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16. Abstract

All geotechnical predictions involve uncertainties. These are accounted for traditionally by conservative factors of safety. Risk based design, on the other hand, attempts to quantify uncertainties and to adjust design conservatism accordingly. Such methods cannot rationalize all the uncertainties of geotechnical engineering, but they provide increased efficiency at modest cost. A risk-based approach is presented which is intended for practical use. Design problems from engineering practice are used to illustrate the approach.

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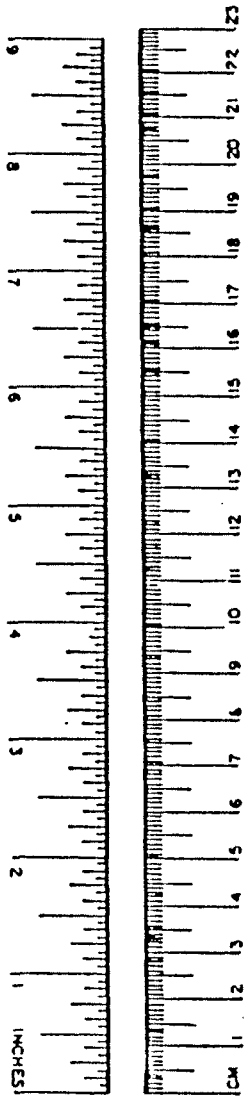
METRIC CONVERSION FACTORS

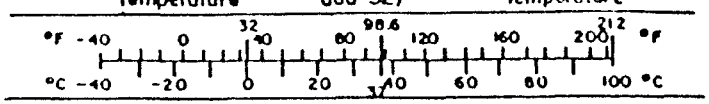
APPROXIMATE CONVERSIONS FROM METRIC MEASURES

SYMBOL	WHEN YOU KNOW	MULTIPLY BY	TO FIND	SYMBOL
LENGTH				
in	inches	2.5	centimeters	cm
ft	feet	30	centimeters	cm
yd	yards	0.9	meters	m
mi	miles	1.6	kilometers	km
AREA				
in^2	square inches	6.5	square centimeters	cm^2
ft^2	square feet	0.09	square meters	m^2
yd^2	square yards	0.6	square meters	m^2
mi^2	square miles	2.6	square kilometers	km^2
	acres	0.4	hectares	ha
MASS (weight)				
oz	ounces	28	grams	g
lb	pounds	0.45	kilograms	kg
	short tons (2000 lb)	0.9	tonnes	t
VOLUME				
tsp	teaspoons	5	milliliters	ml
tbsp	tablespoons	15	milliliters	ml
fl oz	fluid ounces	30	milliliters	ml
c	cups	0.24	liters	l
pt	pints	0.47	liters	l
qt	quarts	0.95	liters	l
gal	gallons	3.8	liters	l
ft^3	cubic feet	0.03	cubic meters	m^3
yd^3	cubic yards	0.76	cubic meters	m^3
TEMPERATURE (exact)				
°F	Fahrenheit temperature	5/9 (after subtracting 32)	Celsius temperature	°C



APPROXIMATE CONVERSIONS FROM METRIC MEASURES

SYMBOL	WHEN YOU KNOW	MULTIPLY BY	TO FIND	SYMBOL
LENGTH				
mm	millimeters	0.04	inches	in
cm	centimeters	0.4	inches	in
m	meters	3.3	feet	ft
m	meters	1.1	yards	yd
km	kilometers	0.6	miles	mi
AREA				
cm^2	square centimeters	0.16	square inches	in^2
m^2	square meters	1.2	square yards	yd^2
km^2	square kilometers	0.4	square miles	mi^2
ha	hectares (10,000 m^2)	2.5	acres	
MASS (weight)				
g	grams	0.035	ounces	oz
kg	kilograms	2.2	pounds	lb
t	tonnes (1000 kg)	1.1	short tons	
VOLUME				
ml	milliliters	8.03	fluid ounces	fl oz
l	liters	2.1	pints	pt
l	liters	1.06	quarts	qt
l	liters	0.26	gallons	gal
m^3	cubic meters	36	cubic feet	ft^3
m^3	cubic meters	1.3	cubic yards	yd^3
TEMPERATURE (exact)				
°C	Celsius temperature	9/5 (then add 32)	Fahrenheit temperature	°F



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1. RISK-BASED DESIGN

All engineers design in the face of uncertainty. Uncertainty about material properties, about conditions in service, about engineering models, and many others. Traditionally, geotechnical engineers have accounted for uncertainty by specifying capacities greater than projected demand. The ratio of capacity to demand--the factor of safety--is usually chosen by experience.

This approach has certain drawbacks. For example, there is a tendency to be conservative about each estimate needed for design. As a result, the overall factor of safety in a design is unknown. Also, conservatism in estimates of soil properties and loads is subjective. Thus, predictions are often not repeatable. Finally, uncertainty varies from situation to situation. A fixed factor of safety implies a different likelihood of failure in each different situation.

Risk-based design can be used to overcome some the limitations of fixed factors of safety. The term "risk-based design" means nothing more than attempting to quantify uncertainties inherent to an engineering problem and dealing with them in an engineering-like manner.

Risk-based design in geotechnical engineering complements normal data analysis and modeling. At the data analysis stage, uncertainties are identified and simple statistical tools are used to quantify them. At the modeling stage, mathematical techniques are used to assess the affect of uncertainties on performance predictions. The end result of risk-based design if a quantified measure of confidence that a facility will perform satisfactorily. This measure, called the "reliability index," describes the relative error in performance predictions compared to the margin of safety designed into a facility.

2. UNCERTAINTY IN GEOTECHNICAL ANALYSIS

Four principal uncertainties enter calculations of the response of a soil deposit to design loads (Figure 1):

1. Soil (spatial) variability,
2. Measurement noise,
3. Measurement or model bias, and
4. Statistical error due to limited measurements.

These are the uncertainties affecting calculated predictions.

Soil variability is important because the engineering properties of soils fluctuate from one location to another. Major variations such as stratification are accommodated in calculations, but smaller scale variations are not. Nontheless, these latter variations can be 30 to 50 percent of the average, as reflected in the large scatter observed in test data.

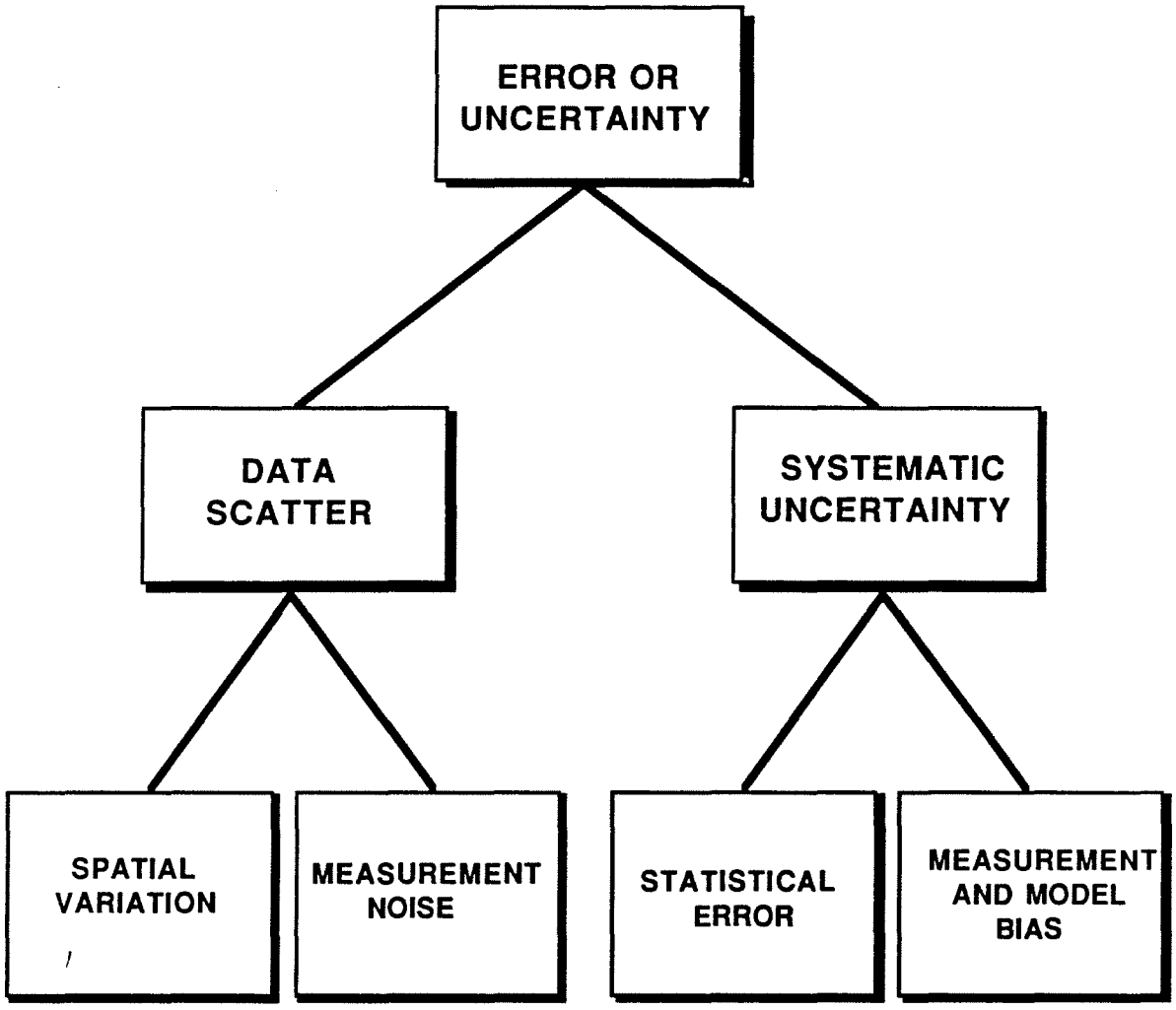


Figure 1 Sources of error or uncertainty in soil property estimates

The second source of uncertainty, measurement noise, also contributes to data scatter. Measurement noise is random error caused by operator or instrumental effects, or by small but real variation in the soil deposit (e.g., stones or shells). Statistical methods are used to separate the fraction of data scatter due to real variability from that due to measurement noise.

The last two sources of uncertainty, measurement bias and statistical error, cause systematic errors. The distinction between data scatter and systematic error is shown in Figure 2. Systematic error is a consistent difference between the actual value of some property and the measured or estimated value.

Measurement bias is common in geotechnical engineering. It is caused by such things as soil disturbance, or differences between how a property is measured and how a structure imposes load on the soil mass. For example, field vane measurements of undrained strength introduce a rotation of principal planes, vertical failure surfaces, and other conditions which differ from those existing under a foundation or embankment. Therefore, strengths back-calculated from embankment failures systematically differ from those measured with the vane. Statistical error is caused by limited numbers of measurements. For example, the average values of two sets of measurements vary somewhat because the measurements themselves vary. Each average differs slightly from the true average across a site. Such error is said to be 'statistical,' as it derives from the statistical variation among sets of measurements.

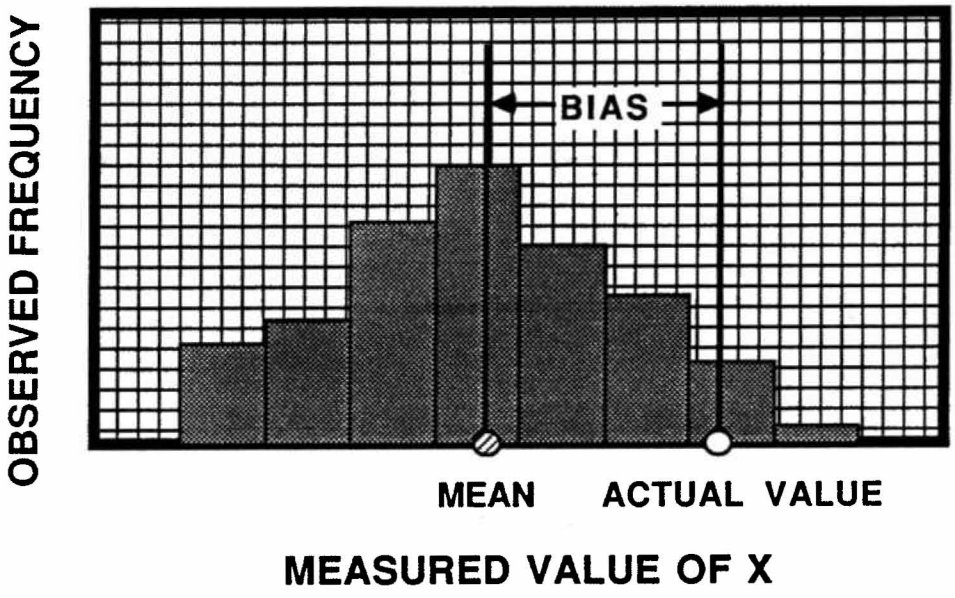


Figure 2 Distinction between data scatter and bias error.

Data scatter and systematic error each introduce uncertainty to geotechnical calculations. Yet, the effects of each differ. Risk-based design separately assesses the four sources of uncertainty and separately evaluates how each affects predictions. The implication of uncertainty for design conservatism and the manner in which one best accommodates uncertainty both depend on the mix of contributions from the four sources.

3. DESCRIBING UNCERTAINTY

Estimates of soil properties for most purposes are adequately represented by two numbers, (a) a best estimate, and (b) a measure of uncertainty. The average value and the standard deviation, respectively, are used to express these two attributes.

3.1 'Best Estimate' = Average

The average (or mean) of a set of measurements $\underline{x} = \{x_1, \ldots, x_n\}$ is denoted m_x, and defined as

$$m_x = \frac{1}{n} \Sigma\, x_i = \text{mean} \qquad (1)$$

In effect, the mean is the center of gravity of a set of measurements along the x-axis. For this reason it is used as the best single-valued estimate of x, being neither conservative nor unconservative. Figure 3 shows a histogram of standard penetration test (SPT) data taken in an alluvial sand deposit. The mean of these measurements is m_x = 8.9 bpf.

3.2 'Uncertainty' = Standard Deviation

The standard deviation of the measurements $\underline{x}$ is their variation with respect to the mean, expressed as the square root of the moment of inertia of the data about the mean

$$s_x = \sqrt{\frac{1}{n-1} \Sigma\, (x_i - m_x)^2} = \text{standard deviation} \qquad (2a)$$

s_x measures the dispersion or uncertainty about the value of x. The standard deviation of the SPT data of Figure 3 is s_x = 4.4 bpf.

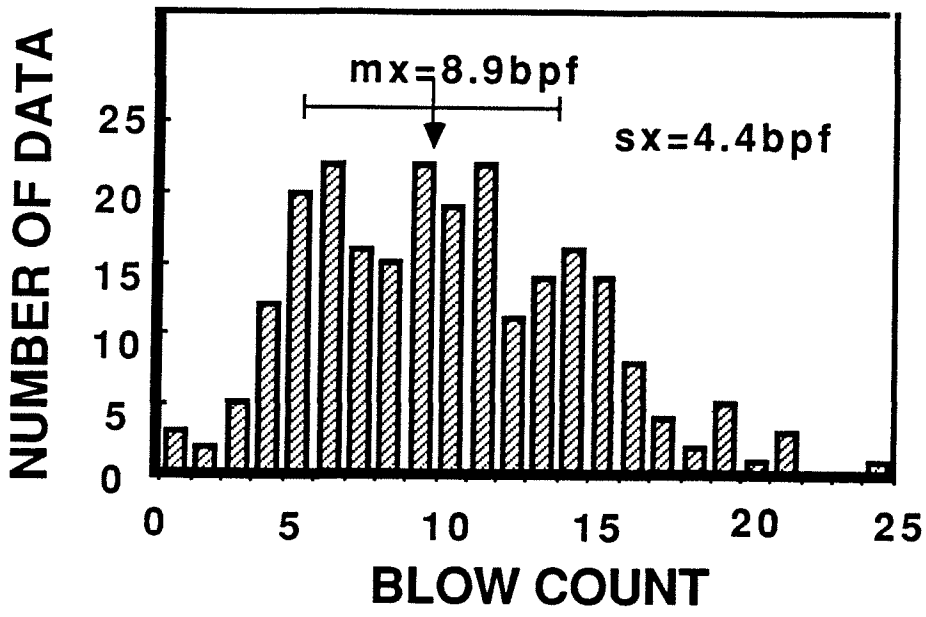


Figure 3 Histogram of standard penetration test blow counts in a silty sand.

