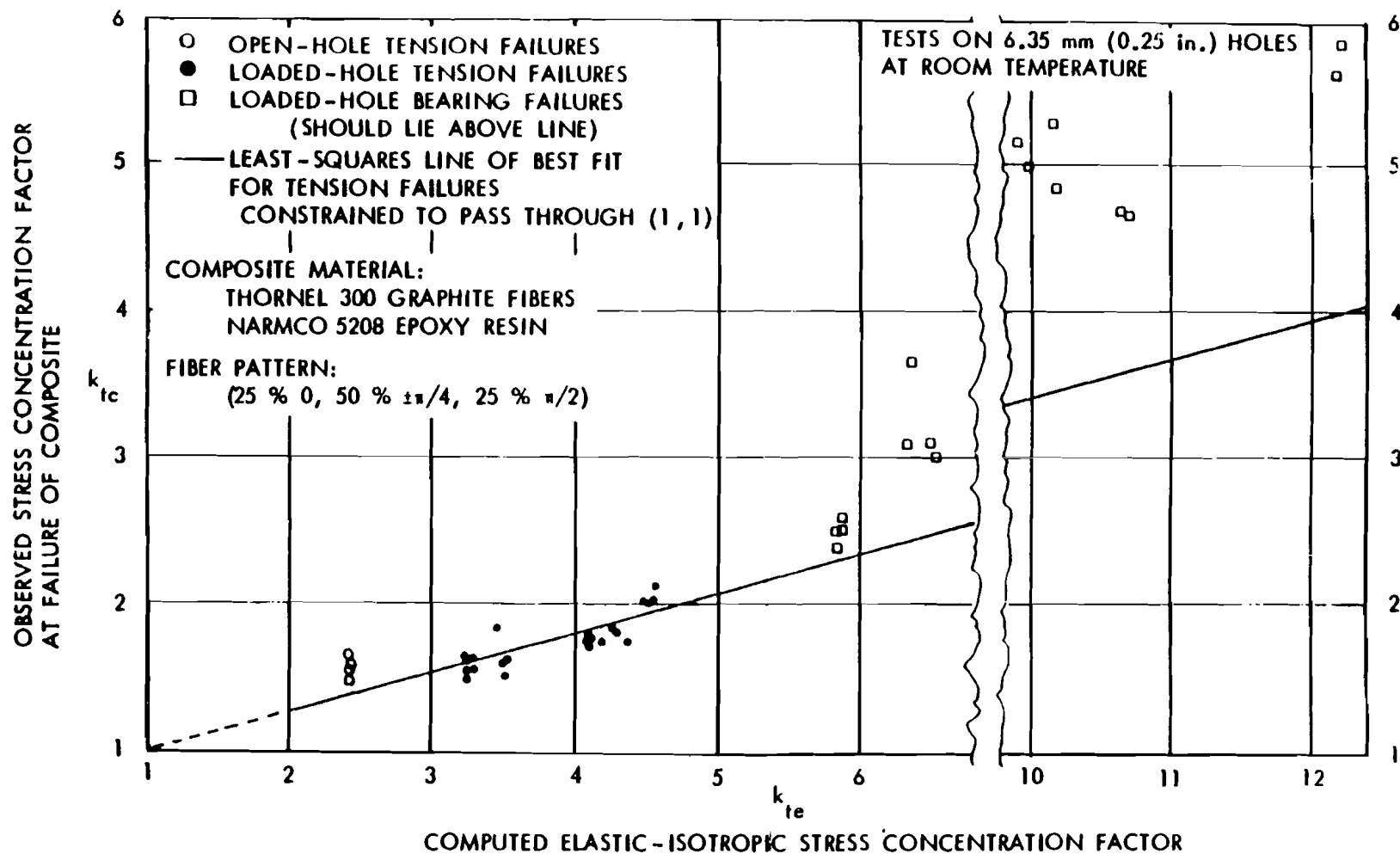
The tests recorded in figures 3.3.19 and 3.3.20 had been preceded by earlier tests on the Narmco 1004 / Marganite 11 carbon epoxy composite. Those results are shown in figures 3.3.21 and 3.3.22. The correlation factors in that case were 0.25 again for the quasi-isotropic pattern and 0.46 for the laminates with 50 percent 0-degree plies. The slightly less relief shown for the smaller bolts at the right of these figures is due to an interaction with the higher bearing stresses, even though the failures were actually by tension. The associated w/d ratios were almost at the transition between bearing and tension failures. Figures 3.3.21 and 3.3.22, likewise, confirm the general validity of the form of eq. (3.3.14). In any case, there is sufficient scatter so that a better understanding of the subject would be needed before a more complicated formula could be proposed.

Further results are given in figure 3.3.10 of reference 3.3.3 for yet another carbon epoxy composite and the correlation factors match very closely with those given above for the same fiber patterns. The results for testing at $w/d = 2$ for unloaded holes were spuriously low because of a drastic change in failure mode involving much more delamination. The hypothesis in eq. (3.3.14) is therefore seen to be invalidated by substantial changes in failure mode, but no such changes are known to occur for the usual joint geometries when using the fiber patterns recommended in figure 3.3.5.

It may seem questionable to seek a linear correlation factor between orthotropic composite laminates and a theoretical isotropic baseline. However, doing so works, as can be seen from figures 3.3.20 and 3.3.22. The reason why is believed to be that, provided the mode and location of failure do not change, the conversion between orthotropy and isotropy would simply be a scale factor on the abscissae. That can equally be accounted for by an equivalent change in the slope of the lines.

Figures 3.3.19 to 3.3.22 show that higher observed stress concentration factors are associated with the stiffer, orthotropic, laminates than with the quasi-isotropic ones. Indeed, the increase in stress concentration factor is almost, but not quite, as great as the increase in the unnotched laminate strength. Actually, the 37-1/2 percent 0-degree laminates do give slightly stronger joints overall, but the strengths of the bolted joints made from laminates in the recommended area shown in figure 3.3.5 do not vary strongly with the fiber pattern. That area is a kind of plateau for both tension-through-the-hole failures and bearing failures.

Knowing the kind of relief associated with the delaminations and fiber pullouts around both holes in composite laminates, it is possible to elucidate the predictions in figures 3.3.16 and 3.3.17. The optimum geometry in figure 3.3.16 for a quasi-isotropic carbon epoxy laminate would be a w/d of about 3.5, with an associated structural efficiency slightly in excess of 40 percent. That represents a two-fold increase in strength with respect to the 21 percent efficiency deduced above for perfectly brittle materials. That substantial improvement also confirms the earlier statement that all reliable methods for



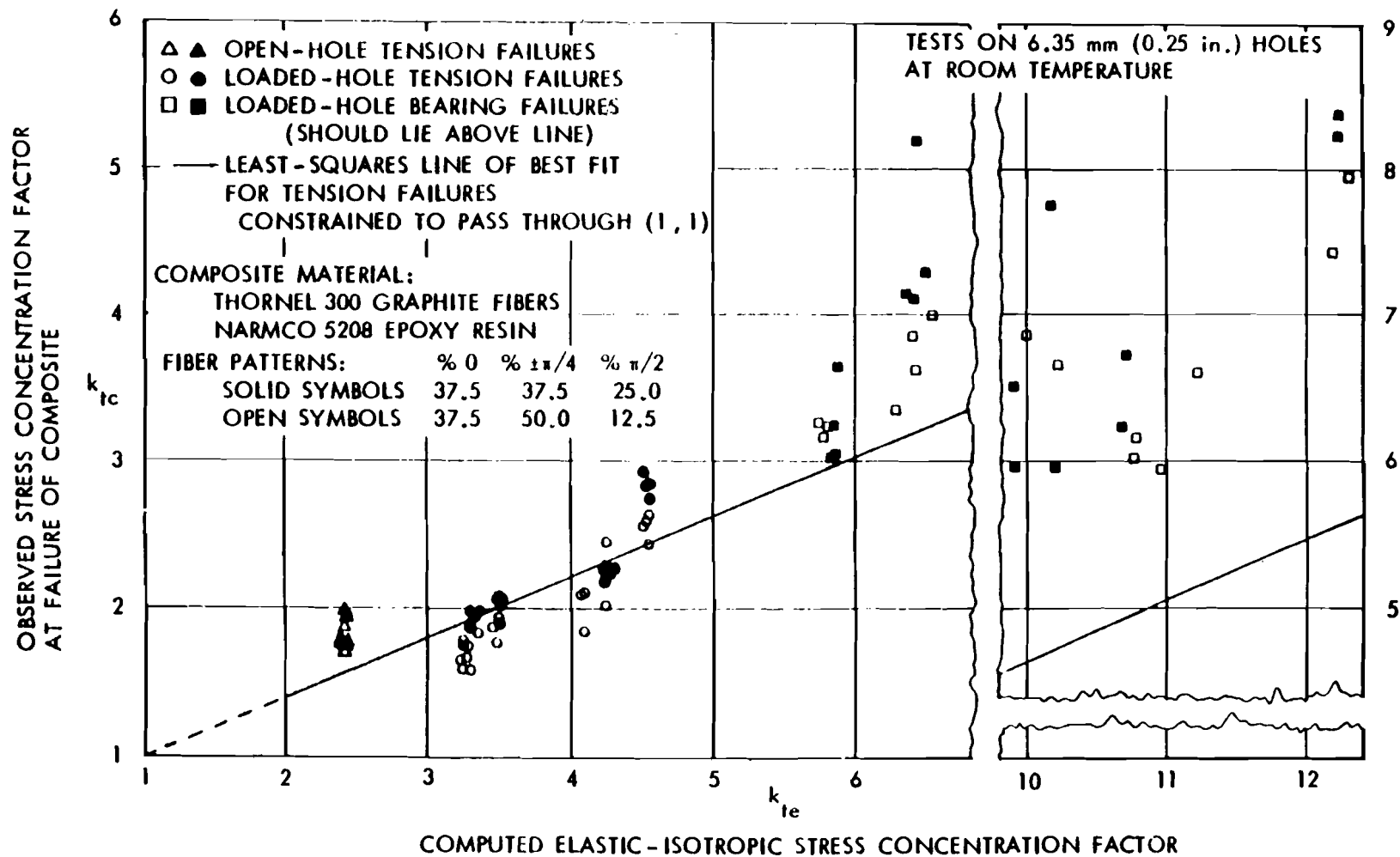


FIGURE 3.3.20. STRESS CONCENTRATION FACTORS AT FAILURE FOR BOLTED JOINTS IN THORNEL 300/NARMCO 5208 GRAPHITE-EPOXY (ORTHOTROPIC PATTERNS)

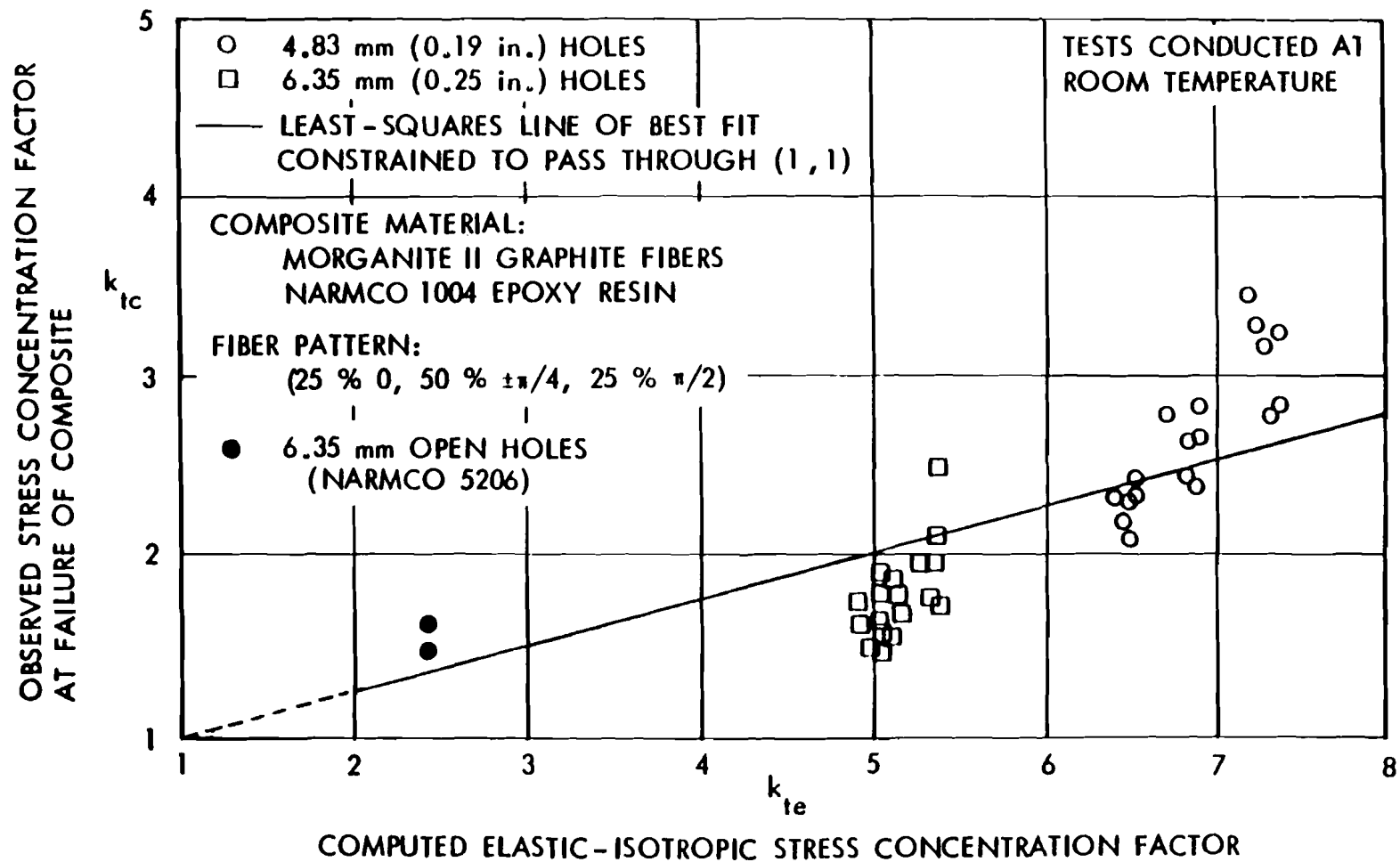


FIGURE 3.3.21. STRESS CONCENTRATION FACTORS AT FAILURE FOR BOLTED JOINTS IN MORGANITE II/NARMCO 1004 GRAPHITE-EPOXY (QUASI-ISOTROPIC PATTERN)

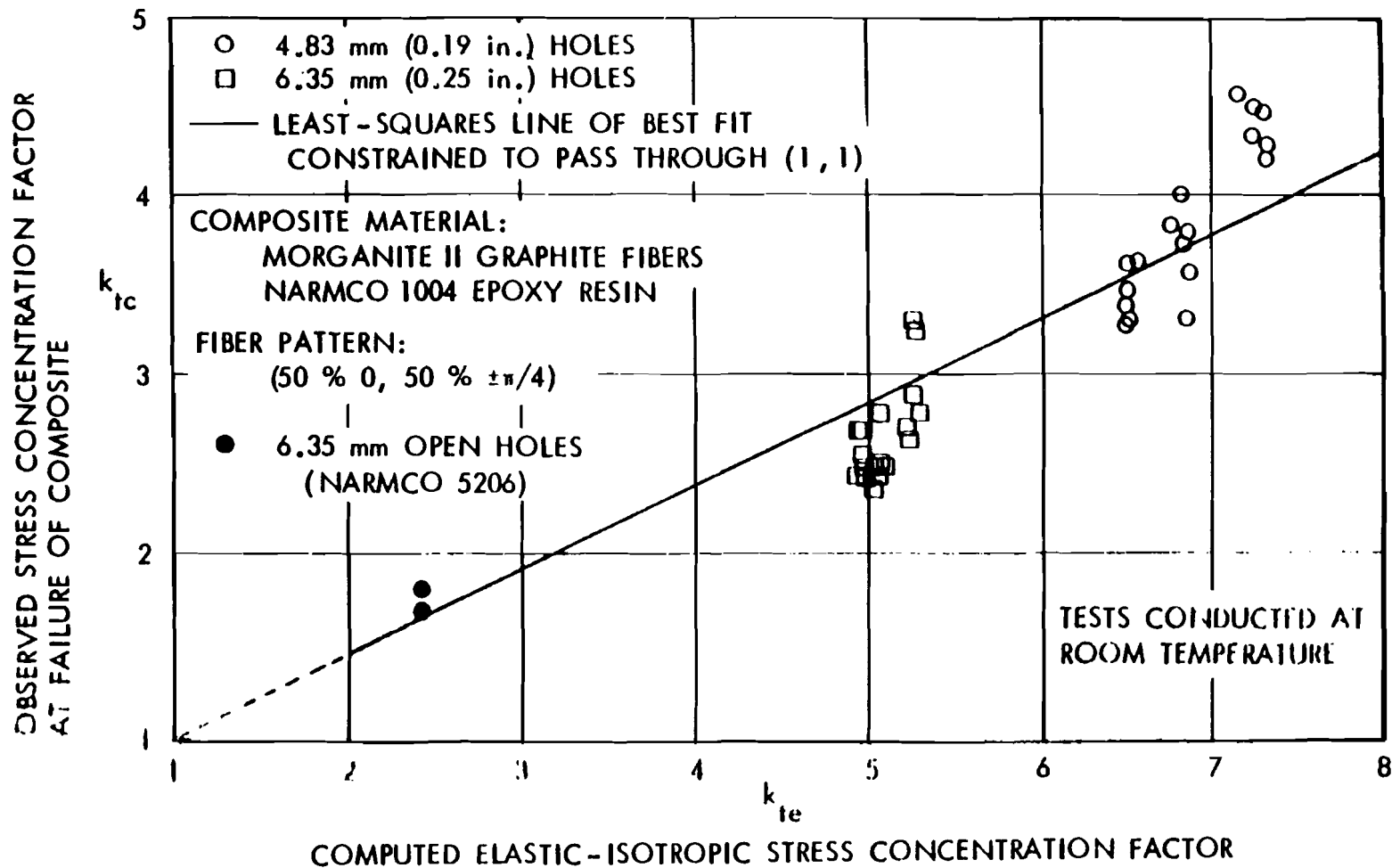


FIGURE 3.3.22. STRESS CONCENTRATION FACTORS AT FAILURE FOR BOLTED JOINTS IN MORGANITE II/NARMCO 1004 GRAPHITE-EPOXY (ORTHOTROPIC PATTERN)

predicting the strength of bolted joints in composite structures must rely on a substantial empirical correction factor, whether it is obvious or not; a minor perturbation will not suffice.

3.3.6 Experimental Test Programs.

The above assessment of the behavior of bolted joints in carbon epoxy laminates, in particular, makes possible the planning of an efficient test program with which to determine the necessary joint strength parameters. Surprisingly few tests are needed for the well-behaved fiber patterns. The tests will be described, starting with the tensile tests, and progressing to the compressive ones which have distinctly different stress trajectories, as shown in figure 3.3.23. All bearing should be initially in double shear. Reference 3.3.1 contains a thorough description of the test procedures.

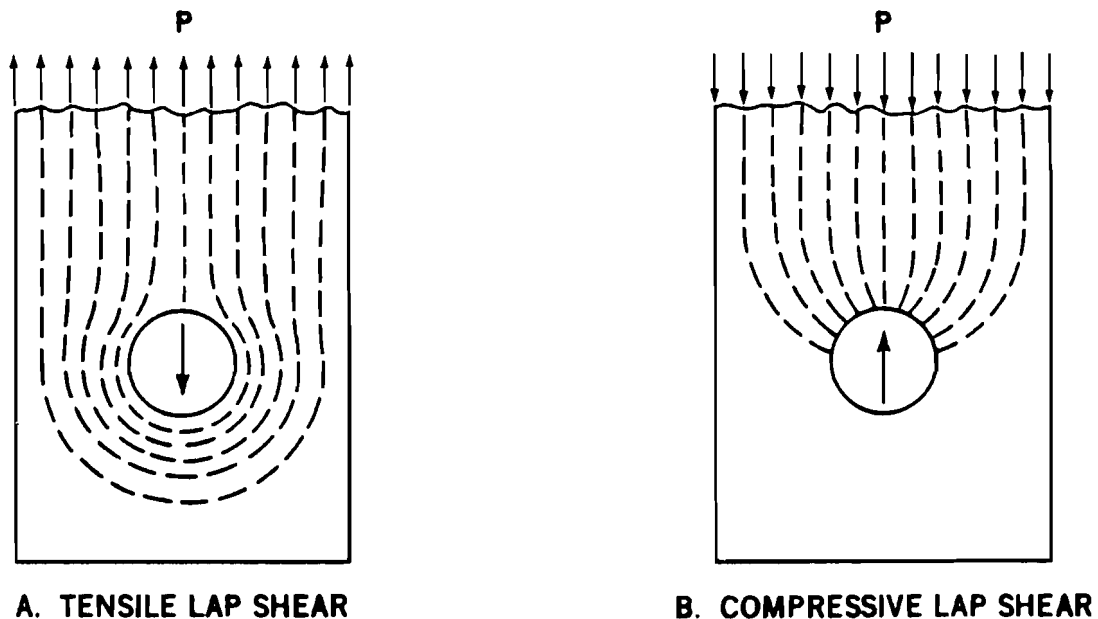


FIGURE 3.3.23. STRESS TRAJECTORIES AROUND BOLTS FOR TENSILE AND COMPRESSIVE LAP SHEAR

The first quantity to be determined is the unnotched laminate strength, in both tension and compression. Theoretical attempts to establish these strengths seem to be slightly nonconservative and the experimental results are often very conservative, because of difficulties with the grips at the area of load introduction or because of insufficiently smooth edges on the test coupons. It is not all that critical to have a precise value for this quantity; however it is important to use the same value (for each load condition) throughout, in the denominators for the data reduction and numerators for the strength predictions. That way, any small errors are self-compensating.

For carbon-epoxy laminates having fiber patterns within the shaded area in figure 3.3.5, the ultimate tensile bearing strengths are achieved for a laminate having a width w and edge distance e both equal to 8 times the fastener diameter d . The transition between bearing and tension failures is known to occur for strip widths somewhere in the range of 5 to 6 diameters, as shown in figures 3.3.19 and

3.3.22. A slightly smaller value would suffice for fiberglass and Kevlar , but the same large value is more appropriate for boron-epoxy. The compressive bearing strength, except in the case of Kevlar fiber reinforcement, is usually found to exceed the tensile bearing strength, so that latter can be used conservatively if compressive data are unavailable.

The bearing tests described above should be repeated for a short edge distance, $2d$ or $3d$, to verify that the laminate pattern is not prone to premature failure due to shearout. That is often the case for laminate patterns containing no 90° plies or no $+45^\circ$ plies. The subject of shearout failure is discussed more fully in a succeeding section of this article.

The recommended w/d ratio is to ensure a tension-through-the-hole failure is 3 - well below the 5 or 6 at the transition and close to the optimum for single-bolt joints. The e/d ratio should also be 3. Compressive net-section failure can occur only for bypass loads (as at unloaded holes) and not for pure bearing loads. The value 3 above, for w/d , should apply for the advanced composites, but fiberglass reinforcement might give a better result with 2 or 2.5.

The unloaded hole tests should be run at w/d for 4, taking advantage of a light clamp-up by a bolt in the hole. The bolt can be either loose or a tight fit in the hole for tensile loading, but separate tests should be run for each case under compressive loading. The reason for the larger value of w/d than the 3 above is that bypass loads are more significant for multi-row joints in which the more usual w/d ratio is 5, as explained below.

These results should be checked for consistency in regard to modes of failure and relative severity of the different stress concentration factors, in the context of figure 3.3.24. Further tests at different w/d ratios should be run if anomalies are uncovered. The values of the correlation coefficient can then be determined.

The above series of tests should be repeated for a range of bolt diameters encompassing the design applications. The "nonlinear" stress concentration relief is related to the ply thickness and fiber pattern, becoming proportionally less significant for larger fastener diameters. The difference in stress concentration factors for the usual sizes of fasteners (less than $3/4$ inch in diameter) is significant, but not dramatic, as shown in figure 3.3.25, based on data in reference 3.3.7. Data for large lugs, of a few inches in diameter, are not yet available.

The tests referred to above should be run first in double shear, with lightly torqued bolts having hexagon heads and washers under the bolt head and nut. Tests in reference 3.3.1, reproduced here in figure 3.3.26, indicated that the complete lack of clamp-up (as with purely pin-bearing) is associated with about a fifty percent reduction in bearing strength. These findings have also been confirmed by other investigations. The reason for this large difference is that, without transverse compression (or at least confinement), the first fibers to fail can be displaced sideways and shed their load onto the remaining fibers. With that confinement, however, the first fibers to be overloaded can still transmit compressive bearing loads because there is nowhere for those fibers to broom out of the way.

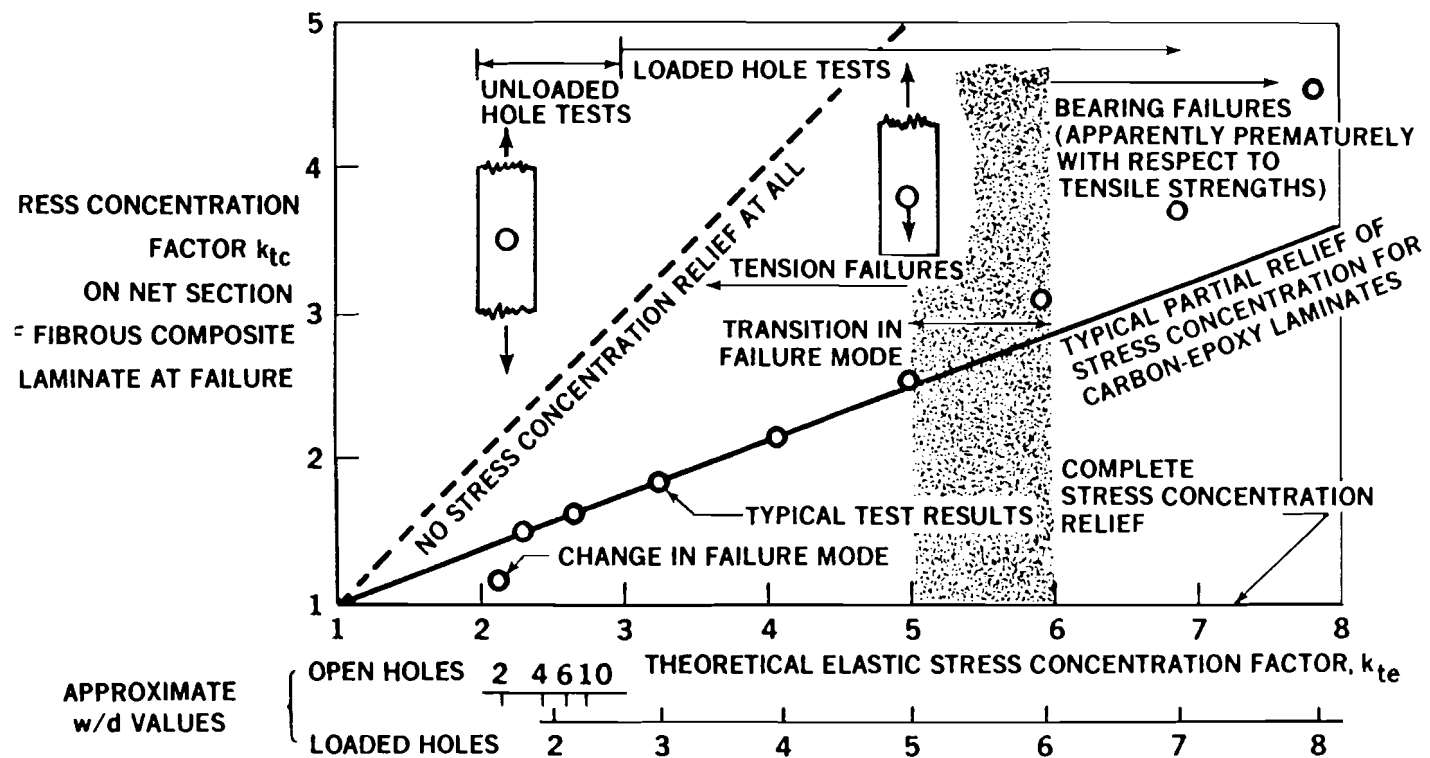


FIGURE 3.3.24. RELATION BETWEEN STRESS CONCENTRATION FACTORS OBSERVED AT FAILURE OF FIBROUS COMPOSITE LAMINATES AND PREDICTED FOR PERFECTLY ELASTIC ISOTROPIC MATERIALS

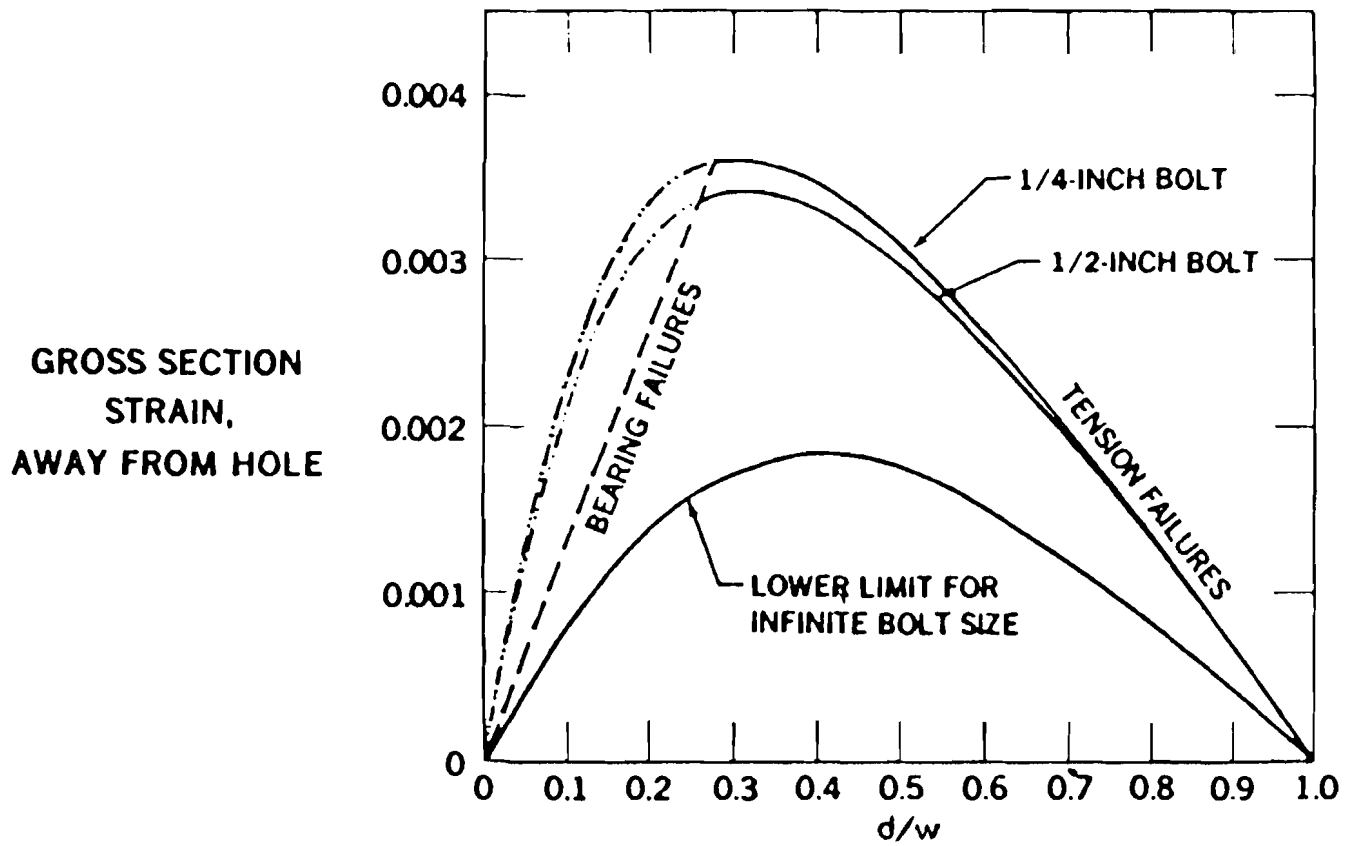


FIGURE 3.3.25. INFLUENCE OF BOLT SIZE ON COMPOSITE JOINT STRENGTH

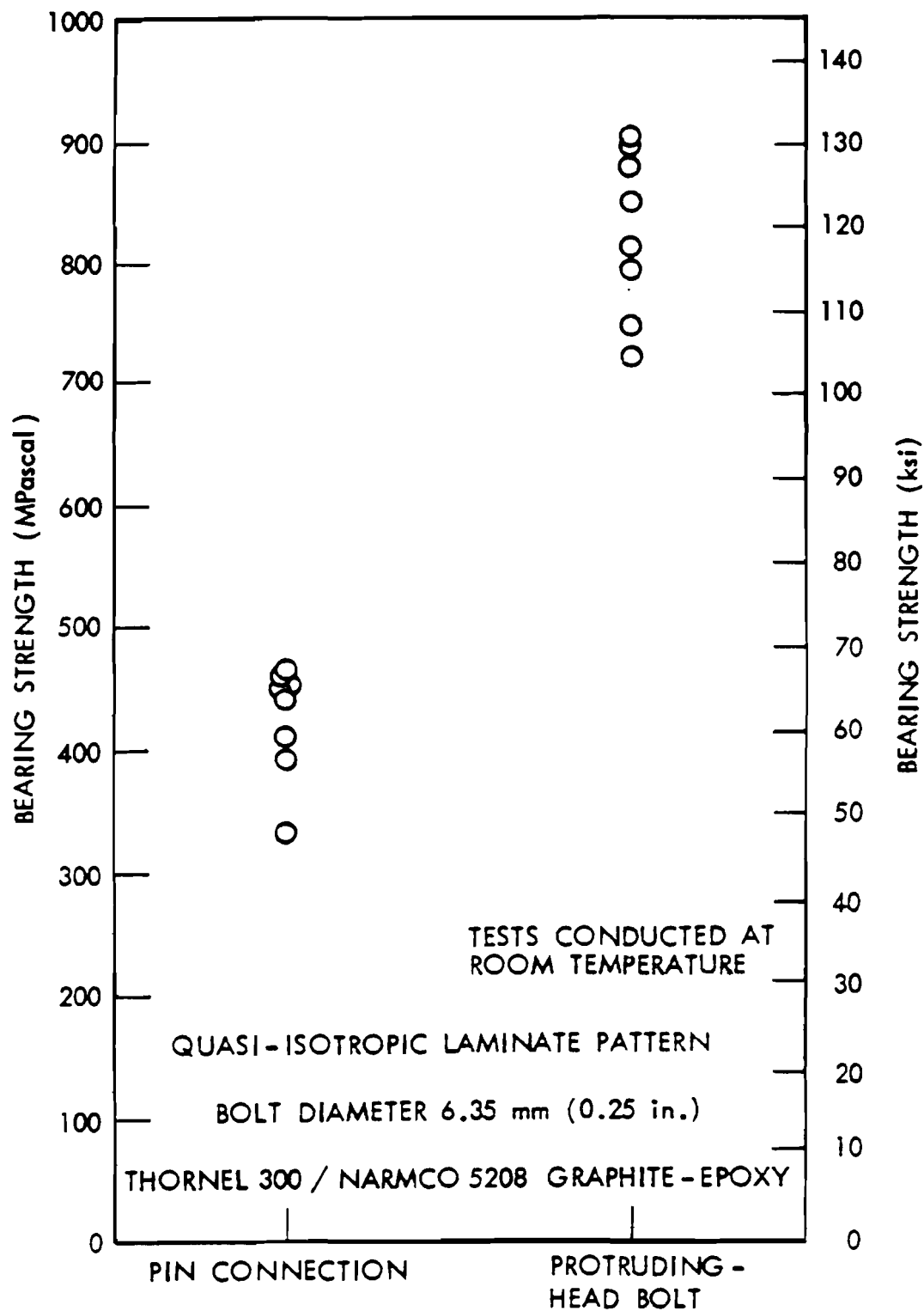


FIGURE 3.3.26. COMPARISON BETWEEN BEARING STRENGTHS FOR PIN-LOADING AND REGULAR (TORQUED) BOLTS

Load deflection curves measured on bolted composite joints sometime reveal a little blip (or a slight cracking) may be heard long before failure. The joint strength can be further increased by substantial clamping torques instead of the more usual value of one half of that for aluminum alloys. The torques on bolts in composites are customarily restricted because of the low interlaminar compression strength (30 ksi or so). Reliance on additional torque is to be discouraged because, whereas a mistake in torquing a bolt in metal structure merely decreases the fatigue life without decreasing the initial static strength, such a mistake in a composite structure would cause a substantial loss of static strength.

On a related subject, it was found in reference 3 that composite splice plates developed only about 80 percent of the strength attained by composite skins sandwiched between two members. This loss of strength appears to be due to a reduced clamp-up as the bolt head rotates when the bolt is bent under load. Interestingly, that same loss of strength is cited below for the case of bolts rotating under single-shear loads. The mechanisms are evidently the same.

Countersunk fasteners will not develop as high a bearing strength in composite laminates as do the basic protruding-head bolts with washers, because of the decreased transverse clamp-up. Therefore, special tests may be required for such installations, although a scheme is proposed in reference 3.3.8 to treat the shank as fully effective at double-shear strengths and to discount entirely the bearing on the bolt head. However, the question of tension-through-the-hole strength in such a case is more complicated.

Single-shear bolted joints are associated with substantial reductions in strength from two sources. The first is the non-uniform bearing stress associated with rotation of the bolt. The second is the bending of the members being joined in typical one-row test coupons. However, structural applications usually are free from the second loss of strength and special test fixtures are described in reference 3.3.1 to eliminate the effects of the eccentricity in load path. The residual loss of strength due to bolt rotation alone is indicated to be of the order of 20 percent, by the test results in figure 3.3.27, taken from reference 3.3.1. Therefore, in addition to the basic double-shear tests described above, additional tests may be needed for single-shear configurations. The performance of single-shear bolted joints in fibrous composites is particularly sensitive to the size of the bolt head and to the presence or absence of washers on both sides of the joint.

If any of the tests described above are to be used for multi-row joint applications, it is necessary (or at least desirable) to acquire load versus deflections records for each load bolt test. Sample results are shown in figure 3.3.28. Such curves establish the joint stiffnesses which are needed to compute the load sharing in multi-row joints. A formula is given in reference 3.3.3 to accomplish this for double-shear joints, but no method has yet been found to predict single-shear stiffnesses without the help of test data. Figure 3.3.28 shows also how bearing failures are associated with larger relative deflections prior to failure than are observed with tension-through-the-hole failures, but that the tension failures occur at higher loads. The data of the kind shown in figure 3.3.28 also indicate the relative levels of the ultimate failure load and the lesser load at which significant damage and nonlinear behavior began. Without such load-deflection data, the load sharing can be estimated only on the basis of educated guesses and while a few experts can do that very well, such a procedure may not be sufficiently convincing for purposes of certification.