Some engineers prefer to work with relative rather than absolute uncertainty. The relative uncertainty, measured by the ratio of standard deviation to mean, is called the coefficient of variation

$$\Omega_X = s_X/m_X = \text{coefficient of variation} \qquad (2b)$$

The coefficient of variation of the data in Figure 3 is $\Omega_N = s_N/m_N = 4.4/8.9 = 0.49$.

In calculations it is often convenient to deal with the square of the standard deviation rather than s_X itself. The square of the standard deviation is called the variance

$$s_X^2 = \text{variance} \qquad (3)$$

Given the similarity of m_X and s_X^2 to mechanical moments, the mean and variance are often called the first and second (statistical) moments of x.

3.3 'Association' Between Uncertainties = Correlation Coefficient

When dealing with two or more soil properties, uncertainties in estimates may be associated with one another. The uncertainty in one estimate may not be independent of the uncertainty in the other estimate. For example, in estimating 'cohesion' and 'friction' parameters of a Mohr-Coulomb strength envelope (Figure 4), if the slope of the envelope $\tan\phi$ is estimated too high, the intercept c will be too low. The reverse is true if $\tan\phi$ is estimated too low. Thus, the uncertainties about correct values for $\tan\phi$ and c are associated with one another.

The degree of association between two uncertain estimates is expressed by the correlation coefficient. The correlation coefficient for a set of paired data $\underline{x},\underline{y} = \{ (x_1,y_1), \ldots, (x_n,y_n) \}$ is denoted r, and defined as

$$r = \frac{1}{n} \Sigma \left(\frac{x_i - m_x}{s_x}\right) \left(\frac{y_i - m_y}{s_y}\right) = \text{correlation coefficient} \qquad (4)$$

In effect, the correlation coefficient is a product moment of inertia. It expresses the degree to which two parameters vary together. The correlation coefficient is non-dimensional because deviations of x and y from their respective means are measured in units of the respective standard deviations.

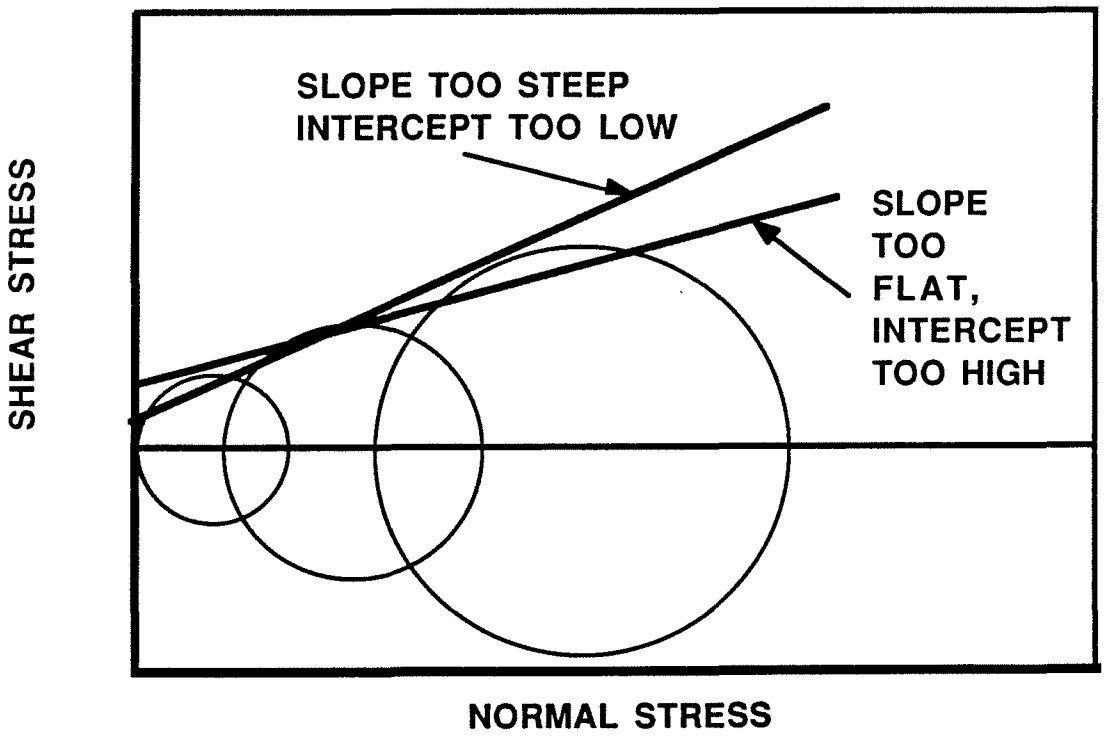


Figure 4 Correlation of uncertainties in Mohr-Coulomb strength parameters.

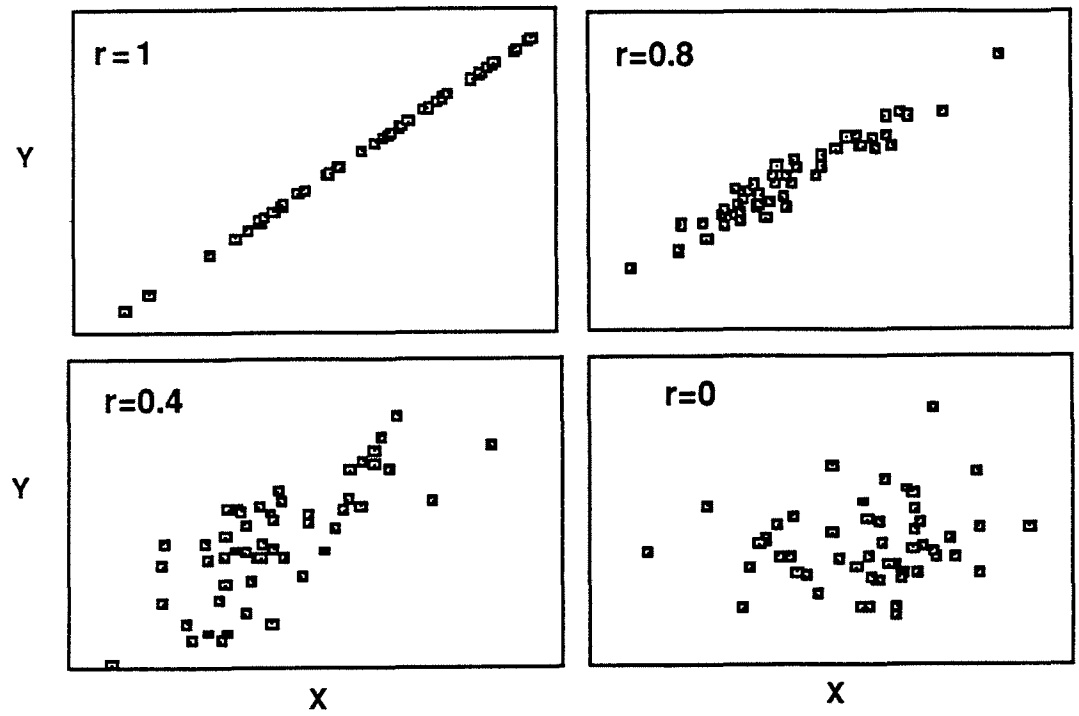


Figure 5 Examples of various levels of correlation between x and y.

For data which vary together--higher values of x are associated with higher values of y--r; is positive. X_i is greater than m_x, y_i tends to be greater than m_y, and vice versa. Thus, the product in equation 4 tends to be positive. For data which vary inversely--higher values of x are associated with lower values of y--r is negative. The product of equation 4 tends to have one positive term and one negative term. The range of r is ± 1: r-+1 implies a linear relation between x and y with positive slope, r = -1 implies a linear relation with negative slope, and r = 0 implies a "shotgun blast" (Figure 5).

The corresponding dimensional form of 4, that is, using the absolute deviations of x and y rather than normalized deviations, is called the covariance and denoted

$$C = \frac{1}{n} \Sigma (x_i - m_x)(y_i - m_y) = \text{covariance} \qquad (5)$$

From the definitions of 4 and 5

$$r = \frac{C}{s_x s_y} \qquad (6)$$

4. UNCERTAINTY IN A CALCULATED PREDICTION

Engineering analysis uses mathematical models of one form or another to predict facility performance. Estimates of soil properties are used as input to these models, and calculations of settlement, factor of safety, or other performance variables are obtained as output. In risk-based design soil properties are estimated by a mean and standard deviation, and this mean and standard deviation are translated through an engineering calculation to obtain a corresponding mean and standard deviation on the performance variable.

Mathematically, an engineering calculation can be represented by the model

$$y = g(x) \tag{7}$$

in which x = a soil property or other input parameter, and y = a calculated prediction (Fig. 6). In risk-based design the mean m_x and standard deviation s_x of the input parameter are translated through g(x) to obtain a mean m_y and standard deviation s_y of the calculated prediction. The mathematical approach to translating (m_x, s_x) to (m_y, s_y) is based on a linear approximation. For most geotechnical purposes this linear approximation is sufficiently accurate to meet practical needs.

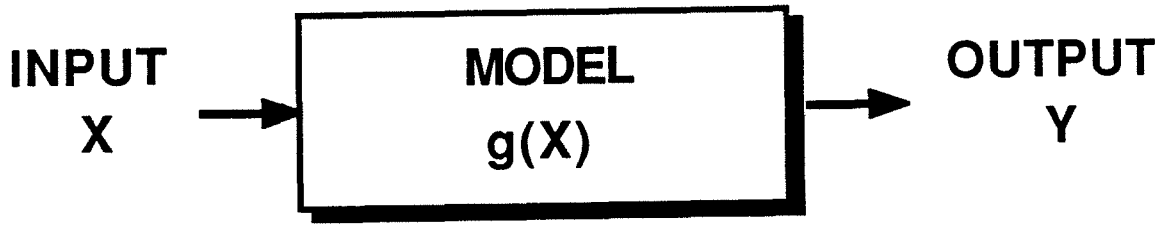


Figure 6 Error propagation through an engineering model.

4.1 Best Estimate (Mean) Prediction

Operationally, best estimates of soil properties are translated through g(x) using a linear approximation. This approximation replaces g(x) by its tangent at m_x (Figure 7). Applying probability theory leads to the result,

$$m_y \cong g(m_x) \qquad (8)$$

in which $\cong$ indicates a tangent approximation. In words, the mean or best estimate of the prediction y is the function of the mean or best estimate of the parameter x. This is the normal deterministic solution using best-estimate soil properties as input.

Figure 8 shows a calculation of bearing capacity for an unembedded footing in which data on friction angle come from laboratory tests, and bearing capacity is calculated using Terzaghi's bearing capacity factor N_γ. This factor is related empirically to ϕ' by the equation $\log_e N_\gamma \approx (-2.107+0.173\,\phi')$ (Ingra and Baecher, 1983).

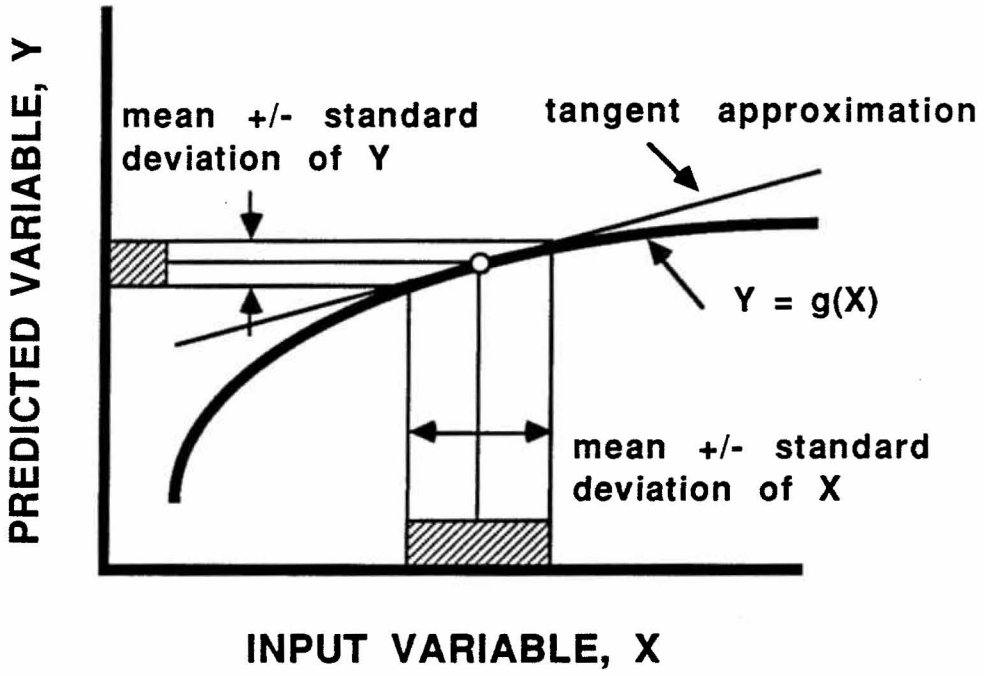


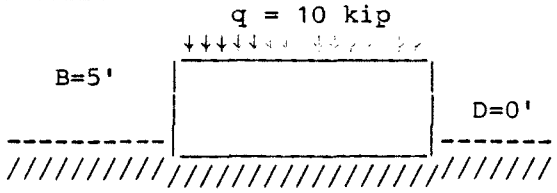
Figure 7 Tangent approximation for calculating mean and standard deviation of Y from mean and standard deviation of X.

CALCULATION SHEET

PROBLEM: bearing capacity CALCULATED BY: _____

DATE: _____ CHECKED BY: _____

PROBLEM



SOIL PROPERTIES

$m_\phi = 36.4°$ (n=5)

$s^2_\phi = (1.14°)^2$

$m_\gamma = 120$ pcf

s^2_γ = nil

$\log_e N_\gamma = -2.107+0.173\phi$

$$q = (1/2)\ \gamma\ b\ N_\gamma$$

(a) BEST ESTIMATE (MEAN) OF BEARING CAPACITY

$$\begin{aligned} m_q &= (1/2)\ (m_\gamma)\ (b)\ (m_{N\gamma}) \\ &= (1/2)\ (120)\ (5)\ (66.0) \\ &= 19.8 \text{ kip} \end{aligned}$$

(b) UNCERTAINTY (STANDARD DEVIATION) OF BEARING CAPACITY

$$\begin{aligned} s^2_{\ln N\gamma} &= \left(\frac{dq}{dN\gamma}\ \frac{dN\gamma}{d\phi'}\right)^2 s^2_\phi \\ &= [(1/2)\gamma B]^2\ [0.173\exp(-2.107+0.173\phi)]^2\ s_\phi^2 \\ &= [(300)^2(11.4)^2(1.14)^2 \\ &= (3.9\text{kip})^2 \end{aligned}$$

(c) RELIABILITY INDEX

$$\begin{aligned} \beta &= \frac{m_{qv} - q_{vo}}{s_{qv}} \\ &= \frac{19.8 - 10}{3.9} \\ &= \underline{2.5} \end{aligned}$$

Figure 8

4.2 Uncertainty (Standard Deviation) in Predictions

By similar reasoning, the standard deviaton on an input soil property x is translated through g(x) to find a corresponding standard deviation on the prediction y. The tangent approximation leads to the relation

$$s_y \cong \left(\frac{dy}{dx}\right) \quad s_x \quad . \tag{9}$$

In words, the standard deviation of the prediction y is the product of the standard deviaton of the parameter x and an influence factor equal to the derivative of y with respect to x (Figure 8). The relation is exact when g(x) is linear.

When the prediction depends on a set of parameters $\underline{x}=\{x_1,\ldots,x_n\}$ the equivalent forms of Eqs. 8 and 9 are,

$$m_y \doteq g(m_{x_1},\ldots,m_{x_n}) \tag{10}$$

$$s_y^2 \doteq \Sigma\,\Sigma \frac{dy}{dx_i}\frac{dy}{dx_j} C_{xi,xj} \tag{11}$$

in which $C_{xi,xj}$ is the covariance of x_i and x_j from Eqn. 5 (for a more general discussion see Ditlevsen, 1981). Note that, when two variables x_i and x_j are independent, their correlation coefficient and covariance are zero. Similarly, the covariance of a variable x_i with itself equals its variance s_{xi}^2. Thus, in the case that all x_i and x_j are independent, 11 reduces to,

$\ldots,y_m\}$ is calculated from a set of soil properties $\underline{x} = \{x_1, \ldots,x_k\}$ are

$$m_Y \cong g(m_X), \text{ and}$$

$$\Sigma_y \cong G^t \Sigma_x G,$$

in which $m_Y=\{m_{y1},\ldots,m_{yn}\}$ is the vector of means of the y_i; $m_X=\{m_{x1},\ldots,m_{xk}\}$ is the vector of means of the x_j; Σ_Y is the covariance matrix of $\underline{y}$ having ij-th term $C_{yi,yj}$, and Σ_X is the covariance matrix of $\underline{x}$.

$$s_y^2 \cong \sum \left(\frac{dy}{dx_i} \right)^2 \; s_{x_i}^2 \qquad (12)$$

Two special cases of 12 deserve note because they are common in practice. For the case in which y is a linear combination of a set of independent parameters $y = \Sigma\, a_i x_i$ the variance of y is exactly

$$s_y^2 = \Sigma\, a_i^2\, s_{x_i}^2 \qquad (13)$$

This situation occurs in the example of Section 7. For the case in which y is a power function of a set of independent parameters, $y = \Pi\, x_i^{a_i}$, the variance of y is approximately

$$1+\Omega_y^2 \triangleq \Sigma\, (1+a_i^2 \Omega_{xi}^2) \qquad (14a)$$

which for small coefficients of variation reduces to

$$\Omega^2_y \doteq \Sigma\, a_i^2\, \Omega_{x_i}^2 \qquad (14b)$$

This case occurs in the example of Section 8.

4.3 Reliability Index β

In traditional analysis the adequacy of a design is expressed by the factor of safety

$$F = \frac{\text{capacity}}{\text{demand}} \qquad (15)$$

Depending on the level of uncertainty in a calculation, the same numerical value of F can imply different levels of reliability. A high F with correspondingly high uncertainty can imply less reliability than a small F with correspondingly small uncertainty. This situation is shown schematically in Figure 9. The bell-shaped curves represent the probability distributions of two calculated factor of safetys. The peaks occur at the respective means; the widths reflect the respective standard deviations. The areas under the curves beneath F=1.0 are proportianal to the likelihoods of failure. The smaller this area, the greater the reliability of the design. In this case, the design with the smaller best estimate F is actually the more reliable.

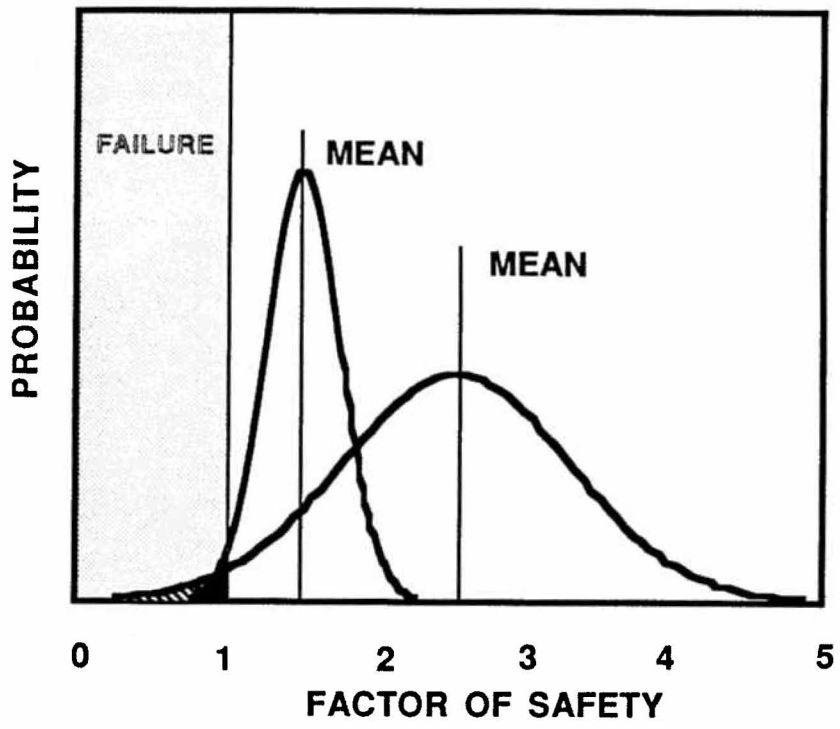


Figure 9 Two cases of uncertainty in predicted factor of safety

To express reliability, we would like to combine the best estimate of F and the standard deviation of F in a single number. One convenient way is the reliability index

$$\beta = \frac{m_F - 1.0}{s_F} \qquad (16a)$$

The reliability index measures the number of standard deviations of the predicted F separating the best estimate from the limiting value F=1.0. Rewritten in a more general form

$$\beta = \frac{m_y - y_f}{s_y} \qquad (16b)$$

in which y_f is the "failure" value of y, and $m_F = (m_y/y_f)$. β measures the number of standard deviations separating the best estimate m_y from some unacceptable value y_f. In Figure 8 the design load on the footing is 10ksf.

Since the best estimate of the bearing capacity is m_y = 19.8ksf and the standard deviation is s_y = 3.9ksf, the reliability index equals β = 2.5. That is, m_y = 19.8ksf is 2.5 standard deviations above the required capacity 10ksf.

For most geotechnical applications β ranges from 1.5 to 3.0 (Table 1). Lower values of β imply lower reliability. β=0 means the best estimate of performance just equals the failure criterion, that is, $m_y = y_f$. β>0 means that $m_y > y_f$, because the standard deviation is always positive.

5. SEPARATING THE COMPONENTS OF UNCERTAINTY

In Section 4, no distinction was made among different types of uncertainty in considering how standard deviations translate through calculations. A standard deviation was merely assumed for an input parameter, and a corresponding standard deviation on the output prediction was derived. In fact, the four components of uncertainty discussed in Section 2 affect predictions in different ways. As a result they should be considered separately. The present section uses results of Section 4 to consider how each component of uncertainty individually translates through an engineering calculation. At the end, the components are brought together in a single standard deviation on a calculated prediction, y.

The four components of uncertainty in a soil property estimate as discussed in Section 2 are:

1. Spatial variability,
2. Measurement noise, (17)
3. Measurement bias, and
4. Statistical error.

Spatial variability and measurement noise appear as data scatter. Measurement bias and statistical error are systematic errors. Each component can be represented by its own standard deviation, denoted by the subscript 1 through 4, respectively

s_{x_1} = standard deviation due to spatial variation,
s_{x_2} = standard deviation due to measurement noise, (18)
s_{x_3} = standard deviation due to measurement bias,
s_{x_4} = standard deviation due to statistical error.

Based on 12, the total uncertainty in a soil property estimate x is the sum of the four component variances,

Table 1 -- Typical Reliability Indices for Geotechnical Facilities

Facility	Typical Ω	Typical F	Typical β
Earth retaining structures	0.13±	1.3 to 1.5	2.0 to 2.5
Earthworks	0.15±	1.5 to 2.0	2.0 to 3.0
Offshore Foundations	0.20±	1.5 to 2.0	1.5 to 2.5
Onshore Foundations	0.25±	2.0 to 3.0	2.0 to 3.0

(after Meyerhof, 1976)

$$s_x^2 = s_{x_1}^2 + s_{x_2}^2 + s_{x_3}^2 + s_{x_4}^2 \quad (19)$$

That is, the four components of uncertainty add together through the square of their respective standard deviations.

In analyzing the uncertainty in a prediction $y = g(x)$ the four component standard deviations of 18 are individually translated through $g(x)$, as shown in Figure 10. This leads to four component standard deviations on y corresponding to spatial variation, measurement noise, measurement bias, and statistical error. Only at this stage are the components combined to assess the overall uncertainty in y. If the components are combined beforehand using Eqn. 19, and then the overall uncertainty in x used to calculate the uncertainty in y, a wrong answer is obtained.

5.1 Spatial Variation

Soils are geological materials with physical properties that vary from place to place. This spatial variability, combined with random errors in testing, produces the scatter observed in soil engineering data.

In Section 3, the mean and standard deviation were used to describe the variability in a set of soil property data. These are useful measures, but they combine data in such a way that spatial information is lost. For example, consider the two sequences of measurements in Figure 11. These data have the same mean and standard deviation but reflect different soil conditions. The first data exhibit a distinct trend while the second are erratic. This difference is not captured by the mean and standard deviation alone.

Two new measures are needed to describe spatial variation. The first is the trend of the data. In principle, spatial variation can be characterized precisely, but only if many, many tests are made. Thus, for engineering purposes a simplification is adopted which necessitates only more realistic amounts of data. Spatial variability is separated into two parts: a deterministic trend and residual variability about the trend. This model is written

$$x_i = t_i + u_i \quad (20)$$

in which x_i = the soil property at location i, t_i = the value of the trend at i, and u_i = residual about the trend. The trend is characterized by some equation (e.g., a line or curve), and the residuals are characterized statistically (e.g., by a mean and standard deviation).

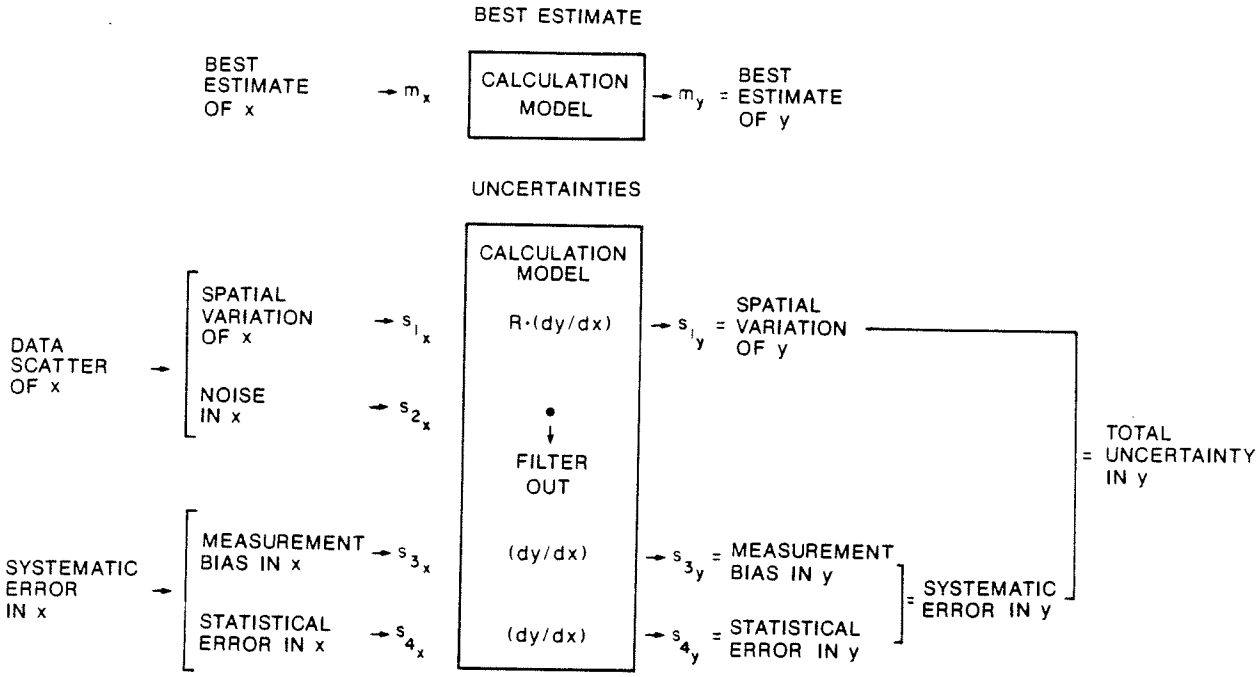


Figure 10 Propagation of principal sources of uncertainty through a model.

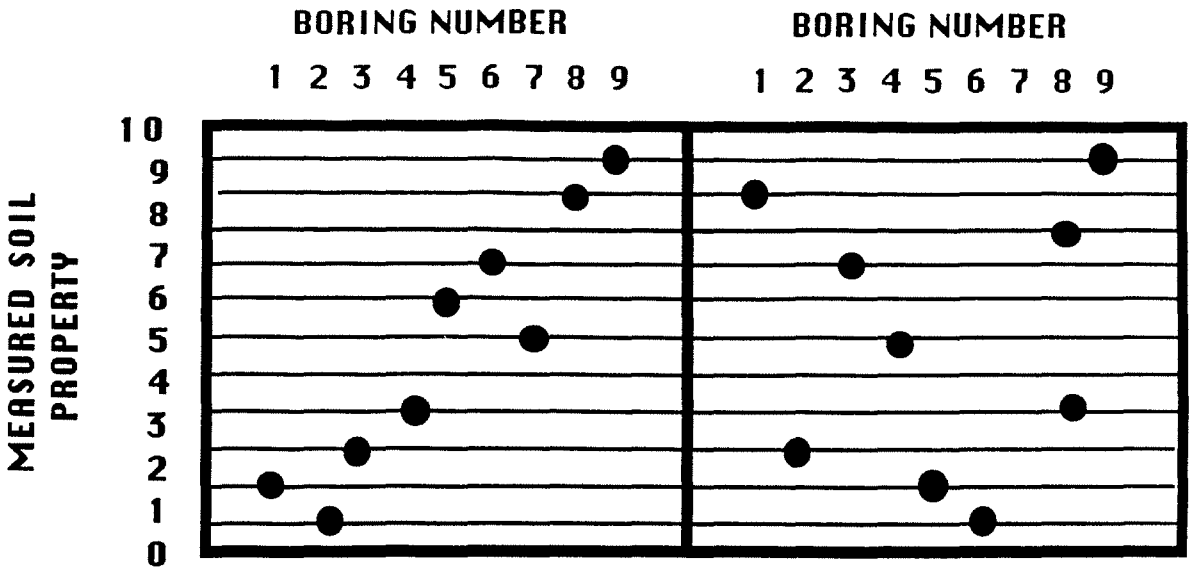


Figure 11. Hypothetical data showing difference in spatial pattern but same mean and standard deviation.