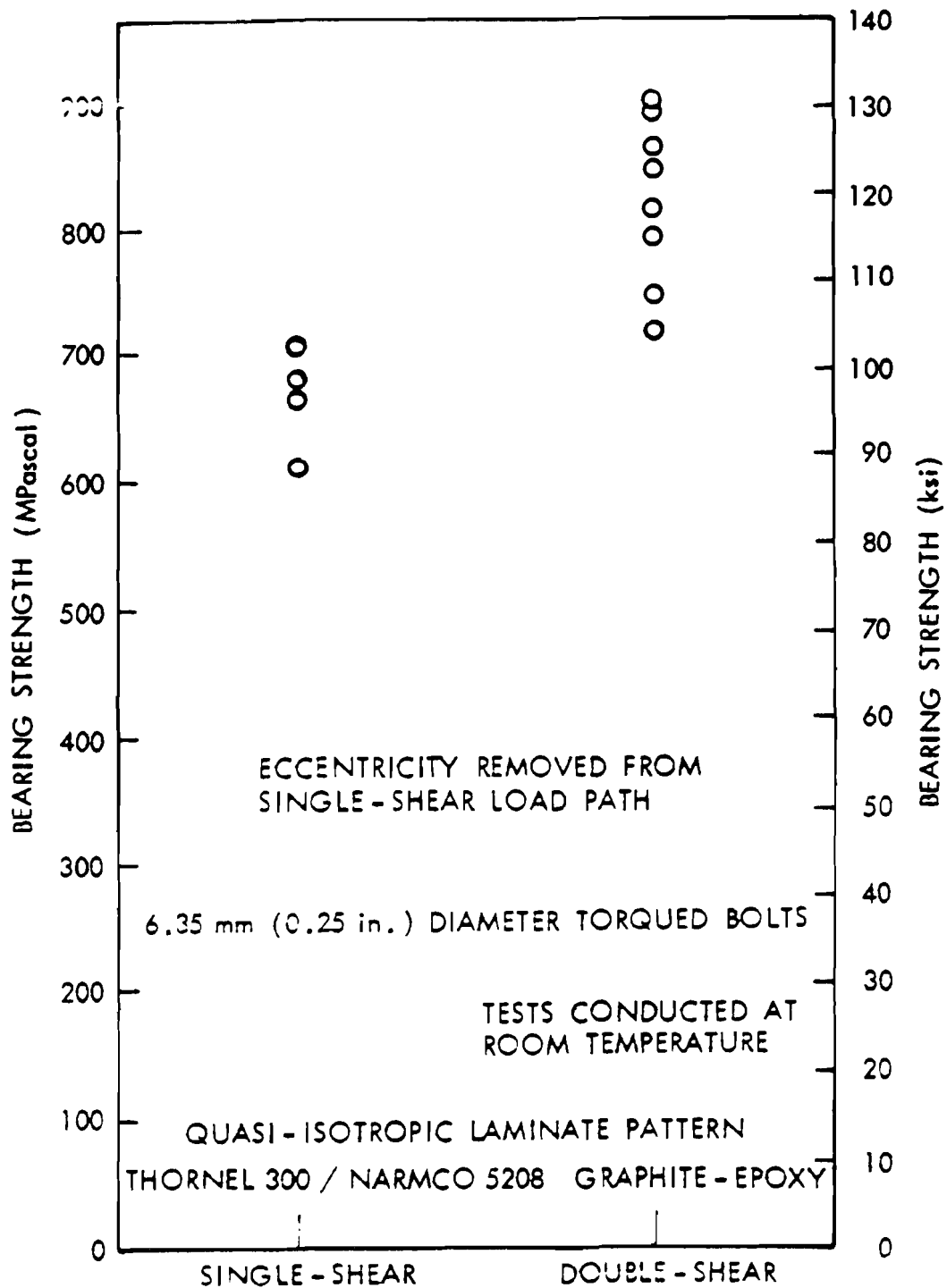


FIGURE 3.3.27. COMPARISON BETWEEN BOLT BEARING STRENGTHS IN SINGLE AND DOUBLE-SHEAR FOR GRAPHITE-EPOXY LAMINATES

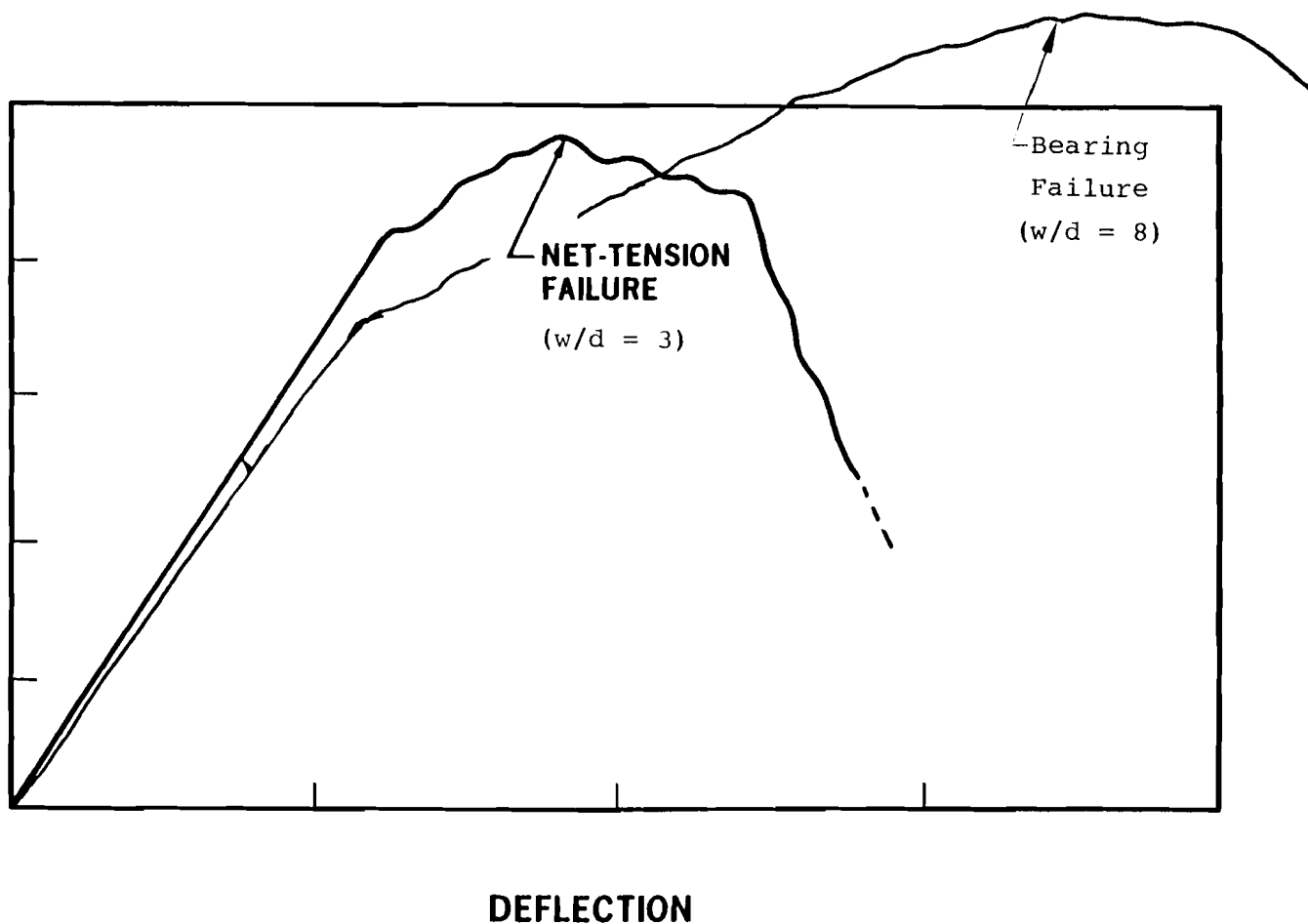


FIGURE 3.3.28. LOAD-DEFLECTION CURVES FOR COMPOSITE BOLTED JOINTS

3.3.7 Effects of Fiber Pattern on Strength of Bolted Composite Joints.

One of the advantages claimed for fibrous composite structures is that their orthotropic properties can be tailored for any specific application. That design philosophy would imply a need for bolted joint strength data for a large variety of laminate patterns. Design of composite structures would then be quite a complex and costly task. However, the motivation for such a philosophy derives mainly from stiffness considerations only; the desire to also develop adequate strength, particularly at bolt holes and cutouts, tends to restrict the choice of fiber patterns to those recommended in figure 3.3.5. Most, if not all, of the major U.S. aerospace companies have similar limits on their design procedures, even though they are not always enforced.

The reason for these restrictions on practical laminate patterns for aircraft use is that today's 350°F* cured resin matrices are not quite adequate to develop the full potential of the high-strength fibers and filaments. It is, therefore, necessary to maximize the number of resin interfaces between changes in direction of the layers of fibers and, obviously, specifying a significantly orthotropic lay-up pattern prevents this because some parallel plies must inevitably be bunched together instead of being thoroughly interspersed. Similarly, the use of thicker plies or the unnecessary bunching together of thinner plies has been

found by test to impose significant reductions in laminated strengths (which incidentally are not accounted for in most laminate strength theories). Even the restriction of fiber angle changes to 45 degrees at a time is known from test to produce stronger laminates than those developed by the same plies in a stacking sequence that permits 90 degree changes; for example (0, 45, 90, -45) lay-ups are to be preferred over (0, 90, 45, -45) patterns.

The logical preference for near-isotropic fiber patterns in basic unnotched laminates is even more pronounced at bolt holes. This can be seen from figure 3.3.29, in which the bearing strength of high tensile strength (HTS) carbon-epoxy laminates are plotted for various fiber patterns. Additional test data given in reference 3.3.3 confirms the existence and height of this plateau for a tough-resin carbon-epoxy material. There is an obvious peak strength plateau around the quasi-isotropic pattern in figure 3.3.29. This is particularly important since the bearing strength is thus insensitive to minor changes in the fiber pattern in that region. Those patterns with an excess of 0° plies -- more than 50 percent -- do not develop their full bearing strengths because of premature failure by shearout, even at large edge distances (8 to 10 times the bolt diameter). The location of the transition in failure mode is shown in reference 3.3.1 to vary with the fibrous reinforcement and probably also varies with the resin content. It would probably take much thinner plies or much tougher resins, that are used today, to eliminate those premature failures in the top portion of figure 3.3.29.

Figure 3.3.30 presents the corresponding test results for specimens having a short edge distance ($e/d = 2$). These strengths have rather inappropriately come to be called the shearout strengths. In spite of large specimen widths ($w/d = 8$ to 12), all of the failures in the lower part of this diagram were by tension-through-the-hole. The only shearout failures occurred in the top portion of the diagram, for which shearout failures were shown in figure 3.3.29 to still occur even with edge distances as large as $e/d = 8$ to 10. It may seem that the absence of shearout failures for the testing in the lower portion of figure 3.3.30 would imply that an e/d ratios of 8, which is the value beyond which no further increases can be realized. An edge distance of $3d$ is shown in figures 3.3.37 and 3.3.38 of reference 3.3.1 to develop some 90 percent of the full bearing strength of the laminate. Reference 3.3.1 contains also the corresponding shearout contours for boron-epoxy and mixed carbon-epoxy laminates. Much the same change in failure mode was observed, with a slightly relocated transition in behavior.

*The more ductile 250°F cured resins used with fiberglass secondary structure would provide a considerable improvement for carbon epoxy laminates for subsonic transport aircraft. Such laminates are known to be free from the problem of propagation of delaminations and what little damage does occur is usually more visible. Also, the lower temperature cure does not leave the matrices full of millions of micro cracks because of the resin shrinkage during cool-down after cure. Consequently, the in-service durability of the laminates would be enhanced significantly.

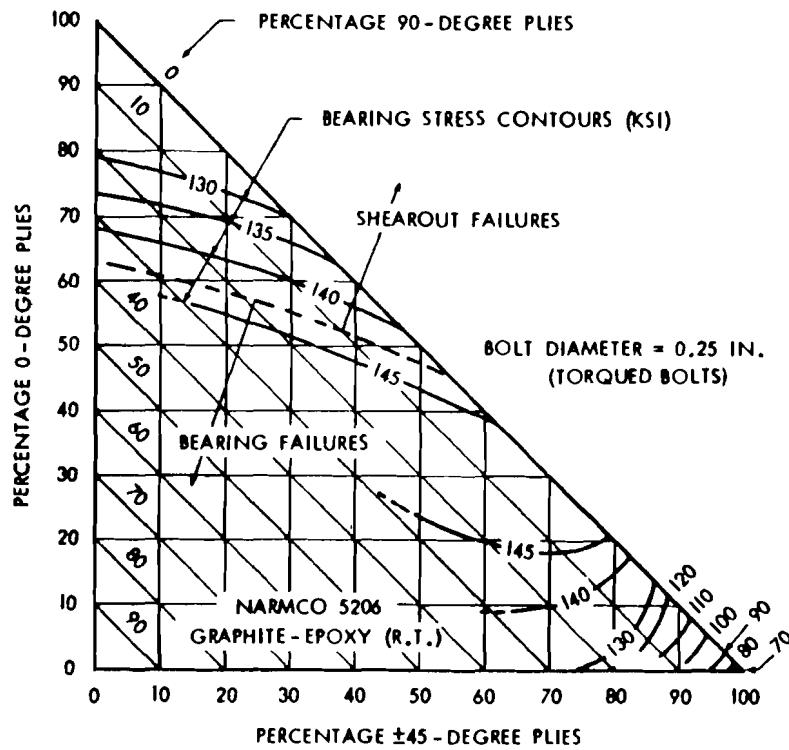


FIGURE 3.3.29. BEARING STRESS CONTOURS FOR BOLTED GRAPHITE-EPOXY JOINTS

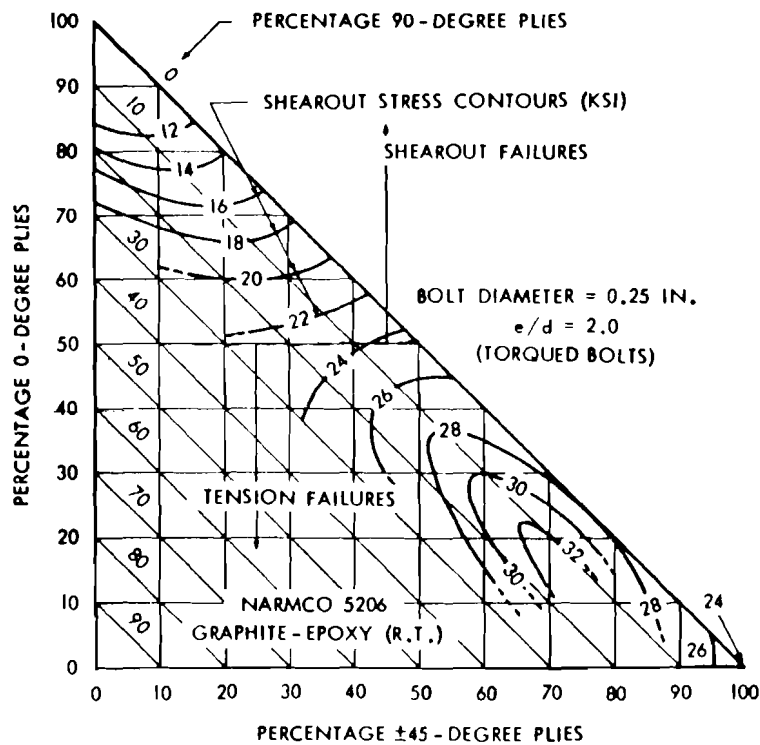


FIGURE 3.3.30. SHEAROUT STRESS CONTOURS FOR BOLTED GRAPHITE-EPOXY JOINTS

3.3.8 Bearing-Bypass Interactions and Joint Efficiency Charts for Composite Joints.

At every row of bolts except the last, in multi-row bolted joints, there is an interaction between the bearing and bypass loads. The bearing load refers to the load actually being sheared through a particular bolt row, while the bypass load refers to the remainder of the load at that location; the remainder is reacted at other rows of bolts. This concept is explained in the upper right corner of figure 3.3.31, for tensile loading.

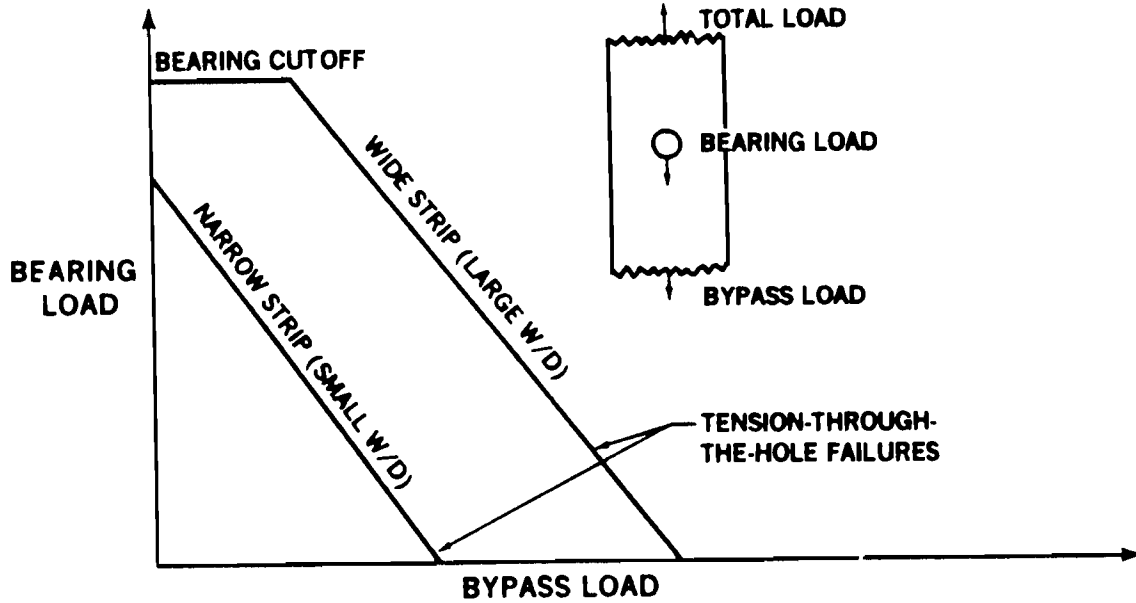


FIGURE 3.3.31. BEARING/BYPASS LOAD INTERACTION FOR LOADED BOLTS IN ADVANCED COMPOSITES

Now, if the joint geometry is such as to produce a tension-through-the-hole failure for both bearing loads and bypass loads separately (that is if w/d is small enough), it is reasonable to propose a linear interaction for all combinations of such loads between those extremes. This is shown by the uninked line (small w/d) in figure 3.3.31. Mathematically, this can be expressed by

$$\sigma_{\max} = k_b \sigma_b + k_t \sigma_t \leq F_{tu} , \quad (3.3.16)$$

in which the coefficients represent the alleviation factors and geometric stress concentration factors described above. Thus, for the bearing load,

$$k_b = \frac{1}{(w/d - 1)} \left\{ 1 + C \left[k_{be} (w/d - 1) - 1 \right] \right\} , \quad (3.3.17)$$

with k_{be} given by eq. (3.3.4), for example. Actually, it is easier to work with two different expressions for k_{te} , taking care not to confuse their use. Similarly, for the bypass load,

$$k_t = 1 + C (k_{te} - 1) , \quad (3.3.18)$$

with k_{te} being given here by eq. (3.3.10), for example, and not to be confused with the expression for k_{te} in eq. (3.3.1) that refers to bearing loads. The linear form of eq. (3.3.16) has been confirmed by many series of tests, one of which is depicted in figure 3.3.32. It has also been predicted by detailed analytical studies in reference 3.3.9.

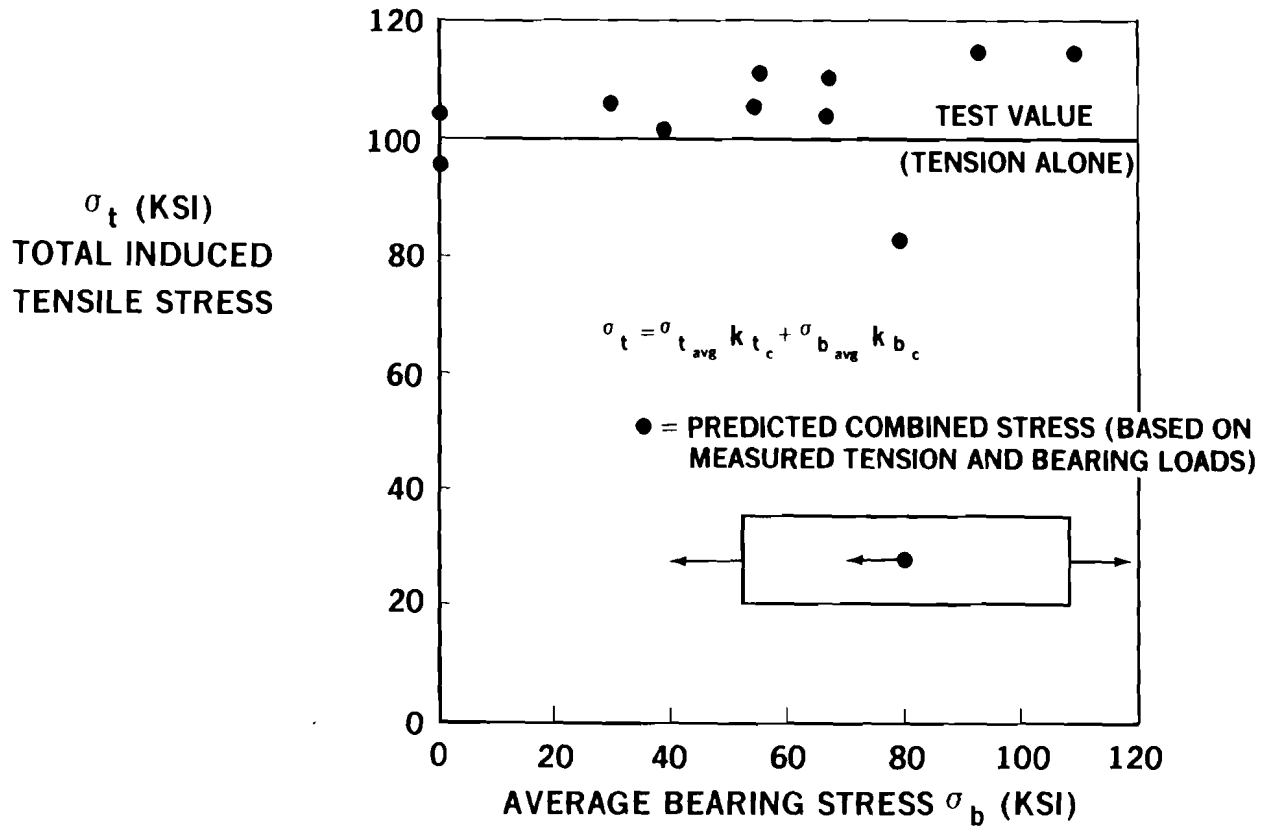


FIGURE 3.3.32. TEST/THEORY CORRELATION (BEARING-TENSION INTERACTION AT BOLT HOLE IN COMPOSITE)

Now, even if the w/d ratio is too large to exhibit a tensile failure under a pure bearing load, it will still do so for the bypass load and, by inference, for combinations of bypass and small bearing loads. The bilinear interaction in figure 3.3.31 is established as follows, considering the tension failure line extended to intercept the bearing load axis at an elevated intercept given by

$$\sigma_b = F_{tu} / k_b . \quad (3.3.19)$$

The intercept on the bypass load axis is given by

$$\sigma_t = F_{tu} / k_t . \quad (3.3.20)$$

The bearing-strength cut-off is given by

$$\sigma_b = (F_{tu} - k_t \sigma_t) k_b \leq F_{br} \quad (3.3.21)$$

and is shown by the horizontal line at the top of figure 3.3.31. The analytical studies in reference 3.3.9 indicate that perhaps the cutoff might have a small slope but that is not an issue of real concern. The most efficient joints have, for their most critical conditions, one bolt subjected to a pure bearing load with no bypass load in combination with the minimum bearing load. Maximum joint strength is attained by so proportioning the joint as to cause tension, rather than bearing, failures at all bolt holes; the purpose of the bearing cutoffs is just to prevent the spurious mathematical prediction of physically impossible joint strengths and failure modes.

The corresponding interactions for compressive, rather than tensile, loading have a slightly different form, as indicated in figure 3.3.33. The sloping bearing-failure characteristic at the top left of that figure is expressed by a constant maximum compressive bearing strength at the bolt hole, which can be developed either entirely by bearing loads or by the combination of bearing and bypass loads. Thus

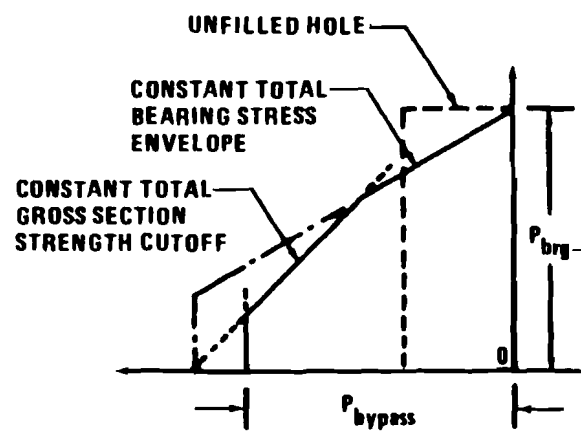
$$F_{br} = \sigma_b + \sigma_c, \quad (3.3.22)$$

in which σ_c is the gross-section compressive stress. There are no geometric coefficients. Strictly, there should be a proportionality constant on the σ_c term to allow for the greater stiffness of the fastener than of the laminate, but that would be a refinement beyond the present state of the art. The two possible vertical cutoffs on compressive bypass load follow from the application of ultimate compressive load F_{cu} to either the gross section for a tight-fit bolt or to the net-section for a loose-fit bolt. Normally, the former would prevail.

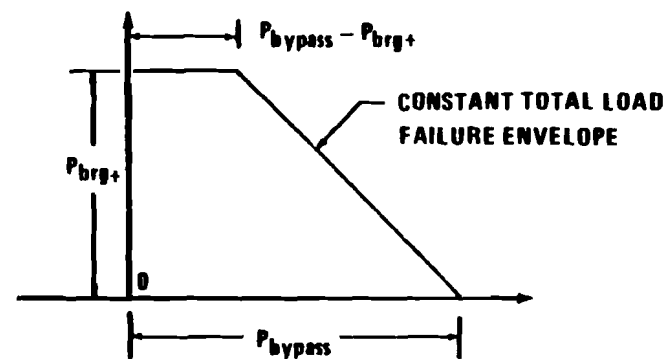
Neither the theory nor testing have yet developed to the point at which such interactions could be developed for the transfer of in-plane shear load through multi-row joints.

It is not necessary to know the precise nature of the load sharing in a multi-row bolted joint to assess the possible structural efficiencies; this can be done by simply postulating various combinations of bearing and bypass load and examining each bolt hole separately. This has been done in figure 3.3.34 for 1/4 inch holes in quasi-isotropic high tensile strength (HTS) carbon-epoxy laminates. Similar diagrams have been prepared for other bolt diameters and fiber patterns and all have the same features and surprisingly similar forms.

The salient features of figure 3.3.34 are as follows. The usual design range for d/w or d/p is about 0.15 to 0.35, so the remainder of the diagram can be given less attention. The best single-hole joints, at a w/d of 3, are surprisingly efficient, with gross-section strains on the order of 0.0035. Single-hole joints with a larger w/d , to enforce a bearing failure, are markedly weaker. Somewhat surprisingly, the strongest joint in which tension failures can be suppressed (to confine the spread of damage) contains only one row of bolts, at a w/d of just over 4. The only way that multi-row joints can improve on the efficiency



COMPRESSION



TENSION

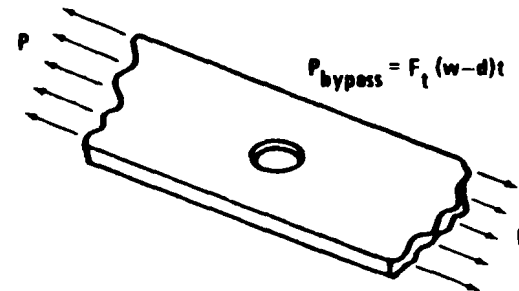
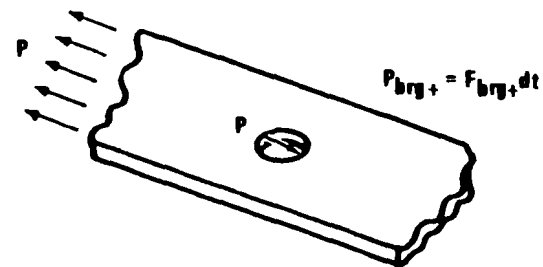
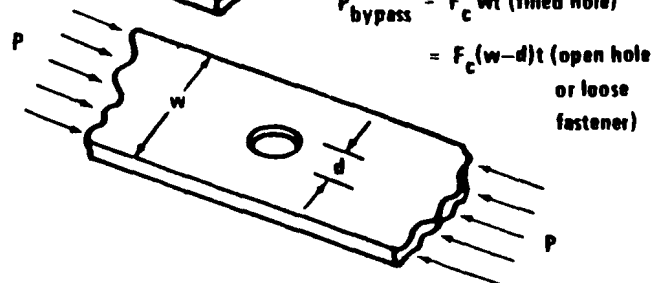
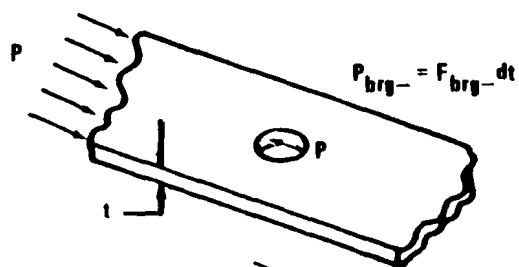


FIGURE 3.3.33. OUTER ENVELOPE OF BEARING-BYPASS LOAD INTERACTIONS

of the best single-row joints is by increasing the bolt spacing and simultaneously substantially decreasing the bearing stress. Normally, these would be mutually exclusive goals. Fortunately, only the outermost rows of fasteners are usually potentially critical and the task is not as ominous as it seems. Successful designs and test results are described in reference 3.3.3. Nevertheless, the design of efficient multi-row bolted joints suitable to overcome the challenges posed in figures 3.3.3 and 3.3.4 is not possible without an accurate analysis with which to establish the load on each individual fastener. Such an analysis is coded in the A4EJ computer program (reference 3.3.4). By this means joints have been tested to failure at strains of 0.005 and load intensities on the order of 50,000 pounds per inch. Figure 3.3.34 also confirms that simply adding a second row of bolts to the best single row joints to halve the bearing stress, would attain a strength increase of only ten percent, as observed by test in reference 3.3.1. Designing really efficient bolted joints in advanced composite structures is still very much an art, even though the scientific investigation of the problem has now provided many guidelines.

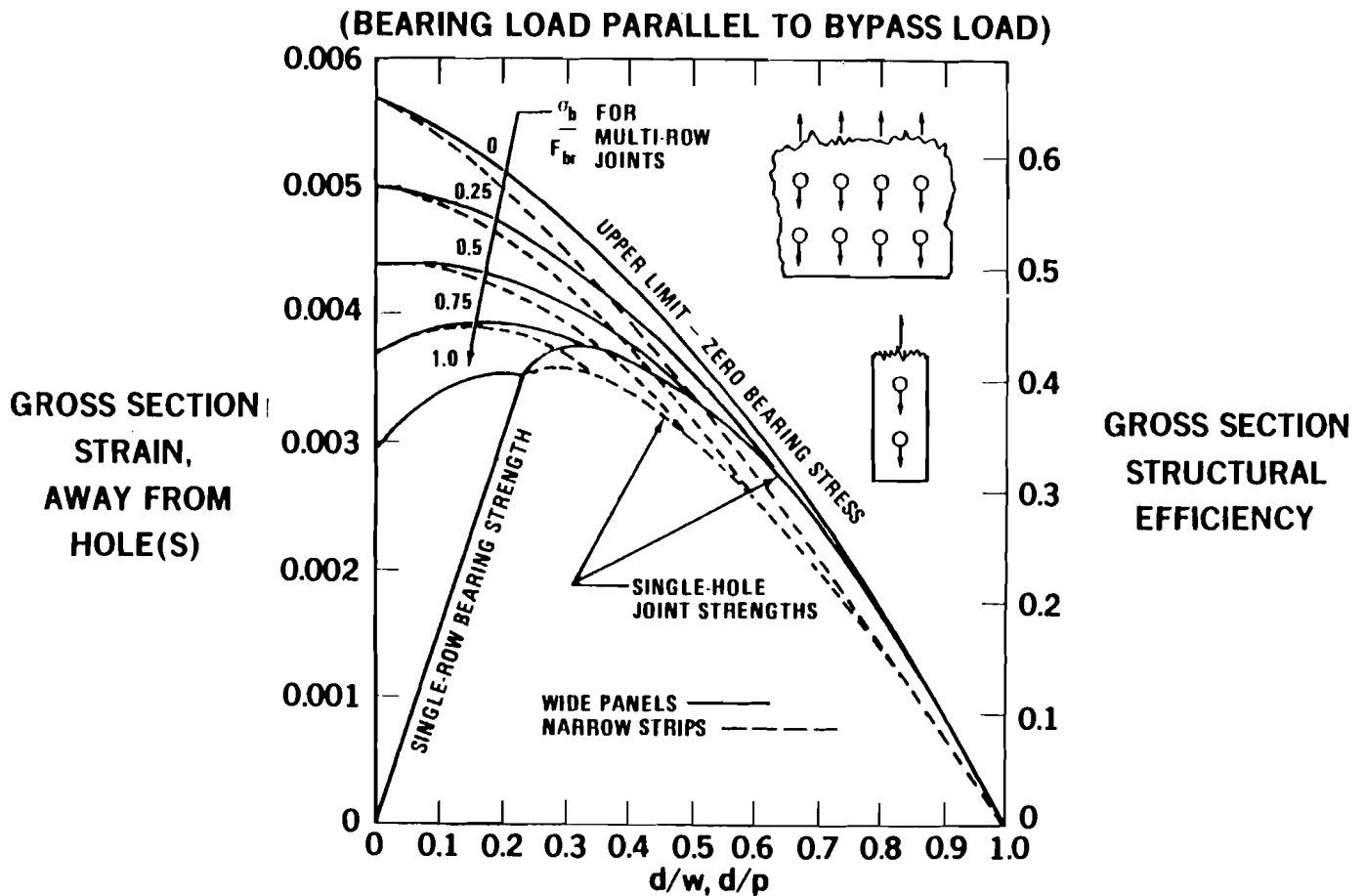


FIGURE 3.3.34. STRESS CONCENTRATION INTERACTIONS IN MULTI-ROW BOLTED COMPOSITE JOINTS

Figure 3.3.34 also makes it possible to assess the appropriateness of simplified design procedures for bolted composite joints. The basis of the design procedure for many projects is typified in figure 3.3.35. The exact values of the intercepts vary, but those bearing and bypass stresses shown are representative of the state-of-the-art for structures designed in the early 1980's. Figure 3.3.35 lacks any consideration of w/d effects and comparison with figure 3.3.34 would suggest that figure 3.3.35 is more appropriate for larger w/d ratios than the smaller ones that would cause a tension failure, even at high bearing stresses. Nevertheless, the lines shown in figure 3.3.34 identified by A and B, which correspond with points A and B in figure 3.3.35, indicate that the simplified procedure is reasonable for general purpose designing. The simpler procedure of figure 3.3.35 is usually conservative. The more difficult high-load-transfer joints employing design strains in excess of 0.004 require a more sophisticated analysis. Furthermore, the only regime in which the simplified method could be significantly nonconservative - that for single-row joints at too large a bolt pitch - is easily precluded by a secondary check on the bearing strength, as given by the radial line from the origin in figure 3.3.34. In short, figure 3.3.35 is useful to guide the design of the majority of bolted joints in composite structure.

3.3.9 Load Sharing in Multi-Row Bolted Composite Joints.

The use of the characterization of the structural efficiency of multi-row bolted joints that is shown in figure 3.3.34 requires the determination of the bearing and bypass loads on each fastener in the joints. The bypass load will inevitably decrease steadily from the first row of bolts to the last, but the bearing loads on the bolts can peak at one or both ends or the middle, depending on the proportions and stiffnesses of the joint.

The portion of a typical multi-row bolted shown in figure 3.3.36, extending between bolt stations (k) and (k+2) shows most of the features that are included in the load-sharing analysis that is documented in reference 3.3.4. The member properties can vary along the length of the joint by a step-by-step approximation. Running shear load, prescribed to be uniform, can act on either or both members. The fasteners can be loaded in single shear, as shown, or double shear. Provision is included for the thermally induced stresses and strains associated with the joining of different materials. The solution consists of the very simple expressions for equilibrium of each element of each member, as well as of each fastener, in conjunction with expressions for the compatibility of deformations. The bolt load is defined in terms of the relative motion between the members. The entire analysis is coded as the A4EJ Fortran IV computer program which is documented in Volume 11 of reference 3.3.4. The solution can be entirely linear, or nonlinear, depending on the materials or load intensity. Composite members are prescribed to have linearly elastic behavior until failure, except for local deformations at each fastener, which are included in the fastener stiffness characteristic. Metal alloy adherents are characterized by the Ramberg-Osgood nonlinear formulation (reference 3.3.10). The fastener characteristic is as shown in figure 3.3.37 and consists of two straight lines in combination with the possibility of initial clearances, as shown, or preloads due to misalignments at other holes.

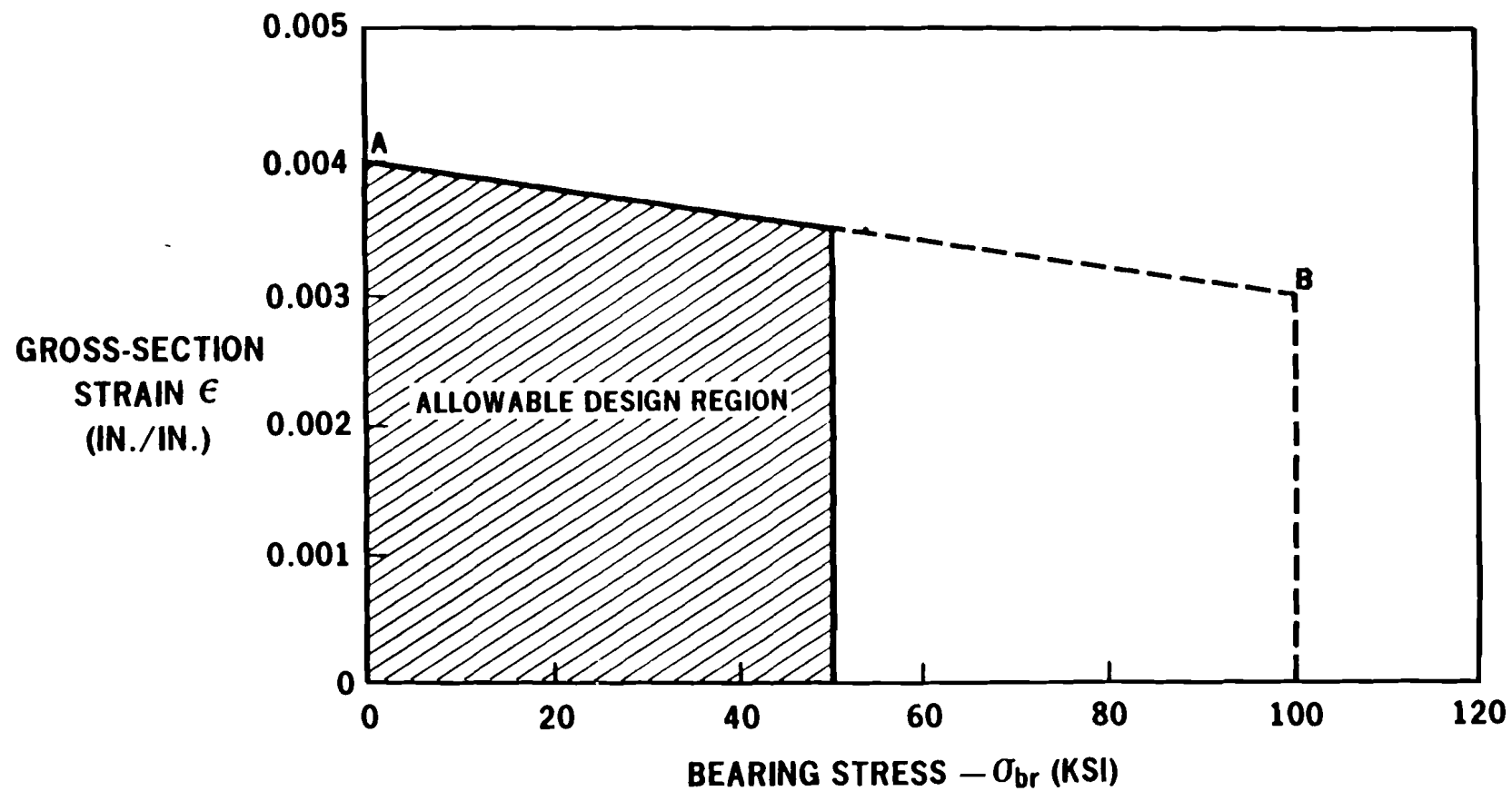


FIGURE 3.3.35. DESIGN TECHNIQUE FOR BOLTED GRAPHITE-EPOXY STRUCTURES

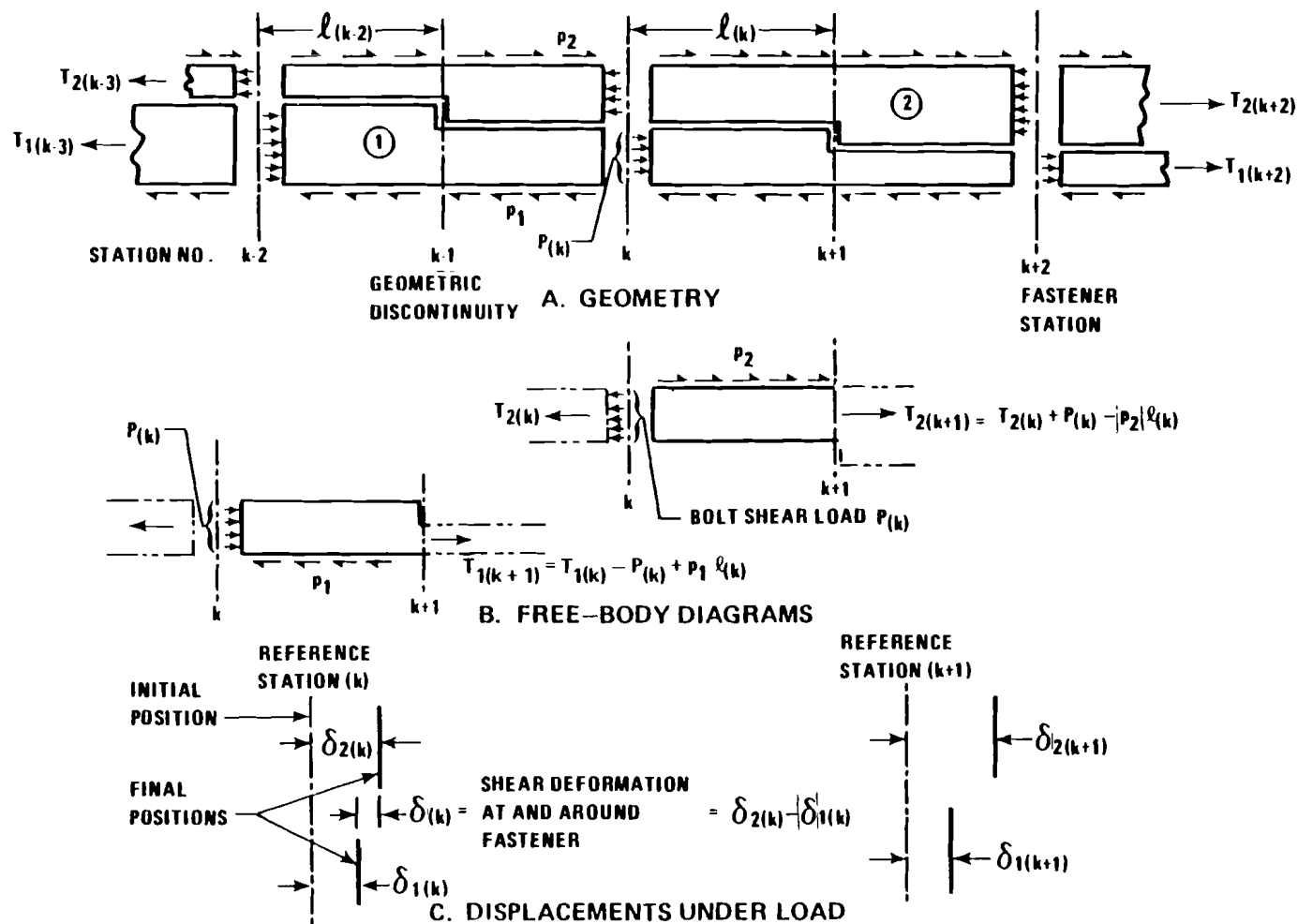


FIGURE 3.3.36. LOADS AND DEFORMATIONS ON ELEMENTS OF BOLTED JOINTS

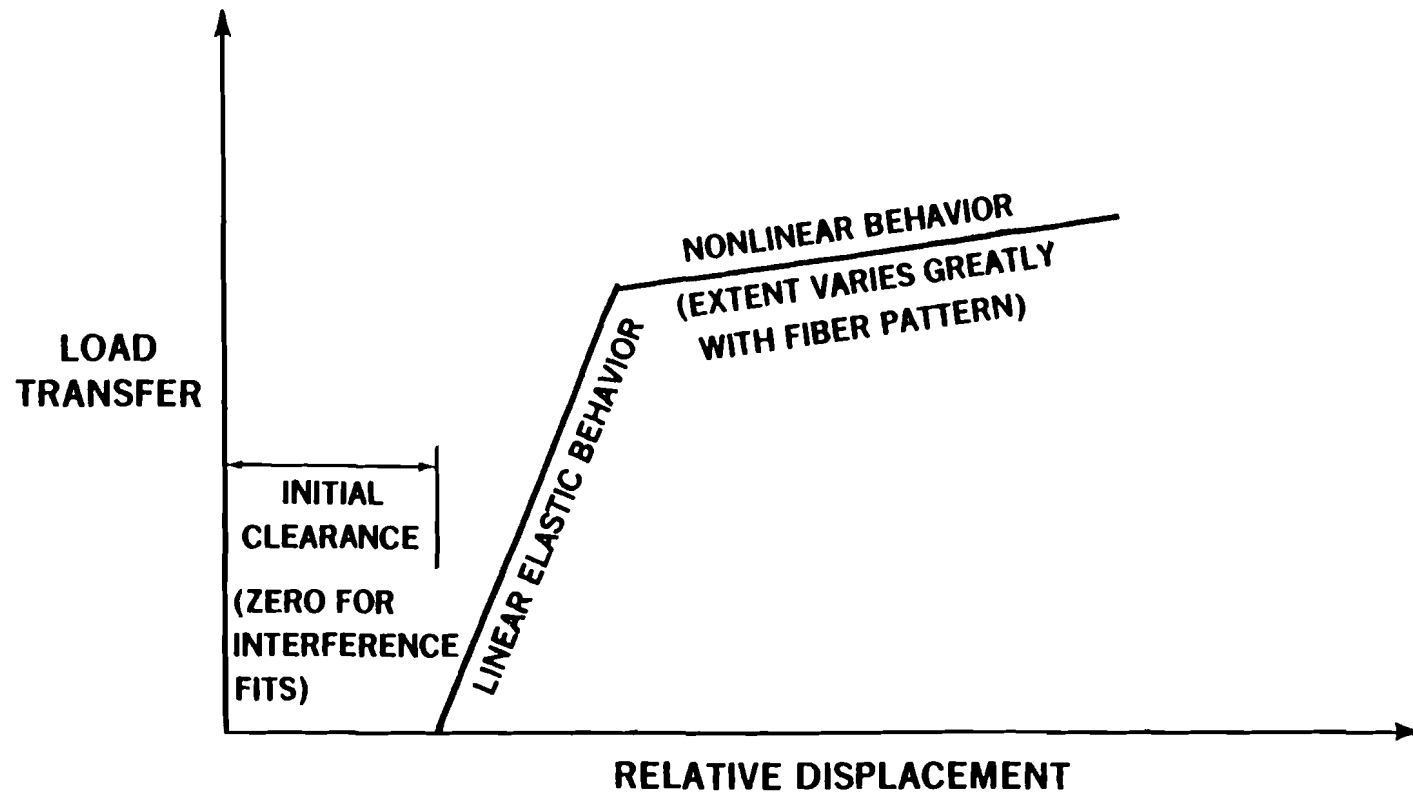


FIGURE 3.3.37. FASTENER LOAD-DEFLECTION CHARACTERISTICS

Provision is made for different strengths and nonlinear behaviors between tensile and compressive shear loads. The elastic stiffness is prescribed to be the same for each case and can be expressed by a formula in reference 3.3.3 for double shear loads. Figure 3.3.38 shows the entire fastener load-deflection mode. That deflection includes fastener rotation (in single shear) and bending, as well as local distortion of the cross-section as shown in figure 3.3.39. The instrument with which such characteristics are recorded is described in reference 3.3.11. Typical curves are shown in figure 3.3.28.