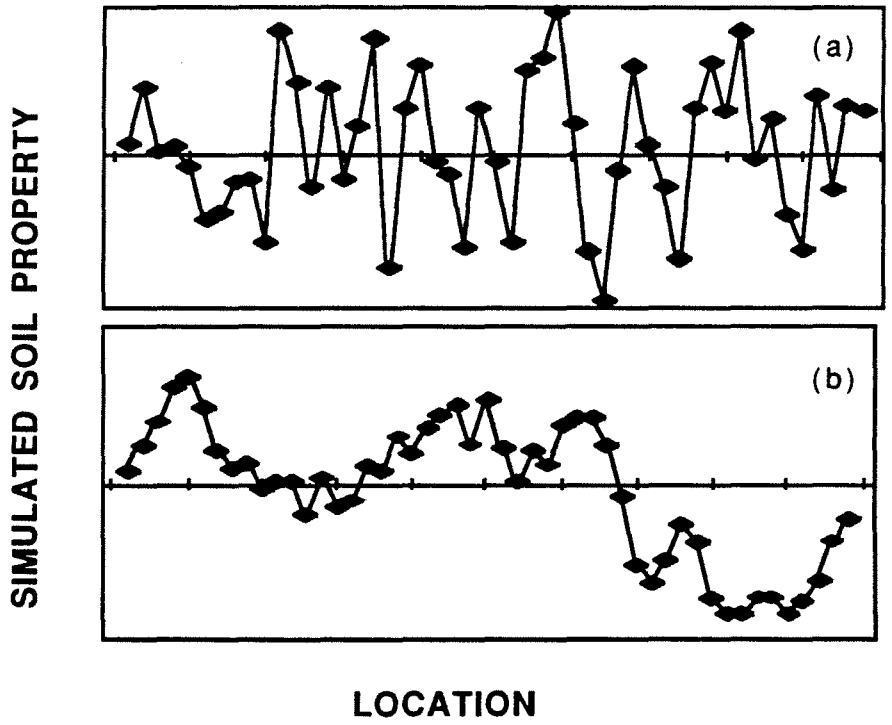


Figure 12 Spatial variation of soil property data showing, (a) rapid fluctuation and (b) smooth fluctuation; mean and standard deviation constant.

The second new measure is something statisticians call the autocorrelation function. In a loose sense, the autocorrelation function measures the waviness of the residual variations about a trend. Consider the data in Figure 12. In both Figures 12a and 12b the amplitude (i.e., standard deviation) of the residual variation is the same. However, in Figure 12a the residuals fluctuate rapidly. Their predominant wavelength is short. In Figure 12b the residuals fluctuate less rapidly. Their predominant wavelength is long. This difference in waviness is captured in the autocorrelation functions of the two sets of data.

As a general rule, when a measurement at some location i in a soil deposit is observed to lie above the trend line at that point, the measurements at nearby locations also lie above their corresponding trends, and vice versa. This is autocorrelation. The longer the apparent 'wave length' of the residuals about their trend, the farther autocorrelation extends. The waviness of residual soil data reflects spatial structure that is ignored in fitting a trend. That is, it is spatial variation at too detailed a level to be accounted for deterministically.

More formally, autocorrelation is the property that residuals off the mean trend are not statistically independent. They are associated with one another, and the degree of association depends on the distance separating two measurements in the field. As in Section 3, the degree of association is measured by a correlation coefficient. The correlation coefficient defined in Eqn. 4 gave the correlation between two different soil properties, for example, undrained strength and water content. Autocorrelation gives the correlation between two measurements of the same property made at different locations.

Mathematically, the autocorrelation function $R_x(\delta)$ is defined as

$$R_x(\delta) = \left(\frac{1}{n}\right) \Sigma \left(\frac{u_i}{s_u}\right) \left(\frac{u_{i+\delta}}{s_u}\right) \qquad (21)$$

in which u_i = the residual at location i, $u_{i+\delta}$ = the residual at location $i+\delta$, s_u = the standard deviation of u, and n = the number of data pairs having separation distance δ. $R_x(\delta)$ expresses the correlation of two residuals as a function of their separation distance. By definition, $R_x(0)=1.0$.

In the same way that covariance (Eqn. 5) is related to correlation (Eqn. 4), the autocovariance function $C_x(\delta)$ is related to $R_x(\delta)$ by the variance of the residuals s_u^2

$$C_x(\delta) = \left(\frac{1}{n}\right) \Sigma \left(u_i\right) \left(u_{i+\delta}\right) \qquad (22)$$

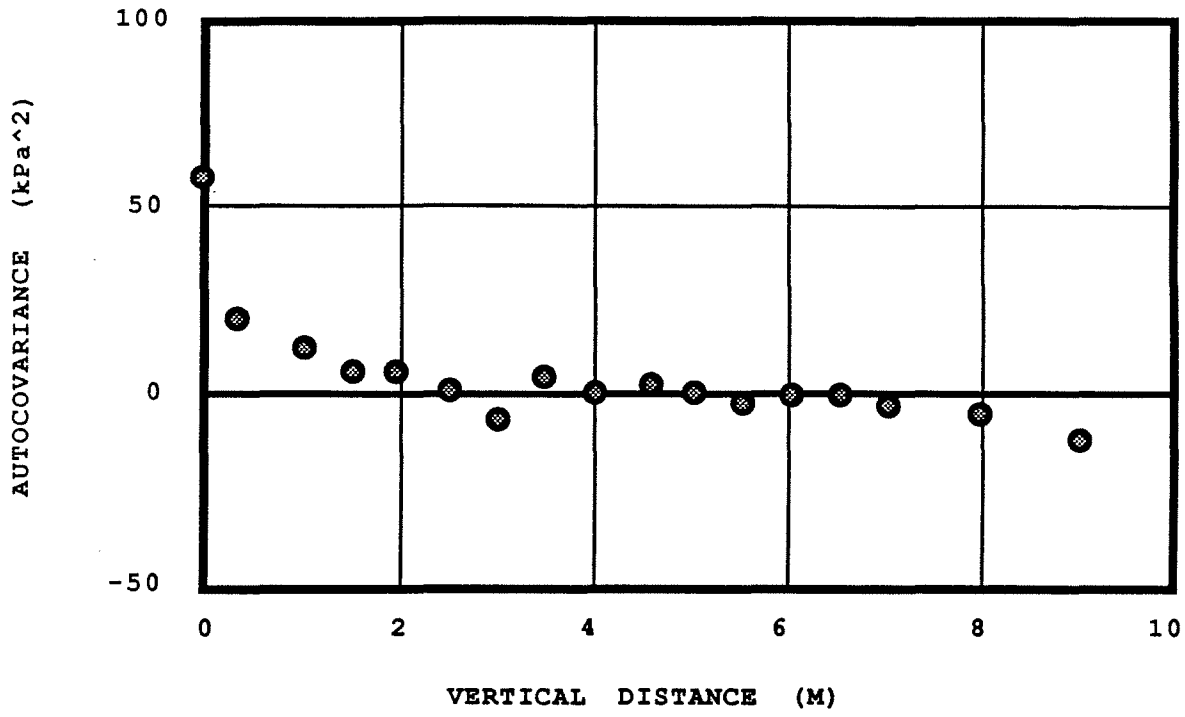


Figure 13 Vertical autocovariance function for field vane data in a soft marine clay.

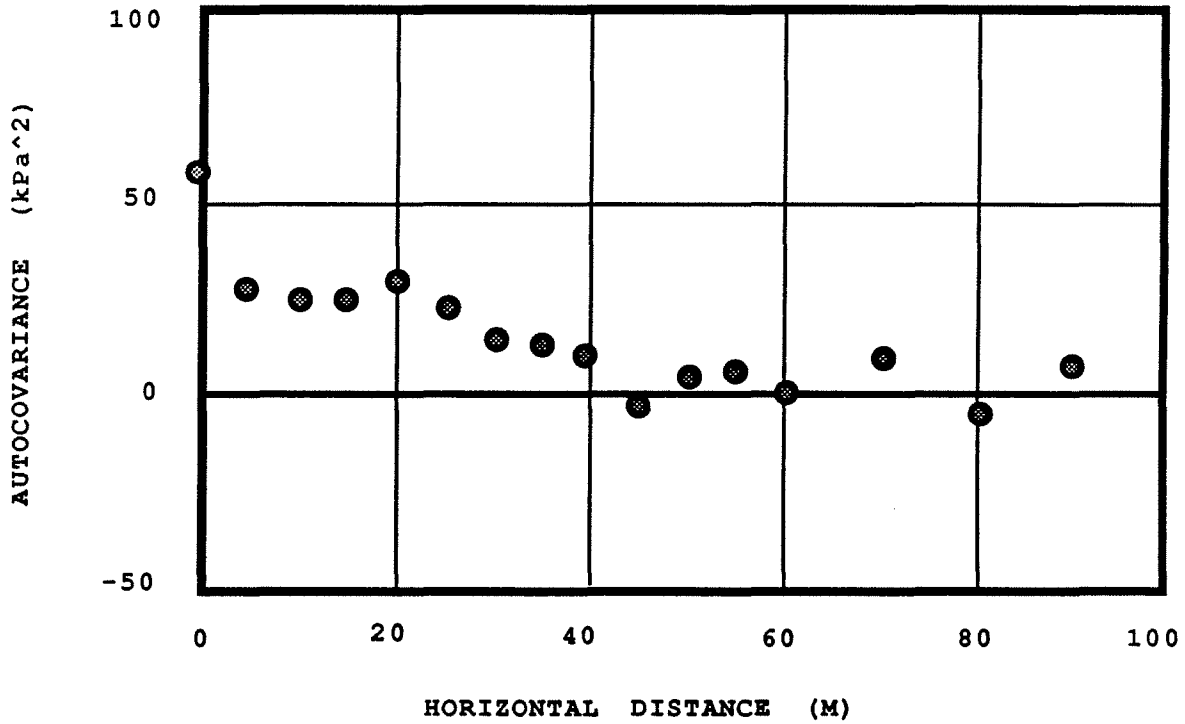


Figure 14 Horizontal autocovariance function for field vane data in a soft marine clay.

The relationship between the autocorrelation function and the autocovariance function is the same as that between the correlation coefficient and the covariance.

Figure 13 shows the vertical autocovariance function estimated for a set of field vane measurements taken from 27 borings in a soft clay deposit. At zero separation distance, $C_x(0)=s_x^2$. For these data s_x^2 is about $50kPa^2$. Figure 14 shows the horizontal autocovariance function for the same set of data.

Horizontally, the autocorrelation may be isotropic or anisotorpic depending on how a deposit was formed, but in practice isotropy is often assumed. Also, autocorrelation is typically assumed to be the same everywhere within a deposit. This assumption, called stationarity, is equivalent to saying that the deposit is statistically homogeneous.

Manually, autocorrelation is estimated from a set of data following a five-step procedure:

1. Form every possible pair of measurements.
2. Group the pairs into sets having approximately the same separation distance.
3. For each set plot a scattergraph of (u_i) vs. $(u_{i+\delta})$.
4. Calculate the correlation coefficient of each graph using Eqn. 4.
5. Plot the correlation coefficients as a function of separation distance.

The autocovariance function is estimated by replacing Eqn. 4 in step 4 by Eqn. 5. An example of the procedure using the SPT data from Figure 3 is shown in Figure 15. Alternately, micro computer programs based on more sophisticated statistical techniques are available which automatically compute estimates of the autocorrelation or autocovariance function (e.g., DeGroot, 1985).

5.2 Measurement Noise

Random measurement error is that part of data scatter attributable to instrument or operator induced variations from one test to another. Sometimes the measurement is higher than the real value of the property, sometimes it is lower, and on average may systematically differ from the real value. The systematic difference between the real value and the average of the measurements is said to be measurement bias (cf., Figure 2), while the variability of the measurements about their mean is said to be random measurement error.

Typically, random errors tend to be small and they tend to distribute equally on both sides of zero. Measurement error is the cumulative effect of an indefinite number of small perturbations simultaneously affecting a measurement. Because these errors do not reflect inherent variability in the soil itself, if their magnitude can be accurately estimated they can be removed from the total uncertainty in x and a more confident prediction of facility performance can be made.

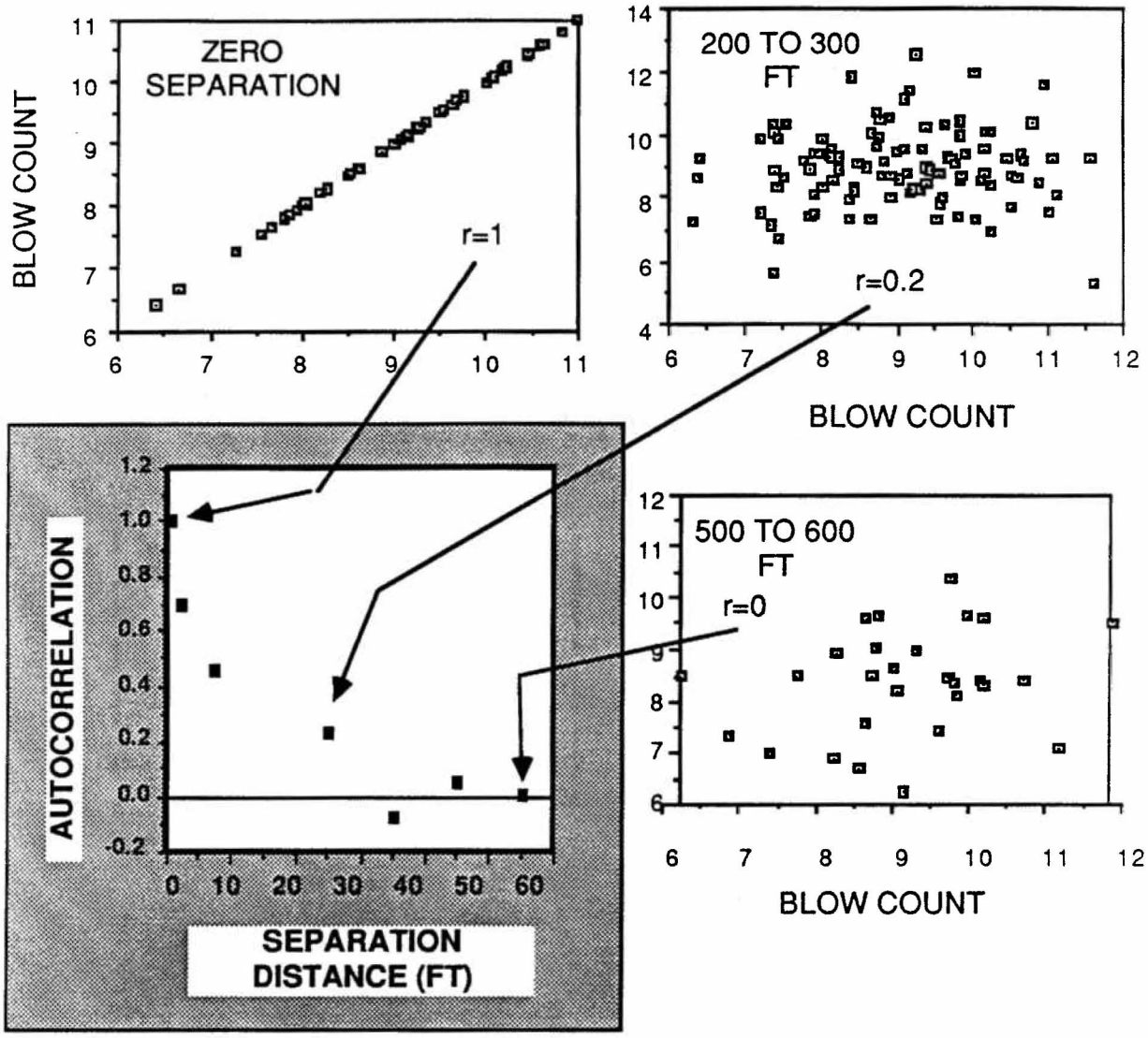


Figure 15 Estimation of autocorrelation function from data: Surrounding scatter plots show correlation of blow count data at different separation distances, which are transferred to form autocorrelation function.

The common model of measurement error is

$$z = x + e \tag{23}$$

in which z = the measurement, x = the soil property being measured, and e = a random error with zero mean. Empirically, the value e takes on at one measurement is assumed to be unrelated to the value it takes on at any other. That is, values of e are independent from one test to another.

Random measurement error can be estimated in a variety of ways. As a general rule, direct techniques such as replicate testing are difficult to apply to soil data because measurements are made destructively. Indirect methods based on the autocorrelation function are usually preferred. These techniques are based on the observation that real spatial variability of soils and random measurement errors have different signatures in the autocorrelation function, and this difference in signature can be used to estimate the relative contribution of each to the total data scatter.

Inserting the measurement model z=x+e into the definition of the autocovariance function (Eqn. 22), and algebraically rearranging, leads to a relation between the autocovariance function of the measured data z and the autocovariance functions of the real soil variability x and of the random measurement error e

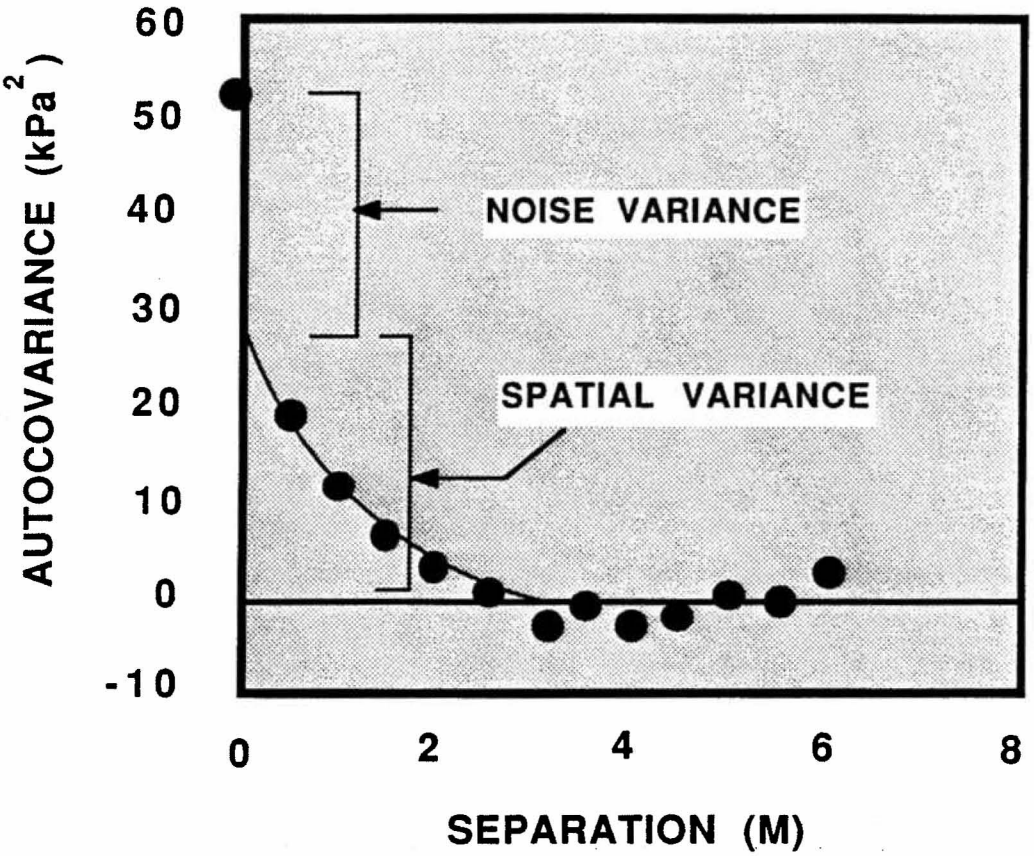


Figure 16 Estimation of noise in field vanes.

$$C_z(\delta) = C_x(\delta) + C_e(\delta) \qquad (24)$$

The autovariance function of z is the sum of the autocovariance functions of x and e.

For most soils, the waviness of real spatial variability about a trend leads to an autocovariance function $C_x(\delta)$ which equals s_{x1}^2 at the origin, and monotonically decreases to zero as δ increases. On the other hand, since the random errors e are independent from one test to another, their autocovariance except at the origin is zero. At the origin, $C_e(0)=s_{x2}^2$. Thus, $C_e(\delta)$ is a spike at $\delta=0$ and zero elsewhere. The relative contribution of s_{x1}^2 and s_{x2}^2 to the total data scatter s_z^2 can be estimated by extrapolating the autocovariance function of the observed data back to the origin. When this is done for the autocorrelation function in Figure 13, the variance of random measurement noise appears to be about $20kPa^2$, or about 40 percent of the data scatter variance $s_z^2=50kPa^2$ (Figure 16).

5.3 Size Effect Factor R_v

The volume of soil influenced by an in situ test or contained in a laboratory specimen is small compared with that influenced by a prototype structure. To make predictions of how the prototype will perform, an estimate is needed of the average properties within this larger, representative volume of soil, and of the variability among the averages of representative volumes.

This is done by considering the representative volume to be composed of a large number of elements each the size of a test specimen. From the statistical formulas of section 3, the mean and standard deviation of the properties of specimen sized elements are found, and then using the spatial structure described by the autocorrelation function, a mean and standard deviation for the larger volumes can be calculated. These calculations are summarized in a size-effect factor, R_v, which can usually be expressed by simple formulas or read from tables. The derivation of R_v, however, is beyond the scope of the present manual.

Empirically, the variability of average properties among small soil elements is larger than among large elements. Less averaging-out takes place in small volumes, so the average property fluctuates considerably from one element to the next. In large elements the reverse is true. High values balance out against low values within a single element and the average fluctuates little from one element to the next. The extent of averaging of properties within a large volume of soil depends on the structure of the spatial variation. More precisely, the extent of averaging depends on the autocorrelation function.

Consider the problem of calculating the variability of average SPT blow count among borings in a homogeneous soil. Figure 17 shows a set of six boring logs. One N value is randomly chosen from each boring and the standard deviation among them is calculated. Then two N values in each boring are randomly chosen, the average taken, and the standard deviation of these boring averages calculated. From this calculation a smaller standard deviation results. Continuing, the greater the number of N values included in the average of each boring, the smaller the standard deviation of the boring-averaged N across the six borings.

The same thing happens when the properties in individual elements within a large volume of soil are averaged. As the number of elements increases--that is, as the volume over which averaging takes place increases--the variation among the averages of the large volumes of soil goes down. In this way, for example, one would expect that the average soil properties beneath large footings would be less variable than the average properties beneath small footings.

This effect of size on the spatial variation among average properties within representative volumes of soil is represented by a size-effect factor, R_v. The size-effect factor is defined as the ratio of the variance of the average soil property within a large volume of soil, s_v^2, to the variance among specimen-sized volumes, s_{x1}^2

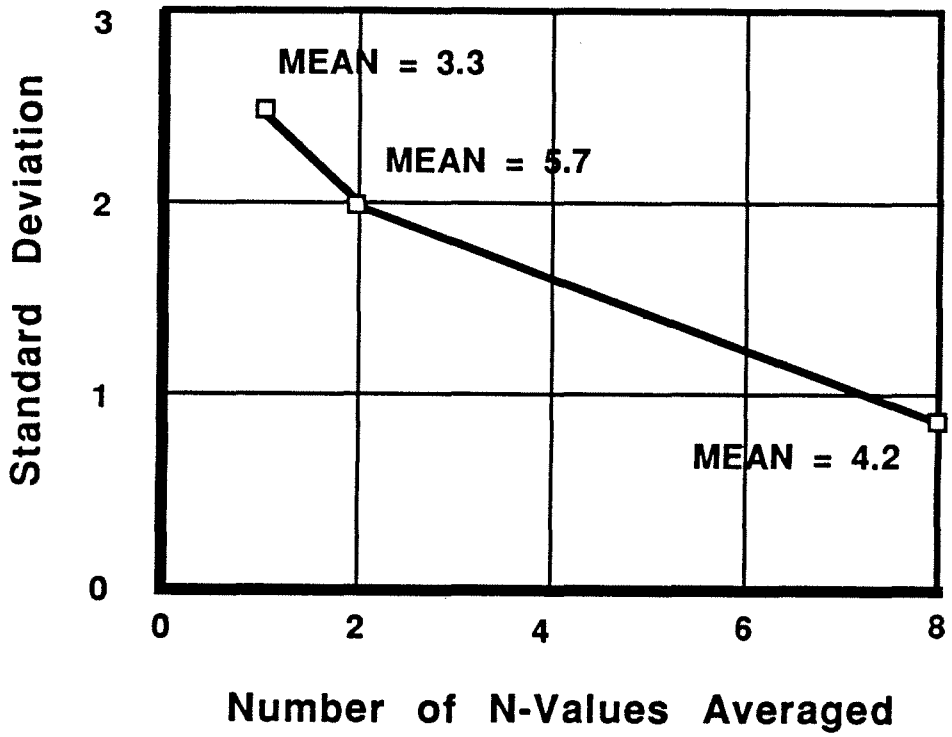


Figure 17 Averaging of blow count data in six borings.

$$R_v = \frac{s_v^2}{s_{x_1}^2} \qquad (25)$$

in which v = the representative volume. The ratio of variances rather than standard deviations is used because it is more convenient in subsequent uncertainty analysis.

As a first approximation, the size effect factor R_v for averaging of soil properties over a line is approximately,

$$R_v \cong (2\delta_o/L) \qquad \text{one-dimensional case} \qquad (26)$$

in which L is the distance of the line, and δ_o is the so-called autocorrelation distance. The autocorrelation distance is that distance at which $R_x(\delta_o)$ reduces to 1/e, in which e is the base of the natural logarithms (1/e=0.37). The autocorrelation distance is simply a convenient way to express the significant extent of correlation in the soil properties. The approximation is good for $L \geq 4\delta_o$, but a satisfactory upper bound for $L \geq 2\delta_o$.

For averaging over 2-dimensional rectangles or 3-dimensional prisms--under fairly general conditions--Eqn. 26 becomes,

$$R_v \cong (2\delta_1/L_1)(2\delta_2/L_2) \qquad \text{2-dimensional case} \qquad (27)$$

$$R_v \cong (2\delta_1/L_1)(2\delta_2/L_2)(2\delta_3/L_3) \qquad \text{3-dimensional case} \qquad (28)$$

in which L_1, L_2, and L_3 are the lengths in each dimension, and δ_1, δ_2, and δ_3 are the corresponding autocorrelation distances in those dimensions.

5.4 Systematic Error

Thus far the analysis of uncertainties has concentrated on data scatter. We saw that data scatter uncertainties manifest as variability across a site, for example, variability of settlement from one footing to another. Another type of uncertainty is also important: systematic errors. Uncertainties due to systematic errors do not manifest as variability across the site, but appear as a difference between the predicted average performance and the average performance which occurs in the field. Systematic errors are biases. Usually they occur because errors are introduced in estimating mean values of soil properties, loads, and other input variables.

The most important systematic errors in soil property estimates are measurement bias and statistical error. Measurement bias is caused by

inadequacies in the way soil test results are interpreted. Statistical error is caused by limited numbers of tests.

5.5 Measurement Bias

In testing soils, whether in the field or laboratory, a system of boundary conditions is applied to a specimen and response measured. From this response and a set of physical assumptions--i.e., a model--soil properties are backcalculated. These properties are then used with another model to predict performance. Nonrandom errors are introduced to this process at several points, and it is these nonrandom errors which give rise to measurement bias.

Among the more common measurement biases in estimating soil properties are, (1) inappropriate boundary conditions, (2) incorrect model assumptions, and (3) sample disturbance. The first occurs, for example, when the stress system imposed on a specimen differs in principal directions from that imposed by a prototype structure. The second occurs, for example, when elastic theory is used to backcalculate deformation moduli for a soil which actually deforms nonlinearly.

For most practical situations, the systematic error introduced by measurement can be adequately represented by a bias correction factor, B, such that the measurement z is related to the real soil property x by an extension of Eqn. 23

$$z = \left(\frac{1}{B}\right)(x) \tag{29}$$

Applying Eqn. 12 the variance in x due to uncertainty about the correct value of B is

$$s_{x_3}^2 = m_x^2 s_B^2 \tag{30}$$

in which s_B^2 is the uncertainty in the appropriate value of B.

The evaluation of s_B^2 is usually made by comparing predicted performance of prototype structures with the actual performance measured in the field. For example, for measurements of undrained strength of soft clays made with the field vane device, s_B^2 can be found from the scatter of calibration data such as those collected by Bjerrum (Fig. 18).

5.6 Statistical Uncertainty

Because a limited number of measurements are made at any depth, their average may be above or below the actual spatial average even if there is no measurement bias. Statistical theory allows an assessment of the probable

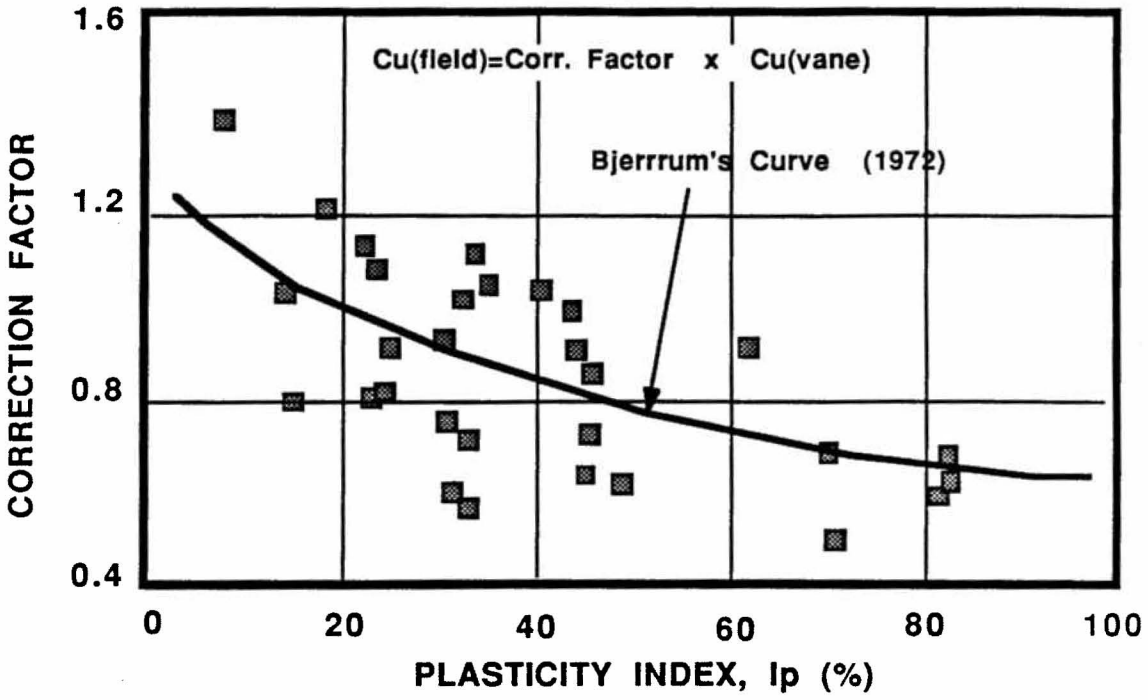


Figure 18 Field vane correction fator after Bjerrum (1972), with additions by Ladd (personal communication, 1983).

magnitude of this error. Typically, statistical error is expressed as a variance or standard deviation on the estimated parameter. For example, the statistical error on the estimate of the average soil property would be expressed as a variance on the mean.

The larger the number of measurements at any depth, the lower the statistical error. In general, the variance of statistical error decreases approximately in proportion to the reciprocal of the number of observations, n. Doubling the number of tests typically reduces the statistical error by about $1/\sqrt{2}$, so the marginal benefit of increased testing suffers diminishing returns. From rudimentary statistics, the variance of the statistical error of the mean is approximately

$$s_{m_x}^2 = \frac{s_{x_1}^2 + s_{x_2}^2}{n} \qquad (31)$$

If repeated samples of n tests from the same soil deposit are made, and if each of the tests is statistically independent of all others, and if for each sample the mean is calculated, then the variability of those means would have variance $s_{m_x}^2$.

5.7 Recombinining the sources of Uncertainty

Having considered each component of uncertainty by itself, it is now time to bring the components together in an aggregate standard deviation on the predicted variable y.

The effect of the components of uncertainty in x on the prediction y is found by Eqn. 9. Each component standard deviation s_{x1} through s_{x4} is multiplied by (dy/dx) to determine the corresponding component standard deviations s_{y_1} through s_{y_4} on y, then the standard deviations are combined through their squares (i.e., variances) as in equation 12.