The mathematical description of the load transfer characteristic in figure 3.3.38 is as follows:

$$P = 0 \quad \text{for } \delta_{o-} \leq \delta \leq \delta_{o+} , \quad (3.3.23)$$

$$P = K_{el}(\delta - \delta_{o+}) \quad \text{for } \delta_{o+} \leq \delta \leq \delta_{el+} , \quad (3.3.24)$$

$$P = P_{el+} \quad \text{for } \delta = \delta_{el+} . \quad (3.3.25)$$

$$P = P_{el+} + K_{pl+}(\delta - \delta_{el+}) \quad \text{for } \delta_{el+} \leq \delta \leq \delta_{pl+} , \quad (3.3.26)$$

$$P = P_{ult+} \quad \text{for } \delta = \delta_{pl+} , \quad (3.3.27)$$

$$P = -K_{el}(\delta_{o-} - \delta) \quad \text{for } \delta_{el-} \leq \delta \leq \delta_{o-} , \quad (3.3.28)$$

$$P = P_{el-} \quad (-ve) \quad \text{for } \delta = \delta_{el-} , \quad (3.3.29)$$

$$P = P_{el-} - K_{pl-}(\delta_{el-} - \delta) \quad \text{for } \delta_{pl-} \leq \delta \leq \delta_{el-} , \quad (3.3.30)$$

and

$$P = P_{ult-} \quad (-ve) \quad \text{for } \delta = \delta_{pl-} . \quad (3.3.31)$$

The negative relations are needed both for compressive external loads and to cover the possibility of internal (self-equilibrating) loads induced by thermal mismatch between dissimilar members fastened together at one temperature and operated at a different temperature. They are also needed in the iterative analysis, described below, to cover those intermediate analyses prior to convergence.

Load transfer due to friction associated with tightly clamped assemblies can easily be accommodated in such a model. However, particularly in the case of fibrous composite with a resin matrix, the reliance upon fractional load transfer in long-life aerospace structure is questionable. The resin would creep to relieve the preload. Also, in the many structures which have countersunk fasteners to achieve a flush exterior, there is often not much clamp-up possible even in the short term. Leaving aside the questions of estimating the magnitude of any fractional load transfer, the modifications to eqs. (3.3.23) to (3.3.31) would be

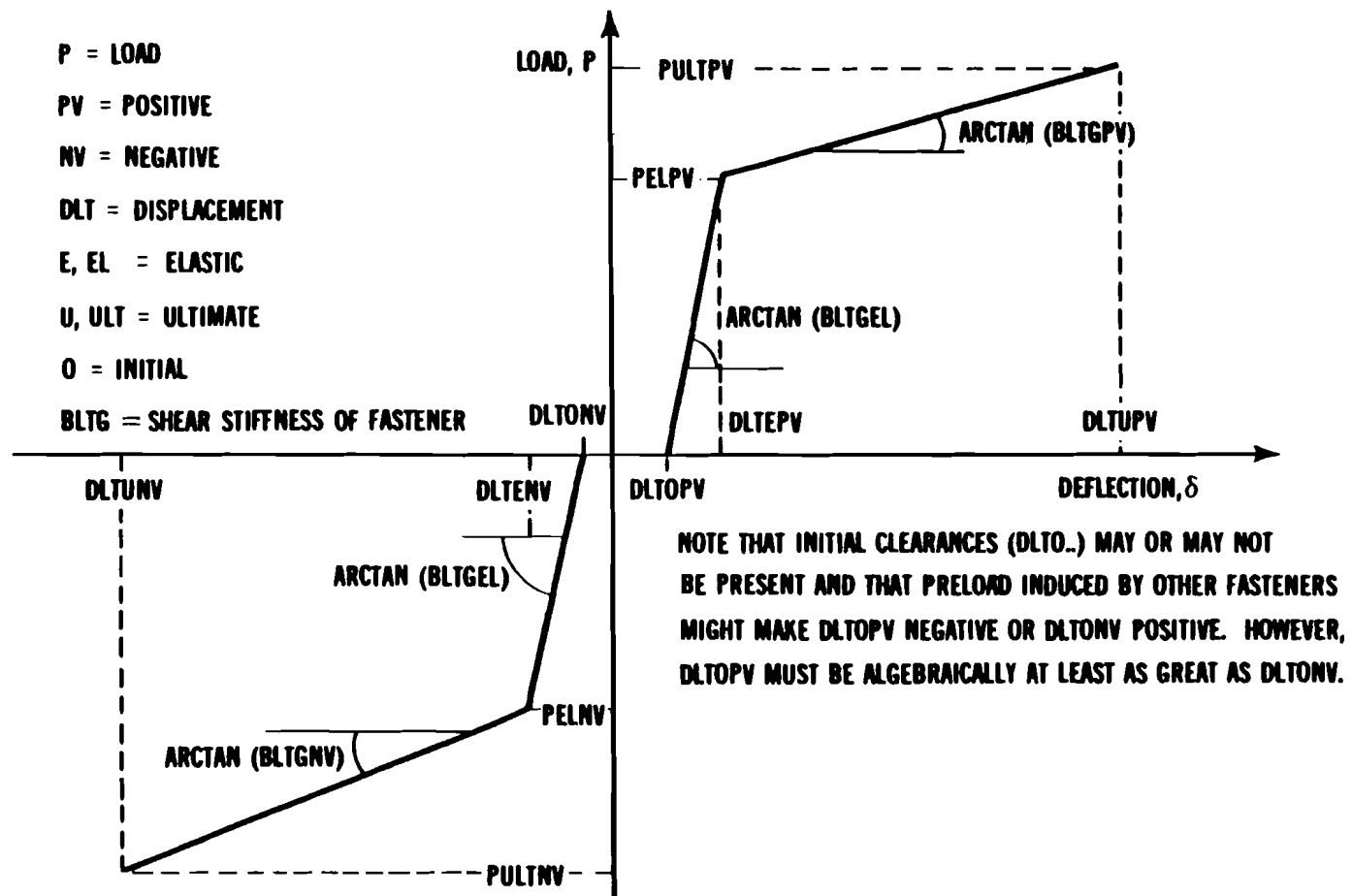


FIGURE 3.3.38. IDEALIZED FASTENER LOAD-DEFLECTION CHARACTERISTICS

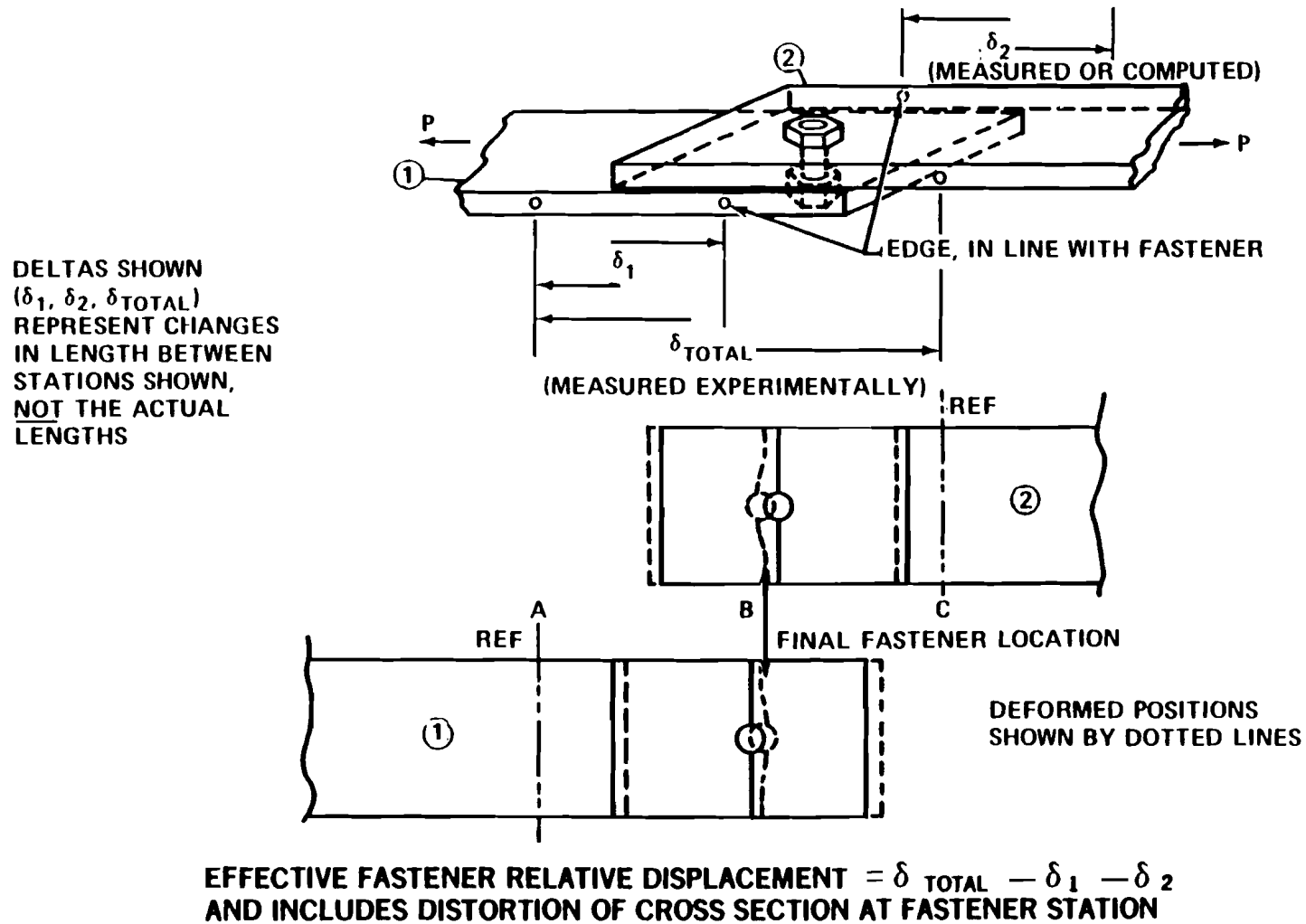


FIGURE 3.3.39. DEFORMATIONS IN MECHANICALLY FASTENED JOINTS

$$P = P_{fr} = \text{constant} \quad \text{for } 0 \leq \delta \leq \delta_{o+} \quad (3.3.32)$$

in place of eq. (3.3.23), and

$$P = P_{fr} + K_{el}(\delta - \delta_{o+}) \quad \text{for } \delta_{o+} \leq \delta \leq \delta_{el+} \quad (3.3.33)$$

instead of eq. (3.3.24). Likewise, eq. (3.3.32) would be complemented by

$$P = -P_{fr} \quad \text{for } \delta_{o-} \leq \delta \leq 0 \quad (3.3.34)$$

and eq. (3.3.28) would become

$$P = -P_{fr} - K_{el}(\delta_{o-} - \delta) \quad \text{for } \delta_{el-} \leq \delta \leq \delta_{o-} . \quad (3.3.35)$$

The other eqs. would remain unchanged.

The preceding discussion concerns the load-transfer characteristics of either a single-fastener joint or of one fastener isolated out of a multirow joint. The subject of calculating just how the load is shared between multiple fasteners when there are redundant load paths is discussed below. After that, it is necessary to consider the failure criteria, station by station, to identify just where any failure would occur and at what load.

A variety of analysis techniques is available for the solution of the problem of load sharing between multirow fasteners. Most are rendered unsuitable for this task because of a restriction to linear behavior. The method used for the A4EJ program was chosen because it is compatible with the equivalent nonlinear adhesive-bonded analysis A4EI, with which it was later integrated, to produce the A4EK program for bonded-bolted joints.

The known boundary conditions are at both ends of the joint, while the unknown conditions are likewise not confined to one end. Therefore, in solving the problem iteratively, it is appropriate to start at one end of the joint (preferably that which is more critically loaded to improve accuracy) and to assume a value for some unknown there. The assumption may be of the total joint strength or the displacement at the first fastener (or the edge of the joint) induced by a specified load. By then calculating progressively along the joint, satisfying both equilibrium and compatibility requirements, one can interpret the consequent predictions at the other end of the joint. The initial assumption can then be modified as necessary until all boundary conditions are satisfied at both ends of the joint. Actually, unless the initial assumption is very close, or has been refined by prior iterations, the progressive calculations will diverge and the appropriate change to the previous estimate is recognized in terms of the nature of the divergence.

The establishment of the typical eqs. for each step of the joint is explained in figure 3.3.36. The conditions of equilibrium are that, for member 1 between stations (k) and (k+1)

$$T_{1(k+1)} = T_{1(k)} - P_{(k)} + p_1 \ell_{(k)} \quad (3.3.36)$$

in which T_1 is the member internal load, being positive for tensile loads, P is the shear load transfer by that fastener and is positive for tensile lap shear, as shown. The remaining term $p_1 \ell$ is usually absent from test coupons but represents the running shear load that may be present in such real structures as the splice at the root of a wing under bending loads. The sign convention for p_1 , and p_2 likewise, is that each load is taken to be positive when it acts in the same direction as a tensile load in the member outside the joint area. Similarly, for the same element of member 2,

$$T_2(k+1) = T(k) + P(k) - p(k) \cdot \quad (3.3.37)$$

Figure 3.3.36 illustrates a single-shear application. In a double-shear joint, the two portions which would make up members 1 or 2 would be combined and the fastener load transfer P would be changed from single to double shear values. The form of eqs. (3.3.36) and (3.3.37) would not be altered. Similarly, in what follows, the subscripts 1 and 2 refer to any appropriate summation of properties as well as of loads.

To ensure that the analysis complies with compatibility requirements, one must use the mechanical (and possibly thermal) properties of the members. Because of uniqueness requirements, it is mathematically not permissible to use any ideal elastic-plastic material characterization, even though that could sometimes be done for the fasteners, in terms of figure 3.3.37. For fibrous composite materials, a linearly-elastic gross-material behavior will suffice but, for ductile materials like aluminum alloys, one must include some nonlinear behavior to permit loading beyond the material proportional limit. The stiffness of each segment of the joint can be calculated to any level of accuracy desired but any improvement in accuracy beyond what can be achieved by the well publicized Ramberg-Osgood model or even a two-straight-line (bilinear) model is likely to be very small and not worth the added complexity. The nonlinearity of purely +45° fibrous composites can be represented by the Ramberg-Osgood model. Other fiber patterns, containing some 0° fibers aligned with the load direction, can be treated as linearly elastic to failure.

The extension of the members between stations (k) and $(k+1)$ is given by

$$\delta_1(k+1) - \delta_1(k) = \alpha_1 \Delta T \ell(k) + \epsilon_1 \ell(k) \quad (3.3.38)$$

and

$$\delta_2(k+1) - \delta_2(k) = \alpha_2 \Delta T \ell(k) + \epsilon_2 \ell(k) \quad (3.3.39)$$

since for uniformly stepped members, the strain can be considered to be uniform between stations, except for the local deformation around the fasteners, which is accounted for in eqs. (3.3.23) to (3.3.31). In eqs. (3.3.38) and (3.3.39) the thermally-induced strains are derived from the temperature differential

$$\Delta T = T_{\text{operating}} - T_{\text{assembly}} \quad (3.3.40)$$

and the mechanically induced strains ϵ_1 and ϵ_2 are deduced on the basis of the member loads and the material behavior.

Any running load is considered to be applied uniformly along the length of the joint and, in calculating the strains in eqs. (3.3.38) and (3.3.39), the stretching is taken to be that which would be associated with the average member load in each segment. Since the fastener loads are treated as step discontinuities at each station, it follows from eqs. (3.3.36) and (3.3.37) that the average loads causing the stretching of each member between stations (k) and (k+1) are

$$\overline{T_1(k)} = T_1(k) - P(k) + p_1 \ell(k)/2 \quad (3.3.41)$$

and

$$\overline{T_2(k)} = T_2(k) + P(k) - p_2 \ell(k)/2 . \quad (3.3.42)$$

The corresponding stresses are

$$\sigma_1(k) = \overline{T_1(k)} / w_1(k) t_1(k) \quad (3.3.43)$$

and

$$\sigma_2(k) = \overline{T_2(k)} / w_2(k) t_2(k) . \quad (3.3.44)$$

For linearly elastic materials, such as fibrous composites, the equivalent strains would be

$$\epsilon_1(k) = \sigma_1(k) / E_1(k) \quad (3.3.45)$$

and

$$\epsilon_2(k) = \sigma_2(k) / E_2(k) . \quad (3.3.46)$$

In terms of the Ramberg-Osgood material characterization for ductile materials,

$$\epsilon_i = \frac{\sigma_i}{E_i} + \frac{\sigma_i^n}{(E F_{0.7}^{n-1})_i} = \frac{\sigma_i}{E_i} \left[1 + \left(\frac{\sigma_i}{F_{0.7}} \right)^{n-1} \right], \quad (i = 1, 2). \quad (3.3.47)$$

Care must be exercised in using the formulations in eq. (3.3.47) because some of the published mathematical models do not agree precisely with the material properties given in MIL-HDBK-5. The discrepancies are usually small but it is necessary to input into the solution, strength and failure properties at each station which are precisely consistent with eq. (3.3.47).

Having determined the member strains, per eqs. (3.3.45) to (3.3.47) as appropriate, the relative displacement between the members, at the next station, follows from eqs. (3.3.38) and (3.3.39) as

$$\delta_{(k+1)} = \delta_{2(k+1)} - \delta_{1(k+1)} \quad (3.3.48)$$

as shown in figure 3.3.36. This, in turn permits a new increment of fastener load transfer to be evaluated on the basis of eqs. (3.3.23) to (3.3.31).

Actually, for the example shown in figure 3.3.36, there would be no transfer of load at station (k+1). Therefore, between stations (k+1) and (k+2), eqs. (3.3.36) and (3.3.37) would be modified to read

$$T_1(k+2) = T_1(k+1) + p_1 \ell_{(k)} \quad (3.3.49)$$

and

$$T_2(k+2) = T_2(k+1) - p_2 \ell_{(k)}. \quad (3.3.50)$$

However, eqs. (3.3.38) through (3.3.48) would be applied in precisely the same manner for the next increment of the joint.

The bearing-bypass characteristics, with which the program is coded, are described in figure 3.3.33. Such properties vary from station to station and are part of the required input data. The program solves by iteration for either the load at which the joint becomes critical at some station, or for the internal stress distribution associated with a specified external load. The program is coded only for tensile or compressive shear loads. There is no provision for in-plane shear loads yet, mainly because of the absence of bearing/bypass interaction failure data. The input data could certainly be modified to yield the internal loads for a specified externally applied in-plane shear load, in much the same way that the A4EI adhesive-bonded joint program handles in-plane shear loads.

3.3.10 Optimization of Multi-Row Bolted Composite Joints.

The analysis program A4EJ has been used in reference 3.3.3 to analyze and predict the failure loads for many different composite bolted joint specimens. Apart from a minor problem, whereby the bearing strength of composite splice plates was found to be less than that of the skin sandwiched in between - which required reanalysis - the predictions were in good agreement with the test results in most cases. The only significant discrepancy was associated with premature failure at spot faces in tapered splice plates where, in the future, tapered washers will be used.

The same program was used to perform an analytical assessment of the strengths and weaknesses of four different joint geometries. The result of these analyses are shown in figure 3.3.40 for four different four-row bolted joints.

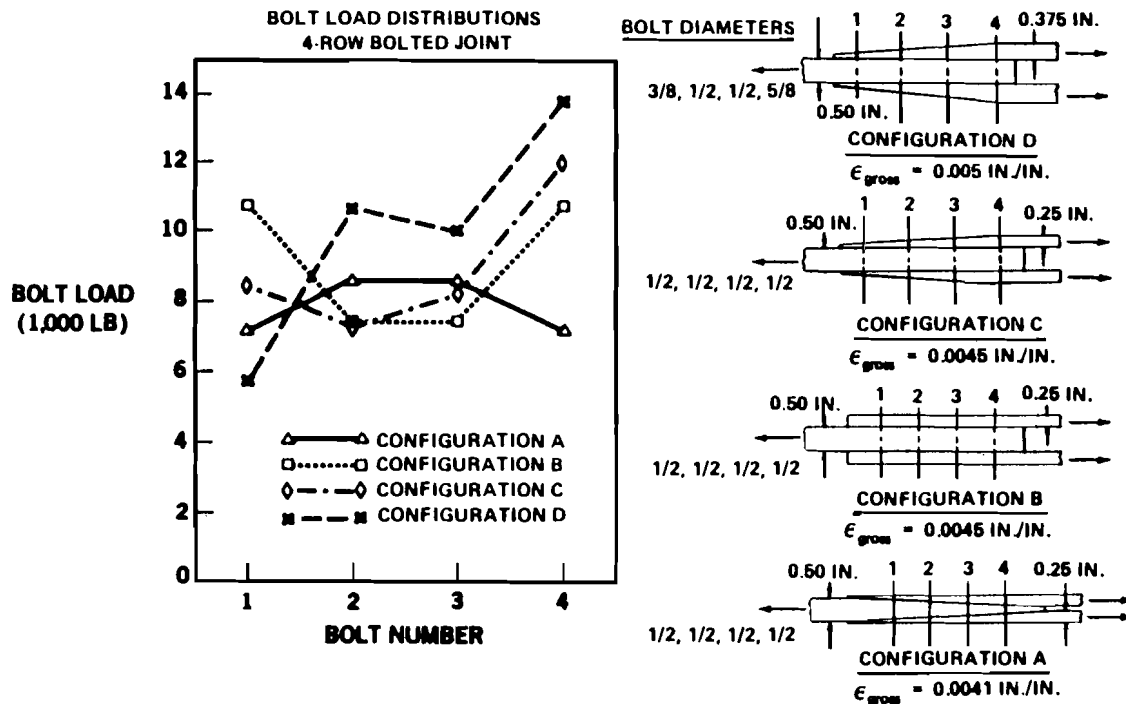


FIGURE 3.3.40. EFFECT OF JOINT CONFIGURATION ON BOLT LOAD DISTRIBUTION

Despite the theoretical appeal of the scarf joint, configuration A, that was the weakest joint of all. The prime reason is the reduction in thickness of the members between the ends of the overlap and the outermost rows of fasteners. Note how the load transfer is highest for the middle two fasteners, because of the much greater flexibility of thin ends of the tapered members that, in turn, softened the stiffnesses of the outermost bolts. The baseline case of uniform adherents, configuration B, was predicted to have the next to highest strength. It had obvious potential for improvement because the bolt load distribution was such that, at the outermost bolts, the peak bearing and bypass loads were combined. Configuration C, which has tapered but unreinforced splice plates, was slightly weaker than the baseline because the tapering reduced the loads transferred at the left end without any change in the conditions at the right end of the joint. The strongest joint was configuration D, in which the splice plate

was tapered to reduce the load transferred at the first row of fasteners (at which station the bypass load was highest) and simultaneously the splice plates were thickened to permit the maximization of the load transfer at the last row of fasteners (where there was no bypass load left in the skin).

The maximization of the total load transferred by the combination of the maximum bypass load with the maximum bearing load in the skin at the most critical station, the first, is quite evidently consistent with the assessment in figure 3.3.34. The consequent reinforcement needed for the splice plate, where at the other end of the joint the bearing and bypass loads in the splice plate are both maximized at the same station, is a small penalty to pay for the increased structural efficiency of the entire skin, which is very much larger. Note that the fastener diameters were varied to assist in this refinement of the joint strength.

What is important about these findings is that the strongest joint has the simplest skin, because it is uniform and unreinforced. All of the sculpturing is confined to the very much smaller splice plates, that are easier to modify. That this configuration should be the strongest should not be the least surprising; it has been standard practice in metal aircraft structures for many years and is used on the DC-10 wing skin splices at the side of the fuselage.

3.3.11 Fatigue of Bolted Composite Joints.

The biggest difference between bolted joints in metal alloys on the one hand, and fibrous composites, on the other, is that those factors that influence the fatigue life of the metals, influence the static strength of the composites, instead. As a general rule, those design details used to enhance the fatigue lives of bolted metal structures are necessary to maximize the static strength of bolted composite structures. There is no all-forgiving massive yielding to better redistribute the loads in composite structures.

Fibrous composite structures are renowned for excellent fatigue performance. This is the result of their heterogeneous two-phase composition. Continuous cracks do not grow through the thickness of composite laminates the way they grow perpendicular to the principal stress in metals, each fiber must be broken individually and none are notched by the fracture of any adjacent fiber. Indeed, even the overloading next to a broken fiber is first alleviated by failure at the resin-to-fiber interface.

Perhaps the closest similarity between the fatigue of metals and composites is the growth of cracks in metal alloys under tension-tension loading and the growth of delaminations in fibrous composite laminates under cyclic compressive loading. However, in the latter case, the rate of growth from subcritical to critical size is so rapid that the behavior is better described as a threshold rather than stable growth problem. The life will be virtually infinite, provided that the loads are low enough not to open up the periphery of the delamination to make it spread.

It should be surprising, therefore, if fibrous composite exhibited stimulate the classical S-N curves so characteristic of the fatigue of metals.

The fatigue behavior of bolted composite joints is characterized in figure 3.3.41, which looks anything but a classical S-N curve. Each form of behavior is related to specific conditions about each joint. The rapid loss of strength shown for the loose-fit bolt is exacerbated by any load reversal, with the bolt pounding back and forth in the hole. The loss is far less severe for joints not subject to load reversal. The more normal behavior particularly for net or some interference fit fasteners is that the strength increases asymptotically as the internal delaminations grow and relieve the stress concentrations. That enhancement of strength has not been observed when the laminate was damaged by the installation of the interference fit bolts. Damage can occur either by pounding the bolt home or by twisting it in; in either case the ends of the fibers in contact with the bolt can be bent and separated from the resin matrix. The best way to install an interference-fit bolt is to use an undersize sleeve that is expanded to fill the hole properly as bolt is installed. If each hole is filled slightly, there may never be a decrease in residual strength, as suggested in figure 3.3.41. If there is wear out it is probably the result of a particularly severe load spectrum or of unusually efficient bolted joints that have permitted the composite to be strained much higher (at limit load) than the more normal 0.002 fiber strain.

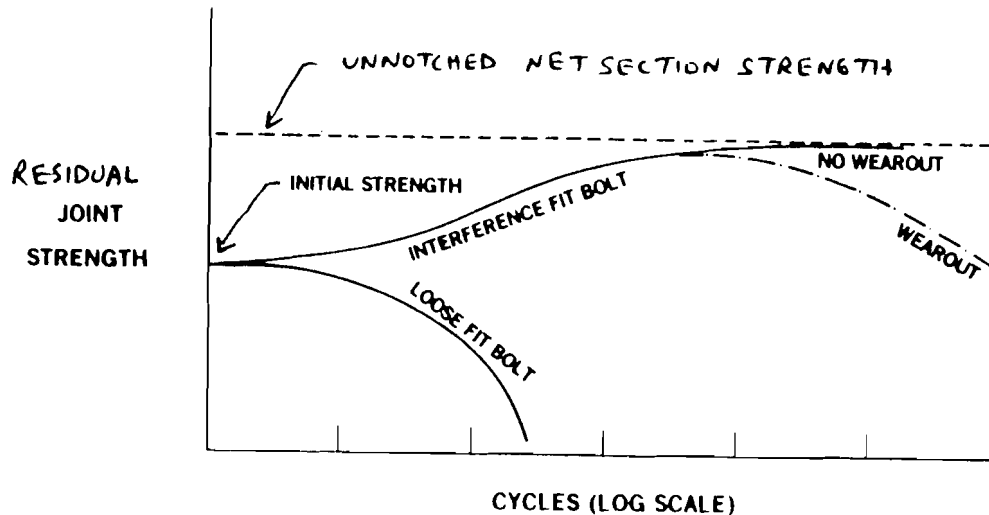


FIGURE 3.3.41. FATIGUE OF NOTCHED COMPOSITES OR BOLTED COMPOSITE JOINTS

3.3.12 Conclusion.

This chapter has provided an explanation of various aspects of mechanically fastened joints in fibrous composite laminates. The topics covered include:

1. a description of observed behaviors
2. an explanation of the physical mechanisms of behavior within the laminates
3. theoretical predictions of the behavior of perfectly elastic bolted joints

4. empirically derived correlation factors with which to predict the actual behavior using the theoretical elastic behavior as a reference
5. some measured strengths for joints in a variety of fiber patterns in carbon-epoxy laminates
6. an indication of the best fiber patterns to use in bolted composite joints
7. the identification of optimum joint geometries with which to maximize the strength of composite laminates containing bolt holes
8. the interaction between bearing and by-pass loads at each bolt in multi-row joints
9. the prediction of the strength of multi-row joints on the basis of tests on only single-bolt coupons
10. an outline of the minimum series of tests with which to characterize the behavior of bolted composite joints, and
11. an explanation of the phenomenon associated with the repeated loading of bolted joints in fibrous composite structures

The intent of this chapter has not been to catalog a mass of uninterpreted data, of which much would be rendered obsolete by newer materials. Rather, the emphasis has been placed on explaining why fibrous composite behave the way they do around bolt holes. That understanding can be translated into the design and analysis of efficient bolted joints in any composite material, given only a minimum of test results from a brief series of experiments which have been described herein.

Bolted joints impose significant limits on the strength of fibrous composite structures and the use of the more efficient joints, described here, is necessary if large primary composite structures are to compete with well-designed metal alloy structures.

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CHAPTER IV. COMPOSITE STRUCTURES

4.0 INTRODUCTION.

The design of a structural component requires configuration of the available material into a shape or system of shapes which will perform the intended function of the structure for the specified length of time. The design process itself is complex because it is a function of the specific problem and of the philosophy of the particular design organization. In this chapter, the design process will be dissected into its constituent parts for the purpose of identifying those areas where designing with composites differ from designing with metals.

Previous chapters have been devoted to the analysis of composite components and, specifically, have shown that the principal difference between composites and metals is that the properties vary significantly with direction. This difference gives rise to the requirement that the designer carry numerous material constants throughout the analysis as well as consider a large number of possible failure modes. While this does not put a significant burden on the designer, since high-speed computers can handle and keep track of the necessary data, it does, however, require him to alter his thinking process when approaching the design of a composite structure.

In this chapter, the overall design process, as it applies to both composites and metals, will be reviewed. Those areas where composites differ from metals will be focused upon in greater detail. The design process will be reviewed from the initial stages (including asking the question: what must this component do?) to the developmental stages of a preliminary design. This is followed by prototype hardware design and fabrication, and finally, design substantiation. It will be seen that, at each step of the design cycle, composites require different types and degrees of information relative to the component's function. There are also design restrictions in addition to material properties and manufacturing limitations.

The design process will be illustrated by way of examples. It will be shown that for any given problem, multiple solutions will exist. The designer's chore then becomes one of selecting a "best" configuration and material for the given problem.

Finally, this chapter will review briefly the computer codes which are currently available and which can be employed to model complex structures with composites. It will be noted that analysis techniques are continually being improved and, as composites see more use, additional analysis codes will become available. Therefore, the codes and capabilities presented here must be viewed within the timeframe of this writing.

4.1 DESIGN LOGIC.

Before discussing the peculiarities of designing with composites, one must first consider the design process in general terms. Remember that, with composites, one can design the material and structure simultaneously and that the material properties themselves become design variables which can be controlled. Herein lies a major departure from metal design.

When designing any component, the task must be approached in a step-by-step manner. These steps often blend together and become hard to distinguish. Nevertheless, at various points during the design, it is necessary to ask for or provide answers to such questions as:

What is the function of this part?
What restrictions are placed on the part?
 Dimensions, weight, stiffness, environment, etc.
What kinds of materials might work and what are their properties?
What should the part dimensions and configurations be?
Does the part function well or should it be modified?
Should this part be put into production?

The design process can be presented in flowchart form as shown in figure 4.1.1.

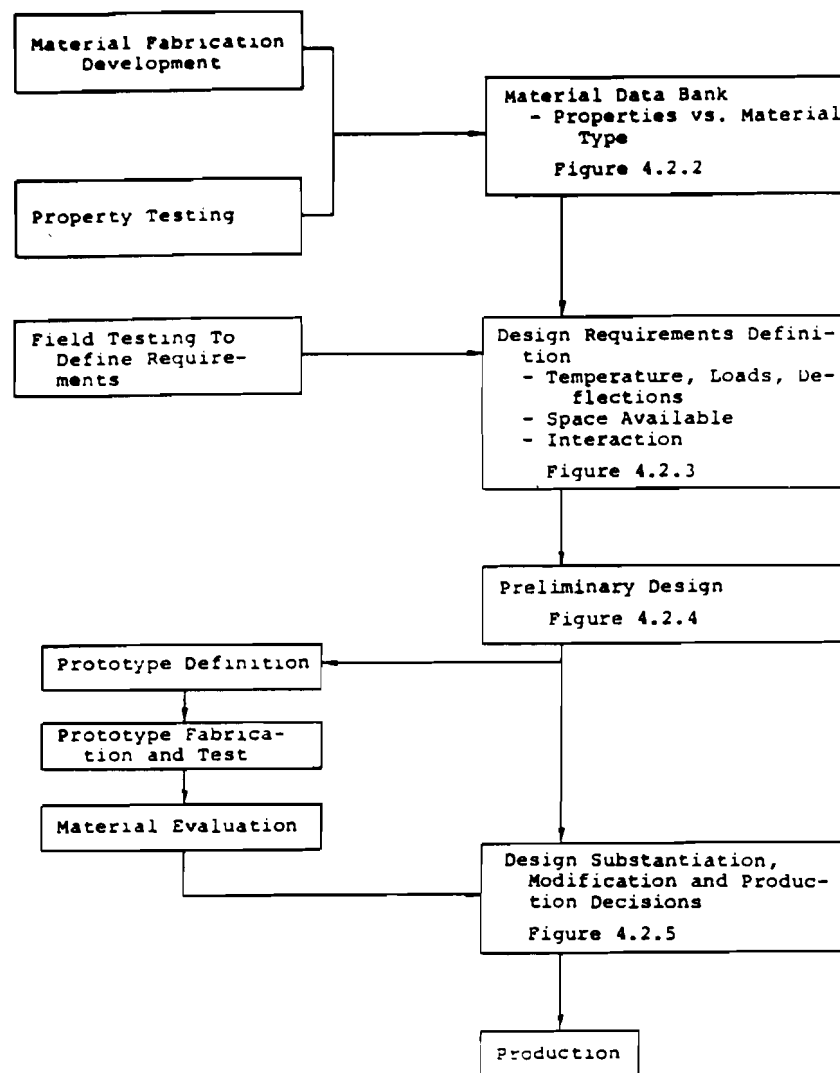


FIGURE 4.1.1. DESIGN PROCESS

Although one could argue the details of the design process shown in figure 4.1.1 the major blocks of the figure serve as an outline for the most general design process. The design process begins with the establishment of a material property data bank coupled with the analysis tools needed to synthesize variations of the known material configurations. The options include the freedom to synthesize the material in order to achieve a "best" design, or to rely solely upon material characterization data for those material configurations being considered.

A prime example of constraining the designer to a specific material configuration occurs when an organization has developed large amounts of data, particularly environmental and cyclic load effect data, for that particular material configuration. Since these properties are different to synthesize, and expensive to measure, any departure from the available material will require a substantial investment in reverifying the adequacy of an alternate material. Thus, although an alternate configuration may provide improved weight or material cost savings, the costs associated with requalifying the material may overshadow the performance benefits unless the volume of potential material usage is large.

Nevertheless, the designer should always try to define the best material configuration. Only by identifying that configuration will it be possible to identify the penalty or benefit associated with using an alternate configuration. Obviously, there would be little to be gained from an alternate material if the final result is only a few percent weight savings over a well characterized material. On the other hand, if the potential exists for a large weight savings or performance improvement, then there may be justification for embarking upon an experimental program to fully characterize an alternate material.

4.2 MATERIAL PROPERTY DATA.

Designers tend to get involved with materials at the material property level and not at the fabrication level. They frequently do not give much attention to how the manufacturer makes the material, but do care about the properties of the material. For each material, that one considers designing with, one must have a course of material property characterization data as shown in figure 4.2.1.

In the case of composites, it is insufficient to simply identify a material as a fiber reinforced plastic. The data bank on material "XYZ" must contain some documentation on the type of fiber, fiber orientation and quantity, type of matrix and processing history. Ideally, one would then like to have each of the blocks in figure 4.2.1 completed with test characterization data on material "XYZ".

Not all of the desired data will be available for any given material system. As has been alluded to throughout this volume, material property data present a moving target. We must always be looking for and collecting new pieces of data to upgrade our knowledge on the material of interest. This gets further complicated by the introduction of new fibers and matrices.

Materials manufacturers continually introduce new fibers and matrices into the composites market. One can expect that the data on new materials will be rather sparse. However, one can also expect that the available data will highlight that data which the manufacturer believes are the "best" features of the new material, namely, the features which might give the material an edge over the

competition. Designers may be attracted to these materials to take advantage of these "better" features. However, one must constantly guard against a potential pitfall, the unlisted or unknown properties of these new materials.

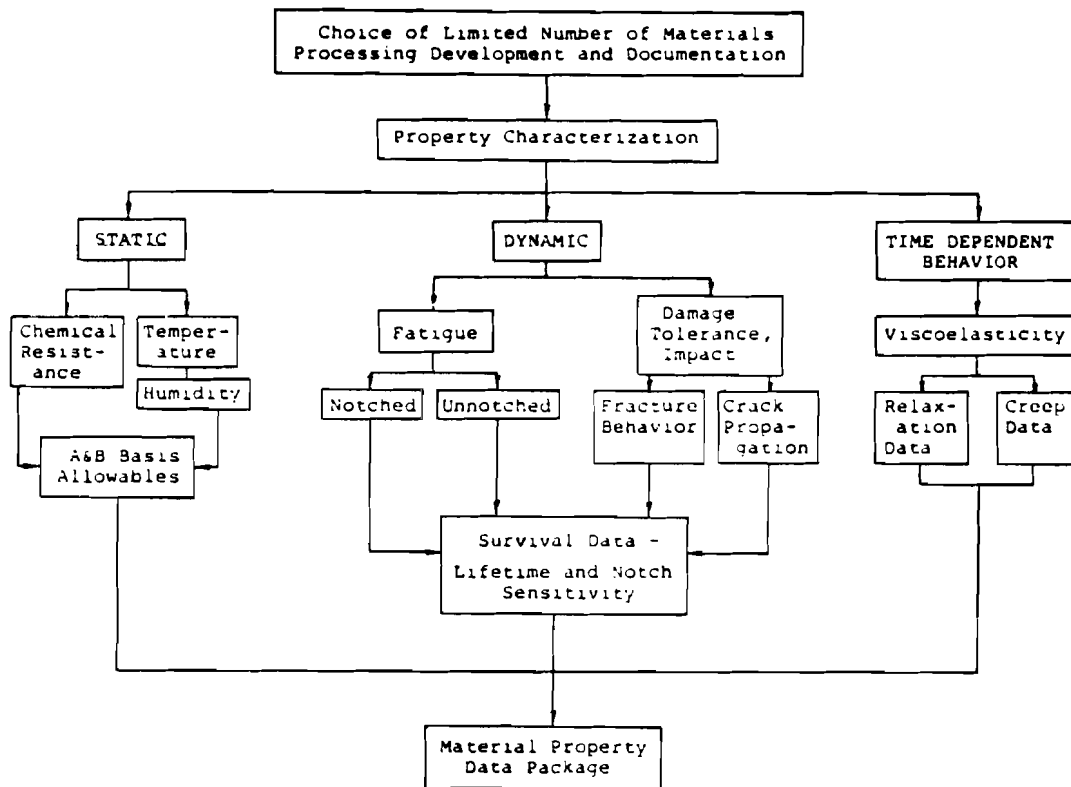


FIGURE 4.2.1 MATERIAL DATA BANK

Since characterization data on new materials will be skimpy, one will have to assume many of the properties. For example, transverse modulus and strength of unidirectional composite, and compressive strength of a fiber. These assumptions may be critical and we must be extremely careful in our assessments. An example of this is provided by Kevlar. Today, it is well known and understood that the compressive strength of Kevlar composites will be low. However, it was some time after Kevlar was introduced before this fact was well known and understood to be a characteristic of the fiber and not simply a sizing or processing problem.