For the spatial variability component, $s_{x_1}^2$ is multiplied by the appropriate size-effect factor R_v to correct for the difference in scale between test specimen and prototype (Eqn. 25)

$$s_{y_1}^2 = R_v \left(\frac{dy}{dx}\right)^2 s_{x_1}^2 \tag{32}$$

For the measurement bias component, $s_{x_3}^2$ is taken from Eqn. 30

$$s_{y_3}^2 = \left(\frac{dy}{dx}\right)^2 m_x^2 s_B^2 \tag{33}$$

For the statistical error component, $s_{x_4}^2$ is taken from Eqn. 31

$$s_{y_4}^2 = \left(\frac{dy}{dx}\right)^2 \left(\frac{s_{x_1}^2 + s_{x_2}^2}{n}\right) \tag{34}$$

Measurement noise reflected in $s_{x_2}^2$ increases statistical error because it increases data scatter and thus increases the variability of the calculated value of m_x from one set of tests to another. However, the direct affect of measurement noise on the uncertainty in y is eliminated because it does not reflect real variability in soil properties and can be statistically removed from the data scatter using techniques such as those in Section 5.2.

The components of uncertainty in y are combined as in Eqn. 19 to give an overall variance on predicted performance s_y^2

$$s_y^2 = s_{y_1}^2 + s_{y_3}^2 + s_{y_4}^2 \qquad (35a)$$

$$= \left(\frac{dy}{dx}\right)^2 \left(R_v\, s_{x_1}^2 + m_x\, s_B^2 + \frac{s_{x_1}^2 + s_{x_2}^2}{n}\right) \qquad (35b)$$

in which the direct effect of $s_{x_2}^2$ is eliminated by having been removed from the original data scatter. The following sections illustrate how Eqn. 35 is used in applications.

6. APPLICATION TO SHALLOW FOOTINGS

In this section the procedure outlined above is applied to settlement calculations for the design of shallow footings on sand.

6.1 Site Conditions

The site is underlain by fine dry sand to a depth of 10m (Hilldale, 1971). Fifty SPT borings were made across the site and a limited number of laboratory tests were performed to correlate blow count with friction angle. The trend of depth-averaged blow counts was corrected by Gibbs and Holtz's method, and the autocovariance function used to estimate the contribution of noise to total data scatter, as described in Section 5.2. For the upper levels of the profile which most strongly influence the settlement of shallow footings, the average blow count is 16.6 bpf, and the average corrected blow count is about constant with depth at 25 bpf (Figure 19).

The standard deviation of the vertically averaged corrected blow count in the upper levels is about s_N=11bpf. Thus, the coefficient of variation is Ω_N=(11bpf/25bpf)=0.44. Using the autocorrelation function technique to estimate noise suggests that noise contributes about 50 percent of this data scatter, measured in variances.

6.2 Best Estimate of Footing Settlement

The results of applying the methods described in Section 5 to predictions of settlement are shown in Figure 20.

The footing is 10' wide and embedded 5', with a design load of 3 TSF, as shown. Settlement is predicted by the Peck and Bazaraa formula

$$\rho \doteq \left(\frac{2\Delta q}{N}\right) \left(\frac{2b}{1+b}\right)^2 \left[1 - \frac{1}{4}\, D/b\right] \qquad (36)$$

CALCULATION SHEET

PROBLEM: SPT data analysis CALCULATED BY: _____

DATE: _____ CHECKED BY: _____

DESIGN PROFILE: SPT data in a clean, wind-deposited sand.

DATA SCATTER

Data:

n = 50 measurements

m_N = 16.6 between elevations 590' and 610'

m_{Nc} = 25 bpf, corrected blow count approximately constant with depth for first 20'

s_N = 11 bpf (total data scatter of vertically averaged blow counts)

Measurement Noise (from autocorrelation analysis):

$s_{N_2} \approx$ 7.8 bpf

Spatial Variability:

$$s_{N_1}^2 = s_N^2 - s_{N_2}^2$$

$$\approx (11\text{bpf})^2 - (7.8\text{bpf})^2$$

$$\approx (7.8\ \text{bpf})^2$$

SYSTEMATIC ERROR

Statistical Error $S_{3_N}^2$:

$$S_{m_N}^2 \approx S_N^2/n$$

$$\approx (11\text{bpf})^2/50$$

$$\approx (1.6\text{bpf})^2$$

Measurement Bias S_4^2:

<ignored>

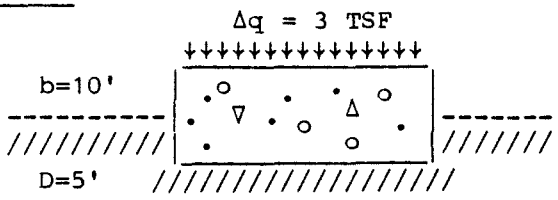
Figure 19

CALCULATION SHEET

PROBLEM: footing settlement _____ CALCULATED BY: _____

DATE: _____ CHECKED BY: _____

PROBLEM



allowable ρ = 1"

SOIL PROPERTIES

N = vertically averaged corrected SPT blow count

m_N = 25 bpf

s_N = 11 bpf

n = 50 measurements

(a) BEST ESTIMTE (MEAN) OF SETTLEMENT

$$m_\rho = \left(\frac{2\Delta q}{m_N}\right)\left(\frac{2b}{1+b}\right)^2 [1-\frac{1}{4} D/b] = \left(\frac{2\cdot 3}{25}\right)\left(\frac{2\cdot 10}{1+10}\right)^2 [1-\frac{1}{4} 5/10]$$

$$= \underline{0.70"}$$

(b) UNCERTAINTY (STANDARD DEVIATION) OF SETTLEMENT

Spatial Variability

$$\Omega_{\rho'} = \Omega_N$$

$$\Omega_{\rho'} = \Omega_N = \frac{\sqrt{11^2/2}}{25} = 0.31$$

$$s_{\rho_1} = \Omega_{\rho'} m_\rho = (0.31)(0.70") = \underline{(0.22")}$$

Figure 20 sheet 1/3

CALCULATION SHEET

PROBLEM: footing settlement CALCULATED BY: GBB
DATE: CHECKED BY:

[(b) UNCERTAINTY OF SETTLEMENT con't]

Systematic Error (statistical only, model bias neglected)

n=50 borings thus the statistical error on the mean blow count at any elevation is,

$$s_{N_4} = \sqrt{\frac{(11bpf)^2}{50}} = 1.6bpf$$

Total Uncertainty from Equations 41 and 42.

(c) RELIABILITY INDEX

$$\beta = \frac{m_\rho - \rho_o}{s_\rho} = \frac{|0.70" - 1"|}{(0.33)\ (0.7)} = 1.3$$

(d) OBSERVED SETTLEMENT

	Observed Settlement	vs.	Predicted Settlement
m_ρ =	0.35"		0.70"
S_ρ =	0.12"		0.23"
Ω_ρ =	0.34		0.33

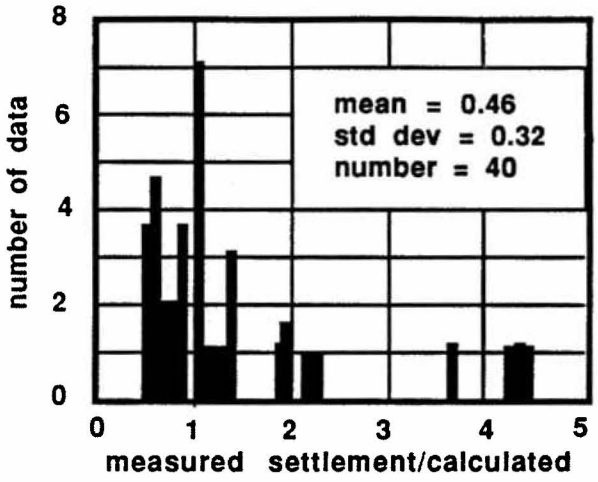
Figure 20 Continued sheet2/3

CALCULATION SHEET

PROBLEM: footing settlement CALCULATED BY: GBB
DATE: CHECKED BY:

(e) ESTIMATE BIAS DUE TO SETTLEMENT MODEL:

Correlation of Peck and Bazaraa method and actual footing settlements shown in histogram at right.



From observed data, the ratio of measured settlement to cacluated settlement has mean and standard deviation,

SR = settlement ratio
= measured settlement/caclulated settlement
m_{SR} = 1.45
s_{SR} = 1.32
Ω_{SR} = 0.91

Mean (best estimate) settlement corrected for model bias (from Eqn. 8):

$\rho^c = \rho \cdot SR$
$m_\rho{}^c = m_\rho \cdot m_{SR} = (0.7")(1.45) = 1.02"$

Standard deviation (uncertainty) of corrected settlement (from Eqn. 15):

$\Omega_{\rho c}{}^2 = \Omega_\rho{}^2 + \Omega_{SR}{}^2 = 0.31^2 + 0.91^2 = 0.96^2$
$s_\rho = (0.96)(1.02") = 0.98"$

Figure 20 Continued sheet3/3

in which,

Δq = applied stress (37)
b = footing width
ρ = settlement
D = embedment depth
N = depth averaged blow count

The mean settlement m_ρ is found by substituting mean values of all the parameters in Eqn. 36, in the same way that the deterministic solution would be obtained. In the present case, the only uncertain parameter is SPT blow count, N, for which m_N is substituted. Inserting m_N into Eqn. 36 gives the best estimate of settlement, m_ρ = 0.7 in..

6.3 Spatial Variability of Settlement

The uncertainty in the settlement prediction is represented by the standard deviation s_ρ. This is calculated by propagating the four sources of uncertainty in the input parameter N through Eqn. 36, using Eqn. 12, and then recombining the output according to an equation of the form Eqn. 19.

The first step in calculating s_ρ is assessing the magnitude of the four contributions to uncertainty in N: spatial variability, measurement noise, measurement bias, and statistical error. The first two appear as data scatter and must be separated from one another. The second two are systematic errors and can only be estimated by calculation (i.e, they do not appear in data scatter or in any other explicit form).

The scatter in SPT data for the site, by empirical observation, has a standard deviation of 11bpf. Since about half the data scatter measured as a variance appears to be noise, the standard deviation of the spatial variability alone is

$$s_{N_1} = \sqrt{0.5\ (11\text{bpf})^2} = 7.8\ \text{bpf} \qquad (38)$$

Applying Eqn. 14 to the settlement formula yields the result that the coefficient of variation of ~should be proportional to the coefficient of variation of N. This conclusion could be reached by inspection by noting that ρ is proportional to N, so the proportional uncertainties should be the same. Given that the autocorrelation distance is large compared to the footing width, the assumption was made that $R_v \approx 1.0$. As a result

$$\Omega_\rho = \Omega_{N_1} = \frac{7.8}{25} = 0.31 \qquad (39)$$

and the standard deviation, as shown on Figure 20, is $s_{\ln}$=0.22 in..

6.4 Systematic Error in Settlement Predictions

In the calculations of Figure 20 the measurement bias s_{N_3} is ignored because blow counts are measured directly rather than inferred through a model or set of calculations. Thus, no error of interpretation was assumed to be introduced by the way measurements are analyzed. This is obviously a simplification because it ignores the model bias in going from N values to settlement predictions. We return to this problem below.

The statistical error in the mean value of N, expressed as a variance, is approximately equal to the data scatter variance divided by the number of independent measurements, n. In the present case, there are n=50 blow count measurements at any depth, so the standard deviation of the statistical error is approximately

$$s_{N_4} = \sqrt{\frac{(11\text{bpf})^2}{50}} = 1.6\text{bpf} \qquad (40)$$

6.5 Total Uncertainty

The total error in the settlement prediction is found by combining the sources of uncertainty according to equation 19. The main causes of uncertainty are spatial variation and statistical error. Measurement noise has been statistically removed from the prediction, and measurement bias has been ignored. This gives

$$s_\rho^2 = s_{\tilde{1}}^2 + s_{\tilde{4}}^2 \qquad (41)$$

Dividing both sides by the mean settlement squared, $m_{\tilde{\rho}}^2$, gives

$$\Omega_\rho^2 = \Omega_{p_1}^2 + \Omega_{p_4}^2 = 0.31^2 + 0.06^2 = 0.33 \qquad (42)$$

The histogram of actually observed footing settlements is shown in Figure 21. The mean settlement was about half that predicted, but the variability among footing settlements was close to the spatial variability predicted. The difference between mean predicted settlement and mean observed is due to two factors. First, in service, the footings were subject to less than the design loads. Second, the settlement model of Eqn. 36 itself contains bias. As shown in Figure 22 this latter bias can be accounted for by regression analysis, and incorportated in the uncertainty analysis as an s_3 term. As can be seen, for many settlement methods, model uncertainty is very large.

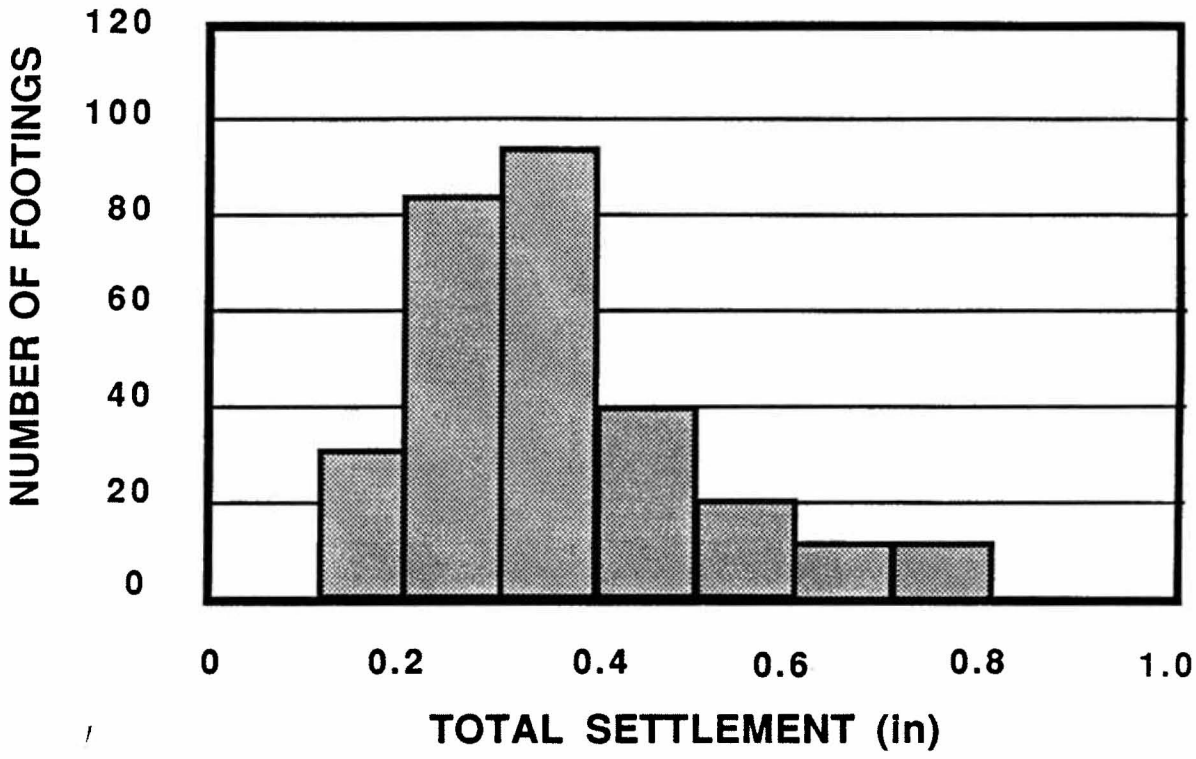


Figure 21. Observed footing settlements.

CALCULATION SHEET

PROBLEM: 2-D slope stability analysis CALCULATED BY: _____

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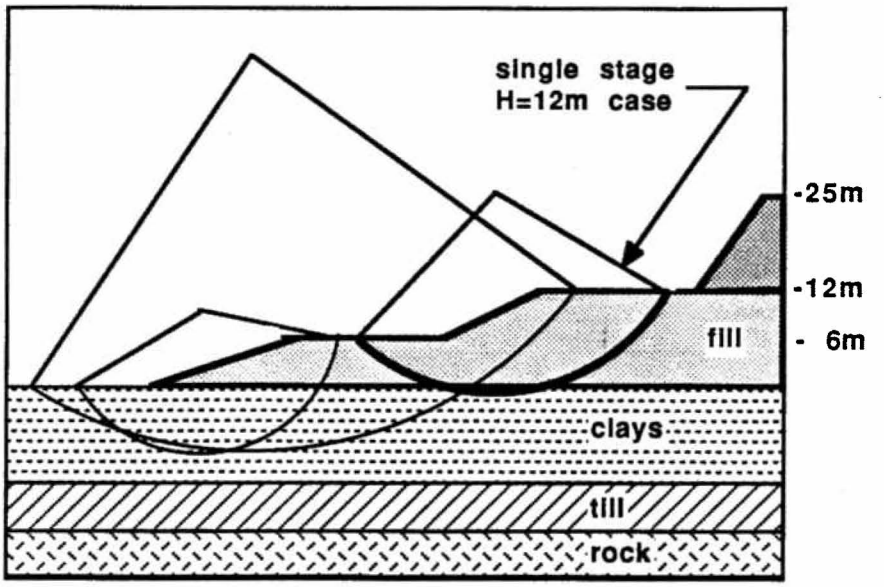
Problem:



(a) BEST ESTIMATE (EXPECTED VALUE) OF FS AGAINST INSTABILITY

m_F = 1.45 [by modified Bishop method]

(b) UNCERTAINTY (VARIANCE) IN FACTOR OF SAFETY

assuming the soil parameters to be statisticall independent,

$$s_F^2 = \sum_i \left(\frac{\partial F}{\partial x_i}\right)^2 s_{x_i}$$

in which x_1 = friction angle, ϕ
x_2 = density of the fill, γ_{fill}
x_3 = depth to till, D_{till}
x_4 = undrained strength of the lacustrine clay, $c_u(L)$
x_5 = undrained strength of the marine clay, $c_u(M)$

Figure 22 1/2

CALCULATION SHEET

PROBLEM: 2-D slope stability analysis CALCULATED BY: _____

DATE: _____ REFERENCE: _____

Parameter	$\Delta FS/\Delta x_i$	Variance			$(\Delta FS/\Delta x_i)^2 \cdot V(x_i)$		
		Spatial	Syst.	TOTAL	Spatial	Syst.	TOTAL
Φ'	0.01	1.0	3.0	4.0	0.0001	0.0003	0.0004
γ_{FILL}	0.06	1.0	1.0	2.0	0.0036	0.0036	0.0072
D_{crust}	0.008	0.96	0.036	1.0	0.0013	--	0.0013
D_{till}	0.056	0.0	1.0	1.0	0.0000	0.0031	0.0031
$c_u(L)$	0.0215	74.8	24.9	99.7	0.0346	0.0115	0.0461
$c_u(M)$	0.0137	40.0	7.6	47.6	0.0075	0.0014	0.0089
			s_F^2	=	0.0471	0.0199	0.0670
			s_F^2 @ R_v=0.2	=	0.0094	0.0199	0.0290

(c) RELIABILITY INDEX

$$\beta = \frac{m_F - 1.0}{s_F}$$

$$= \frac{1.45 - 1.0}{\sqrt{0.029}}$$

$$= 2.66$$

Figure 22 (continued) 2/2

6.6 Reliability Index for Settlement Prediction

The reliability index is calculated from Eqn. 16 as

$$\beta = \frac{m_\rho - \rho_o}{s_\rho} = \frac{|0.70" - 1"|}{(0.33)(0.7)} = 1.3 \qquad (43)$$

7. APPLICATION TO EARTH EMBANKMENT DESIGN

The second case involves end-of-construction stability of a low embankment on soft ground.

7.1 Site Conditions

The site is underlain by 20m of soft marine and lacustrine clays lying on glacial till, in turn lying on crystalline bedrock. The site characterization program included borings, field vane measurements, block samples and laboratory testing. A 25m embankment is to be constructed on the clay, and the principal source of information on undrained strength for end-of-construction analysis is the field vane data shown, leading to the design profile of Figure 23.

7.2 Stability Calculation

The tentative design configuration is shown in Figure 22. Three design cases were analyzed, a 6m single-stage dyke, a 12m single-stage dyke, and a 25m two-stage dyke. In the two-stage construction the foundation clays are allowed to consolidate under a 12m fill which is then raised to 25m. The worst case or design condition is end-of-construction, which is analyzed assuming undrained conditions. For illustrative purposes, only the analysis of the 12m dyke is presented.

The principal uncertainties in the stability calculations are the undrained strengths of the foundation clays, the engineering properties of the embankment fill materials, and the geometry of the subsurface stratification. These are shown in Figure 24, with their respective systematic and spatial variances.

The derivatives of factor of safety F with respect to the uncertain parameters were calculated numerically using simplified Bishop circular arc and Morgenstern-Price wedge-type failure geometries. For each design geometry a base-case analysis used all parameters at their means. This gives the best-estimate F. For each parameter, additional calculations were made to numerically determine the derivative of F near the mean. The derivative was calculated as the ratio of change in F to change in input parameter, $\Delta F/\Delta x$. The square of the derivative dF/dx with respect to each principal uncertainty

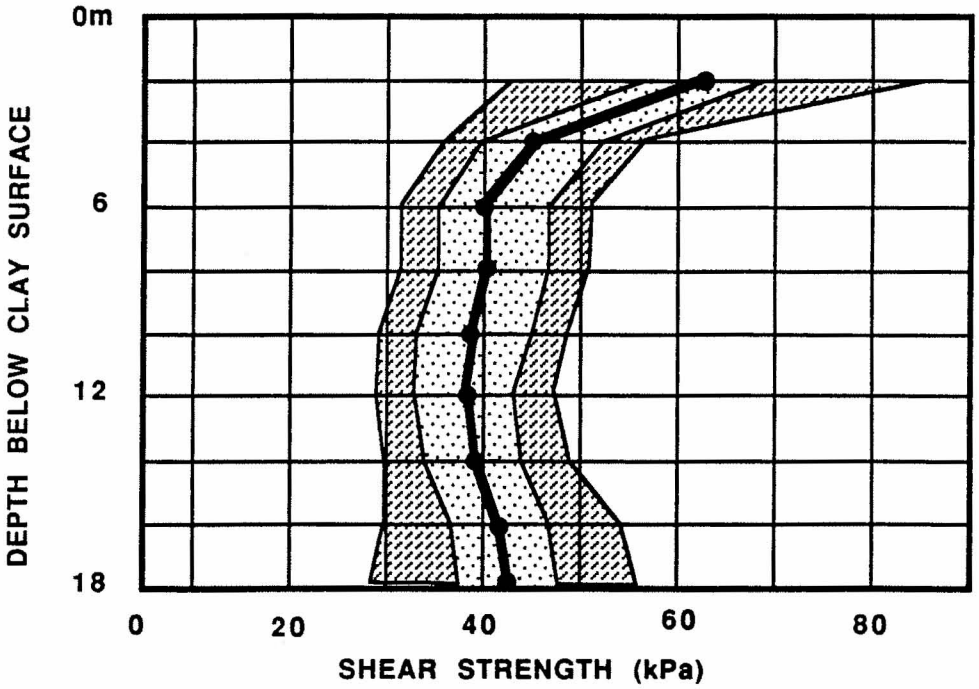


Figure 23 Best estimate strength profile with standard deviation envelopes on mean and spatial variability.

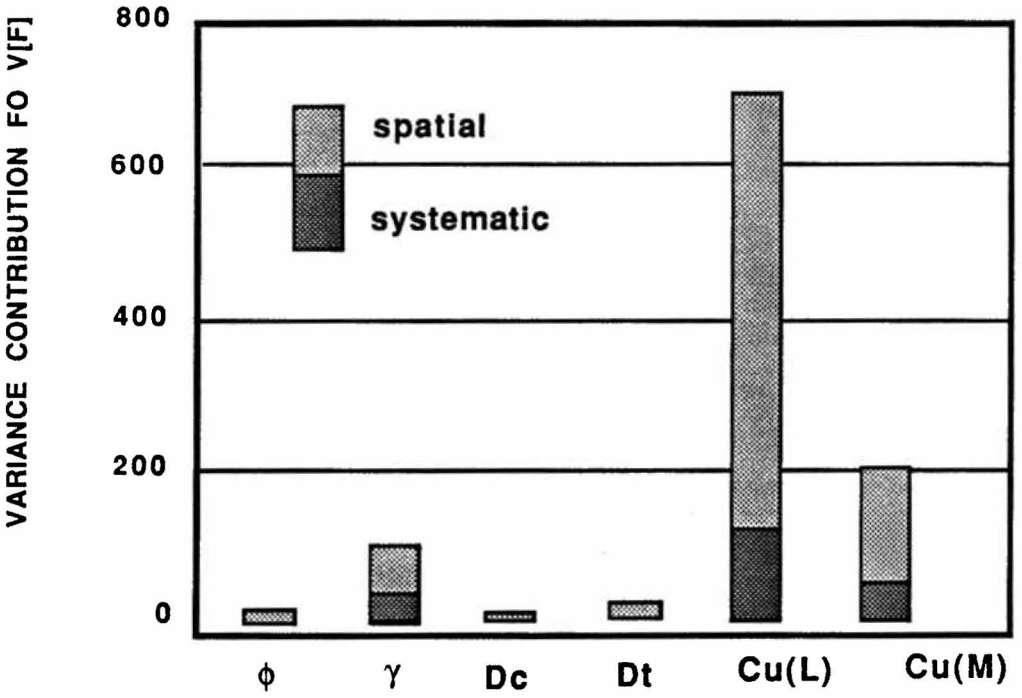


Figure 24 Variance components for factor of safety against strength instability of dykes.

was multiplied by the corresponding variances of Figure 24 to obtain the contribution of each uncertain input parameter to systematic and spatial uncertainty in the calculated factor of safety. These are shown as variance contributions. By Eqn. 12, the sum of these variance contributions over all the input uncertainties gives the overall spatial and systematic variances in the calculated value of F. From Figure 24, this total variance is $s_F^2=0.067$.

The critical failure circle for the two-berm 12m case has a 125m radius. Since the deposit was anisotropic and the failure circle cut across different soil layers, numerical integration was used to obtain a more precise size effect factor R_v than could be obtained from Eqn. 26. This led to a reduction factor $R_v=0.2$ for the spatial component of variability. Thus, the variance of F is calculated as

$$\begin{aligned} s_F^2 &= 0.2\, s_{F_1}^2 + (s_{F_3}^2 + s_{F_4}^2) \\ &= 0.2\,(0.0471) + 0.0199 \\ &= 0.029 \end{aligned} \tag{44}$$

and the reliability index is

$$\beta = \frac{m_F - 1.0}{s_F} \tag{45}$$

$$= \frac{1.45 - 1.0}{\sqrt{0.029}} = 2.66$$

Contributions of the various uncertainties or errors to s_F^2 are shown in Figure 25. The cross-hatched portion of each contribution is systematic error, which does not reduce with increasing failure dimensions. The uncolored portion is spatial variability, which does reduce with increasing failure dimension. Reduction for the 125m critical failure circle ($R_v=0.2$) is shown as a heavy horizontal bar in Figure 25.

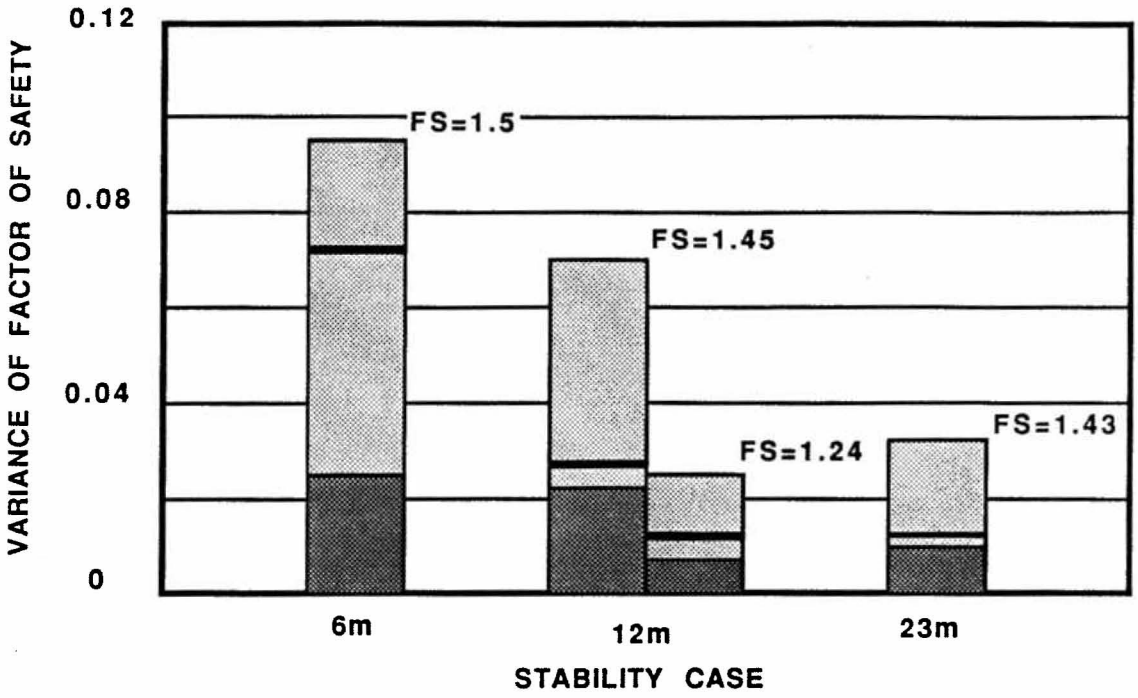


Figure 25 Spatial and systematic components of variance on factor of safety.

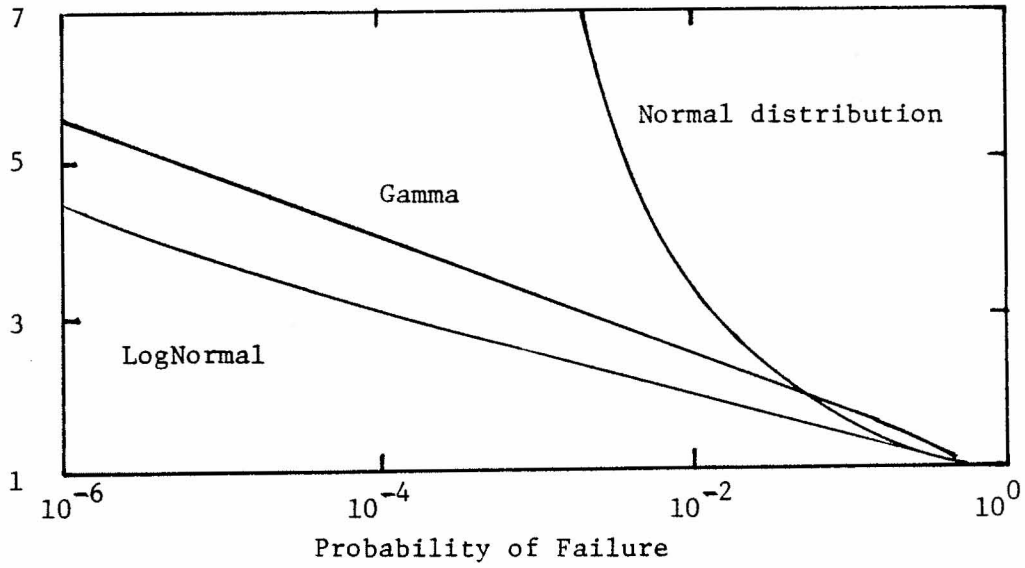


Figure 26 Nominal probability of failure for three distribution assumptions, for coefficient of variation equal to 0.30.