If one thinks back, the designer attempting to compare Kevlar to glass or T-300 graphite might make the assumption that the compressive strength would be at least near the tensile strength. Considering glass and T-300 composites, this assumption would certainly appear valid. Naturally, then any application in which the compressive stresses were significant could result in Kevlar appearing to be much better suited than it really is.

The point here is that Kevlar composites have different strengths under tension and compression. Designers, must make note of all of the assumptions regarding materials modulus and strength characteristics and be prepared for experimental data which do not support these assumptions. Knowing which assumed properties could easily affect the design can quickly identify the focus for early material testing.

The minimum data needed to begin a design are the static room temperature properties of a unidirectional or laminated material, also properties as a function of direction are required. (See Table 2.8, Volume I, unidirectional material properties table.) If the properties of the unidirectional material are available one can utilize existing computer codes to synthesize material properties for lay-ups of interest. It is a satisfying fact to know that, when properly applied, the analytical models and a simple failure criterion provide excellent predicted elastic constants and thermal expansion properties and good estimates of material strength.

The difference between composites and metals in terms of material property requirements is one of magnitude and environment. Since properties are dependent on load direction, the material properties test matrix must be expanded to allow measurement of properties in all critical material directions. Hence, more test data are required. Another characteristic of composites is increased sensitivity to its physical environment. Properties which are matrix dominated tend to be influenced by environmental factors such as temperature and moisture to a much greater degree than metals. This is not a drawback other than the limits it places upon maximum operating temperature and the requirement for material characterization data to include these effects so that they can be accounted for in the selection of design allowables. The important point is that the designer should be aware that properties such as transverse layer modulus will show significant changes for even moderate temperature increases.

It is generally true that, during preliminary design stages, there will not be a sufficient amount of material characterization data available. Even when good static properties exist, the data bank will fall short when one tries to consider fatigue and impact conditions. Therefore, prototype testing will nearly always be essential for new designs or designs with replacement material.

4.2.1 Design Allowables.

There is a major question facing each designer concerning design allowables. That is, how does one treat limit load, ultimate load, factors of safety, etc. These questions must be answered specifically for each use and therefore can not be answered in general terms.

As with metals, A and B basis design allowables should be sought and different factors of safety applied for limit and ultimate load. Composites supply an additional factor which can be thought of as parallel (in some cases) to yielding and ultimate strength in metals. Namely, the load on the material when the first ply fails and the load when the full section fails. These are generally termed the first ply failure load and ultimate load.

In an attempt to be strictly conservative, one can design to the first ply failure load, as based on test or analysis. However, this leads to numerous problems.

1. For laminates in general, the first ply failure may occur at extremely low load levels resulting in very low design allowables in a material which has very high ultimate strength.

2. For materials such as in (1) above, it is very difficult to detect experimentally that the first ply has failed. In these cases the first failure is usually a layer transverse or layer shear failure. Since these failures are difficult to identify during testing, it becomes difficult to quantify the values against which to design. On the other hand, if we have good layer properties, it is relatively easy to predict first ply failure with standard laminate analysis tools.
3. The ratio between first ply failure stress and ultimate failure stress is not a constant and can vary with loading condition (see figure 4.2.2). In fact, it is possible that when the first ply fails, this result precipitates complete failure. Therefore, all extremes can exist from the case where first ply failure occurs at trivial loads and total failure is at much higher loads, to the case where first ply failure means total failure.

One must be very careful therefore to insure that, regardless of the failure criteria being applied, the design approach is conservative. The problem becomes one of design philosophy. With metals, the strength is independent of load direction and the ratio of yield surface to ultimate failure surface is relatively constant in stress space. Therefore, selecting a given factor of safety on ultimate load and a different factor of safety on limit load can imply equivalent probabilities of failure for any load combination.

Since, with composites, the ratio of limit to ultimate load capability can vary significantly with load combinations, it does not follow that the application of safety factors on limit and ultimate load imply uniform probability of failure under general loading conditions.

No attempt will be made here to resolve the above problem. It will be assumed that the appropriate factors of safety and failure criteria have been defined. Thus, we will circumvent the above problem.

For the purposes of illustration, the design examples which will be discussed in the chapter all make the assumption that the basic ply properties provided in Volume I, Table 2.8 are design allowable properties. It will be additionally assumed that first ply failure governs the design of any laminated configuration. These two assumptions must be considered when attempting to translate the design examples into any similar real life designs.

4.3 DESIGN REQUIREMENTS DEFINITION.

Given sufficient material properties for each composite material of interest, one can next ask: "What must this part do?" Figure 4.3.1 outlines the kinds of requirements which we must define in order to adequately conduct the design.

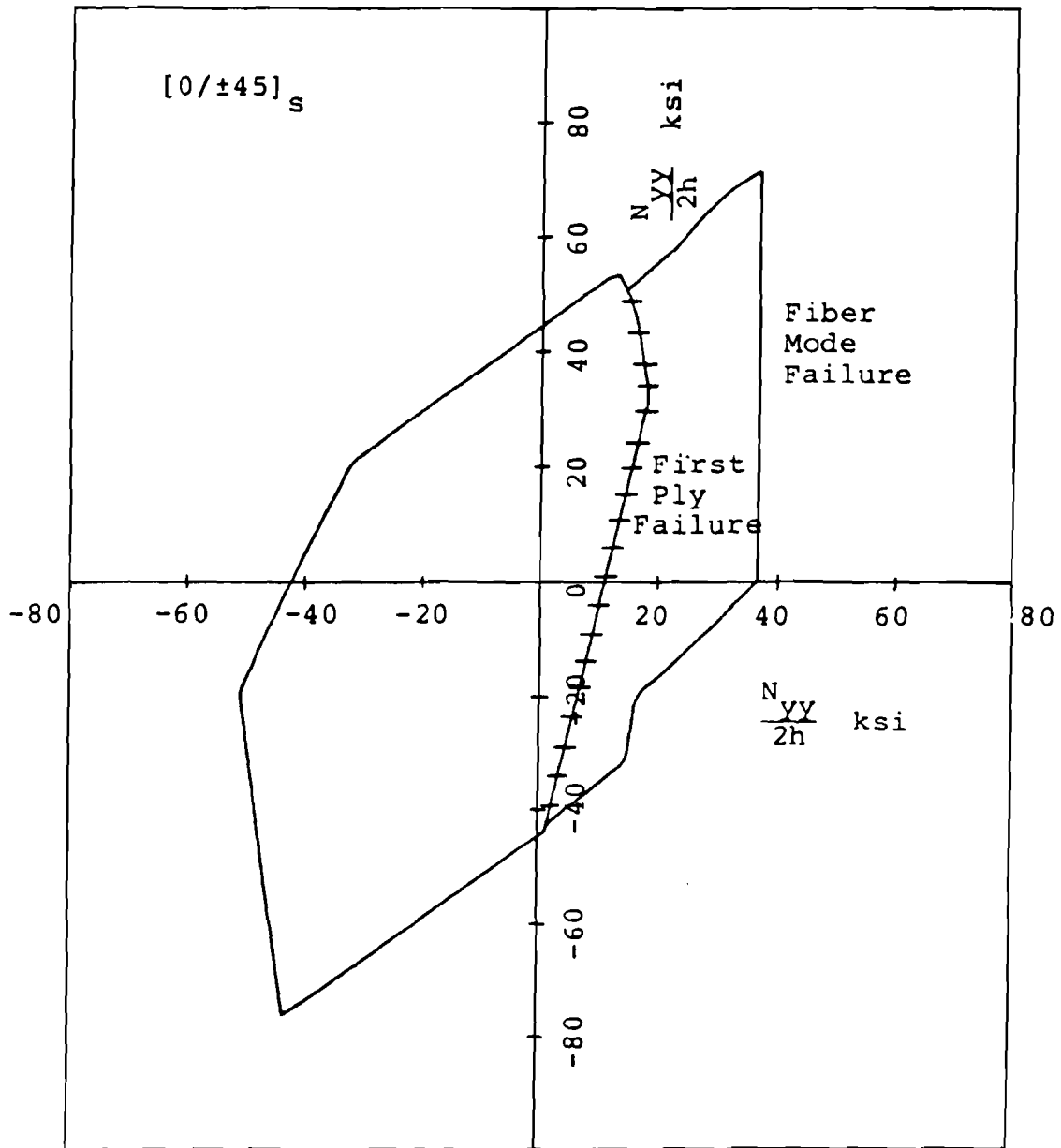


FIGURE 4.2.2 LAMINATE FAILURE SURFACE FOR COMBINED IN-PLANE EXTENSIONAL STRESS STATES

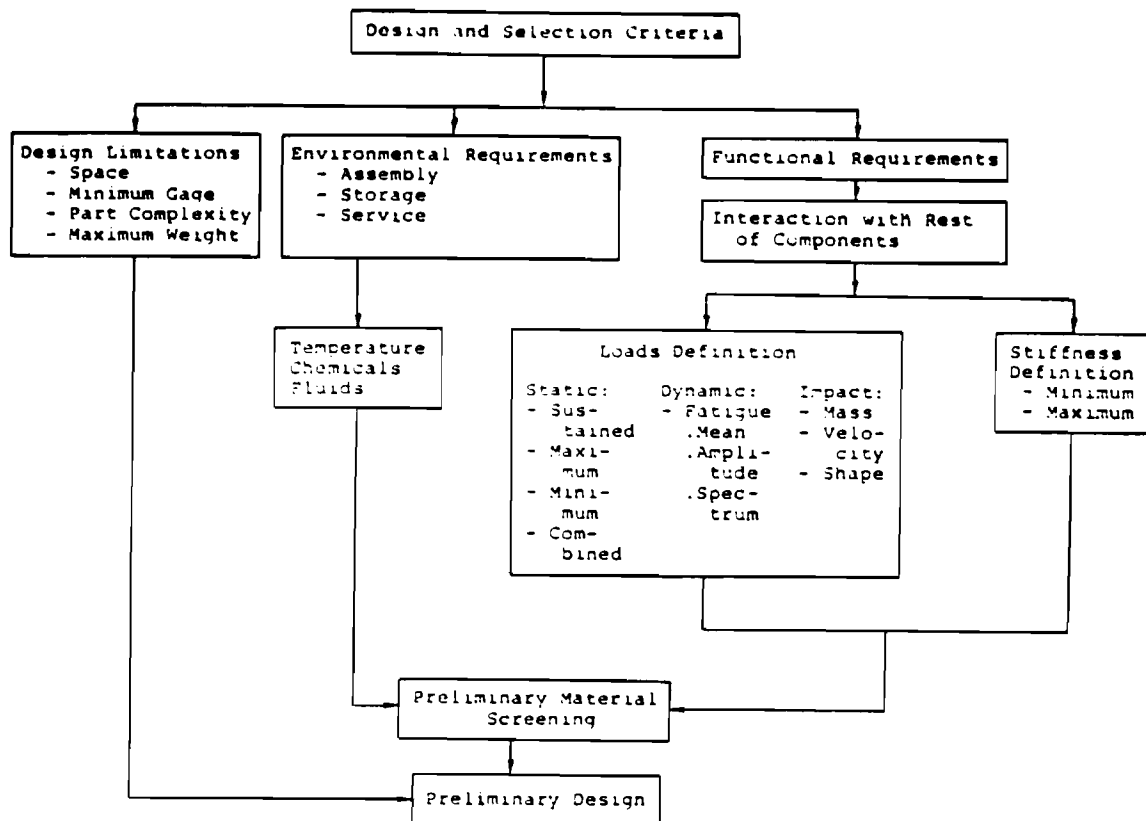


FIGURE 4.3.1. DESIGN REQUIREMENTS DEFINITION

4.3.1 Functional Requirements.

Because the focus is upon composites, one must define the functional requirements and how the component must interact with the remaining structure. This is important for composites because one does NOT want to strive for a one-for-one part substitution in all cases. Since complex shapes can be made with composites, one always wants to take advantage of parts consolidation whenever possible. This means that the following questions must be asked:

"What is the function of this part?"

"If I combine several pieces, will the part still perform this function?"

4.3.2 Environment.

Another aspect of design requirements definition which has more importance for composites than metals is the environment. For example, a maximum operating temperature of 300° F poses no difficulty for steel, but is a disaster for some composites since their properties may drop rapidly with temperature. Likewise, repeated contact with chemicals and fluids must be defined so their influence on mechanical properties can be accessed when selecting design allowables.

4.3.3 Design Limitations.

Design limitations, such as space restrictions and minimum practical material gage sections, are common for metal design. In the case of composites, one must ask for a redefinition of these space restrictions because the best composite configuration for a part may require dimensions that are different from those of the current metal part. For example, composites may not compete with steel in terms of section stiffness (EI) if the part dimensions are the same. If one can increase the cross-sections, composites can compete with steel by increasing the section moment of inertia and, in many cases, retain a weight savings.

4.3.4 Loads and Stiffness.

Loads and stiffness definitions are the same as those required for metal design. The only change introduced by composites is that one must ask how the secondary loads might affect the design. In metals, generally one can ignore secondary loads once we define the critical loading conditions. This is usually not true with composites. Each component design should be reviewed for all expected load conditions to insure that secondary loads do not govern the design.

The definition of static and fatigue load conditions are no more difficult for composite parts design than for metals. They must either be specified by the industry or measured on similar components.

4.4 PRELIMINARY DESIGN.

Given suitable material properties and a definition of the component requirements, the real work for the designer can be initiated. As shown in figure 4.4.1, the designer must mesh the material properties and design requirements with the definition of component configuration and design details to develop a preliminary component specification. This should be done for each material which is being considered and for a range of material configurations. One quickly finds that parametric analyses of simple configurations are an excellent method of focusing the design into the appropriate materials and constructions. By simplifying the component configurations, we can generally utilize simple solutions to size the various materials. These results identify those materials which should then be pursued in the actual configuration using more complex analyses.

Once prototype designs are defined utilizing several materials, the competing designs can be compared based on both weight and projected cost. The designer cannot leave the design loop at this stage. Particularly with composites, one must follow the design through the substantiation phase. Ideally, the design substantiation represents the point where a prototype component is fabricated and tested to verify the adequacy of the design. In practice, however, this stage becomes one of modification and compromise to achieve a component with the production capabilities of the organization.

4.5 DESIGN SUBSTANTIATION.

Assuming success through the first few stages of figure 4.5.1, the fabrication lab provides a component for verification. The next major question is - will the part function as required? Satisfactory performance of the part is, of course, the next major step in bringing the design to a level where it can be considered

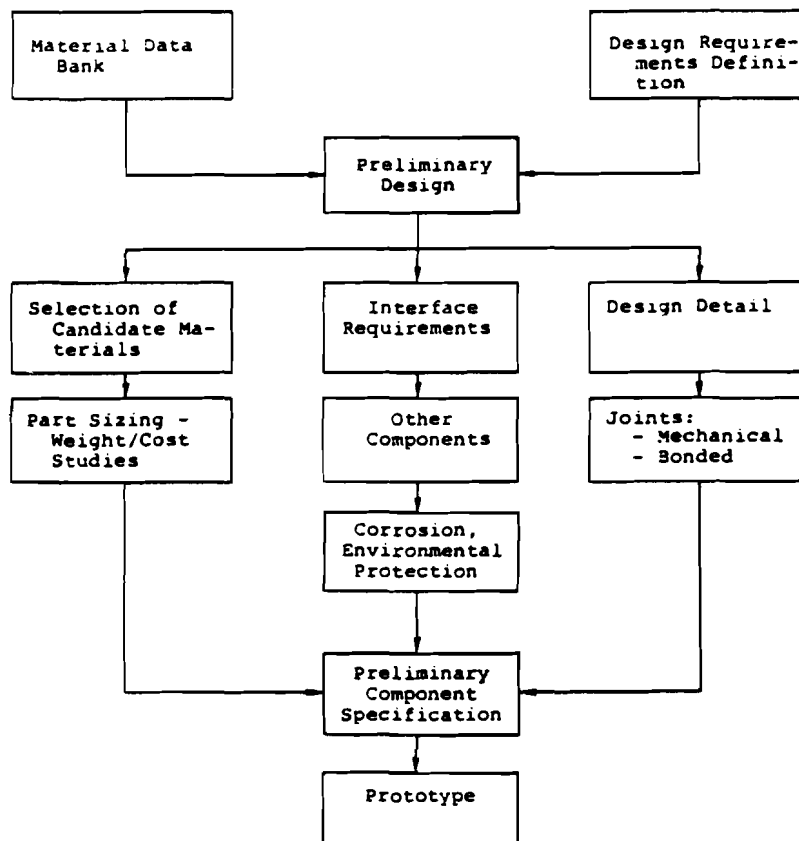


FIGURE 4.4.1 PRELIMINARY DESIGN

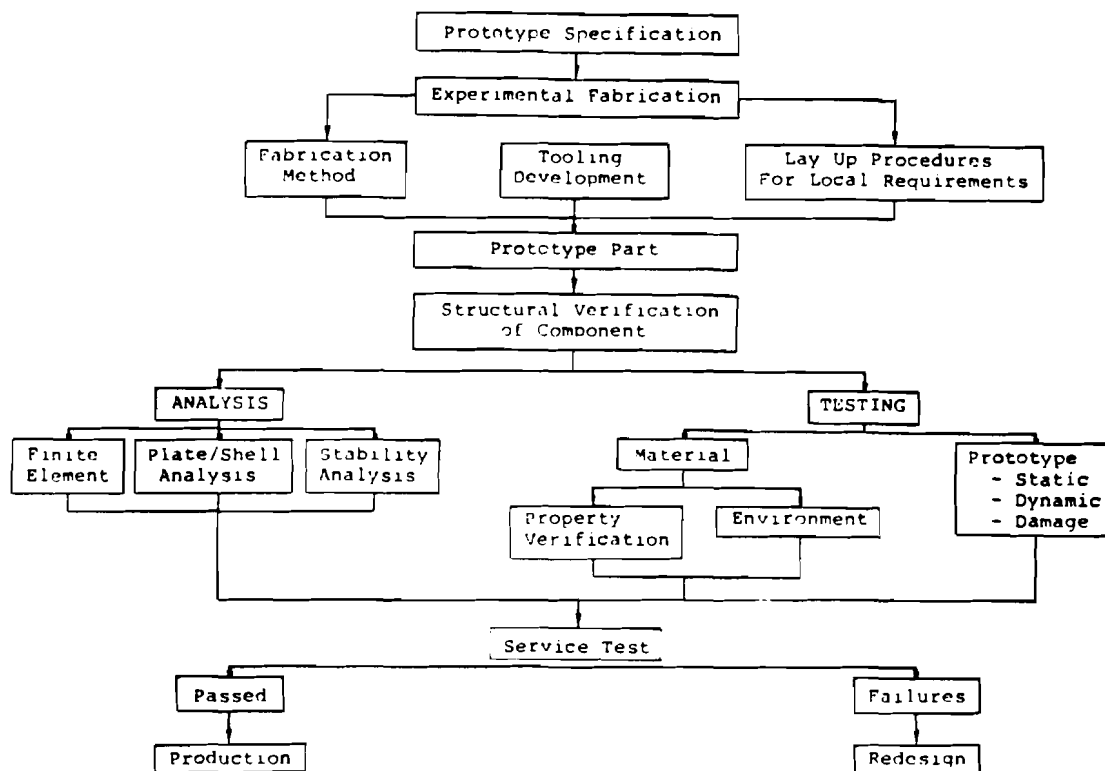


FIGURE 4.5.1 PROTOTYPE AND DESIGN SUBSTANTIATION

for production. Here one must adopt a view which is substantially different from metal design.

It is NOT sufficient to simply take the prototype part and begin subjecting it to a series of tests. The material itself possesses numerous variables, such as number of plies, ply orientation, stacking sequence, and cure cycle. Therefore, one must first establish what material is in the actual component. This can be accomplished by sectioning various areas of the component, inspecting them, and then measuring the properties of the area. That is, whenever possible, it is desirable to measure the actual material properties of the component. This permits one to identify if the material achieved the expected property levels. It also permits a remodeling of the component analytically with the actual material properties. This can be a critical step in designing with composites. If failures occur during testing, it is best to anticipate those failures analytically before they happen.

The failure of a complexly shaped prototype part is likely to be due to numerous modes. Adequate redesign of the part will be achieved only if one can correlate the analysis of the part to the experimental behavior of the part such that the cause or causes for failure is reasonably clear. This is not very different from metals except that failure of composites is made complex and prevention of failure requires combined analysis and experiment.

4.6 DESIGN EXAMPLES.

The following design examples serve to illustrate many of the similarities and differences between designing with composites and metals. Each example is self contained with the following:

- a) Statement of the problem and imposed restrictions.
- b) Governing equations.
- c) Solutions for steel, aluminum, glass epoxy and graphite epoxy. The composite results include several configurations of material to demonstrate that influence.
- d) Brief discussion of the results.

In all cases the basic layer properties used are that given in Volume I Table 2.8. Note that the Table gives unidirectional layer properties. Properties for laminated configurations were predicted using the lamination theory as developed in Volume I and the first ply failure strength was utilized as the governing strength for each laminate.

The following composite design examples increase in complexity from a simple rectangular beam in bending, to cylindrical shells in compression, and finally to a stiffened panel design.

4.6.1 Beam.

Consider a simply supported beam subjected to a single central load, as shown in figure 4.6.1. We will, for simplicity, further restrict ourselves to a beam of rectangular cross section and will neglect the local effects of stress concentration directly under the loading points. The design of this beam requires defining the dimensions of the cross section such that failure will not occur due to bending stresses or shear stresses. Additionally, a deflection

restriction will be imposed to limit the maximum beam deflection to a few percent of the span.

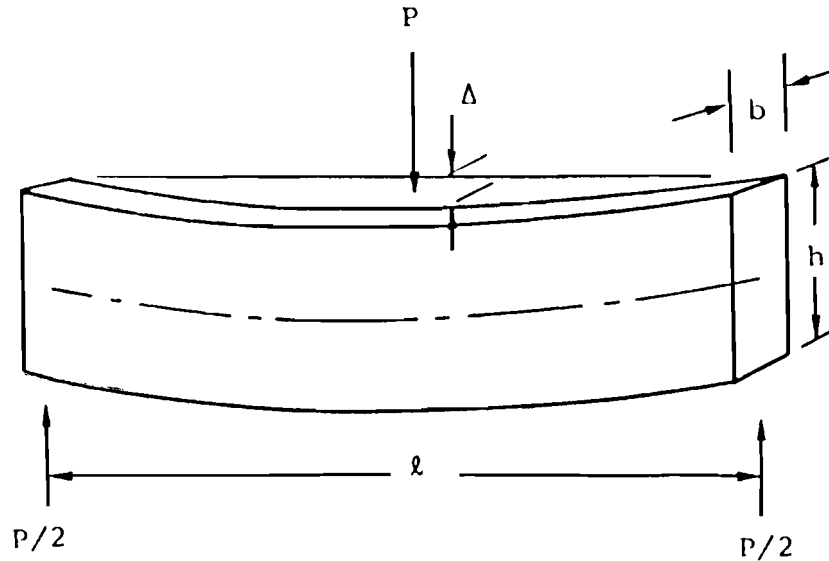


FIGURE 4.6.1 RECTANGULAR BEAM IN 3-POINT BENDING

The imposition of both a strength and stiffness limitation, will result, in general, in a design which is either strength or stiffness limited. Which criteria governs will be a function of the material properties (stiffness and strength) as well as the applied load and beam dimensions.

For the simply supported beam under a central load, P , figure 4.6.1, the maximum bending stress is given by beam theory by the familiar:

$$\sigma_{\max} = \frac{M_{\max} c}{I} = \frac{3Pl}{2bh^2}$$

Not having bending allowables available in our material properties, Volume I, table 2.8, the maximum allowable bending stress is taken as the minimum of the material tensile and compressive strength as measured along the beam axis.

Simple manipulation of the above yields the beam height required to prevent failure due to bending as

$$h_b = \sqrt{\frac{3Pl}{2b\sigma_{\max}}}$$

assuming that the width of the beam, b , is fixed.

Beam theory defines the maximum shear stress, $\tau_{\max}$, for this beam as

$$z = \frac{3P}{4bh}$$

which leads to a definition of beam height required to prevent a shear failure as

$$h_s = \frac{3P}{4b\tau_{\max}}$$

where $\tau_{\max}$ is the design allowable shear stress for the material.

In the case of metals, the governing height is almost always the height required to prevent failure due to bending because the magnitudes of the shear and bending strengths typically are of the same order. In the case of unidirectional composites, the bending strength is very much higher than the shear strength with the result that the governing height in many practical problems could be the height required to prevent shear failure. This effect is not as pronounced for other material configurations since the ratio of longitudinal strength to shear strength is highest for unidirectional composites.

The deflection of the beam due to the load is comprised of both bending and shear deformations. The contribution of shear deformation is significant for beams with low span to height ratios and low shear stiffnesses. The expressions for the deflection at mid-span for the simply supported beam are given by:

$$\Delta_b = \frac{Pl^3}{48EI} \quad , \text{ bending induced deflection}$$

$$\Delta_s = \frac{6}{5} \frac{Pl}{Gbh} \quad , \text{ shear induced deflection}$$

$$\text{and } \Delta = \Delta_b + \Delta_s \quad , \text{ Total deflection}$$

These are the only expressions required to evaluate a beam design. For purposes of illustration, assume that the problem is to design the lightest possible beam which can carry a 5000 pound load over a 20-inch span. Further, assume that dimensional constraints require the width of the beam to be 0.5 inches. Also, assume that the maximum deflection must not exceed 5% of the length of the beam, in this case, 1 inch.

The beam depths required to prevent bending and shear failure can now be computed for various materials. Given the governing depth, the total beam deflection can be computed and checked to insure that the deflection is less than the allowable 5 percent of span. If the deflection is greater than 5 percent then the beam stiffness, and not strength, governs the design and the beam height must be increased until the total deflection is acceptable.

The resulting beam designs are shown in figure 4.6.2. It can be observed that bending stress governs the design for all materials. However, the ratio of thickness to prevent shear failure to that required to prevent bending failure is substantially larger for the composite materials as compared to steel and aluminum.

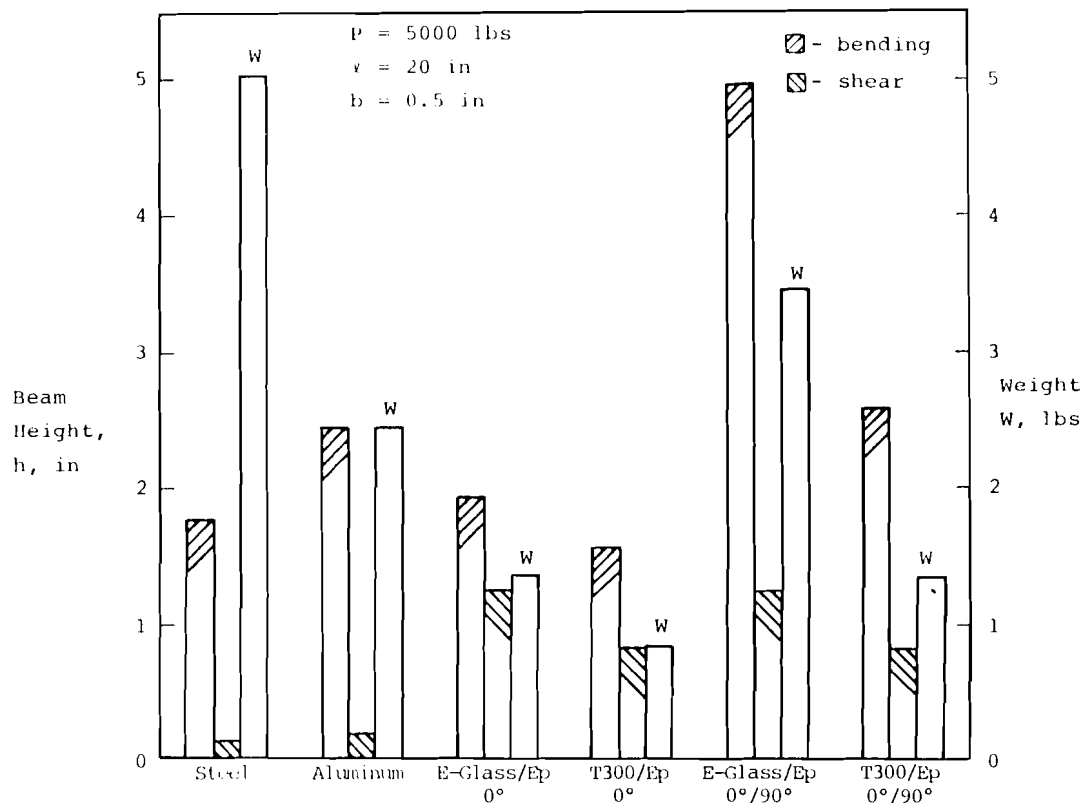


FIGURE 4.6.2 COMPARISON OF BEAM DESIGNS SHOWING DEPTH AND RESULTING WEIGHT

The largest beam depth is obtained for the $0/90$ E-Glass/epoxy due to its poor bending strength. As was stated previously, all material strengths were computed through laminate analysis and the maximum stress criterion and also were based on first-ply failure. The $0/90$ E-Glass material has a low bending strength due to the failure of the plies in the transverse direction. The superiority of the unidirectional material (T300/epoxy and E-Glass/epoxy) is due to its excellent longitudinal strengths. The lightest weight design is observed for the unidirectional T300/epoxy. Except for the $0/90$ E-Glass/epoxy, all the other composite beams are lighter than the aluminum design.

A comparison of the beam designs showing the total and shear induced deflections is shown in figure 4.6.3. All of the designs satisfy the deflection constraint and are governed by strength. If the deflection constraint had been more severe (e.g. $\Delta/l < 2\%$) then the unidirectional E-Glass/epoxy beam depth would have to be increased to satisfy the stiffness criterion. The required depth could then be obtained through an iterative procedure starting from a depth greater than that obtained from strength considerations.

The contribution of the shear deformation term is very small for steel and aluminum, figure 4.6.3, whereas it is significant for the composites. Both the shear mode of failure and the contribution of the shear deformation term to deflection will increase for more severe load conditions.

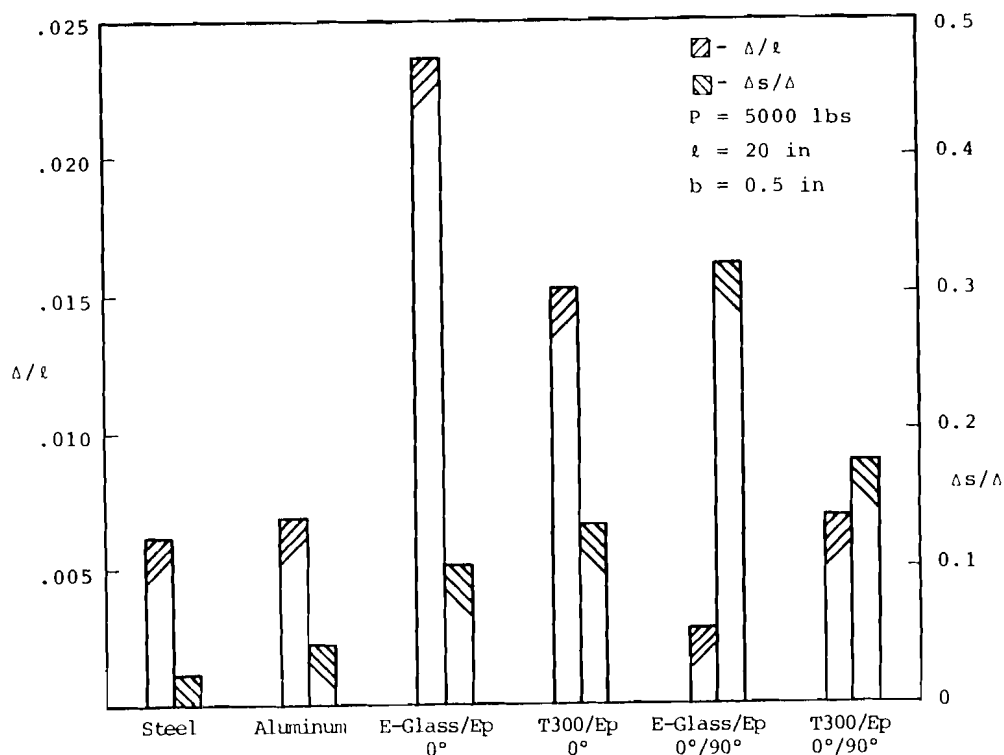


FIGURE 4.6.3 COMPARISON OF BEAM DESIGNS SHOWING TOTAL AND SHEAR INDUCED DEFLECTIONS

This can be demonstrated by considering the same problem, but with a more severe load requirement of 15,000 lb. The results for this load case are shown in figures 4.6.4 and 4.6.5, respectively. For this case the unidirectional glass thickness is governed by shear strength, figure 4.6.4. Again, all of the composite designs except the 0/90 E-Glass/epoxy are lighter weight than the aluminum design.

Note also that for the 0/90 E-Glass/epoxy, the deflection due to shear is greater than the deflection due to bending, figure 4.6.5, thus neglecting the shear contribution would result in greater than an error factor of two in deflection.

Two important conclusions are evident from this example:

1. Shear failures are more likely to occur in composite beams than metal beams.
2. Shear deformations in composite beams are more significant.

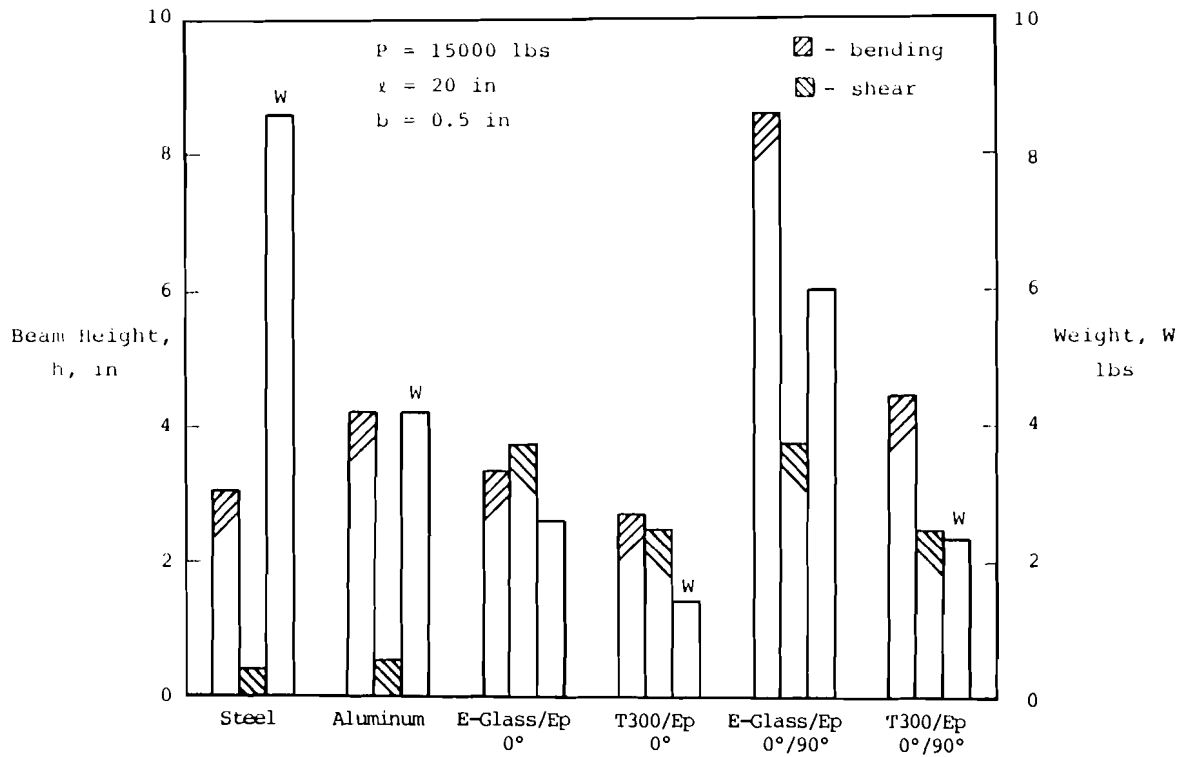


FIGURE 4.6.4 BEAM DESIGNS UNDER SEVERE LOAD SHOWING DEPTH AND RESULTING WEIGHT

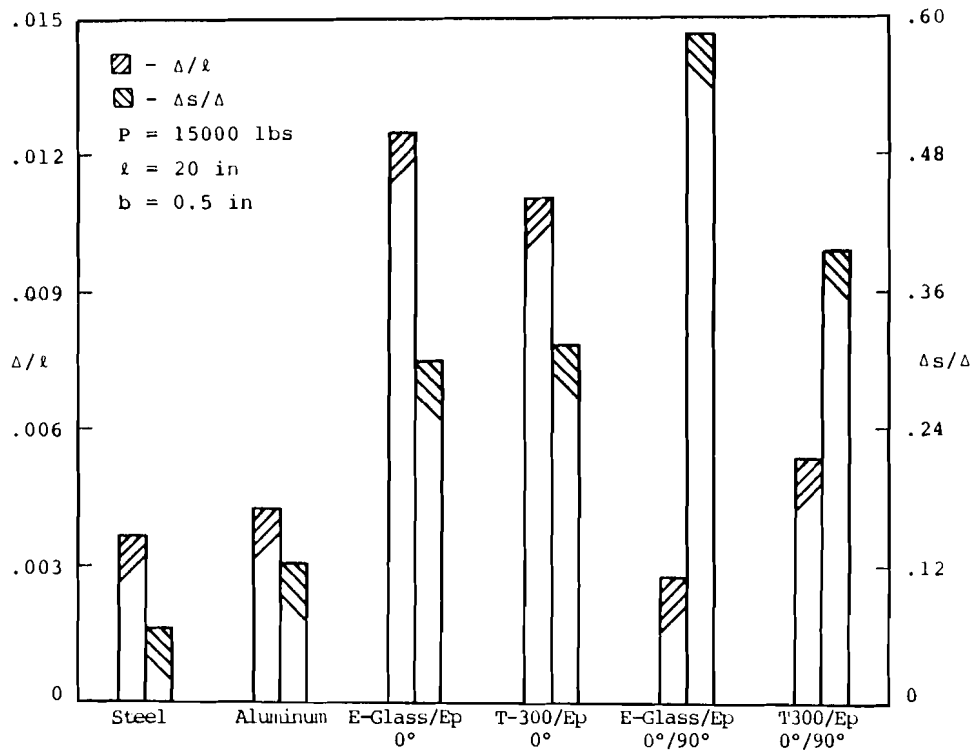


FIGURE 4.6.5 BEAM DESIGNS UNDER SEVERE LOAD SHOWING TOTAL AND SHEAR INDUCED DEFLECTIONS

4.6.2 Cylindrical Shell Under Axial Compressions.

The previous example required evaluating only the amount of material required to prevent stress induced failure and checking to insure that the maximum deflection was not excessive. The fact that the material was a composite required only the additional consideration of shear failure with no additional complexity in the analysis. The reduced shear strength in the material property table should alert the designer to the possibility of this mode of failure. The one inherent property which might not be self evident is the very low shear modulus which results in much higher than normal shear deformations.

In the case of cylindrical shells under compression, a common structural element, the fact that composite moduli vary with direction will be found to influence the shell wall buckling load capability and therefore complicate the analysis. The design of cylindrical shells (figure 4.6.6) to withstand axial compressive loads involves the consideration of compressive strength as well as several possible modes of buckling failure. Buckling can occur in the form of Euler column buckling for long, slender shells or as local shell wall buckling. There exist two possible modes of local shell buckling depending on the relative magnitudes of the shear and extensional moduli.

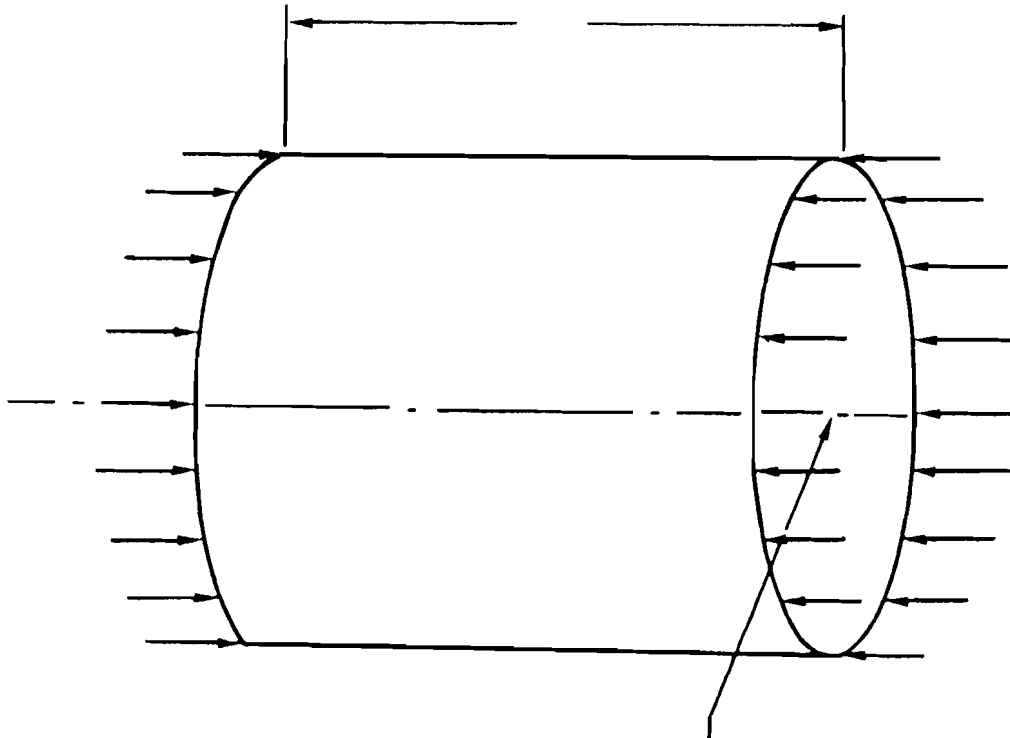


FIGURE 4.6.6 CYLINDRICAL SHELL IN UNIFORM EDGE COMPRESSION

The critical load index, N_{xc}/R can be expressed as a function of the shell geometry and boundary conditions multiplied by an effective shell buckling modulus, E_s . E_s can be expressed as:

$$E_s = \min \left(\frac{\sqrt{E_x E_y}}{1 - \nu_{xy} \nu_{yx}}, \sqrt{\frac{\sqrt{E_x E_y}}{1 - \nu_{xy} \nu_{yx}}} 2G_{xy} \right)$$

The two expressions must be evaluated and the minimum value represents the effective shell modulus and will define the shell buckling mode for that material. See for example, reference 4.1 for other shells and end conditions.

Note that for a given geometry, the desire is to maximize E_s to obtain the highest load carrying capability. Since all of the moduli enter the expressions for E_s and since changing fiber orientation to change one modulus will alter the other, it is not obvious which material configuration will maximize E_s . For example increasing E_x can generally be achieved in a laminate only at the expense of E_y or G_{xy} .

The general approach, then is to compute laminate properties as functions of angle and percentages of plies in various directions. At the same time, E_s is calculated and plotted. In this manner, one can see what combination of variables will maximize E_s .

For a cylindrical shell supported along the unloaded edges, the compressive load at buckling failure is given by

$$\frac{N_{xc}}{R} = \frac{1}{2\sqrt{3}} E_s \left(\frac{t}{R}\right)^2$$

where t is the shell thickness. We also require that the compressive strength of the material not be exceeded. For a given material, the load required to induce a compression stress failure is given by

$$\frac{N_{xc}}{R} = \sigma_x^{cu} \frac{t}{R}$$

where σ_x^{cu} is the compressive strength of the composite along the axis of the cylinder.

For a given cylinder and material, the compressive load carrying capability will be the minimum of the above two expressions, assuming that Euler column buckling is not a concern. A cylindrical shell constructed with all of the fibers in the direction of the load will have the best compressive strength and will be the best configuration if strength governs. However, this configuration will generally have the lowest buckling load.

Local shell buckling will generally govern the design at low loads. At higher loads, strength considerations generally becomes the critical criterion. Each material has its own optimum operating point at which failure due to inadequate compressive strength and failure due to buckling occur simultaneously. This optimum load index $(N_{xc}/R)^*$ depends on the material elastic moduli as well as the strengths and geometry of the shell. For this particular case of a simple cylindrical shell, it is determined by

$$\left(\frac{N_{xc}}{R}\right)^* = \frac{2\sqrt{3}}{E_s} \left(\sigma_x^{cu}\right)^2$$

Consider a cylindrical shell with a radius of six inches and a length of 48 inches, subjected to an axial compressive load N_{xc}/R of 1000 psi. The shell thickness required to prevent buckling is given by

$$\left(\frac{t}{R}\right)_b = \sqrt{\frac{2\sqrt{3}}{E_s} \frac{N_{xc}}{R}}$$

The thickness required from compressive strength considerations is given by

$$\left(\frac{t}{R}\right)_s = \frac{N_{xc}}{R} \frac{1}{\sigma_x^{cu}}$$

For this example we will assume that column buckling is not a factor in the problem, and will only compute local buckling. In a real problem, the designer will have to consider the column length and end conditions to define the column buckling load as well as the local shell buckling load. Naturally, for the local shell buckling formulations to apply, the column must be long enough so that the local buckle waves can develop without end restraint influence. This implies that the column must be at least several diameters long.

The required shell thicknesses and corresponding weights for steel, aluminum and various composite materials are shown in figure 4.6.7. The $\pm 45^\circ$ composite configuration was also compared with steel and aluminum because this configuration possesses a combination of properties that results in a relatively high shell buckling modulus. It can be observed from figure 4.6.7 that the governing criterion in each case is buckling. Hence, the $\pm 45^\circ$ configuration has a lower weight than the unidirectional material for each of the composites considered. In fact, the $\pm 45^\circ$ configurations of all the composites, offer weight savings as compared to aluminum. The best material is the ± 45 VSB-32/Epoxy, which is about 40% lighter than the aluminum design.

The various materials will become strength critical if the load index is increased and the selection of the best material can change. It is necessary, therefore, to know the expected range of load conditions before selecting a material configuration for a component.

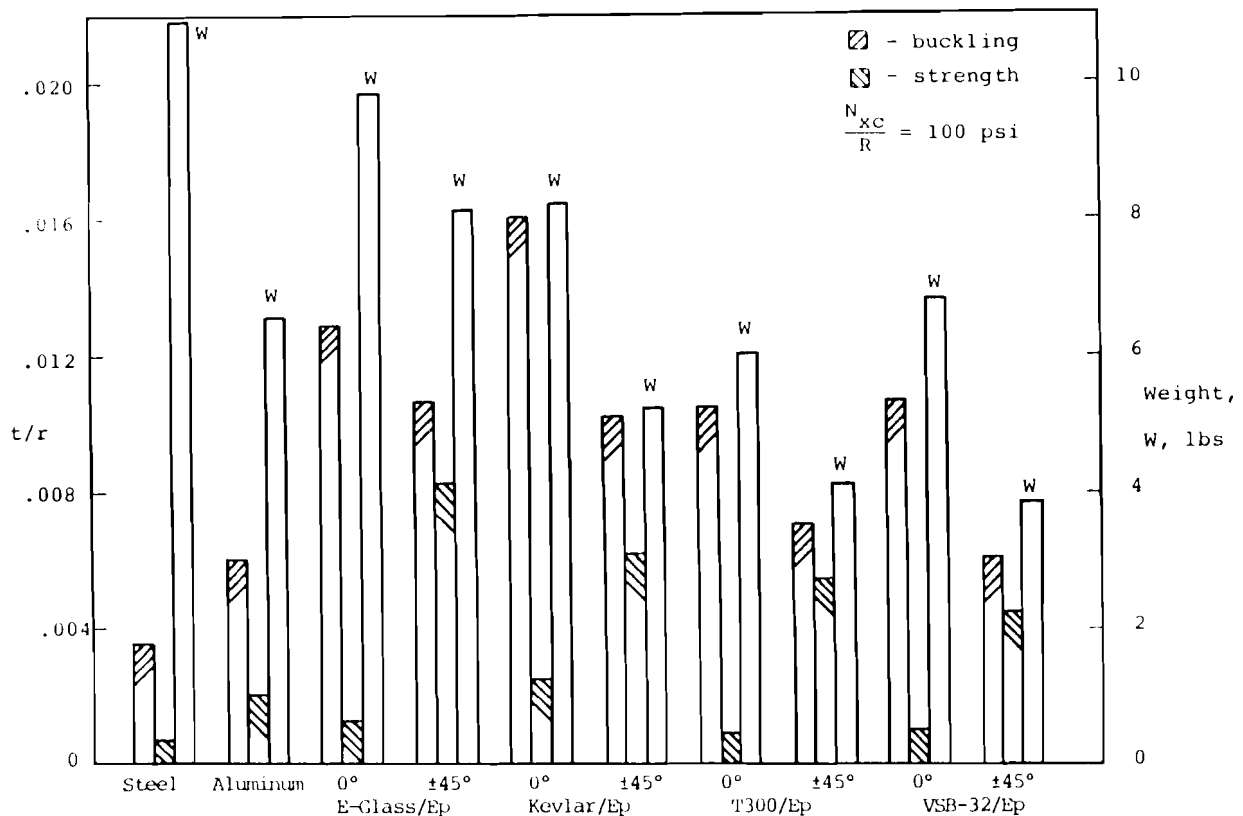


FIGURE 4.6.7 COMPARISON OF CYLINDRICAL SHELL DESIGNS SHOWING THICKNESSES AND RESULTING WEIGHTS

The above examples should highlight the fact that in order to design with composites and achieve the best possible configuration, a large number of cases must be examined. Given the freedom to choose the fiber type and lay-up, we must consider these as additional variables and then search for the set of dimensions, weight, and costs which go with each configuration. This can become, as it sounds, a monumental task. This task becomes particularly challenging when the design allows additional geometric variables.

For example, if the first example, of a simple beam is reconsidered, but allowing the cross section to be a variable then this problem becomes much more difficult. We could consider a hollow box beam, with dimensions of b , h , t to be defined or as an "I" or "H" beam with flange dimensions and properties which differ from the web dimensions. In both cases we would have to include local buckling as a possible failure mode.

The real design cases are more similar to those above than to the simple examples cited. However, the solution to define necessary dimensions are not significantly complicated because the materials are composites. The solutions generally require only solving for more degrees of freedom and considering additional failure modes. Thus, the general case will usually require a computer formulation.

It would be nice to formulate the problems in terms of fiber type on orientation and then obtain the configuration and dimensions for an optimum solution as part of a single solution. However, this is an all but impossible task. The general approach is to formulate the problem with material properties as input and then to optimize the remaining variables. We then obtain solutions for various fibers and fiber orientations and compare results to identify the best approach. In this fashion a "good" solution can be found even though it may not be truly the optimum one.

4.6.3 Design of Stiffened Panels.

The above approach is more clearly illustrated in the following design example. In this example we will consider a problem which is very similar to many practical applications. The objective is to design a compression skin with sufficient stiffness to carry a given compressive load, yet to be as light as possible. Obviously, the best approach to this problem requires adding stiffeners of some type to the skin in order to develop stiffness, rather than simply increasing skin thickness. However, the addition of stiffeners adds numerous geometric variables to the Optimization problem: number of stiffeners, stiffener geometry, stiffener thickness, stiffener material. Additionally, facesheet and stiffener compressive strengths must be considered along with the possible buckling modes for the stiffener and its sections.

This section presents an example of a simply supported blade stiffened panel designed to withstand an axial compressive load. The cross-sectional area and the leading dimensions of the panel are shown in figure 4.6.8.

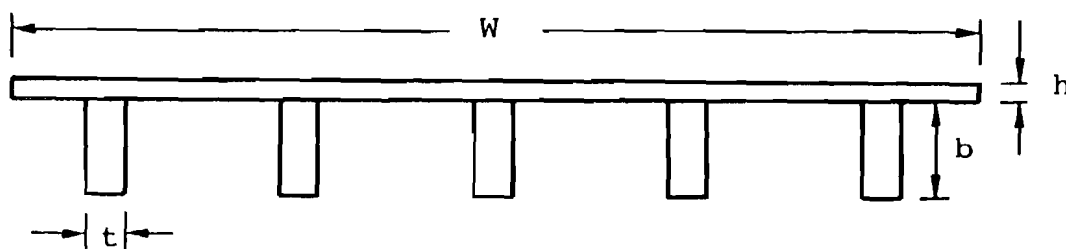


FIGURE 4.6.8 CROSS-SECTION OF BLADE STIFFENED PANEL
FOR AXIAL COMPRESSIVE LOADS

It will be assumed that the overall panel width is fixed to reduce the number of significant variables. The ribs will be assumed to be spaced equally and to be all of the same material. The facesheet and rib materials will be permitted to have different fiber orientations. The designer, therefore, can select a minimum weight design for a given material by optimizing with respect to the following variables:

1. facesheet thickness;
2. rib depth;
3. rib thickness;
4. number of ribs;
5. fiber orientation in rib; and
6. fiber orientation in facesheet.

The design calculations are based on preventing various types of buckling failures such as: a) buckling of the ribs, b) buckling of the segment of the facesheet between successive ribs, and c) buckling of the overall plate. Also, the compressive stresses in the ribs and facesheets must be below the design allowables for each material. The governing eqs. for this design example are summarized in table 4.6.1. The expressions for the buckling of the facesheets between successive ribs and the buckling of the ribs themselves are straightforward and depend on the boundary conditions and plate buckling modulus. The plate buckling modulus, E_p is equivalent to the membrane stiffness of the plate for isotropic materials. For an anisotropic material to have a high E_p , it must have a good combination of longitudinal, transverse and shear moduli. The +45 configuration is usually the configuration with the highest E_p for most composite materials.

TABLE 4.6.1 GOVERNING EQUATIONS FOR STIFFENED PANEL DESIGN

Buckling of Ribs: $(\sigma_{cr})_r = \frac{k_r}{12} \frac{\pi^2}{b^2} E_{pr} t^2$

$$\text{where } E_{pr} = \frac{1}{2} \left[\frac{\sqrt{E_{xr} E_{yr}}}{1 - \nu_{xyr} \nu_{yxr}} + 2G_{xyr} \right]$$

and $k_r = 1.28$ for cantilever plate

Buckling of Facesheets

Between Ribs: $(\sigma_{cr})_f = \frac{k_f}{12} \frac{\pi^2}{b^2} E_{pf} h^2$

$$\text{where } E_{pf} = \frac{1}{2} \left[\frac{\sqrt{E_{xf} E_{yf}}}{1 - \nu_{xyf} \nu_{yxf}} + 2G_{xyf} \right]$$

and $k_f = 4$ for simply supported plate

Overall buckling of Plate:

$$\sigma_{cr} = \frac{k}{12} \frac{\pi^2}{b^2} E_p h^*$$

$$\text{where } E_p = \frac{1}{2} \left[\frac{\sqrt{E_x^* E_{yf}}}{1 - \nu_{xyf} \sqrt{E_{yf}/E_x^*}} + 2G_{xyf} \right],$$

$k = 4$ for simply supported plate

TABLE 4.6.1 GOVERNING EQUATIONS FOR STIFFENED PANEL DESIGN
(Cont.d)

and E_x^* , h^* are determined from, $E_x^* h^* w^* = \Sigma(EA)$

$$\text{and } E_x^* \frac{h^{*3} w}{12} = \Sigma(EI)$$

Compression in Ribs: $\sigma_{xr} = \frac{PE_{xr}}{whE_{xf} + NbtE_{xr}}$

where N = number of ribs

Compression in Facesheets: $\sigma_{xf} = \frac{PE_x^*}{whE_{xf} + NbtE_{xr}}$

Compression in Effective Plate: $\sigma_x = \frac{PE_x^*}{whE_{xf} + NbtE_{xr}}$

Subscripts:

f - facesheet

r - rib

The expression for the buckling of the overall plate is based on the buckling expression for an equivalent simply supported plate with the same longitudinal and bending stiffness. The plate buckling modulus of the equivalent plate is calculated based on the assumption that the ribs only provide an increase in the bending and longitudinal stiffness and do not affect the shear and transverse stiffnesses of the plate.

The approach adopted for the design of this stiffened panel was to convert the problem to an optimization problem. The formulation of the optimization problem is outlined in table 4.6.2. The optimization was done numerically via the Constrained Rosenbrock (Hill Algorithm), reference 4.2.

A compressive load of 5,000,000 lbs. and a panel width of 50 inches were assumed for this example. Since the rib and facesheet can have different material orientations, the best combination has to be first chosen for a particular material.

The optimization algorithm was exercised for T300/epoxy for various material orientations and stiffener configurations, the results are shown in figure 4.6.9. If both the rib and the facesheet are made of the unidirectional material then the required cross-sectional area is high due to the poor buckling modulus of the unidirectional material. If both the rib and the facesheet are made of the $+45$ material, then the compressive strength becomes the governing criterion.

The quasi-isotropic $0/+45/90$ material provides a good combination of strength and buckling properties if the same material orientation is to be used for both the ribs and the facesheet. This design can be improved upon by making the ribs of unidirectional material and the facesheet of the quasi-isotropic material. In this configuration, most of the compressive load will be carried by the stiffer ribs, with adequate buckling resistance provided by the $0/+45/90$ configuration for the facesheet. Of the configurations considered, the best was found to be the $+45$ orientation for the facesheet (higher plate E_p) and the unidirectional material for the ribs, see figure 4.6.9. Since this was the best configuration for T300/epoxy, it was used as a basis against which other materials were compared, figure 4.6.10.

No proof is offered however that this configuration will also represent a good choice for the other fiber systems. A logical way to eliminate the number of analyses is to define a good configuration for one material system and then compare fibers based upon that configuration. If other fibers compare well then the designer should investigate the other fiber system to identify if further improvements can be made by changing the orientation of the other fibers. Obviously one can not be totally sure of an optimum solution unless all configurations of all fiber systems are analyzed. In real design cases this generally becomes too expensive and time consuming.

The optimal weights for the various materials are shown in figure 4.6.10. The optimized panel dimensions are shown in table 4.6.3. The best material choice is the VSB-32/Epoxy material due to its high buckling resistance and adequate strength properties. The T300/Epoxy design is only slightly heavier than VSB-32/Epoxy. The E-Glass/Epoxy material is not very well suited for this application due to its low buckling modulus. Kevlar, T-300 and VSB-32 offer significant weight savings over aluminum and hence, can be used to advantage in

applications where premiums associated with weight savings are high, e.g. aircraft structures.

TABLE 4.6.2 APPROACH FOR STIFFENED PANEL OPTIMIZATION

Objective Function: Weight = PL (wh + Nbt)

Minimize weight with respect to h.b.t. under the following constraints:

1. $\sigma_{xf} \leq (\sigma_{cr})_f$ Facesheet buckling
2. $\sigma_{xr} \leq (\sigma_{cr})_r$ Rib buckling
3. $\sigma_x \leq \sigma_{cr}$ Overall Plate buckling
4. $\sigma_{xf} \leq \sigma_{xf}^{cu}$ Facesheet Compression
5. $\sigma_{xr} \leq \sigma_{xr}^{cu}$ Rib Compression
6. $N \leq 8$
7. $h \geq .015$ Minimum Gage Requirement
8. $t \geq .015$ Minimum Gage Requirement

NOTE: See figure 3.8 and table 3.1 for explanation of symbols

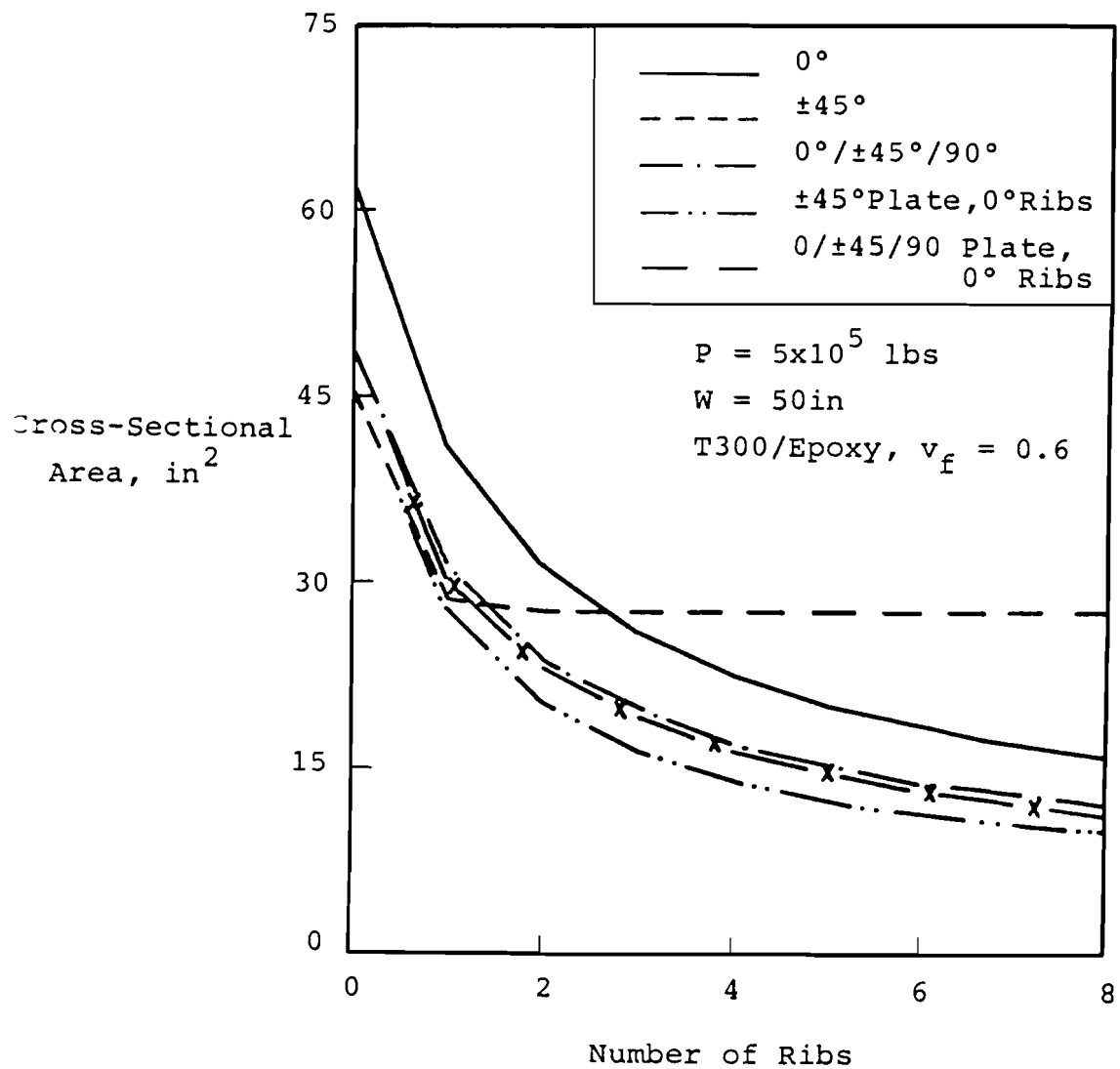


FIGURE 4.6.9 OPTIMAL STIFFENED PLATE CROSS-SECTIONAL AREAS VERSUS STIFFENER CONFIGURATION FOR VARIOUS T-300/EPOXY MATERIAL ORIENTATIONS

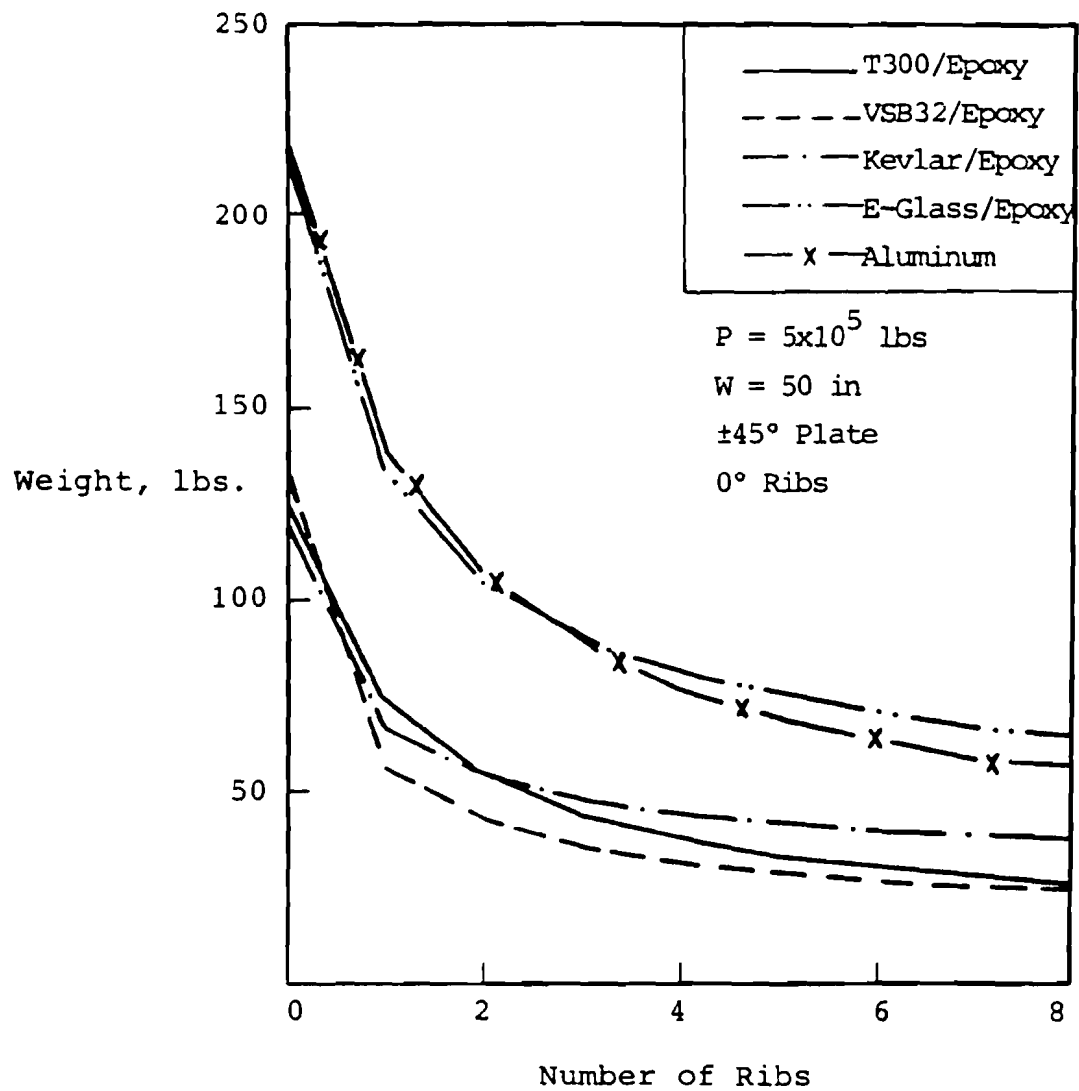


FIGURE 4.6.10. OPTIMAL STIFFENED PANEL WEIGHT VERSUS STIFFENER CONFIGURATION FOR VARIOUS COMPOSITES AND ALUMINUM

TABLE 4.6.3. RESULTS OF STIFFENED PANEL DESIGN

Material	Facesheet Thickness, in	Rib Depth, in	Rib Thickness, in	Cross-Sectional Area, in ²	Weight lbs.
VSB32/Ep	0.070	0.924	0.635	8.17	24.10
T300/Ep	0.127	0.860	0.524	9.97	26.42
Kev/Ep	0.079	3.125	0.484	16.07	37.76
E-Glass/Ep	0.164	1.596	0.808	18.53	64.86
Aluminum	0.190	1.831	0.111	11.46	57.30

- . Number of Ribs = 8
- . $\pm 45^\circ$ Facesheet
- . 0° Rib
- . $v_f = 0.60$

Although VSB-32/Epoxy is the lowest in weight for any number of blade stiffeners, it is not that much lighter than T300/Epoxy or Kevlar/ Epoxy. Therefore, other design factors may provide a logical choice among these three options. At this point we have selected three good candidate designs. The next step would be to evaluate relative costs and to compare the three materials in terms of their environmental stability, etc.

4.6.4 Summary - Design Examples.

The preceding examples of designs with composite materials illustrate, basically, the additional complexities that the designer must face when dealing with composite materials. These complexities are not impossible to overcome, but do render the materials sufficiently different from metals so that the designer must be aware of their existence.

The principal points which must be remembered are the directions of material properties. This means that moduli will vary with direction and that strength may be very dependent upon loading direction. It has been shown and must be remembered that highly structured composite materials, such as unidirectional materials, possess strengths in some directions which are only a few percent of the major strength. Therefore, low secondary loads can govern a design.

The best approach to a composite design is to assume that the secondary strength and/or stiffnesses will govern the design. When a problem is approached in this fashion, the designer will be less likely to overlook a critical property.

Another important feature of composites is the ability to tailor the reinforcement to obtain the most desirable properties for a given application. This ability requires the designer to allow the material configuration to be treated as a design variable. Only by this approach can one be sure that valid conclusions can be drawn regarding comparing various materials.

Obviously, a major problem in designing with composite materials is one of bookkeeping - keeping track of all the material constants, failure modes and potential solutions. Another problem is that simple geometries and boundary conditions only exist in text books. The real world requires cutouts, attachments, thickness changes, all at very inappropriate locations.

The solution to both of these problems is the computer. The computer can be utilized to maintain all of the bookkeeping functions of material properties and failure modes. Additionally, many large scale structural analysis codes can handle composite materials through varying levels of sophistication. A brief survey of some of the popular structural analysis codes which can handle composite materials is given in the following section.

4.7 NUMERICAL METHODS.

Increased interest in composites, sandwich structures, and metals with directional properties has generated more general computational codes which can handle anisotropic material behavior. Finite element and finite difference structural codes are available for stress analysis, stability, vibration, and the dynamic response of composite materials. Additionally, some of these codes can be tied in with, or have builtin, optimization routines.

Most of the available structural analysis codes, which can accommodate anisotropic material properties, treat the material as monogenic. That is, when plate, shell, or solid properties are inputs, they are treated as uniform throughout the element. In other words, the effective laminate properties are utilized and detailed within the laminate, e.g., layer stresses, etc, are not computed. Both to supply properties and to obtain layer stresses, pre and post processors are usually written.

A laminate analysis code can be utilized to enable definition of layer properties and stacking sequences for given structural areas. The output of the laminate code is then generally a file of material properties which is compatible with the given structural analysis code inputs.

The same laminate code can be utilized to interrogate the stress and displacement output files. Knowing the material construction within each element, and the stress and moment or strain and curvatures of that element provide all that is required to obtain individual layer stress and strain states. Finally, failure criteria can then be applied at the layer level.

The post processing step may be considered optional for many cases. Particularly for cases where the stress resultants are reasonably uniform and when strength data exists for the laminate being considered. Post processing is required for cases where the composite strength is not known or at other times when the designer feels that it would be useful to understand additional details of the internal layer stresses. An example of where post processing is a necessity would be that where a high temperature gradient exists through the component thickness. Since the strength of each layer is a function of temperature, it would be necessary to define the stresses within each layer, along with the layer temperature in order to estimate the margin of safety.

A complete survey of structural analysis codes is beyond the intent of this chapter. We will review some of the salient features of the popular commercially available codes. However, the reader should not be biased into thinking that these are the only codes which should be utilized because there are a number of special as well as general purpose codes which are available. A rather complete survey of codes is presented in reference 4.3 and the interested reader should review that reference prior to investing finances into purchasing a code.

Some of the more popular structural analysis codes are listed in table 4.7.1. As indicated by their capabilities, these codes are large, general purpose type codes. Each code has a library of available elements which can range from springs and rods through doubly curved shell elements and multi-noded three dimensional quadrilateral elements. Each of the codes listed possess the capability to handle orthotropic material properties in at least some of the elements. A few of the elements can accept a fully populated stiffness matrix, representing a generally anisotropic material.

In all cases the user must be careful when using plate or shell elements. Very few codes permit inputting both in-plane stiffness, E_t , and bending stiffness, D_t for plate and shell properties. Rather, most codes assume that

$$D = \frac{E_t^3}{12(1-\nu^2)}$$

TABLE 4.7.1. COMMERCIALY AVAILABLE GENERAL PURPOSE FINITE ELEMENT CODES

	STRESS ANALYSIS	THERMAL STRESS	LARGE DEFLECTION	NONLINEAR MATERIALS	NATURAL FREQUENCIES	BUCKLING	MODAL ANALYSIS	TRANSIENT DYNAMICS	FLUTTER	THERMAL ANALYSIS	Developer
ANSYS	✓	✓	✓	✓	✓	✓	✓	✓		✓	Swanson Analysis System Inc.
MARC	✓	✓	✓	✓	✓	✓	✓	✓		✓	MARC Analaysi Research Corp.
											Engineering Mechanics Research Corp.
NISA	✓	✓	✓	✓	✓		✓	✓		✓	
NASTRAN	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	Macneal-Schwendler Corp.

It was shown in Volume I that the above relation is true only when the composite consists of a large number of layers, and those layers consist of a large number of repeated patterns. That is, for any given layer orientation, that orientation must be dispersed through the thickness. Figure 4.7.1 shows examples of the true D_{11} and D_{22} terms versus the above expression. The errors can be significant, therefore the user of a code must be aware of the simplification being made by the code. For a problem where in-plane loading is most significant, one would input the real in-plane properties and let the code be slightly off in its bending calculations.

If one expects, or finds, that the bending stresses are the most significant for a given problem then the desire would be for the code to be utilizing the correct bending stiffnesses.

For a code which computes D 's, this can be accomplished by supplying an artificial in-plane modulus. This modulus must be defined such that the resulting D which is computed is the desired value. That is

$$D_{\text{cole}} = D_{\text{of Laminate}} = \frac{E_{\text{Effective}} t^3}{12 (1 - \nu_{LT}^2)}$$

In this case, by supplying E effective, the appropriate D 's will be utilized but the in-plane stress resultants will be in error.

By allowing both E and t to change, one can force the code to utilize the correct in-plane and bending properties. However, the output will provide the correct in-plane force resultants and bending moments. The stresses calculated in this manner will not be correct. Obviously, the designer must be extremely cautious when analyzing a structure and utilizing the effective properties in various locations. The chances for error are great.

Let us review very briefly some of the salient features of the more popular, general purpose finite element codes.

ANSYS - This code is relatively easy to use, with an extensive element library and the ability to substructure an analysis. Substructuring allows the user to obtain a detailed stress solution in certain regions while retaining a coarse model elsewhere.

Of importance for composites are a number of elements:

1. A 3-D anisotropic solid which accepts a full 6x6 stiffness or compliance matrix as input.
2. A 2-D plane, a 2D axisymmetric and a 3-D element, all of which accept orthotropic material properties.
3. An orthotropic shell element.
4. A laminated, flat triangular shell element.
5. There are currently other laminated shell elements under development which should include interlaminar shear.

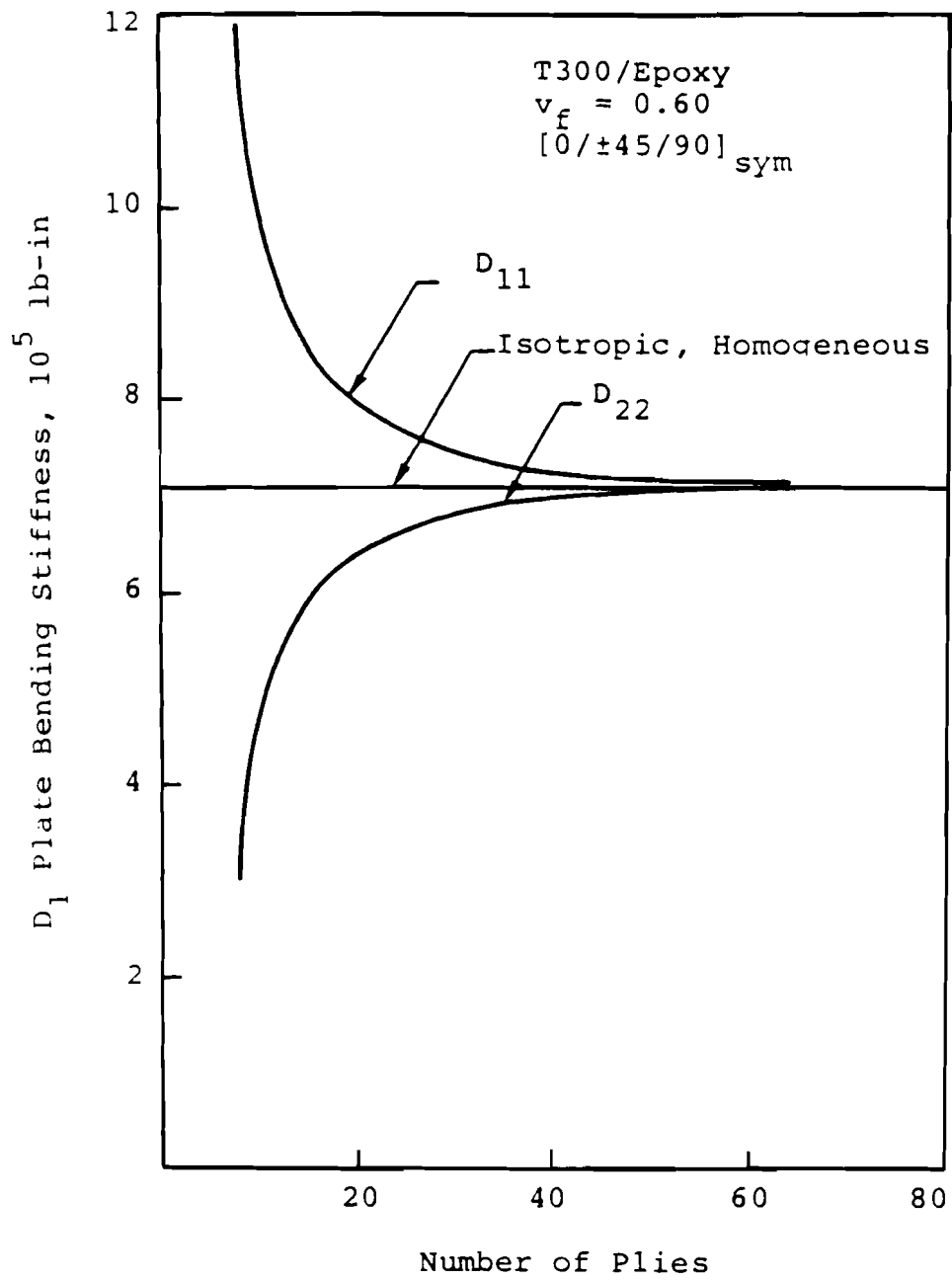


FIGURE 4.7.1 EFFECT OF NUMBER OF PILES ON PLATE BENDING STIFFNESS OF A TYPICAL GRAPHITE/EPOXY LAMINATE

Ansys has very extensive pre- and post-processing capabilities which allow a great deal of flexibility in defining as well as displaying output.

MARC - This is a very general code and is considerably more difficult to use than most other codes. The difficulty arises from the degree of sophistication of the elements and their ability to handle nonlinear material behavior.

Anisotropic material properties may be input for composites but they must be supplied through user written subroutines. Despite the drawback of requiring user written subroutines, MARC offers the most comprehensive material characterization capability.

Additionally, MARC offers a large library of elements for the user. A word of caution, the user should be familiar with programming and use of subroutines for the effective utilization of this code.

NASTRAN - This code is well known to structural analysts as the NASA supported general purpose structural analyses code. The code is relatively easy to use and possesses a substantial element library. Like ANSYS, substructuring is available in NASTRAN. Nastran supports limited capabilities for nonlinear material and geometric behavior. This code has some difficulty with laminates if one must model different membrane and bending stiffnesses.

There are many versions and levels of the Nastran code and associated documentation. The user must therefore be aware that some systems may be supporting different versions of the code.

NISA - This code is perhaps not as well advertised as the previous codes, but it possesses good capabilities relative to composites. Many of its elements are higher order isoparametric elements. It supports a sandwich and a laminated composite shell element, along with a thick shell element which includes interlaminar shear. Like the above codes, NISA also supports a substructuring capability.

The above codes rank in a group of codes which could be termed "General Purpose, Large Scale Finite Element Codes." The advantage of these codes is that they can handle, in some fashion, almost any problem. The general formats for use generally remain similar for various element and analysis options, thus, once the user becomes familiar with one or two element types, it is relatively easy to use other program elements and options.

The disadvantage of general purpose codes is that they are usually costly to execute and require access to large computers. These codes can be cumbersome to use for an organization which does not require ongoing structural analyses. The startup time associated with an engineer refamiliarizing himself with one of these large codes can be significant.

The counterpart to the general purpose code is the large family of special purpose codes. These codes, although limited in their scope, can be very efficient and provide excellent results. A small sampling of some of these codes and their areas of expertise follows:

<u>SPAR</u>	Finite Element with Very Efficient Data Handling
<u>PLANS</u>	Nonlinear Finite Element (orthotropic)
<u>STAGS</u>	General Nonlinear then Shells
<u>VIPASA</u>	Buckling of Stiffened Panels
<u>BOSOR4</u>	Shell of Revolution (finite difference)

Each of these codes, as well as numerous others, has its own area of use specialty. Therefore, the designer should not rely solely upon general purpose codes but should, where appropriate, utilize the more special purpose types.

In analyzing a composite structure, the designer must be careful to insure that the analysis model and code being utilized, have the ability to model the important features of the composite design. Even though a low shear modulus may not induce significant deformations, the analysis model must be capable of including the shear deformation effects. In other words, it is not safe to assume that a material characteristic is unimportant until the analysis shows it to be unimportant.

Specifically, when selecting a code the designer should be careful of the following areas:

1. The code formulation must be sufficiently general to incorporate the material in the design. Additionally, the user must insure himself that he has correctly specified the material properties. An easily overlooked problem is the nomenclature and definition for Poisson's ratio. We CAN NOT assume Poisson's ratio effects to be negligible in composites. If in doubt, it is best to assemble a simple tensile model and compare the output transverse strains to what they should be if the input was correct.
2. Secondary stresses must be obtained accurately. This may require substructuring near joints, fitting, cutouts, etc. Alternately, coarse elements may be used in areas where secondary stresses are relatively unimportant, and a fine model in the important regions.

Figure 4.7.2 illustrates the use of a hybrid element model which permits modeling the major structure yet, by using 2-D elements locally, retain stress definition in important regions. This requirement is not much different than for design with metal structure except for the importance which one must place upon defining the secondary stress levels.

3. Boundary Conditions and Model Symmetries - A common error when modeling with composites is to utilize geometric lines or planes of symmetry to define the analytical model. The coupling terms generated by these laminates will often develop deformations which are not symmetric with respect to geometric symmetries.

Figure 4.7.3 illustrates this effect on a plate subjected to pressure loading. If the material were isotropic, then one eighth of the plate is all that need be

CAUTION: NOT ALL
CODES PROVIDE THE
TYING FEATURES THAT
ARE NECESSARY TO
ACCOMPLISH THIS
TRANSITION

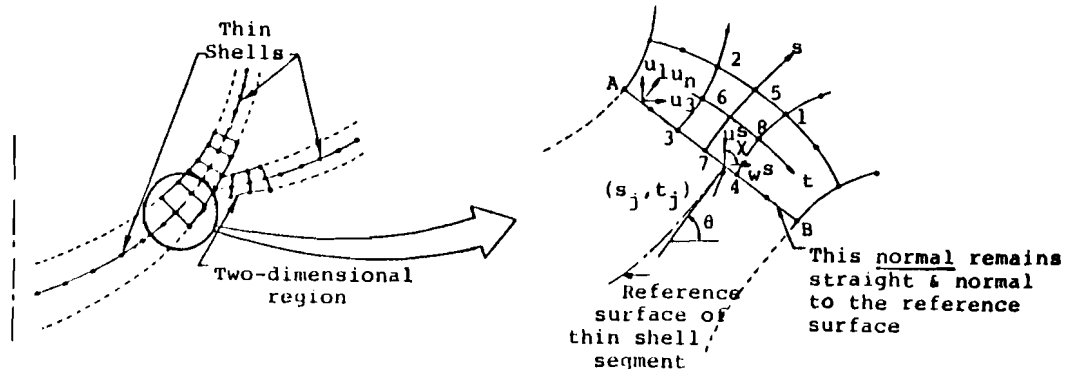


FIGURE 4.7.2 TO OBTAIN SECONDARY STRESS DETAIL IN CRITICAL REGIONS, THE STRUCTURAL MODEL MAY BE TRANSITIONED FROM A PLATE OR SHELL ELEMENT REPRESENTATION TO A SOLID ELEMENT REPRESENTATION, FORMING A 'HYBIRD' MODEL

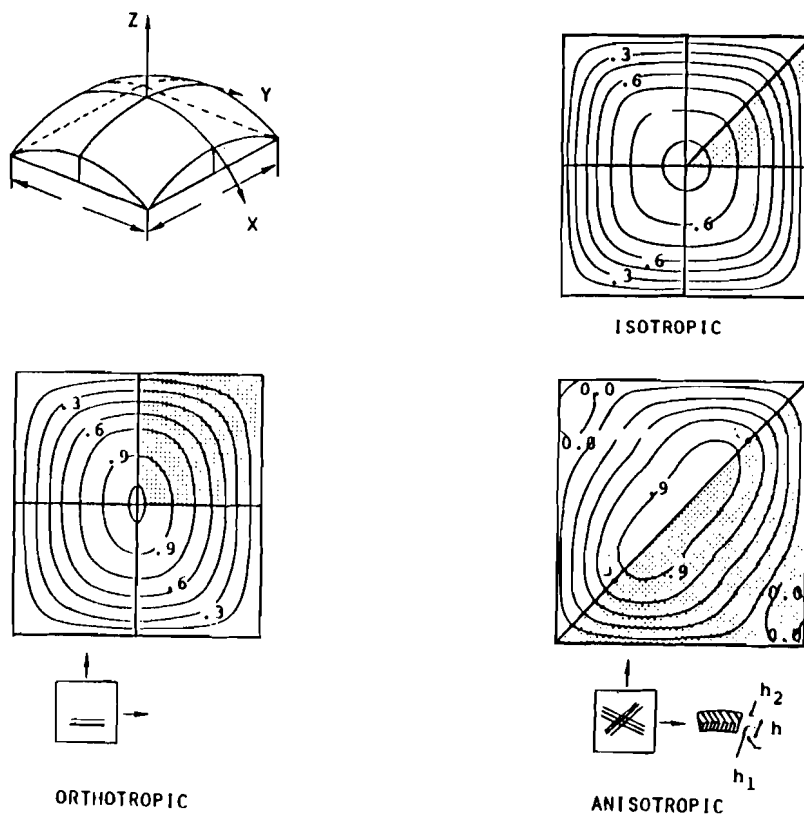


FIGURE 4.7.3 USE OF STRUCTURAL SYMMETRY CONDITIONS FOR COMPOSITE STRUCTURES, AN EXAMPLE OF STATIC PRESSURE LOADING ON SHELL

modeled in order to get a correct solution. For an orthotropic material whose principal axis coincides with the symmetry planes of the plate, one need only model one-quarter of the plate. However, for an anisotropic material one needs to model at least one-half of the plate, and in some cases, the full plate.

The problem which material symmetries presents can be quite serious. An analyst who is unaware of the effects of material properties will proceed to utilize geometric symmetries to section the model into a small segment. If coupling or asymmetries exist in the material, the structural solution will not flag the analyst. The only indication of the effect of the incorrect symmetry conditions will be a requirement for model reaction forces along the symmetry lines. In many codes these forces are not printed out and even in codes where they are printed, they are easily overlooked by the analyst.

Finally, many codes do not have the ability to translate global stress and strain states into individual ply stresses and strain. Therefore, it is often necessary to supply a post processor, laminate loading, code which can break the structural/stress, load, or moment resultants, into local ply stresses. Subsequently, failure criteria can be applied at the ply level.

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APPENDIX A - FABRICATION TECHNIQUES

Due to the wide variety of fiber reinforced composite materials and their applications, a number of fabrication techniques have been developed. These techniques can all be classified under the following general headings: contact molding, compression molding, injection molding, vacuum bag and autoclave molding, filament winding and pultrusion. A variety of material forms suitable to particular fabrication methods are commercially available. These include several types of molding compounds, dry fiber (chopped or continuous), resins, woven fabric, fibrous mat, and preimpregnated fabric or unidirectional tape. While some of these techniques are applied to thermoplastic resin composites (which solidify upon cooling), most have been developed for thermosetting resin composites. Thermosetting resins cure (solidify) by a chemical reaction which is accelerated at elevated temperatures. Each of the general fabrication methods and the particular types of materials employed are discussed in detail below.

CONTACT MOLDING

One of the most common fabrication methods for fiber reinforced composites is called contact molding. As shown in figure A-1, contact molding involves the application of the molding material to an open mold where it is then allowed to cure. Contact molding techniques were developed for the manufacture of large fiberglass components such as boats, swimming pools and custom auto bodies. Since these methods require minimal tooling and equipment costs, they are ideally suited for many low volume, specialty applications. Contact molding consists of two basic approaches: (1) hand lay-up; (2) and spray-up. The two methods differ only in the manner in which the material is applied to the mold. Otherwise, the techniques are very similar.

The first step in fabricating a component by contact molding is to build a mold. Since only one side of the final component (the mold side) has a high quality surface finish, either a male or female mold configuration is selected based upon which side requires a good finish. The mold itself is typically fabricated from materials such as wood, plaster, or metal depending upon the number and type of part being fabricated. It is essential that the mold surface be very smooth and clean to produce a part with a high quality finish and facilitate easy part removal. This is accomplished by fine sanding, polishing, waxing and application of a release agent to the mold surface.

After the mold has been properly prepared, the molding process may begin. The first step in molding a part is the application of a gel coat to the mold surface. This coating is normally a thinned resin and pigment which is sprayed on to give a smooth, shiny surface finish. When the gel coat has hardened, the fiber/resin material may be applied. As mentioned above, this is accomplished by one of two basic techniques: (1) hand lay-up; or (2) spray-up.

When using the hand lay-up method, a layer of wet resin is applied to the hardened gel coat. This step is immediately followed by the application of the first fiber layer. The fibers may be configured in a chopped strand mat, a continuous strand mat, a woven fabric or a woven roving. The fiber layer is generally precut to fit the mold geometry. After the fiber layer is placed in the mold, additional resin is applied to saturate the fibers. This is accomplished by a brush or squeegee used both to apply the resin and work out the air bubbles. When this step is completed, additional fiber layers are applied by

the same procedure until the desired number of layers has been applied. The part is then allowed to cure, normally at room temperature. Curing may be accelerated by the use of heat lamps or by placing the mold in an oven at moderate temperatures. The cure temperature is determined primarily by the curing and flow behavior of the resin/catalyst system being used.

The other method of applying the fibers is referred to as spray-up. In this technique, the fiber/resin mixture is sprayed directly on the mold with an automatic spray-up machine (often called a spray gun). A typical spray-up system (see figure A-1) automatically chops the fibers from a roving to the desired length, mixes the resin and catalyst, and sprays the chopped fibers and resin mixture onto the mold surface. After gel coating, the operator sprays the desired amount of fiber/resin mixture on the mold. It is normally required that the operator stop periodically to compact the lay-up and squeegee out the entrapped air. After the spray-up is complete, the part is cured in the same manner as described above for hand lay-up.

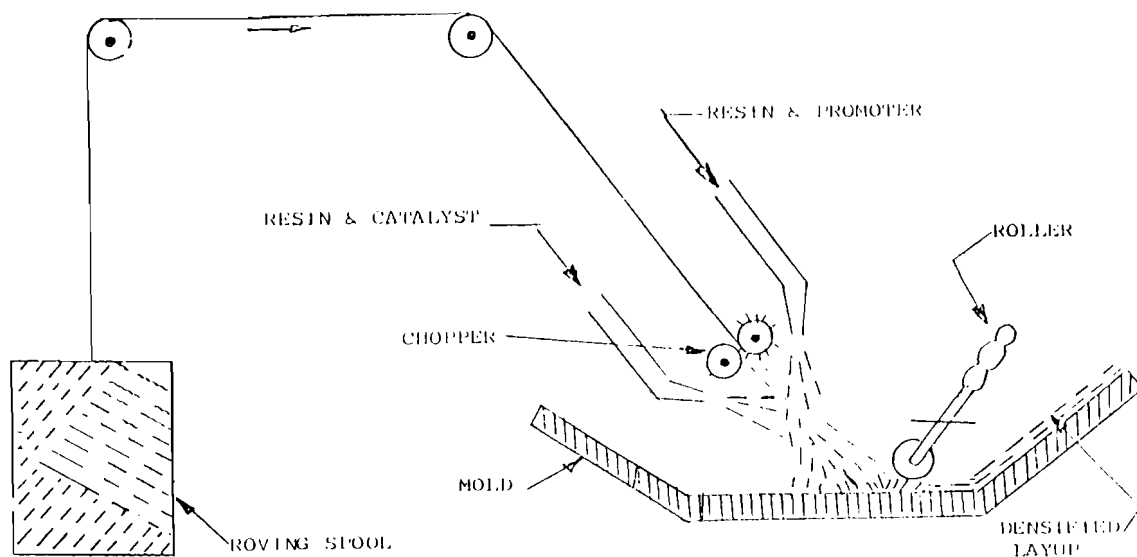


FIGURE A-1. CONTACT MOLDING

The mechanical properties obtained with contact molding are dependent on numerous factors. The selection of resin, fiber, fiber form (mat vs. fabric, etc.) and resin fillers all effect the component performance. In addition, controlling the fiber/resin ratio and fiber direction can also have a strong effect on the resulting material properties. Finally, the quality of workmanship in terms of uniform fiber distribution and elimination of entrapped air is a major factor in determining the ultimate part quality.

The major advantages of contact molding stem from its simplicity, flexibility and relatively low cost. Since the tooling and equipment costs are very low, this technique is practical for low volume production, prototyping and pre-production parts. Due to the low cost simple molds, which are normally used, design changes are easily implemented and start up and lead times are minimized. This

technique is also ideal for very large and complex components. Designs requiring molded-in inserts, structural reinforcements or sandwich construction are all possible with contact molding. Finally, the techniques required can be quickly learned and implemented by semi-skilled laborers.

There are several disadvantages of contact molding. Since the process is highly labor intensive, it can be very expensive when labor costs are high. The method is relatively slow due to the long lay-up and cure times and thus is not suitable for high volume production. Another disadvantage is the fact that only one good surface is produced while the other side is very rough. The amount of material wasted can be very high (up to 50% in some instances). Due to the labor intensive nature of this technique, the part quality is strongly dependent on the skill of the operator and product uniformity is difficult to maintain.

COMPRESSION MOLDING

Compression molding (also called matched die molding) is a fabrication technique well suited for high volume, high speed production. In this process, the part is cured between two mold surfaces under pressure and high temperature. As a result, complex shaped parts can be produced with two high quality surfaces. The elevated temperature produces an accelerated curing reaction with typical cure times from less than a minute to five minutes for thick components.

Figure A-2 shows a typical compression mold. The mold consists of two matching concave and convex mold halves which are brought together to compress the molding compound and to give the desired final part dimensions. To avoid excess shearing of the molding compound when the mold is closed and to facilitate part removal after curing, a slight taper of at least 2-3° between the mold halves is required.

A variety of molding compounds may be used to fabricate a compression molded component; several of the most common types are described below. Bulk molding compound (BMC) is a mixture of the resin, catalyst, fibers and fillers into a putty-type dough or extruded pellets. BMC is also referred to as premix, DMC (dough molding compound) or flowmix. Sheet molding compound (SMC) is a semi-tacky sheet which consists of a chopped strand mat impregnated with a resin and filler. These sheets are cut and laid in the mold. Fabric prepregs are woven fiber fabrics which have been impregnated with a partially cured ("B" - staged) resin to give a tacky sheet which is cut to shape and placed in the mold. Chopped glass preforms consist of chopped fibers and a small amount of resin binder which are combined and shaped on a preform screen to give a preform shape which fits the mold geometry. These are used for complex geometries where conventional molding compounds do not give uniform properties. Finally, wet resin impregnated mats or fabrics similar to that used in hand lay-up contact molding may also be compression molded.

The actual compression molding process is fast and simple. The molding compound is placed into a hot mold which is then closed. Once closed, heat and pressure are applied until the part has cured (normally 1-5 minutes). Typical cure temperatures and pressures are 225°-320° F and 150 - 2000 psi, respectively. If there is a considerable amount of flow during the molding process (particularly when BMC is being used), material non-uniformities may be produced. These non-uniformities are due to flow induced fiber orientations and unreinforced "knit" or "weld" lines where two flow fronts meet.

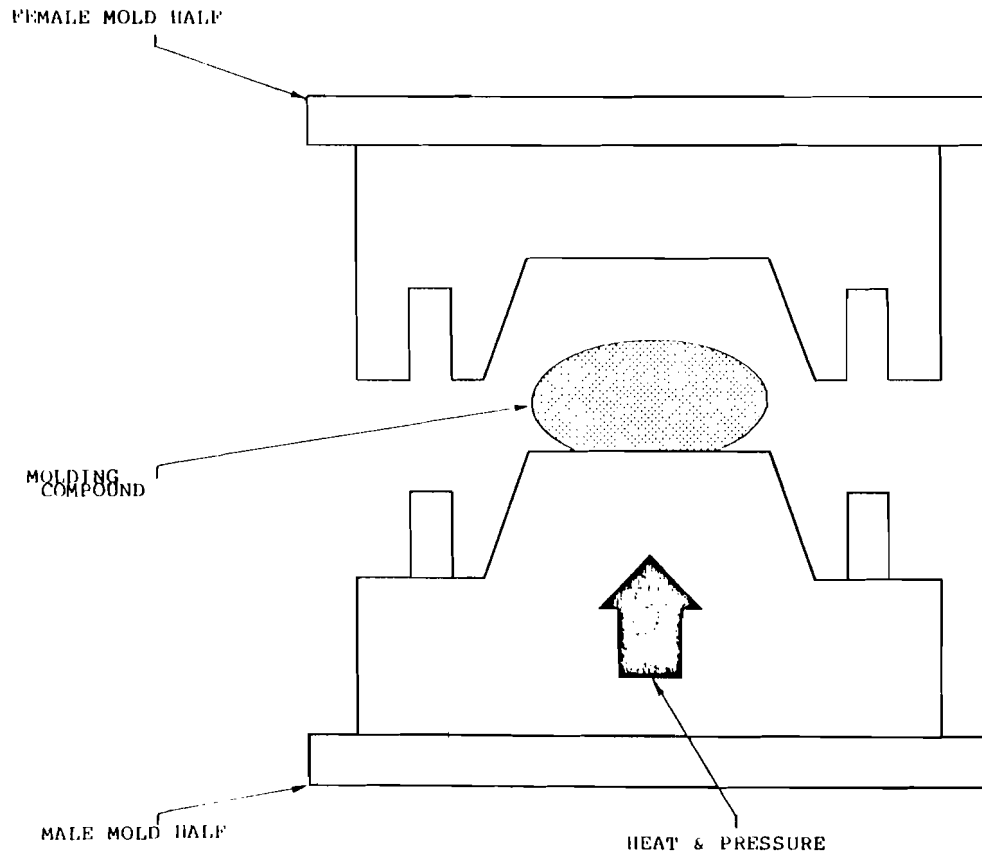


FIGURE A-2. COMPRESSION MOLDING

The primary advantage of compression molding is that it is well suited for many high volume production processes. The process can be highly automated to provide fast production of high quality, uniform components with excellent surface finishes. A wide variety of shapes, sizes and complexity may be produced by compression molding. Molded-in inserts and attachments can also be included in the part design.

The major disadvantage of compression molding is the relatively high start-up expense. Mold design must be very thorough because subsequent design changes are often too expensive to be practical. The tooling costs which include both the mold fabrication and equipment investment tend to be very high. Thus, compression molding is usually impractical for low volume production.

INJECTION MOLDING

As implied by the name, injection molding is the process whereby the molding compound is heated, liquified and injected into a closed mold cavity. This process can be highly automated to give high volume production at very high rates. Multicavity molds are often used to provide even higher production rates.

Bulk molding compound (BMC) in the form of a dry blend, extruded pellets or putty type dough is normally used for injection molding. A typical mold configuration (see figure A-3) consists of a feed hopper, injection chamber, plunger and/or screw, and mold cavity. The molding compound is fed into the injection chamber via the feed hopper. The molding compound is heated to a melt in the injection

chamber. Often a rotating screw is used to advance the material through the chamber and to produce a uniform temperature distribution. When the molding compound achieves the desired temperature and viscosity, a plunger forces it to flow through a series of runners and gates into the mold cavity. For thermoplastics, which solidify upon cooling, the mold is kept relatively cold. For thermosetting resins, solidification occurs at elevated temperatures and thus the mold is kept hot. When the part has solidified, the mold is opened and the part is ejected. The mold is then closed and the cycle is repeated.

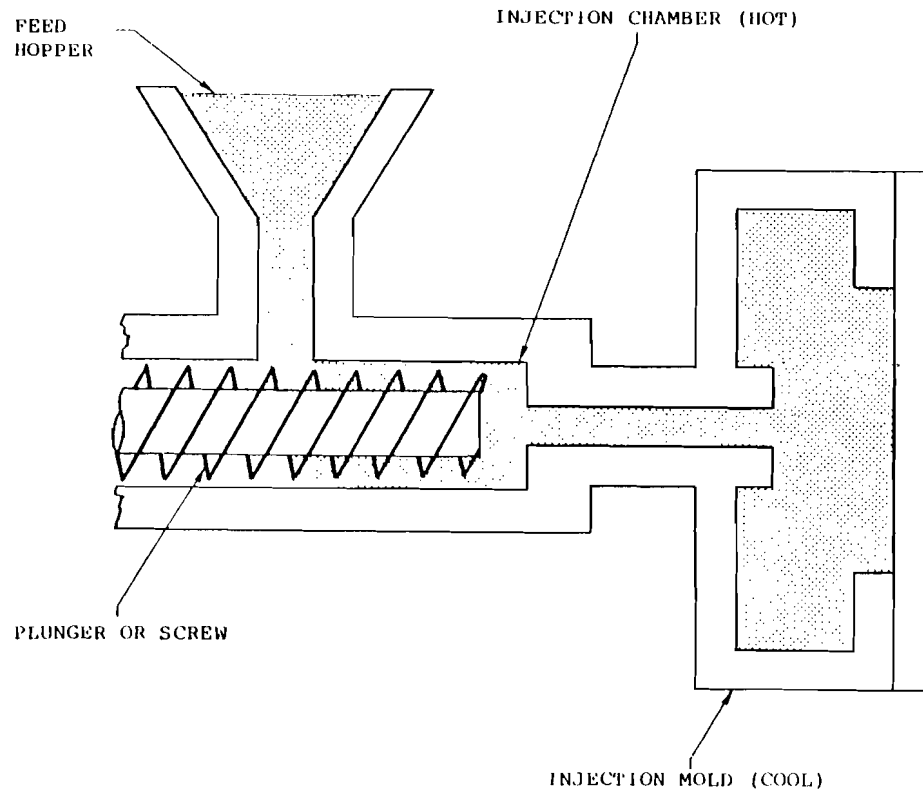


FIGURE A-3. INJECTION MOLDING

Injection molding is normally used for high volume, low cost component manufacturing. As a result it is commonly used for numerous consumer goods such as automotive and appliance parts. In addition, parts of varying size and complexity may be molded by these techniques. Since injection molding of unreinforced plastics has been very common for many years, the techniques and equipment are very sophisticated and highly automated.

There are two basic disadvantages of injection molding. Due to the large amount of flow during the injection process, material nonuniformities such as fiber orientation and weld lines are often produced. An effective mold design can take advantage of these effects to give enhanced properties at high stress areas or at least avoid deleterious effects at critical locations. The other disadvantage of injection molding is that it is limited to materials with very short fiber lengths. This limits the enhancement of mechanical properties which can be achieved by the addition of fiber reinforcement.

VACUUM BAG AND AUTOCLAVE MOLDING

Vacuum bag molding techniques can be applied to a variety of fiber reinforced plastic fabrication processes. These techniques are often used in conjunction with the hand lay-up contact molding method previously described. The techniques are also used for manufacturing aerospace components. The hand lay-up vacuum bag techniques are used with dry fiber (normally in fabric or mat form) impregnated with a wet resin. The aerospace molding techniques, on the other hand, are used for preimpregnated, partially cured unidirectional fiber or woven fabric materials which are generally called "prepregs". These components are often cured at elevated temperatures and pressures in an autoclave.

In the wet resin/dry fiber vacuum bag procedure, the mold configuration and lay-up procedures are identical to the hand lay-up procedures described previously. As shown in figure A-4, the lay-up is covered by a vacuum bag which is sealed around the edges of the mold using a sealant or mechanical clamps and rubber gaskets. In addition, an edge bleeder is placed around the edges of the lay-up to absorb excess resin which may flow during curing.

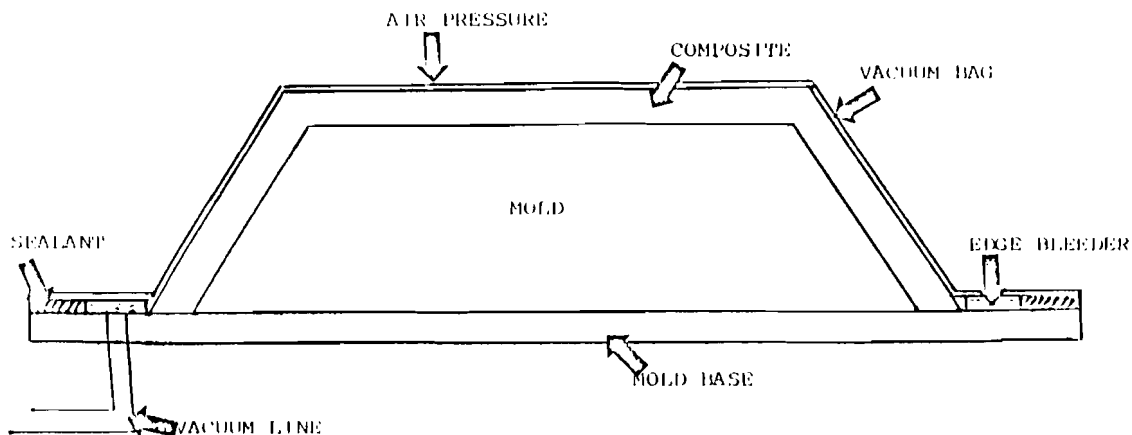


FIGURE A-4. VACUUM BAG MOLDING

A vacuum pump is then attached to the mold to evacuate the lay-up. This helps eliminate entrapped air and provides compaction from the atmospheric pressure. Often manual squeegeeing is also required to completely remove the entrapped air. The removal of excess resin results in a higher fiber content which gives enhanced mechanical properties. The lay-up is usually allowed to cure at room temperature but heat lamps or an oven can be used to reduce the cure time. After curing, the film and sealant are removed. Since the film side of the mold has a high quality surface finish, the need for a gel coat is often eliminated. Finally, the rough edges of the part are normally trimmed to give the required contours.

To fabricate prepreg materials, layers of the prepreg are cut and laid up to give the desired shape and fiber orientations. This layered structure, often called a laminate, is then applied to the mold surface. For complex geometries, the layers are placed ply by ply directly on the mold. Slight heating of the plies with a heat gun is sometimes necessary to improve the conformability. The laminate is then enclosed in the vacuum bag. The exact bagging configuration, is

specified by the prepreg supplier but typically contains an edge dam to assure that the part maintains its shape, a non-stick permeable cloth on the laminate surface, bleeder layers to absorb excess resin, and a vacuum film sealed around the edges. The bag is then connected to a vacuum pump, evacuated, and checked for leaks. A typical configuration is shown in figure A-5. The laminate is then ready to be cured. The vacuum is maintained throughout the cure cycle to remove entrapped air or volatile gases which may be produced. Since the high performance epoxy resins used in these materials require an elevated temperature cure, the curing is done in an oven or autoclave. The autoclave is a heated pressure vessel which is pressurized with a non-flammable gas to 40-100 psi. This additional pressure results in greater resin flow (higher fiber content) and more uniform properties. The exact cure cycle (heating and pressurization sequence) is a function of the resin cure and flow characteristics and is specified by the material supplier. When the cure is complete, the laminate is allowed to cool down under vacuum and then the bagging materials are removed. Often a post-cure is recommended for enhanced properties. This involves placing the component in an oven at elevated temperatures for several hours to allow the resin to complete the cure reaction.

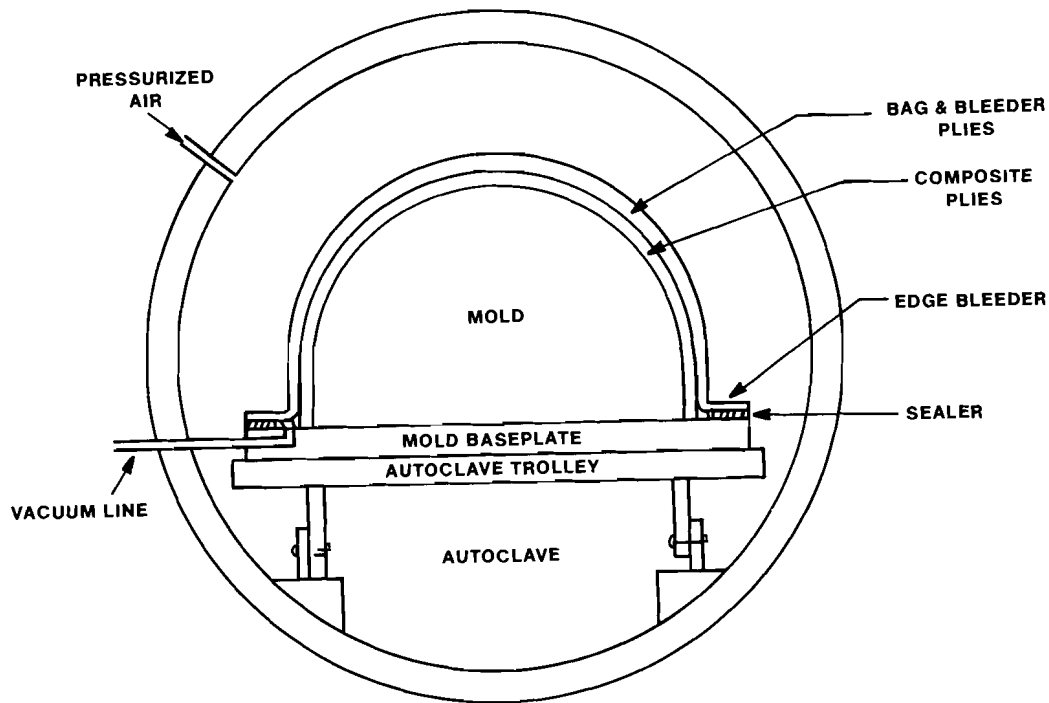


FIGURE A-5. AUTOCLAVE MOLDING

The primary advantage of using vacuum bag fabrication techniques are the material properties which are produced. Due to the removal of entrapped air, resin flow, and compaction, laminates with uniform properties and high fiber content are produced. For hand lay-up methods, the additional cost of vacuum equipment and bagging materials is often well worth the resulting improvement in properties. The primary disadvantage of vacuum bag and autoclave molding is the large amount of labor required. The recent introduction of prepreg cutting machines and robotic lay-up equipment has helped to automate many of these procedures.

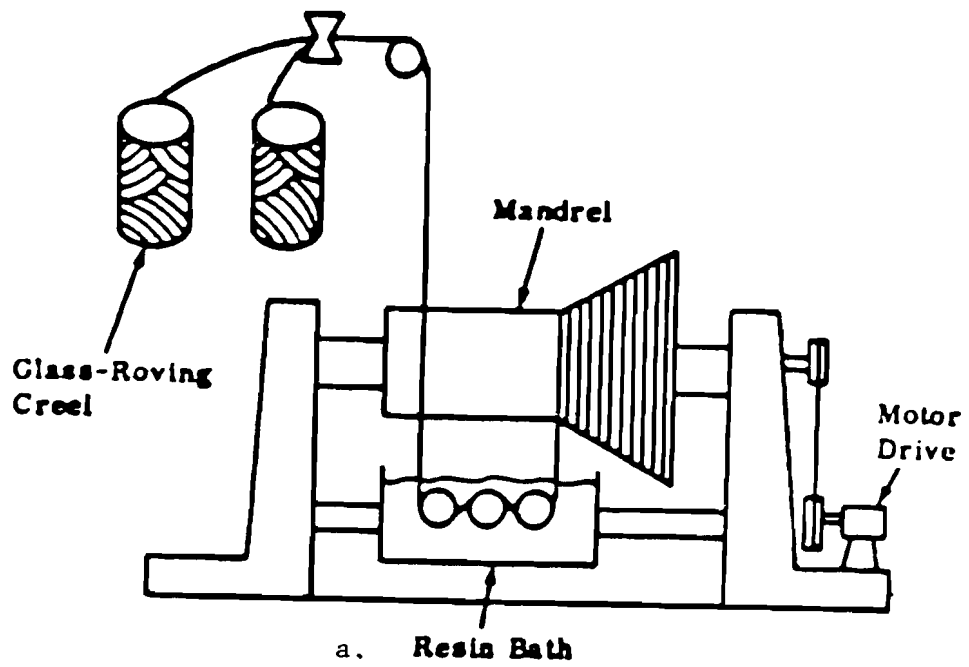
Another disadvantage of autoclave molding is the large capital investment required, especially for large components.

FILAMENT WINDING

Filament winding is a highly automated technique whereby resin impregnated fibers are wound around a mandrel. As a result, this technique is often used for hollow, cylindrical geometries. The recent development of more sophisticated winding techniques are making more complex geometries feasible. Since filament wound components have continuous fiber reinforcement, the resulting mechanical behavior is excellent. Thus, several structural applications such as pressure vessels, pipes and driveshafts are made by filament winding.

As shown in figure A-6, this technique typically involves continuous fibers being wrapped around a rotating mandrel. The process may use either dry fibers fed through a wet resin bath (wet winding) or preimpregnated fibers (dry winding). There are two basic manners in which the structure is wound. Helical winding (figure A-7) involves a rotating horizontal mandrel and a horizontally traversing fiber feed carriage. By controlling the mandrel rotation and feed carriage speeds, the fiber angle is controlled. Angles of 25° - 85° to the mandrel rotation axis are possible with this approach. If longitudinal fibers are required (less than 25° to the mandrel axis), a technique called polar winding is used. In this approach, the fiber feed carriage is rotated about a vertical, stationary mandrel (see figure 6b). Several new, sophisticated filament winding techniques are being developed which employ robotic fiber feed mechanisms, multiple fiber feeds, and mandrels which both rotate and translate horizontally. After winding is complete, filament wound structures may be cured at room temperature, under heat lamps or in an oven. The components are often rotated during cure to give a uniform resin distribution. Filament wound components may also be cured in an autoclave using vacuum bag techniques. After curing, the mandrel is removed and the part is machined to the required dimensions.

Filament winding techniques have many advantages. Due to the use of continuous fibers, excellent mechanical properties are obtained. Since the fiber orientations are easily controlled, there is a very high degree of design flexibility. The highly automated nature of this technique results in high part uniformity. Another advantage is that since closures and fittings may be wound in, the degradation due to cut-outs can be avoided. The major disadvantage of filament winding is the large initial investment required. Another disadvantage is that complex shapes are difficult to wind and require expensive sophisticated winding equipment.



Wet Winding System

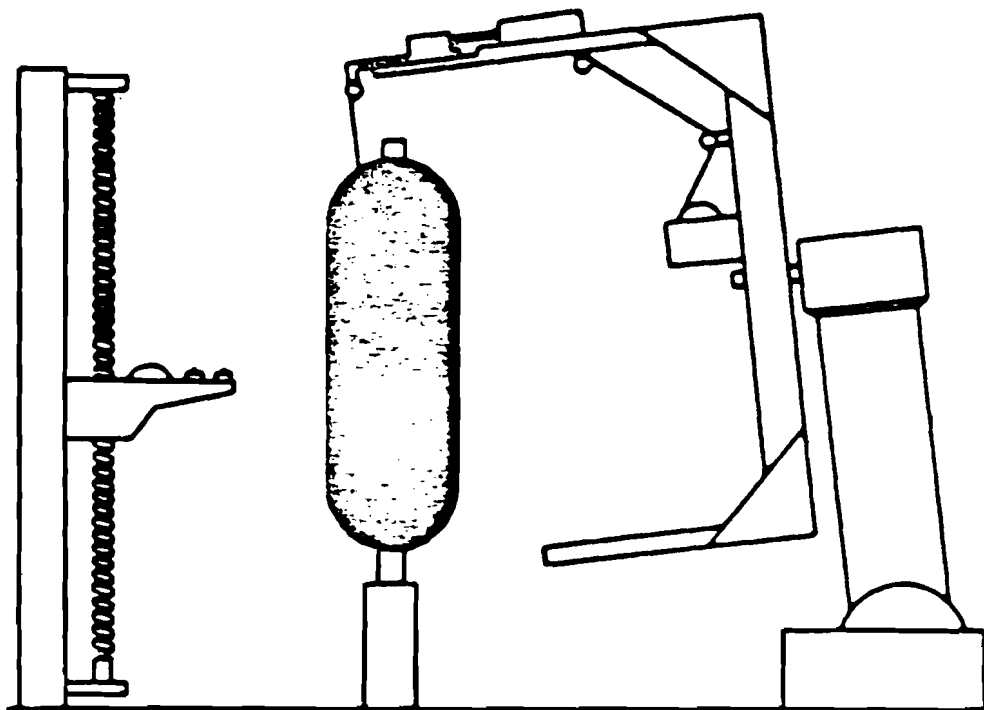


FIGURE A-6. FILAMENT WINDING

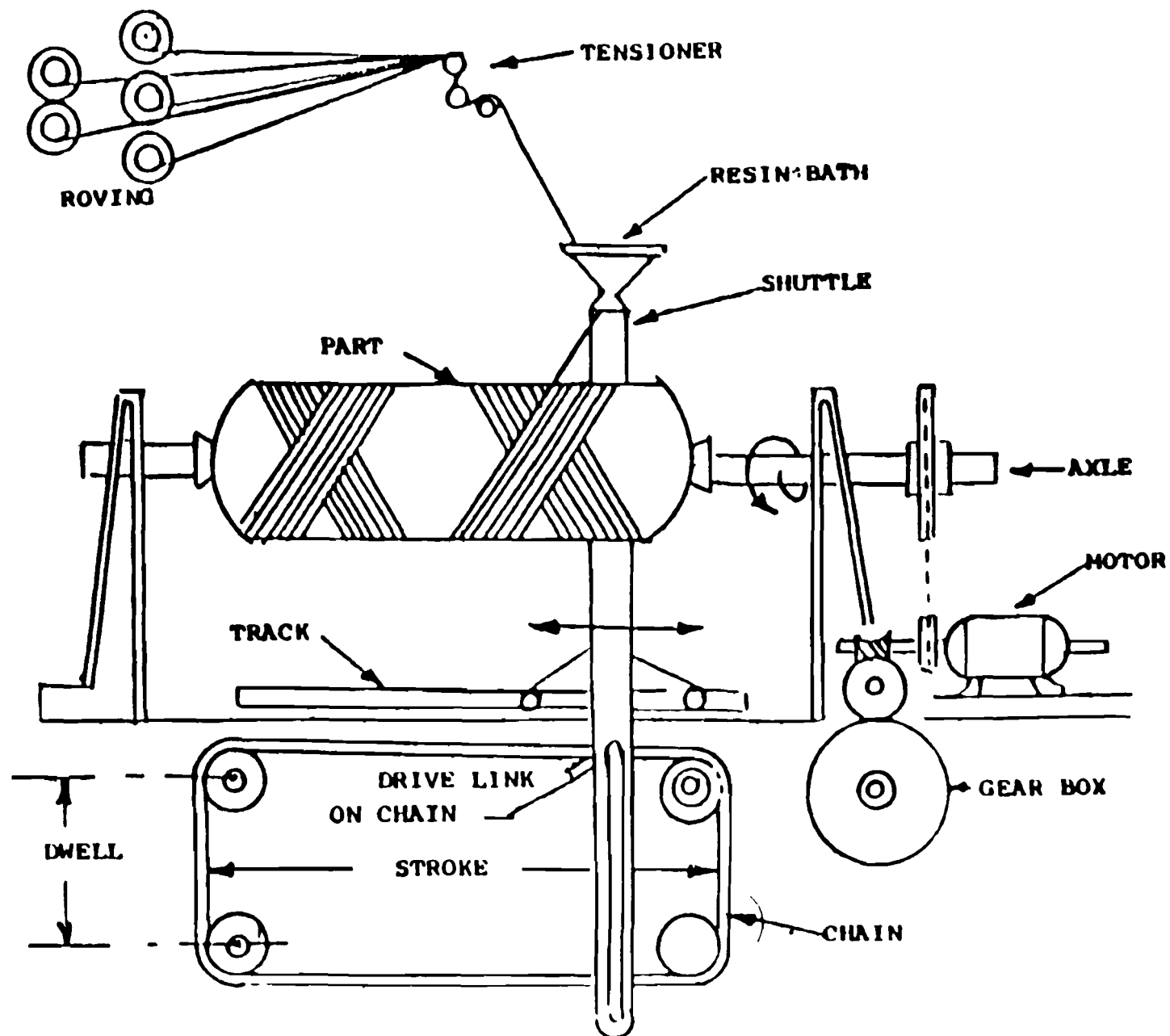


FIGURE A-7. FILAMENT WINDING PROCESS

PULTRUSION

Pultrusion is a process derived from extrusion methods for non-reinforced plastics. Pultrusion is used to manufacture stock sections of constant cross section fiber reinforced products. Common applications of pultrusion methods are the fabrication of rod stock, structural shapes (such as I-beams), pipes, channels, tubing and fishing rods. These components are made by pulling continuous strand roving, mat or cloth through a wet resin bath and then a metal die which forms the material into the desired cross section. After forming by the die, the part is pulled through a heated chamber until the resin cures. Figure A-8 shows a typical pultrusion process. After curing, the section is cut to the desired length.

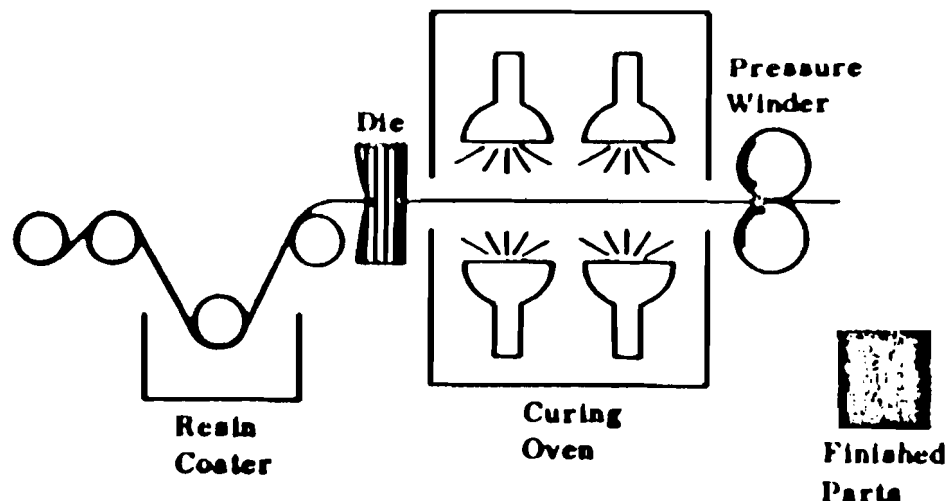


FIGURE A-8. SCHEMATIC DIAGRAM FOR THE CONTINUOUS EXTRUSION PROCESS

The major advantage of pultrusion is that it is the only truly continuous process for fabricating fiber reinforced composites. It provides an excellent method for producing uniform high quality components. The major limitation of pultrusion is that it is restricted to constant cross section bar or sheet stock. More complex geometries are not feasible by this technique.

SUMMARY

In conclusion, it can be seen that a fiber reinforced composite may be manufactured by one of many standard fabrication procedures. The selection of the best procedure for a given application requires the consideration of numerous factors. The selection process must determine the appropriate constituent materials, the actual part design, the required tooling, the fabrication process, and the necessary inspection and testing requirements to assure part quality.

In order to select the appropriate part design and fabrication procedure, several criteria must be satisfied. These criteria can be broken down into two types of requirements: performance and cost. The performance requirements include the in-service mechanical, thermal, electrical and chemical factors which the component must tolerate. The cost requirements include labor and materials costs, required capital investment, and the performance premium obtained with

various materials and/or designs. A thorough investigation of production costs must include an analysis of both recurring and non-recurring costs. Recurring costs include such factors as labor costs for both manufacturing personnel and indirect engineering, quality control and maintenance support personnel. The other recurring cost is the cost of raw materials. Non-recurring costs also include both labor and materials. These labor costs are for engineering design and testing, planning and tooling manufacture. The non-recurring material costs include tooling and prototyping materials. In summary, the selection of the most cost-effective design and fabrication process must include many factors besides the direct fabrication costs.

APPENDIX B - MATERIAL PROPERTY TESTS

The characterization of fiber reinforced composites requires the development of a broad range of information which is largely a function of the end use application. However, for an arbitrary application the following properties should be evaluated for a given material system:

1. longitudinal modulus (tension and compression),
transverse modulus (tension and compression),
longitudinal shear modulus,
major Poisson's ratio (tension and compression),
minor Poisson's ratio (tension and compression);
2. longitudinal strength (tension and compression),
transverse strength (tension and compression),
longitudinal shear strength (in-plane and interlaminar);
3. longitudinal strain to failure (tension and compression),
transverse strain to failure (tension and compression),
longitudinal shear strain to failure (in-plane and interlaminar);
4. coefficient of longitudinal thermal expansion,
coefficient of transverse thermal expansion;
5. coefficient of longitudinal moisture expansion,
coefficient of transverse moisture expansion;
6. longitudinal moisture diffusion coefficient,
transverse moisture diffusion coefficient,
maximum moisture content;
7. longitudinal thermal conductivity,
transverse thermal conductivity,
specific heat capacity;
8. longitudinal impact energy absorption (tension and compression),
transverse impact energy absorption (tension and compression)
longitudinal shear impact energy absorption, (in-plane and interlaminar),
Charpy impact (longitudinal and transverse),
Izod impact (longitudinal and transverse);
9. creep compliance
relaxation modulus,
longitudinal creep rate (tension and compression),
transverse creep rate (tension and compression),
longitudinal shear creep rate (in-plane and interlaminar);
10. longitudinal fatigue (tension and/or compression),
transverse fatigue (tension and/or compression),
longitudinal shear fatigue (in-plane and interlaminar);

11. longitudinal storage modulus (tension and compression),
transverse storage modulus (tension and compression),
shear storage modulus (in-plane and interlaminar),
longitudinal loss tangent (tension and compression),
transverse loss tangent (tension and compression),
longitudinal shear loss tangent (in-plane and interlaminar);
12. longitudinal dielectric constant,
transverse dielectric constant;
13. longitudinal electrical conductivity,
transverse electrical conductivity;
14. density,
surface hardness,
wear resistance,
lubricity;
15. chemical resistance,
resistance to ultraviolet.

Many of these properties are strong functions of temperature and moisture content. Thus, each of these properties must be characterized over the temperature and moisture content range of interest.

A brief discussion of the various test methods that have been developed to measure these properties is contained in this appendix.

CONSTITUENT PROPERTY TESTS

Single Filament Tensile Properties

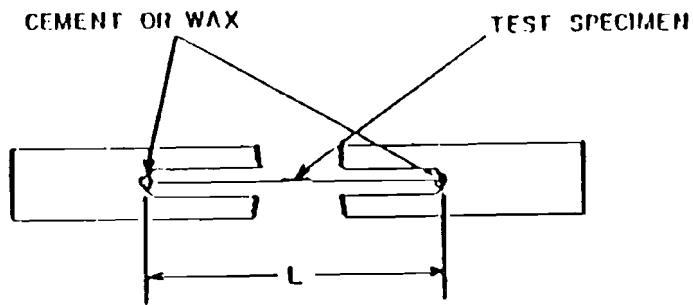
This test method is used to determine strength, modulus, and failure strain of single filaments. The method consists of applying constant strain rate loading to a single filament, center-line mounted on special slotted tabs as shown in figure B-1(A). This method is discussed in ASTM D3379-75, and is limited to fibers with a modulus above 21×10^9 PA (3×10^6 PSI). The fixed gage length of the fiber, L, must be at least 2000 times the nominal filament diameter.

This test procedure consists of loading the specimen to failure at a constant cross-head rate and recording a load-displacement curve.

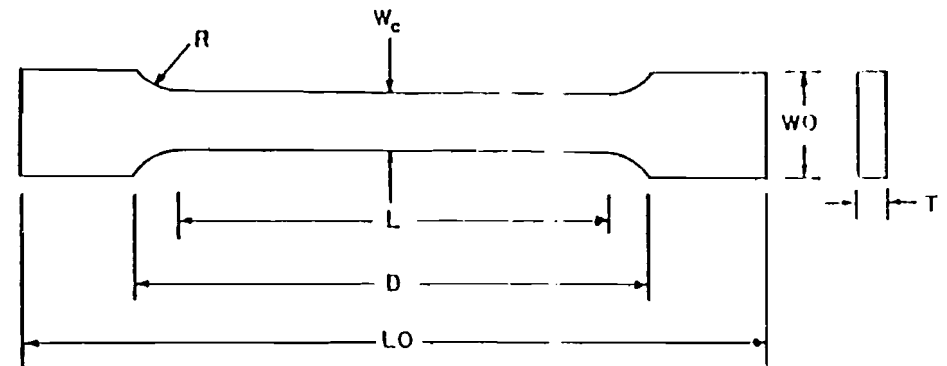
Polymeric Matrix Tensile Properties

For resin systems which can easily be processed in thick sheets, a standard dogbone type specimen, as shown in figure B-1(B), is used for tensile property determination. Details of the exact specimen dimensions are spelled out in ASTM D638-72.

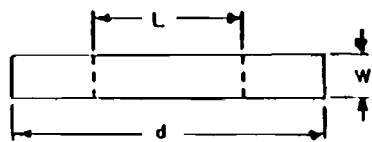
Resin systems which cannot be readily fabricated in thick sheets can often be fabricated into thin sheets or films. The details for testing such materials are discussed in ASTM D882-73. This method is applicable to specimens with a thickness, h, less than 1.0 mm (0.04 in). The specimen is shown in figure B-1(C) where the test gage section is denoted by L.



(a) SINGLE FILAMENT TENSILE TEST



(b) NEAT RESIN TENSILE SPECIMEN



(c) ALTERNATE NEAT RESIN TENSILE SPECIMEN

$d - L \geq 50 \text{ mm (2 in)}$
 $5 \text{ mm (0.2 in)} \leq W \leq 25.4 \text{ mm (1.0 in)}$
 $W/h \geq 8$
 $h = \text{THICKNESS}$
 $L = \text{GAGE LENGTH}$

$W_n = \text{WIDTH OF NARROW SECTION}$
 $L = \text{LENGTH OF NARROW SECTION}$
 $W_0 = \text{OVERALL WIDTH, MINIMUM}$
 $L_0 = \text{OVERALL LENGTH, MINIMUM}$
 $D = \text{DISTANCE BETWEEN GRIPS}$
 $R = \text{RADIUS OF FILLET}$
 $T = \text{THICKNESS}$

FIGURE B-1. CONSTITUENT TEST SPECIMENS

It should also be noted that a dogbone type film specimen can be utilized by molding to the desired dimensions.

Density

The density of polymeric matrix resins and fiber reinforced composites may be determined by measuring the difference between the weight of the specimen in air and in water. The volume of the specimen must be at least 1 CM³ (0.06 IN³), and the surface edges should be smooth. A complete discussion of the test method is covered by ASTM D792-66.

It should be noted that a similar procedure may be used for measuring the density of a single filament. The liquid chosen, however, must be capable of completely wetting the fiber.

Fiber Volume Fraction

There are currently two methods available, for determining the fiber volume content. The first method involves the digestion of the matrix by a liquid media, the resulting fiber weight can then be used in conjunction with the composite and fiber densities to determine the volume percent of fibers. The second method involves a simple determination of the composite density. If the composite is essentially free of voids, then the composite density can be used in conjunction with the fiber and matrix densities to calculate the fiber volume content.

The digestion method is applicable to both polymeric and metal matrix composites. The selection of the liquid for digestion of the matrix is however, very critical. The liquid should be compatible with the fiber (i.e. no fiber degradation), yet be efficient at digesting the matrix material. More details concerning the choice of the liquid for digestion can be found in ASTM D3171-76 for polymeric composites, and in ASTM D3553-76 for metal matrix composites.

Coefficient of Thermal Expansion

Measurement of the coefficient of thermal expansion involves measuring the dimensional change of a specimen over a specified temperature range. The coefficient of thermal expansion is the slope of the strain versus temperature plot.

Appropriate devices, such as an extensometer, may be used to measure thermal strains. This technique is also applicable for polymeric resins which can be fabricated in thick sheets. A detailed procedure for measuring the thermal coefficient of expansion of a cast resin, employing a quartz-tube dilatometer, can be found in ASTM D696-70.

Coefficient of Moisture Expansion

The method used in determining the coefficient of thermal expansion is analogous to that used for determining the coefficient of thermal expansion. Measurement of the coefficient of moisture expansion involves measuring the dimensional change of a specimen over a specified range of moisture weight gains. The difficulty that arises from this technique is that of accurately measuring the dimensional change (i.e., strain), as conventional strain gage adhesives are attacked by moisture.

Composite Test Methods

Characterization of composite material systems requires that experimental tests be performed which yield intrinsic strength and stiffness properties of the material. Much effort has been expended in developing test methods which are appropriate in assessing material variations due to processing or raw material variables. However, these "quality assurance" test methods do not generally yield data which may be directly related to material performance. Although the test methods reviewed in this section were designed to provide intrinsic material property data, the degree of achievement of this goal varies from method to method.

The fundamental principle underlying these test methods can be stated as follows:

The layer of given fiber orientation distribution is the building block of the laminate. Therefore, characterization of layer material properties allows predictions of the properties of any laminate.

In actual practice, the above statement has been demonstrated to hold only for prediction of laminate elastic stiffness properties such as effective modulus or Poisson's ratio. However, prediction of laminate strength properties from intrinsic layer (lamina) strength data has proven more difficult. Therefore, in certain cases it is necessary to resort to direct characterization of laminate strength properties.

Tensile Test Methods

The tensile test is performed in order to determine uniaxial tensile strength, Young's modulus, and Poisson's ratio. The following properties may be determined from the tensile test method:

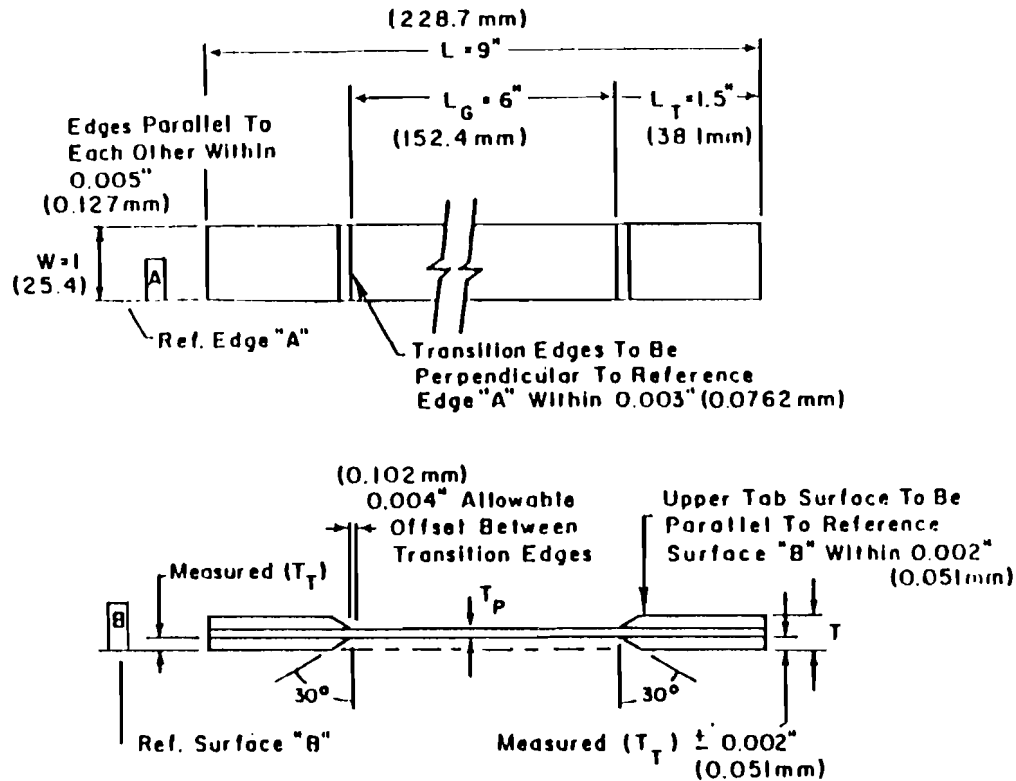
- E_1 = Young's modulus in the principle direction
- E_2 = Young's modulus transverse to the principle direction
- ν_{12} = Major Poisson's ratio
- ν_{21} = Minor Poisson's ratio
- σ_{L}^{TU} = Ultimate tensile strength in fiber direction
- σ_T^{TU} = Ultimate tensile strength transverse to fiber direction
- ϵ_L^{TU} = Ultimate strain in the fiber direction
- ϵ_T^{TU} = Ultimate strain transverse to fiber direction

A typical tensile test specimen is shown in figure B-2. A detailed discussion of the test method is covered in ASTM D3039-76.

Compression Test Methods

Perhaps the most difficult of the intrinsic material properties of composites to measure are the compressive strength properties. This is the case due to the fact that slight specimen geometric variations result in eccentricity of the applied load, thereby enhancing the opportunity for failure to occur due to

geometric instability. Thus, in order to achieve an accurate measure of the compressive strength of a given composite material, rather complex loading fixtures and specimen configurations have been developed.



W	L	L _G	L _T	T	T _T	T _P	
±0.0156	±0.1250	±0.0625	±0.0625	±0.0150	±0.0050	—	Inches
0.396	3.175	1.5875	1.5875	0.381	0.127		mm

FIGURE B-2. GEOMETRY OF TENSILE COUPON

There are basically three generic compression test methods:

I. Specimen geometry characterized by a relatively short and completely unsupported test section.

II. Specimen geometry characterized by a relatively long and fully supported test section.

III. Specimen geometry characterized by a straight-sided coupon bonded to a honeycomb core, which supplies the support necessary for stability.

A summary of the various test fixtures for each of the above categories is shown in table B-1.

TABLE B-1. TYPES OF COMPRESSION TEST FIXTURES

TYPE I

- o CELANESE TEST FIXTURE (ASTM D-3410-75)
- o IITRI TEST FIXTURE
- o NORTHROP TEST FIXTURE
- o NATIONAL BUREAU OF STANDARDS (NBS) TEST FIXTURE

TYPE II

- o SOUTHWEST RESEARCH INSTITUTE TEST FIXTURE
- o LOCKHEED TEST FIXTURE

TYPE III

- o SANDWICH EDGEWISE COMPRESSION TEST
- o SANDWICH BEAM COMPRESSION TEST

Type I Compression Test Methods

Test specimen geometry, and test fixture descriptions (schematics) for each of the Type I compression test methods are shown in figures B-3 through B-7.

The four Type I compression test methods appear to yield acceptable data. However, certain problems may be encountered in their implementation. The Celanese test method requires extreme precision in mounting the specimen in the fixture.

The mass of the IITRI test fixture requires prolonged soak periods for elevated temperature tests. All of the Type I test fixtures require that test specimens be fabricated such that specimen edges and/or tab surfaces are extremely parallel.

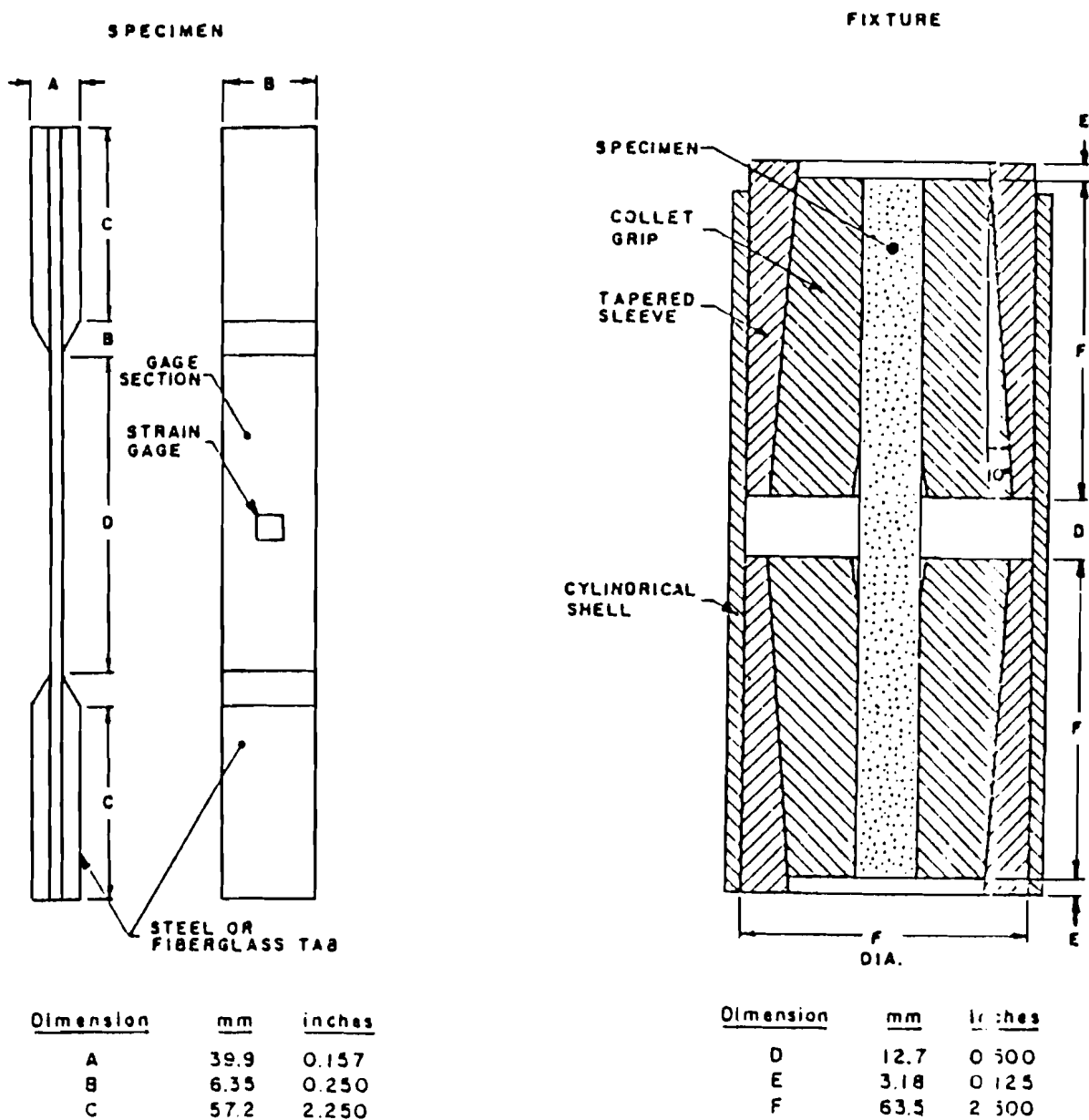


FIGURE B-3. CELANESE COMPRESSION SPECIMEN AND TEST

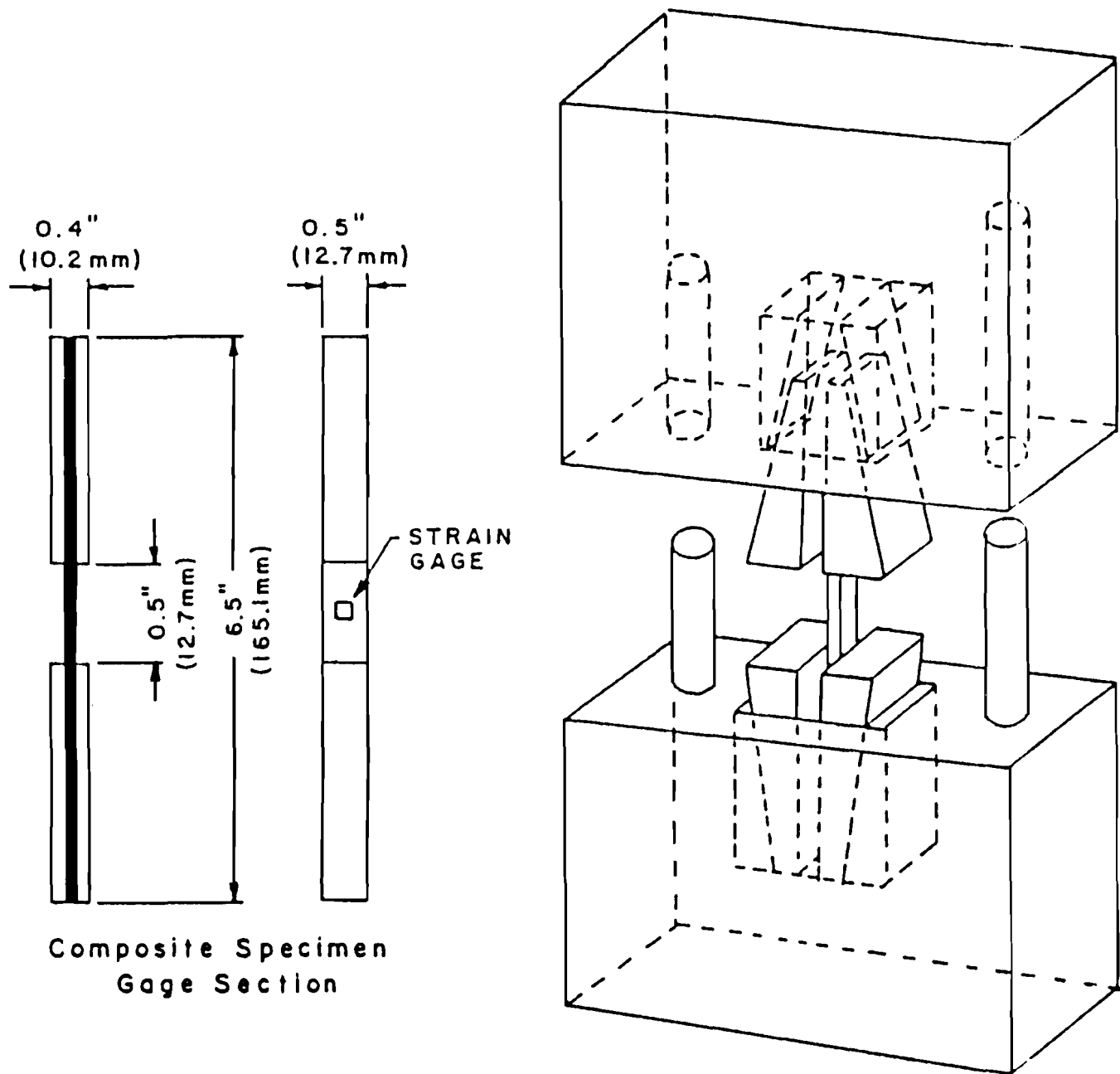


FIGURE B-4. IITRI COMPRESSION TEST FIXTURE AND SPECIMEN

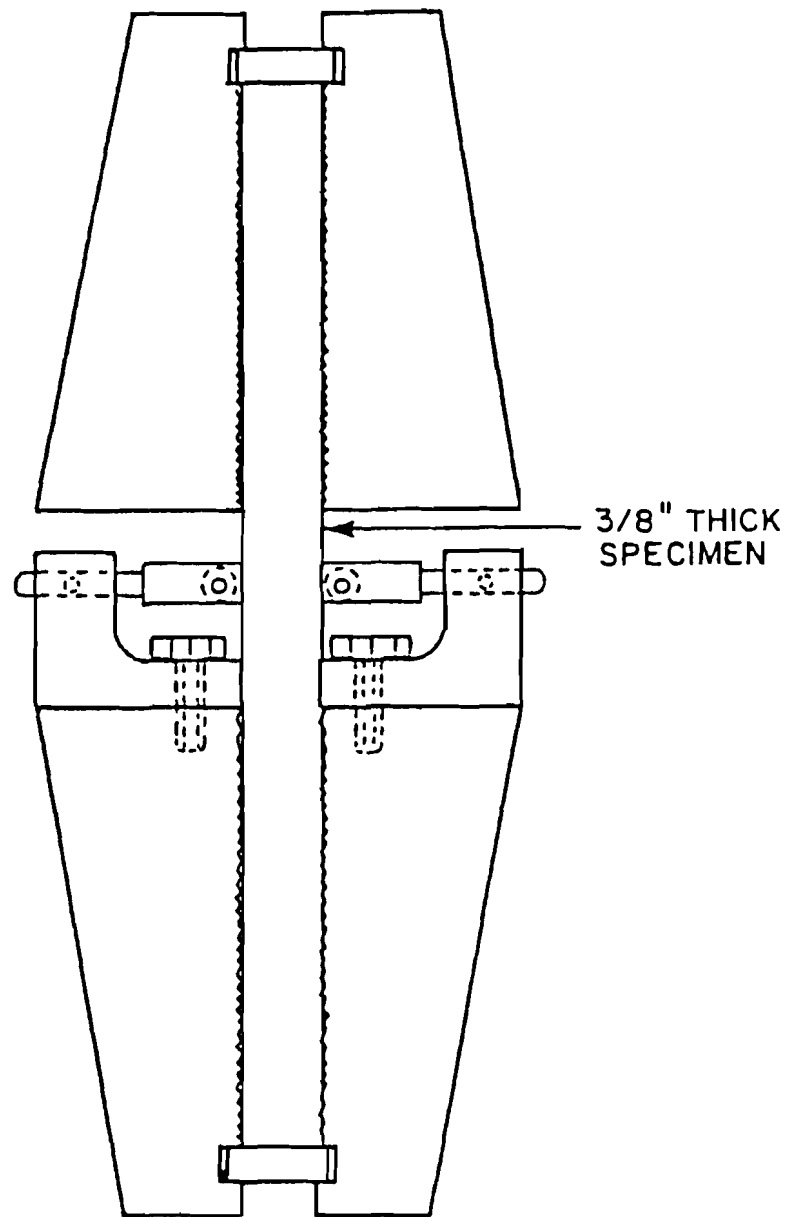
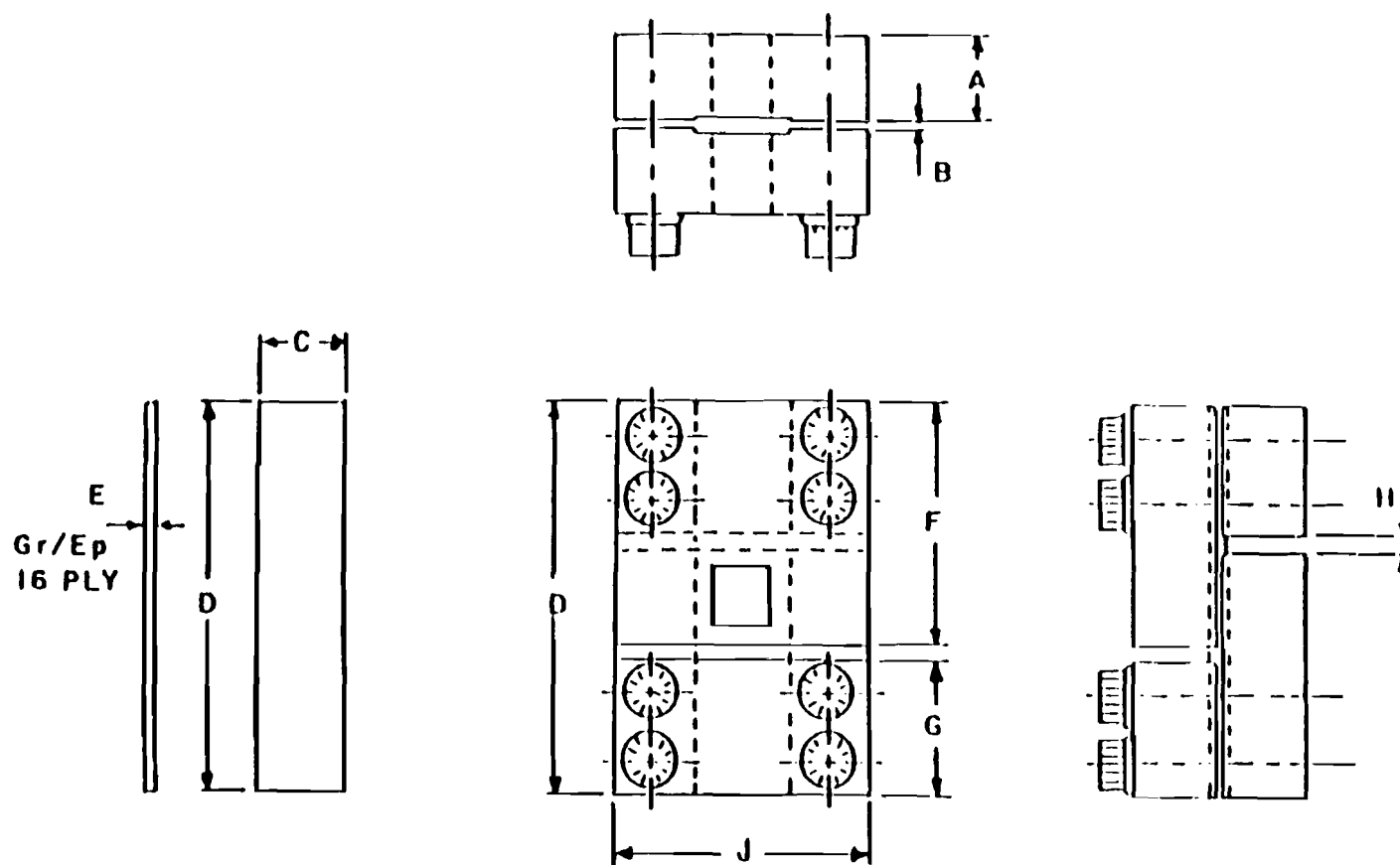


FIGURE B-5. MODIFIED GRIPS FOR IITRI COMPRESSION FIXTURE



Dimension	Inches	mm
A	0.670	17.2
B	0.030	0.76
C	0.750	19.05
D	3.000	76.2
E	0.080-0.096	2.0-2.4

Dimension	Inches	mm
F	1.875	47.6
G	1.000	25.4
H	0.125	3.18
J	2.000	50.0

FIGURE B-6. NORTHROP COMPRESSION SPECIMEN AND TEST FIXTURE

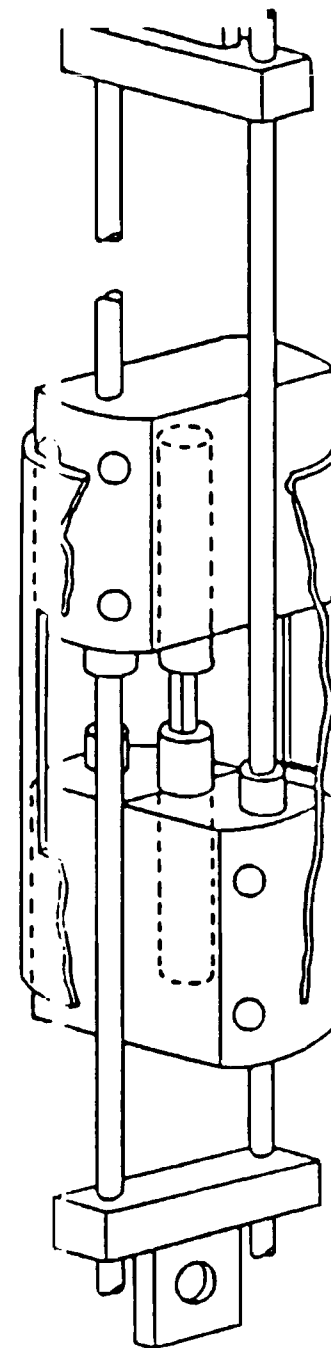
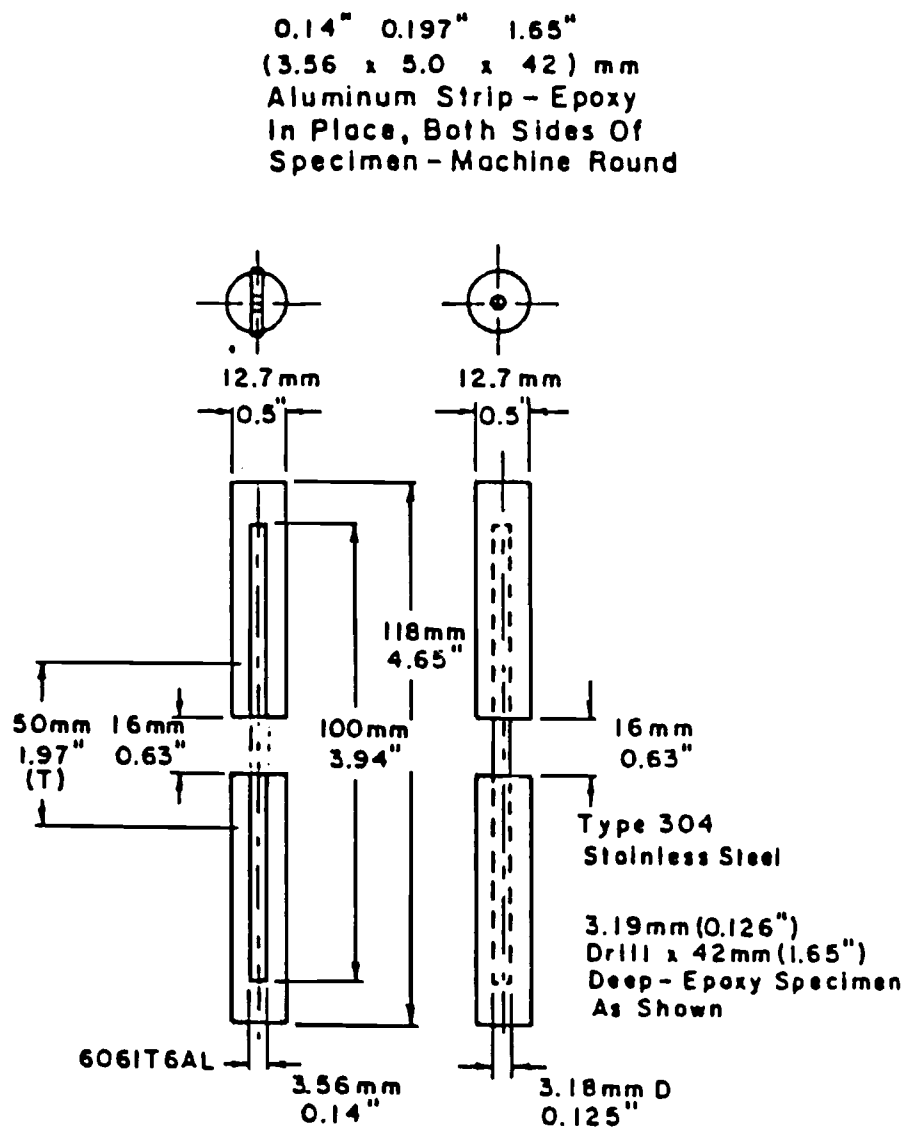


FIGURE B-7. NBS COMPRESSION TEST FIXTURE AND SPECIMEN

Since the tabs are generally adhesively bonded to the specimen surfaces, bonding fixtures are required to insure the desired parallelism. In addition, composite specimens fabricated with vacuum bag techniques may exhibit rather large thickness variations. Compressive tests of specimens prepared for these laminates may lead to erroneous results.

Type II Compression Test Methods

Test specimen geometries, and test fixtures definition for the Type II compression test methods are shown in figures B-8 through B-10.

The Type II compression tests appear to yield laminate data which is compatible with Type I compression data. The values on 0 degree unidirectional compression generated with fully supported coupons, however, appear to be consistently lower than the Type I compression data. There is no precise explanation for this behavior. Perhaps the increased stiffness of the 0 degree unidirectional specimen makes alignment more critical in the case of the Type II compression method.

Type III Compression Test Methods

Type III specimen geometrics and fixtures are shown in figures B-11 and B-12. The type III compression test method involves the loading of a straight-sided coupon bonded to a honeycomb core, which supplies the support necessary for stability.

It should be noted that the sandwich beam can also be used to determine tensile properties.

Development of alternate and modified versions of these compression test methods are continuing throughout the industry. One class of such modified fixtures utilizes sliding plates between the specimen and the lateral support plates of Type III fixtures. These plates are intended to provide lateral support to the ends of the specimens which protrude beyond the ends of the lateral support plates.

SHEAR TEST METHODS

Shear tests are performed in order to establish the ultimate shear strengths, ultimate shear strains, and shear moduli of the composite material system. Characterization of the shear properties of the composite lamina or laminates requires the evaluation of:

- (1) in-plane shear properties (1-2 plane)
- (2) interlaminar shear properties (1-3 and 2-3 planes)

Current test methods are limited to evaluating the lamina in-plane properties (1-2 plane) and the lamina interlaminar properties in the 1-3 plane. Shear tests for the composite laminate have generally been restricted to evaluation of properties in the plane of the laminate. There are four generally accepted test methods for evaluation of lamina in-plane shear properties:

0 (+45) Coupon Test	0 rail shear test
0 off-axis coupon	0 torsion test

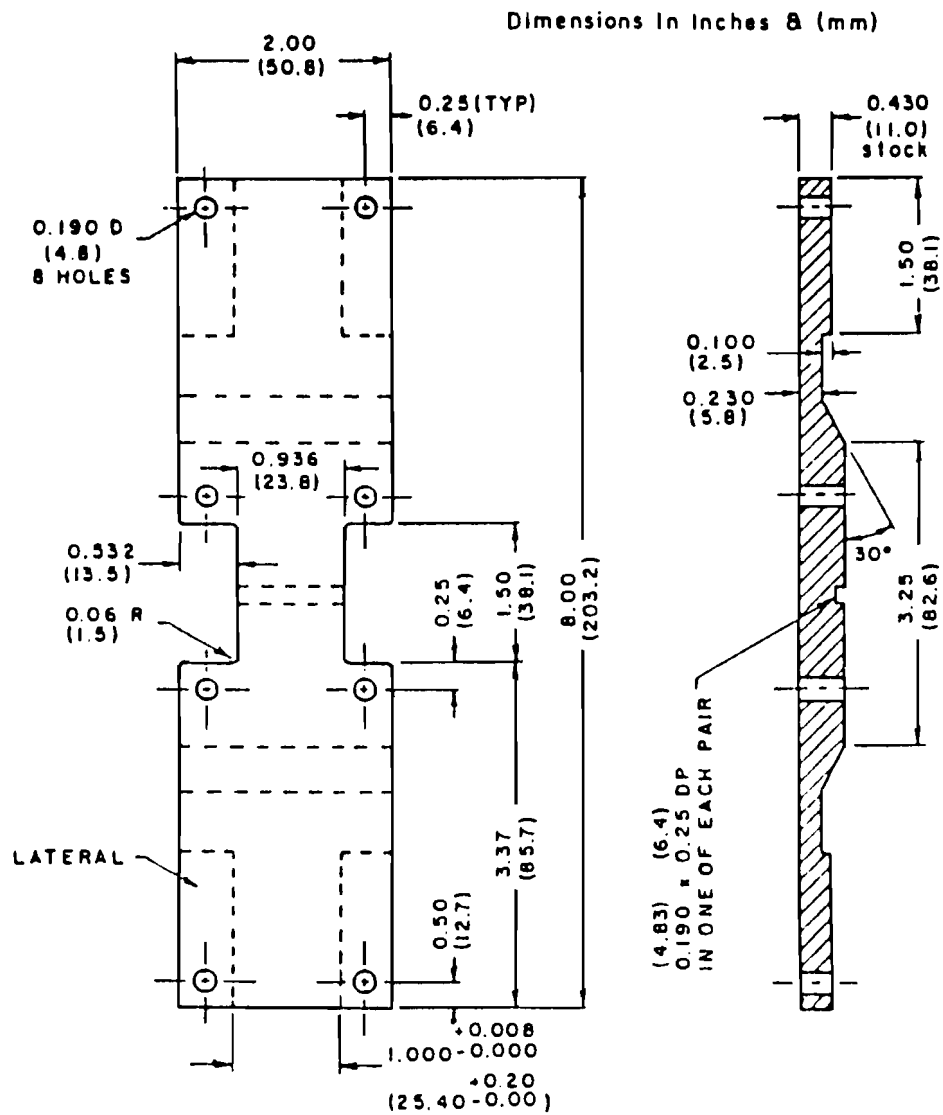


FIGURE B-8. SWRI FULLY-SUPPORTED COMPRESSION FIXTURE

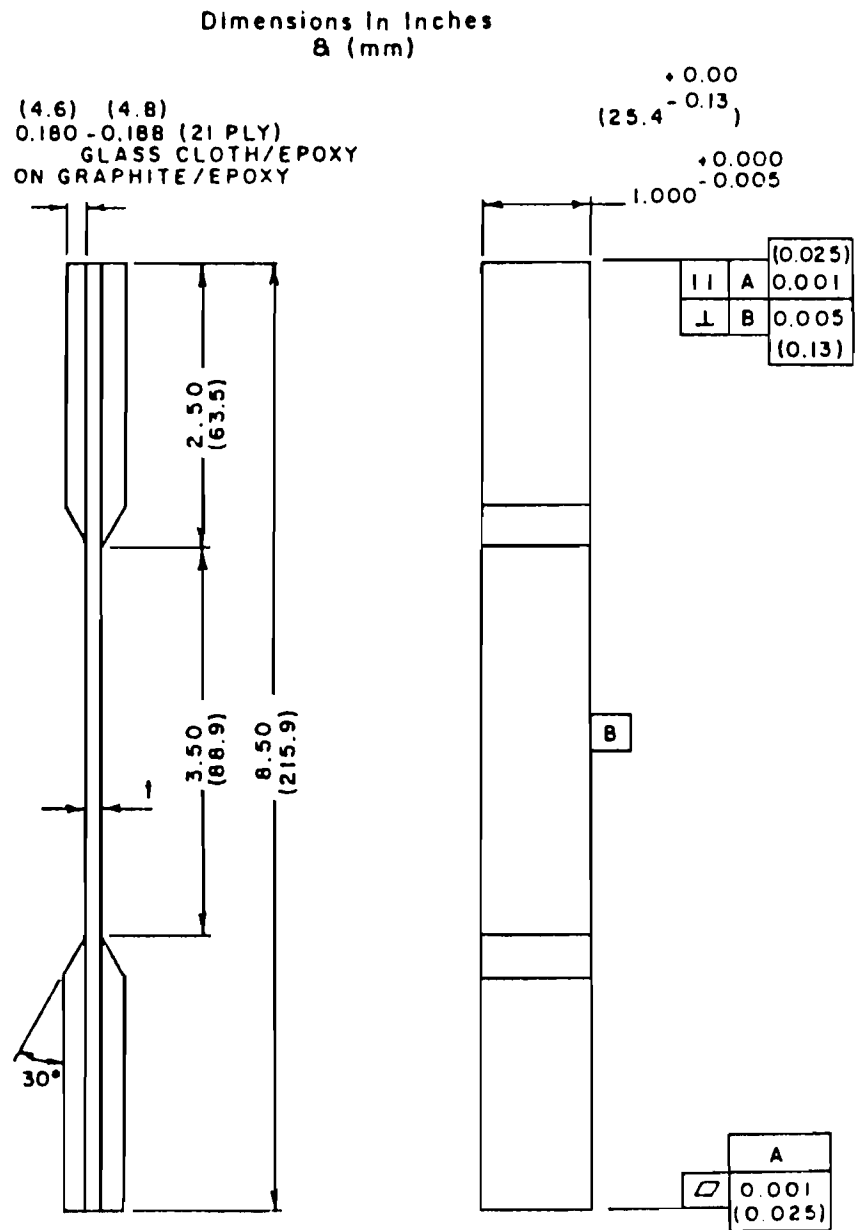
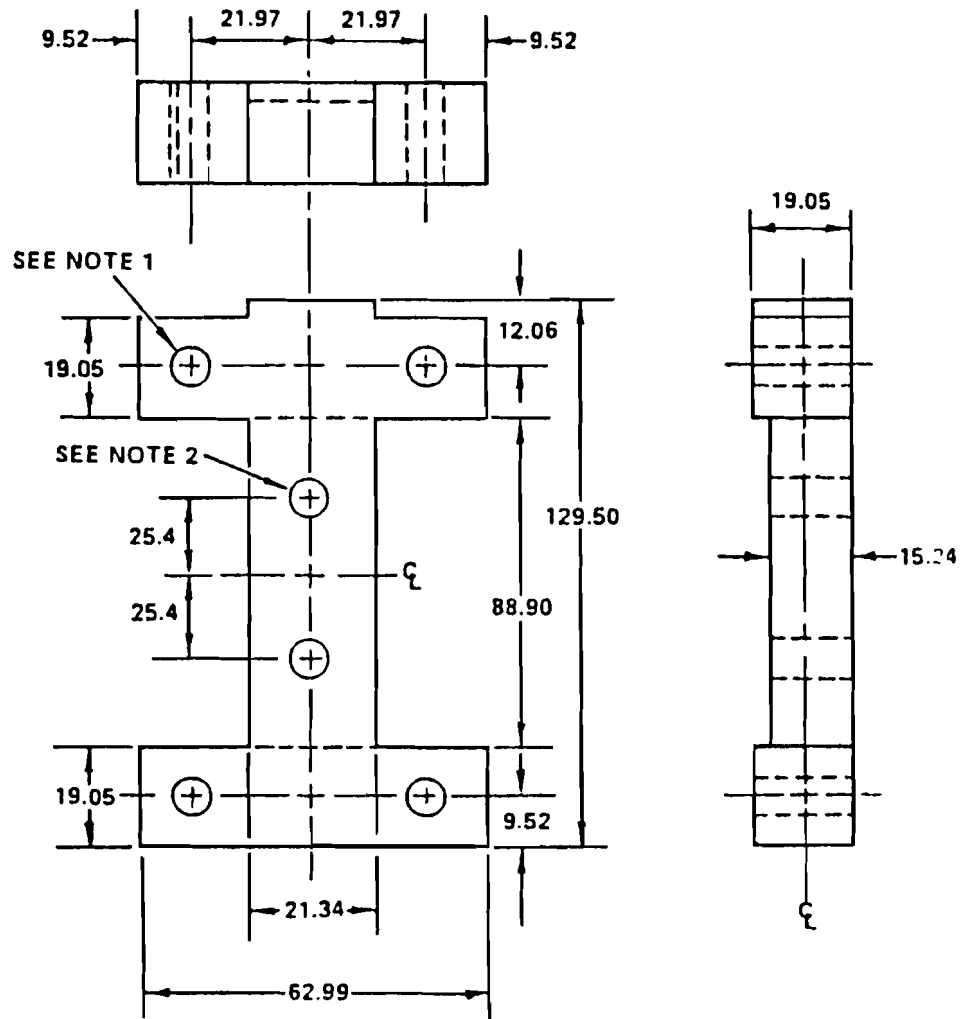


FIGURE B-9. SWRI FULLY-SUPPORTED COMPRESSION SPECIMEN



- NOTE: 1. Two Required: -1 Assembly Drilled and Tapped 1/4" -20 4 Places, -2 Assembly Drilled Only.
2. Hole Diameter as required for Extensometer and/or Strain Gages.
3. Material 1100-1240 MPa Steel.
4. All Dimensions in Millimeters.

FIGURE B-10. LOCKHEED FULLY-SUPPORTED COMPRESSION FIXTURE

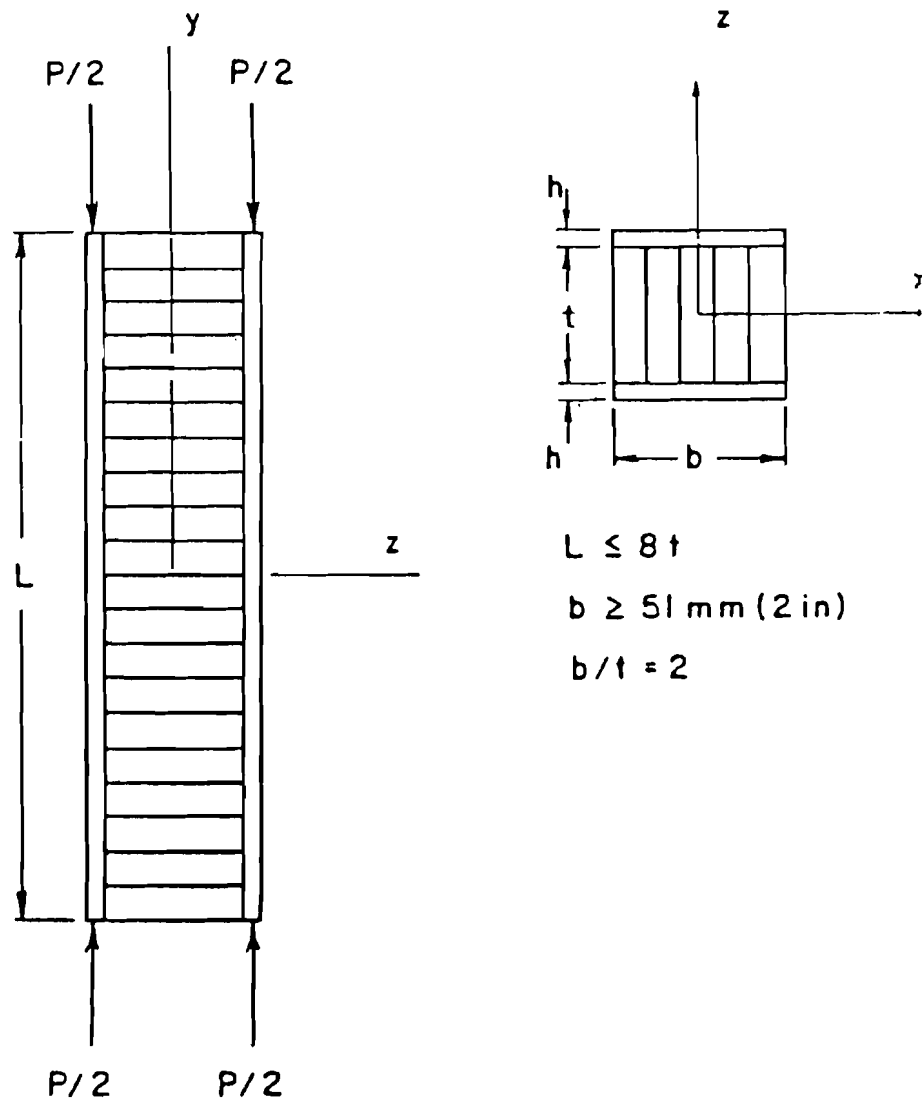


FIGURE B-11. SANDWICH EDGEWISE COMPRESSION TEST SPECIMEN

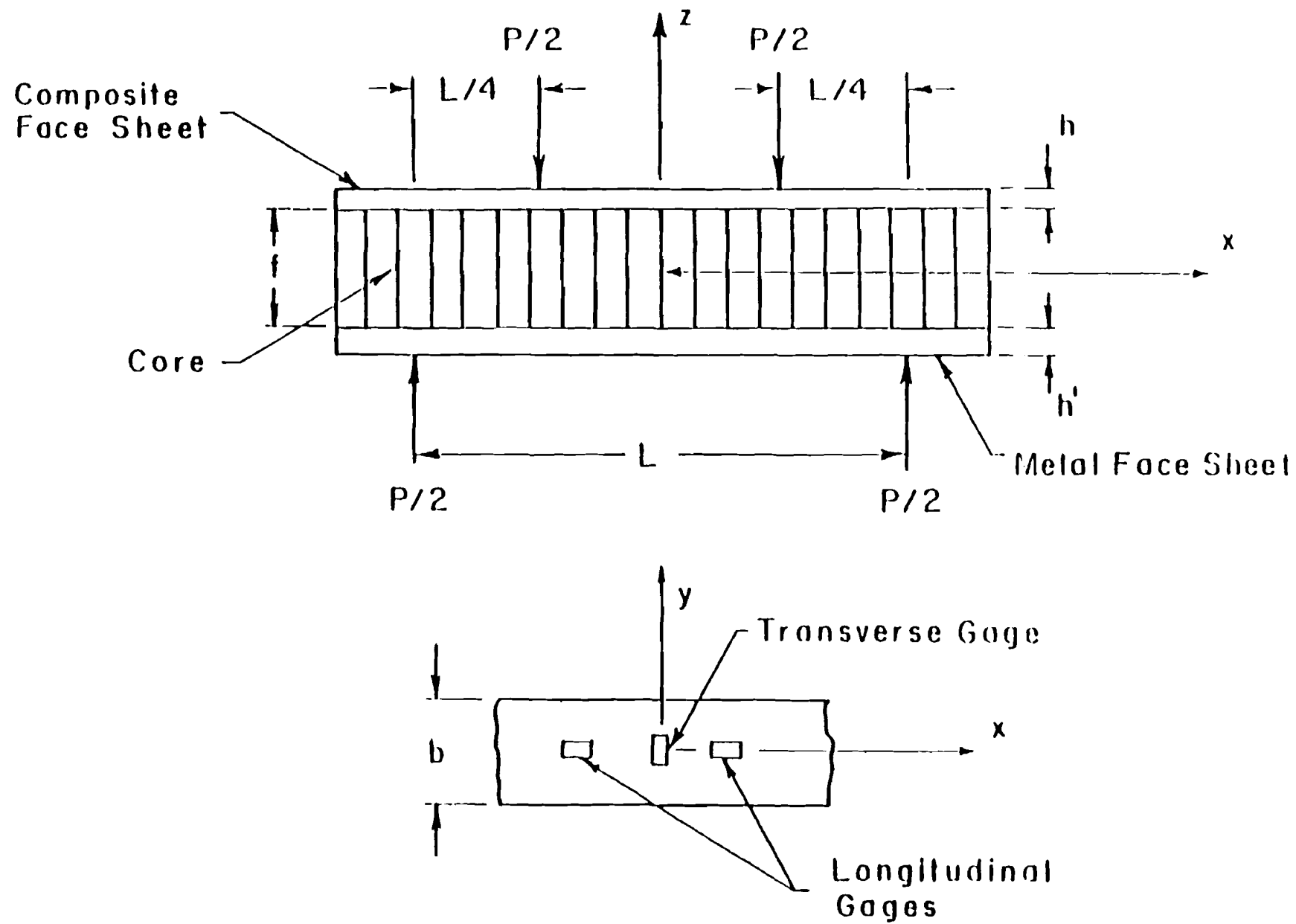


FIGURE B-12. SANDWICH BEAM COMPRESSION TEST SPEIMEN

Details of the (+45) coupon test method are presented in ASTM Standard D3518-76.

Off-Axis Test for Shear Properties

The second method for determination of lamina shear response is the off-axis tensile test. The in-plane shear stress-strain response, as determined by the off-axis test, is consistent for any angle, θ , as shown in figure B-13. However, the ultimate shear strength is a function of fiber orientation of the off-axis specimen. Results for the 150 off-axis test shown in this figure compare quite favorably with the (+45) tensile test. Therefore a 15° off-axis tensile test specimen can be utilized to determine the in-plane shear response of a unidirectional lamina.

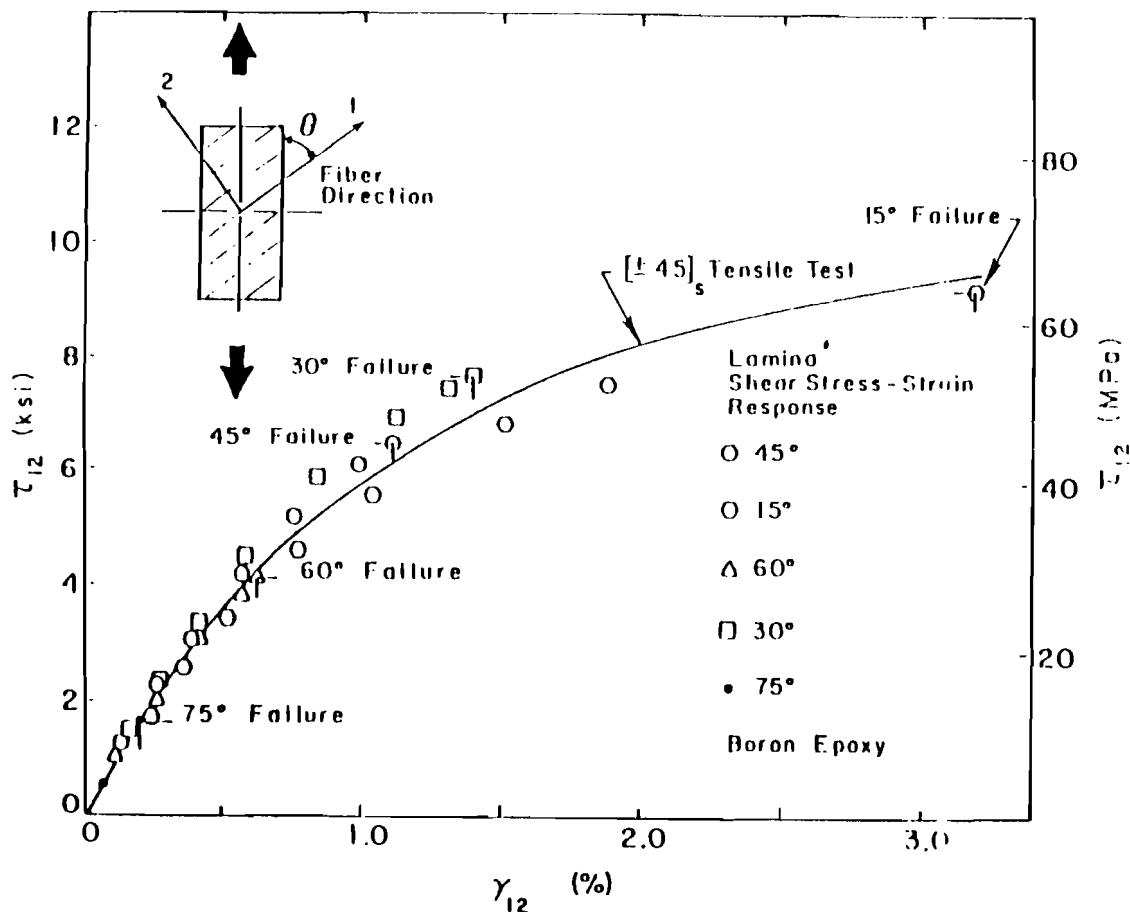


FIGURE B-13. OFF-AXIS TEST FOR SHEAR PROPERTIES

Rail-Shear Test Methods

The third method for determination of lamina in-plane shear response is the rail shear test method. Two test specimens and fixture configurations are generally accepted, the two rail fixture shown in figure B-14, and the three rail fixture shown in figures B-15-17.

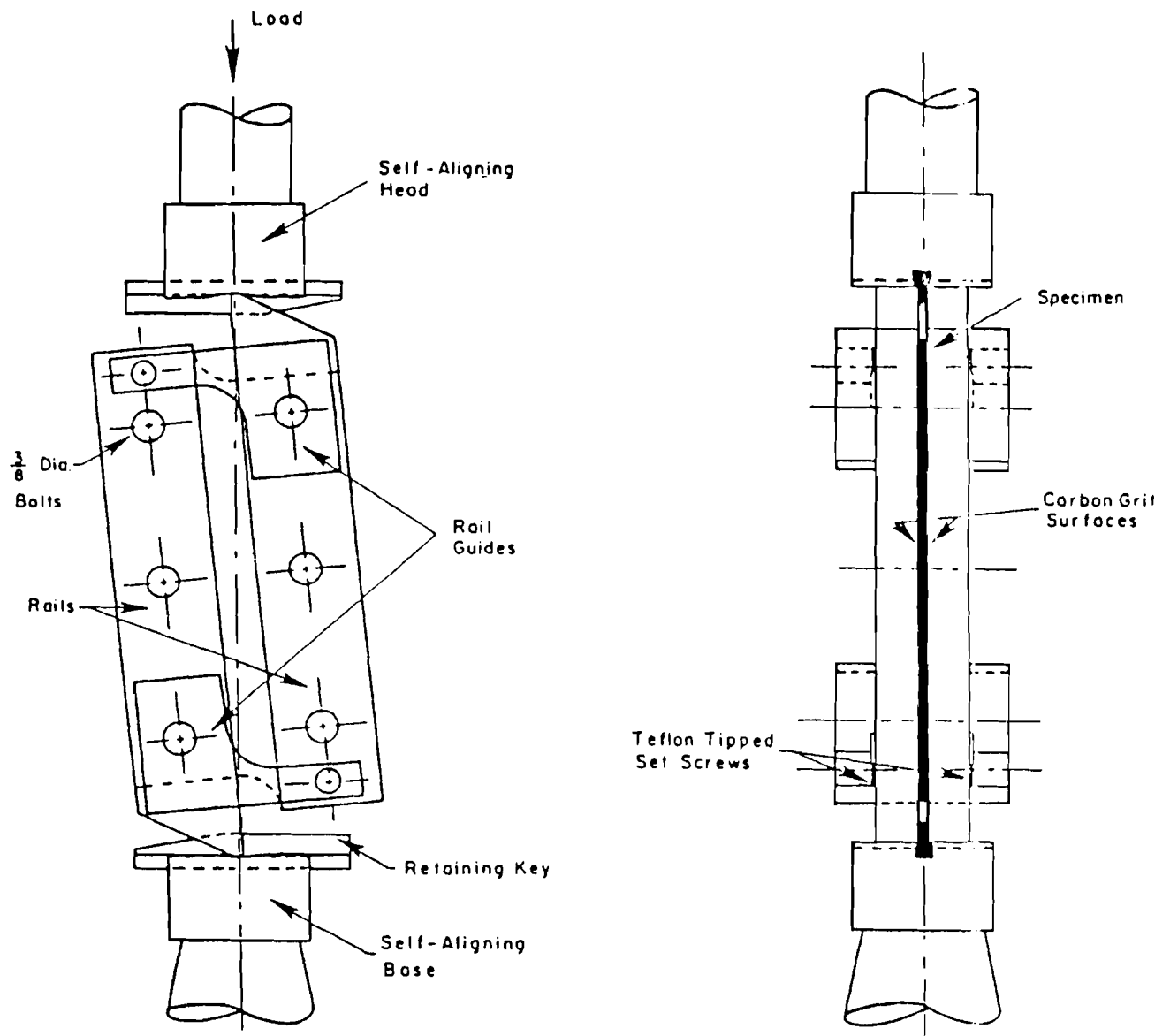


FIGURE B-14. TWO-RAIL SHEAR TEST

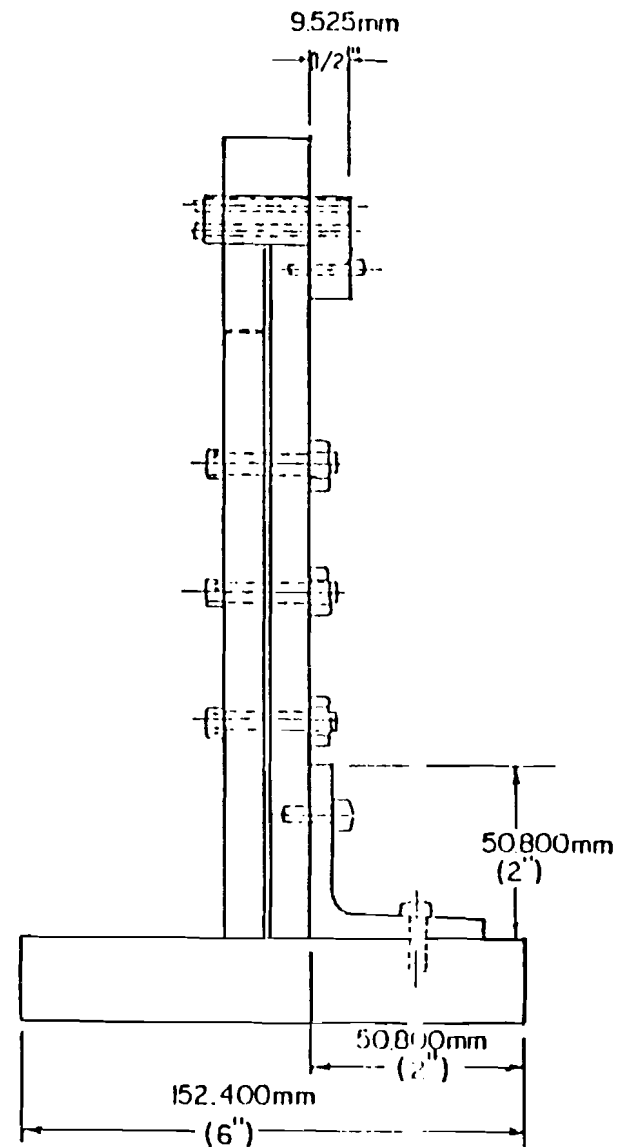


FIGURE B-15. THREE-RAIL SHEAR TEST

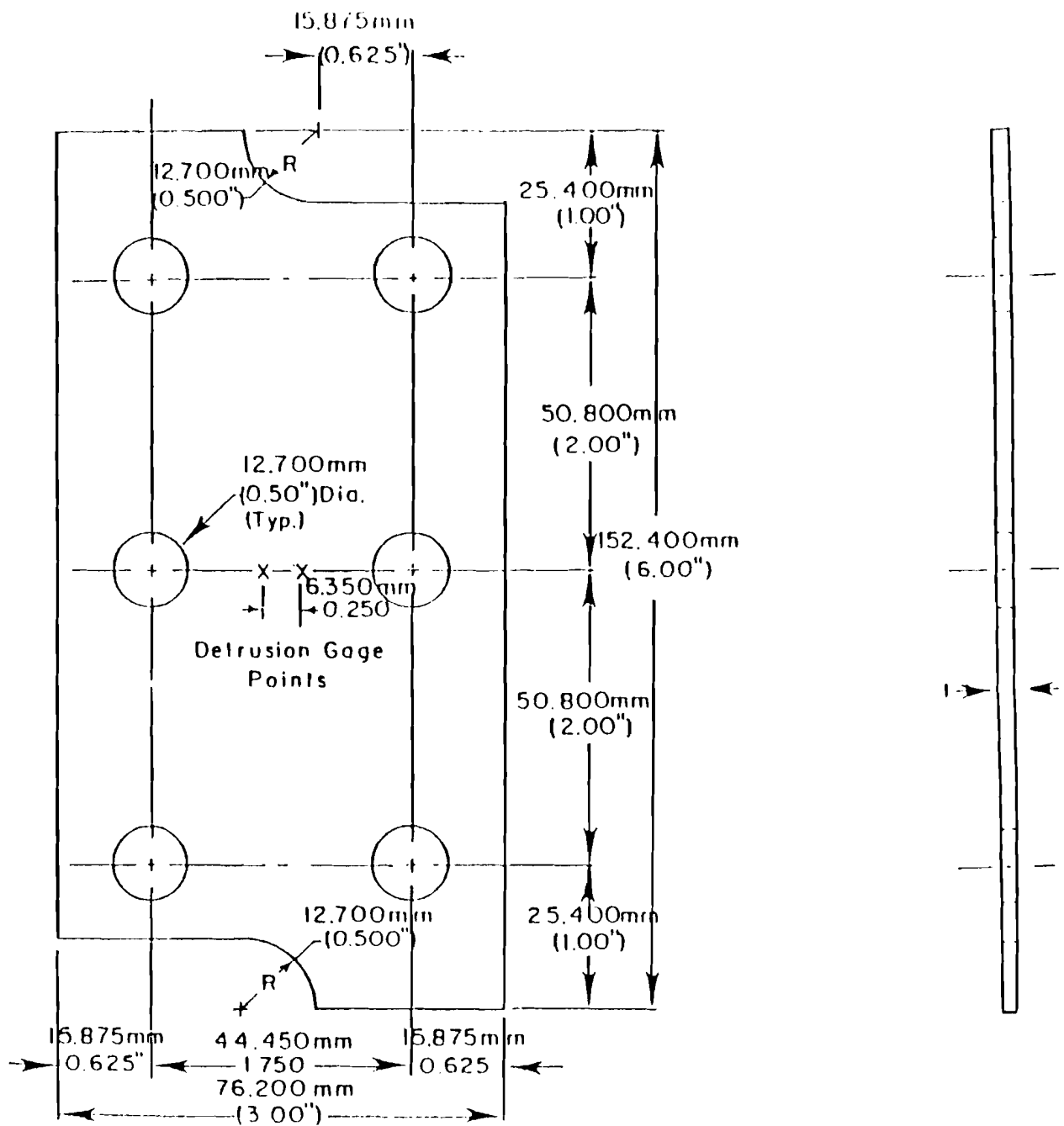


FIGURE B-16. TWO-RAIL SHEAR TEST SPECIMEN

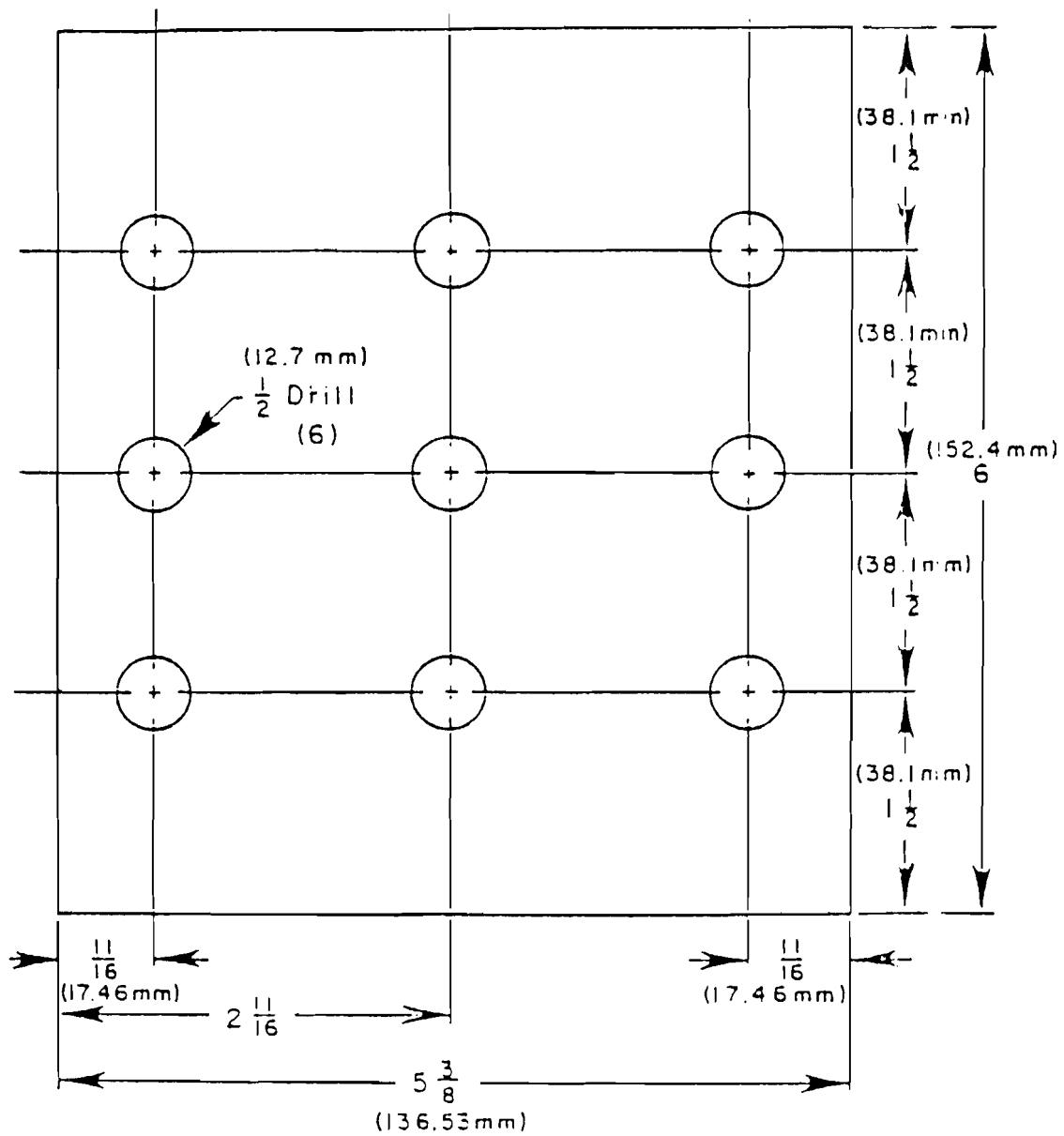


FIGURE B-17. THREE-RAIL SHEAR TEST SPECIMEN

Torsion Test for Shear Properties

The last method for determining in-plane shear properties of a laminate is the torsion method, which can be utilized in conjunction with either a solid rod or a tubular specimen.

Torsion of a thin walled tube provides a means of directly applying pure shear to a fiber reinforced composite specimen. The major drawback to this method, however, is the cost and difficulty associated with fabricating quality tubular specimens. In addition, the method requires specialized equipment and gripping systems. The shear strain is related to the angle-of-twist per unit length. Thus, a shear stress-strain curve can be obtained from a plot of torque versus angle of twist.

Short Beam Shear Test

The short beam shear test is utilized to estimate the inter-laminar shear strength of a composite. Because of its simplicity, the short beam shear test is often used as a quality control tool.

The short beam shear test involves a three point flexure specimen with the span-to-depth ratio, L/h , chosen to produce interlaminar shear failure (see figure B-18). The short beam method is applicable only to polymeric and composite materials which can be treated as homogeneous. In the case of laminated composites, much controversy surrounds the selection of a span-to-depth ratio which results in a true interlaminar shear failure. A detailed discussion of the short beam shear test method can be found in ASTM D2344-76.

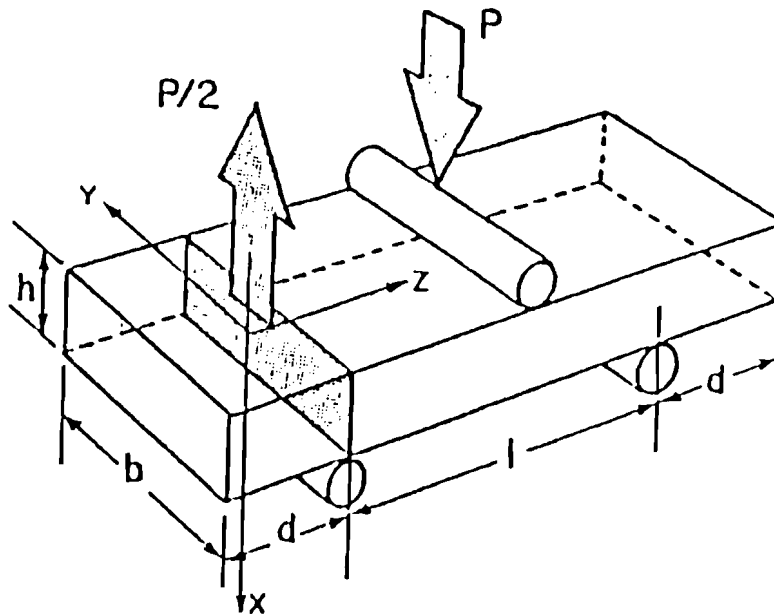


FIGURE 18. CONFIGURATION OF SHORT BEAM IN 3-POINT FLEXURE

The difficulties associated with the short beam shear test have not influenced its popularity. Simplicity of testing and the inexpensive nature of the specimen make the method an ideal tool for quality control or materials screening. Despite its wide use, the value of the short beam shear test as a property measurement technique remains highly questionable.

ASTM STANDARDS

For reference purposes, a listing of the current ASTM test methods for determination of some of the above properties are given below.

MECHANICAL TESTING

- D638-77a, Tensile Properties of Plastics
- D695-77, Compressive Properties of Rigid Plastics
- D2344-76, Apparent Horizontal Shear Strength of Reinforced Plastics by Short Beam Method
- D3039-76, Tensile Properties of Oriented Fiber Composites
- D3518-76, In-Plane Shear Stress-Strain Response of Unidirectional Reinforced Plastics
- D790-71, (reapproved 1978), Flexural Properties of Plastics and Electrical Insulating Materials
- D3410-75, Test for Compressive Properties of Oriented Fiber Composites

FATIGUE

- D3479-76, Tension-Tension Fatigue of Oriented Fiber Resin Matrix Composites
- D671-71, (reapproved 1978), Flexural Fatigue of Plastics by Constant-Amplitude-of-Force

IMPACT

- D256-78, Impact Resistance of Plastics and Electrical insulating Materials
- D1822-68, (reapproved 1973), Tensile-Impact Energy to Break Plastics and Electrical Insulating Materials
- D3029-78, Impact Resistance of Rigid Plastic Sheeting or Parts by Means of Tup (Falling Weight)
- E23-82, Notched Bar Impact Testing of Metallic Materials

CREEP

- D2990-77, Tensile, Compressive and Flexural Creep and Creep-Rupture of Plastics
- D2991-71, (reapproved 1978), Stress-Relaxation of Plastics

PHYSICAL PROPERTIES

- D792-66, (reapproved 1975), Specific Gravity and Density of Plastics by Displacement

- D1505-68 (reapproved 1975), Density of Plastics by the Density-Gradient Technique
D2734-70 (reapproved 1976), Void Content of Reinforced Plastics
D3355-74, Fiber Content of Unidirectional Fiber/Polymer Composites

THERMAL PROPERTIES

- D648-72 (reapproved 1978), Deflection Temperature of Plastics under Flexural Load

THERMAL EXPANSION

- D696-70 (reapproved 1978), Coefficient of Linear Thermal Expansion of Plastics
E228-71, Linear Thermal Expansion of Rigid Solids with a Vitreous Silica Dilatometer

THERMAL CONDUCTIVITY

- C177-76, Steady-State Thermal Transmission Properties by Means of the Guarded Hot Plate

ELECTRICAL PROPERTIES

- D149-75, Dielectric Breakdown Voltage and Dielectric Strength of Electrical Insulating Materials at Commercial Power Frequencies
D150-78, A-C Loss of Characteristics and Permittivity (Dielectric Constant) of Solid Electrical Insulating Materials

WEAR RESISTANCE

- D673-70 (reapproved 1976), MAR Resistance of Plastics
D1242-56 (reapproved 1975), Resistance of Plastic Materials to Abrasion

CHEMICAL RESISTANCE

- C581-74 (reapproved 1978), Chemical Resistance of Thermo-setting Resins used in Glass Fiber Reinforced Structures
D543-67 (reapproved 1978), Resistance of Plastics to Chemical Reagents

DATA SOURCES

Specifications and standards can be obtained from the following sources:

1. All U.S. Department of Commerce, National Bureau of Standards Handbooks are available from the Superintendent of Documents, Government Printing Office, Washington, DC 20402.

2. All other government agency specifications or standards are under the control of the Department of Defense.

All requests for copies of specifications, standards, and qualified products lists should state the title and identifying number and should be submitted to Commanding Officer, Naval Supply Depot, 5801 Tabor Avenue, Philadelphia, PA 19120, Attn. - Code CDS, except:

- a. Copies of specifications, standards and qualified products lists required by contractors in connection with specific procurement functions should be obtained from the procuring agency awarding the contract or as directed by the contracting officer.
 - b. Federal Specifications and Standards and Military Book Form Standards will not generally be furnished by the Naval Supply Depot to Commercial concerns unless required in conjunction with a bid or contract, or for sufficient other justification. Copies of federal documents may be purchased from the Business Service Center, General Service Administration, Washington DC 20405. Most book-form Military Standards may be purchased from the Superintendent of Documents, U.S. Government Printing Office, Washington, DC 20402.
 - c. Only current, "in effect" issues of standardization documents will be available from the Naval Supply Depot. Copies of cancelled or superseded documents required for contracting office of the concerned service.
 - d. Information regarding obtaining DOE standards relative to the Division of Reactor Development and Technology may be obtained from Oak Ridge National Laboratory, P.O. Box X, Oak Ridge, TN 37830.
3. All specifications or standards as issued by the organizations listed below are available directly from the organization at the address shown.

ABS	American Bureau of Shipping 45 Broad Street New York, NY 10004
AIA	Aerospace Industries Association of America
AISI	American Iron & Steel Institute 1000 16th Street, N.W. Washington, DC 20036
Al. Assoc.	The Aluminum Association 420 Lexington Avenue New York, NY 10017
ANS	American Nuclear Society
ANSI	American National Standards Institute, Inc. 10 East 40th Street New York, NY 10016
API	American Petroleum Institute 50 West 50th Street New York, NY 10019
ASME	American Society of Mechanical Engineers 345 East 47th Street New York, NY 10017
ASNT	American Society for Nondestructive Testing, Inc. 3200 Riverside Drive P.O. Box 5642 Columbus, OH 43221
ASTM	American Society for Testing & Materials 1916 Race Street Philadelphia, PA 19103
AVS	American Vacuum Society
AWS	American Welding Society, Inc. 2501 N.W. 7th Street Miami, FL 33125
DOD	Department of Defense
DOE* ERDA	Department of Energy
ICRU	International Commission of Radiation Units
MSFC	National Aeronautics & Space Admin. George C. Marshall Space Flight Center Huntsville, AL 35812

MSS	Military Supply Standard
NAS	Naval Air Station
NBS	National Bureau of Standards
NCRP	National Commission on Radiation Protection NCRP Publications - P.O. Box 4867 Washington, DC 20008
NRC	Nuclear Regulatory Commission
RDT	Reactor Research & Development
SAE	Society of Automotive Engineers, Inc. 400 Commonwealth Drive Warrendale, PA 15096

* Replaced ERDA - Energy Research & Development Admin.
ERDA had earlier replaced AEC - U.S. Atomic Energy Commission

STATISTICAL ALLOWABLES

Material properties are generally derived for aerospace applications on a statistical basis using the following definitions:

- o A-Basis-- The A mechanical-property value is the value above which at least 99 percent of the population of values is expected to fall, with a confidence of 95 percent.
- o B-Basis-- The B mechanical-property value is the value above which at least 90 percent of the population of values is expected to fall, with a confidence of 95 percent.
- o S-Basis-- The S mechanical-property value is the minimum value specified by the governing specification.
- o Typical Basis--- The typical property value is an average value. No statistical assurance is associated with this value.

Summary of Statistical Analysis Procedures for Determining Design Allowables

The following step-by-step procedure can be used to determine A and B design allowables. The procedure discussed below is similar to that used in MIL-HDBK-5A to calculate A and B design allowable strengths.

1. Calculate mean value.

$$\bar{X} = \frac{\sum x_i}{n}$$

where x_i are the individual data points and
 n is the number of data points.

2. Calculate the Standard Deviation

$$S = \sqrt{\frac{\sum (x_i - \bar{X})^2}{n-1}}$$

3. Determination of one-sided tolerance factors, K_A and K_B , for A and B bases, respectively using the curves given in figure B-19. The A basis relates to 99 percent probability of survival with a confidence of 95 percent and the B basis of 90 percent probability of survival with a confidence of 95 percent.

4. Calculate the A and B bases design allowables using the following formulas:

$$A = \bar{X} - K_A S$$

$$B = \bar{X} - K_B S$$

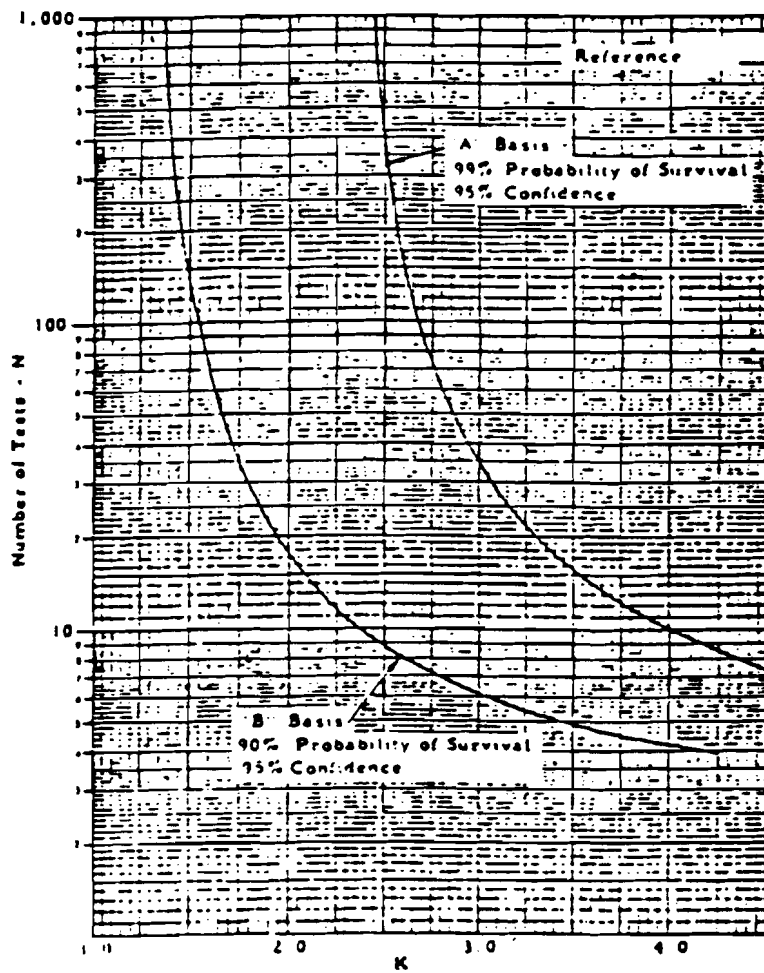


FIGURE 19. ONE-SIDED TOLERANCE FACTORS FOR NORMAL DISTRIBUTIONS, K

Example

The following data points were found in 28 ultimate tensile strength tests for a unidirectional graphite epoxy laminate at room temperature. Determine A and B allowables.

<u>Tensile Strength (ksi)</u>			
<u>Test</u>	<u>Strength</u>	<u>Test</u>	<u>Strength</u>
1	220	15	321
2	316	16	257
3	325	17	239
4	292	18	272
5	292	19	276
6	358	20	259
7	297	21	248
8	354	22	359
9	303	23	285
10	313	24	183
11	326	25	222
12	354	26	283
13	339	27	287
14	253	28	278

Steps 1 & 2: $\bar{X} = 289 \text{ ksi}$ $S = 44.4 \text{ ksi}$

Step 3: $K_A = 3.125$ $K_B = 1.8$

Step 4: $A = 289 - (3.125 \times 44.4) = 150.25 \text{ ksi}$
 $B = 289 - (1.8 \times 44.4) = 209.08 \text{ ksi}$

The "A" and "B" design allowables are 150.25 and 209.08 ksi, respectively.