8. RISK-BASED DESIGN

The approach of previous chapters has been to deal only with means and variances of uncertain variables. Conclusions about safety have been summarized in reliability indices. "Probabilities of failure" have been avoided. The reason is twofold. First, reliability indices can be calculated from the types and amounts of data normally available in geotechnical practice. They do not require extensive computation and are not based on unverifiable assumptions. Second, failures in the field are often caused by incorrect hypotheses or unanticipated conditions, so any probability of failure resulting from calculations alone is necessarily incomplete. Nevertheless, economic benefits can be obtained using risk techniques as a basis for design, because these techniques at least allow an optimization of that part of design which deals with calculational uncertainties.

Optimal design is based on the balancing of risk against cost. Increases in conservatism decrease the probability that a facility will perform inadequately; yet at the same time, conservatism is bought at a cost in construction. The degree of conservatism at which marginal increases in construction cost are just equalled by marginal decreases in risk cost provides the most cost-effective solution.

The traditional way of measuring risk cost is through the product of probability of failure P_f and cost of failure C_f

$$C_r = P_f \cdot C_f \tag{48}$$

As long as the cost of failure is not catastrophic and there is no loss of life, this measure is adequate.

The probability of failure depends on the adequacy of a design against conditions that have been analyzed and against those that have not. Although the latter are responsible for many actual failures, they fall outside analysis. Optimal design addresses only failures due to ordinary or anticipated conditions: partial optimization is the best one can achieve in normal situations.

The relationship of P_f to β for common distributional forms is shown in Figure 26. For $\beta<2.5$ the differences among distributions are small. Therefore, any of these could be used with little error. The most convenient is the normal distribution (Figure 27). For large β (i.e., very small P_f) the situation changes. P_f becomes sensitive to the choice of distribution, yet there is little cogent reason to chose one distribution over another. Conveniently, for most practical problems in geotechnical engineering β is small and the choice of distribution poses little problem.

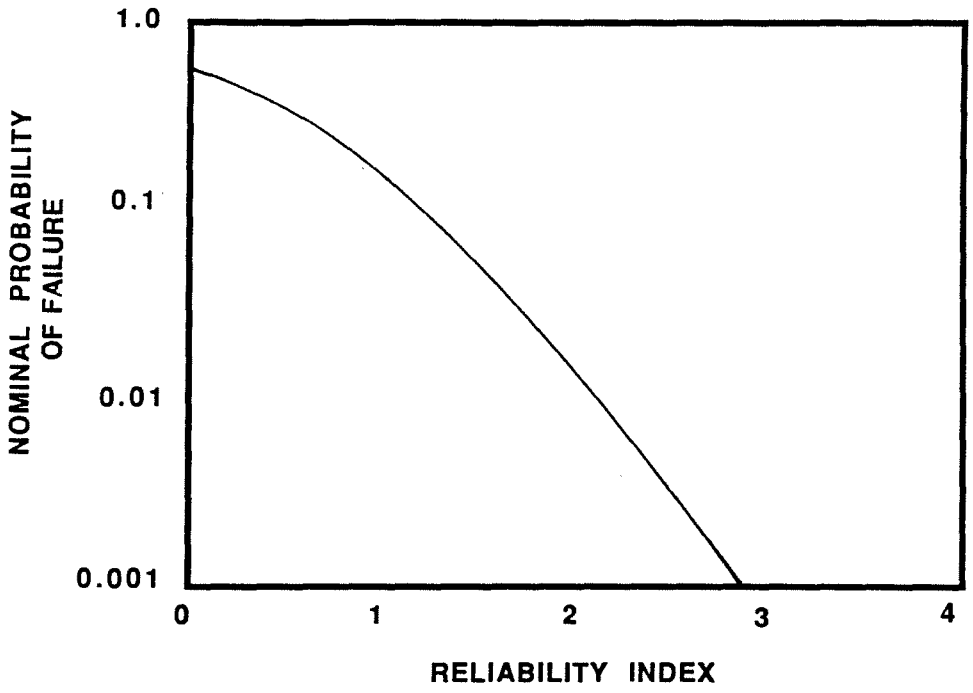


Figure 27 Nominal probability of failure based on normal distribution

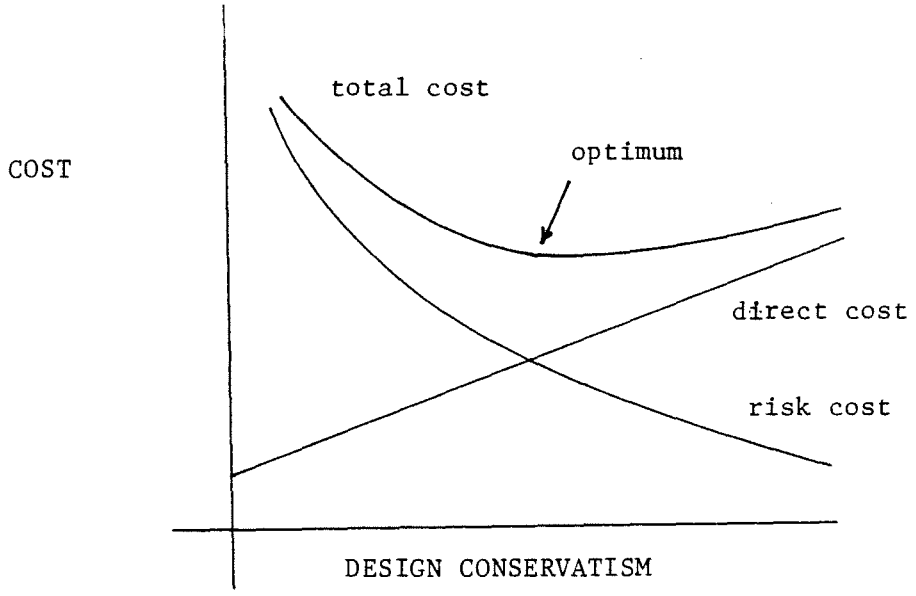


Figure 28 Optimization balances direct cost against risk cost.

8.1 The "Probability of Failure"

An important point to note is that uncertainty in factor of safety has to do with the chance that, if the proper analysis had been made and if the proper parameter values had been used, then the predicted factor of safety would have been less than 1.0. This 'chance' can be expressed as a nominal probability of failure,

$$p_f = Pr\{F<1.0\} \quad , \tag{49}$$

having to do the likelihood that one should be calculating a factor of safety less than 1.0 when actually the calculations indicate $F \geq 1.0$. From Figure 27 this probability that $F<1.0$ is the area under the probability distribution of F within the region $0 \leq F < 1$.

8.2 Risk-Based Design

No matter how design decisions are made, they require a balancing of economy against safety. Only if performance could be predicted precisely could a facility be designed to carry exactly the loads required and no more. In the real world performance cannot be predicted with such precision, and thus a margin of safety must be adopted.

Schematically, optimization is shown in Figure 28. Increasing safety margin means a more conservative design and a larger direct cost. Simultaneously, increasing the safety margin means a lower rate of failure, and thus over a number of facilities a lower cost of clean-up, repair, and other consequences of failure. Since direct cost increases with safety margin and risk cost decreases, their sum has a minimum corresponding to the optimal design.

For the purpose of this manual, the principal thing we ask of risk based design is that it help with the question of how to select appropriate numerical values of the design factor of safety, reliability index, or other safety margin. Actually, there are a number of ways one might go about this task. Safety margins might be chosen such that they are:

- Balanced (i.e., internally consistent),
- Calibrated (i.e., externally consistent), or
- Optimal (i.e., economically efficient).

Each of these is discussed briefly.

8.3 Balanced Factors of Safety

The effect of differing levels of uncertainty on comparitive factors of safety is illustrated by the embankment stability case of Section 7. The project involved staged construction on soft clay. The first stage dyke was constructed to 6m, then raised to 12 m, and after partial consolidation raised

again to 25 m. Figure 29 gives a summary of the mean and standard deviation calculations.

Each design case has a different mean factor of safety and a different standard deviation. To balance the conservatism of design across the three cases, factors of safety should be chosen to give the same reliability index β. A graph for determining consistent factors of safety is shown in Figure 30. Consistent design (mean) factors of safety for the four embankment cases are shown in the last two columns of Figure 31, for $\beta=2$ and $\beta=3$.

8.4 Calibrated Factors of Safety

A second approach to selecting design factors of safety is by comparing nominal probabilities of failure with accepted (i.e., historical) rates of failure of other civil facilities. Fig. 31 shows empirical rates of failure for a variety of civil facilities, plotted against the corresponding consequences of failure. The horizontal axis of the plot shows both financial consequence and life loss. Financial loss and life loss typically occur together, and are not intended to imply a tradeoff of dollars vs. lives.

For a facility such as a large water-retaining embankment the consequence of failure might conceivably be $\$10^8$ or more. Thus, a probability of failure of 10^{-3} would be consistent with accepted civil works risks. Taking this value as the nominal P_f and using the Normal probability distribution leads to a typical reliability index on the order of 3. For a structure such as a shallow footing the failure cost might only be on the order of $\$10^4$ to 10^5 implying a reliability index of about 1.5. These values are approximately consistent with common practice.

8.5 Economic Optimization

The purpose of economic optimization is to find that design factor of safety at which the marginal increase in construction cost necessary to further increase F just balances the marginal decrease in risk cost enjoyed by increasing F. Risk cost is taken as the probability-of-failure times the cost-of-failure, and the total cost to be minimized is taken as the construction-cost plus risk-cost. Presuming construction cost to increase with F and risk cost to decrease, the F at which the respective marginal costs are equal in magnitude yields the least total cost design.

The analysis of section 6 considered the settlement of a shallow footing founded on sand. Fig. 32 shows the reliability index β against excessive settlement (serviceability failure) as a function of footing size, where excessive settlement is arbitrarily taken as 1.0 inch. For sake of illustration, the associated cost of servicability failure is set at \$10,000.

Fig. 32 also shows the relation between construction cost C_c and risk cost $C_r=P_f \cdot C_f$ as a function of footing width. The probability of failure P_f is found from the corresponding reliability index using Fig. 29. The total cost is the sum of C_c and C_r

CALCULATION SHEET

PROBLEM: 2-D slope stability analysis CALCULATED BY: _____

DATE: _____ REFERENCE: _____

SUMMARY OF RELIABILITY RESULTS

	m_F	S_F^2 Spatial	S_F^2 Systematic	S_F^2 TOTAL	S_F^2 w/ R_v	β	P_f	m_F for β=2	m_F for β=3
Single stage, H = 6m	1.500	0.071	0.024	0.095	0.074	1.84	0.031	1.54	1.82
Single stage, H = 12m	1.453	0.047	0.020	0.067	0.029	2.66	0.004	1.34	1.51
Single stage, H = 12m Case II	1.244	0.011	0.006	0.017	0.008	2.69	0.004	1.18	1.27
Multi stage H = 23m	1.427	0.018	0.012	0.030	0.013	3.68	0.0001	1.23	1.34

Figure 29

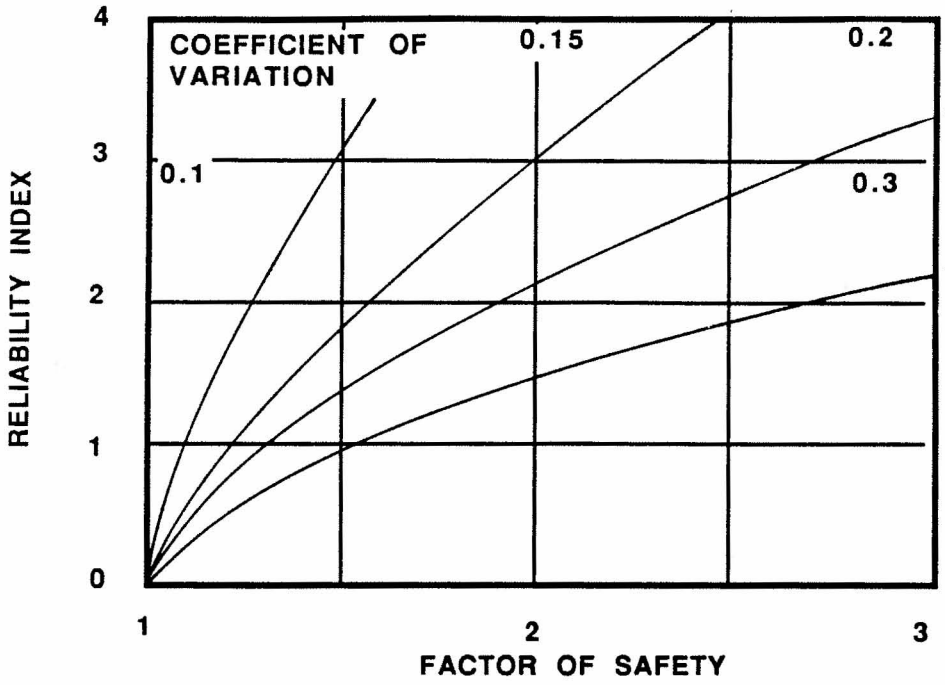


Figure 30 Consistent factors of safety have same value of reliability index.

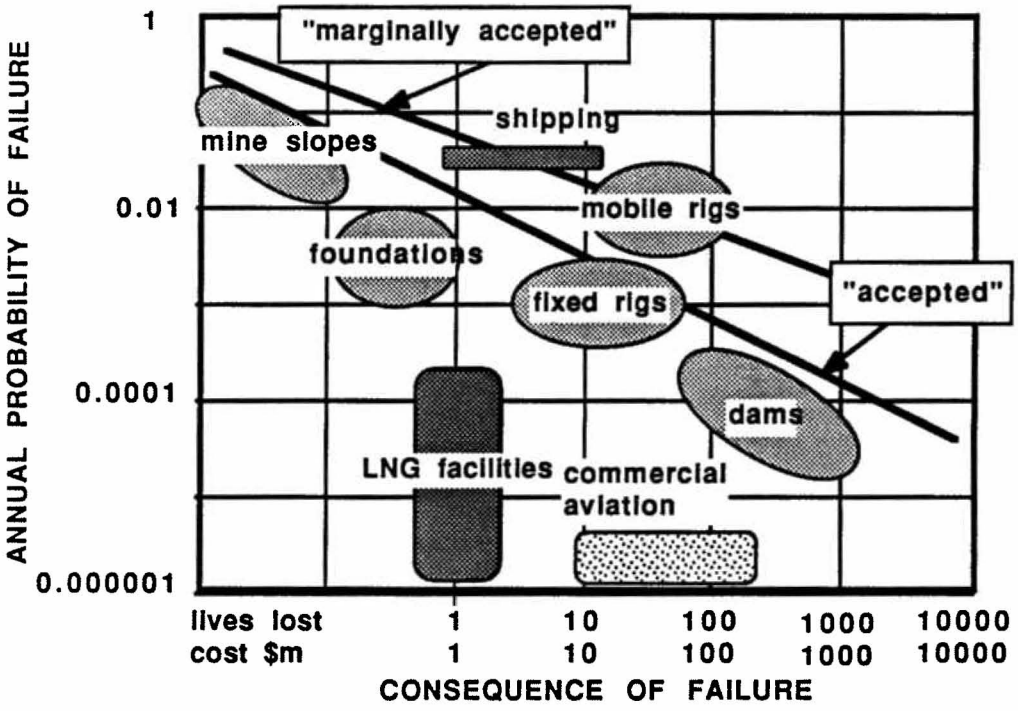


Figure 31 Empirical rates of failure for civil facilities.

SETTLEMENT RELIABILITY ESTIMATES

FOOTING WIDTH	MEAN SETTLEMENT	STANDARD DEVIATION OF SETTLEMENT	RELIABILITY INDEX	NOMINAL PROBABILITY OF FAILURE
B	m_ρ	s_ρ	β	Pf
5	1.0	0.31	0	0.5
7.5	0.83	0.26	0.54	0.25
10	0.7	0.22	1.36	0.08
15	0.52	0.16	3.0	0.002
20	0.42	0.13	4.6	nil

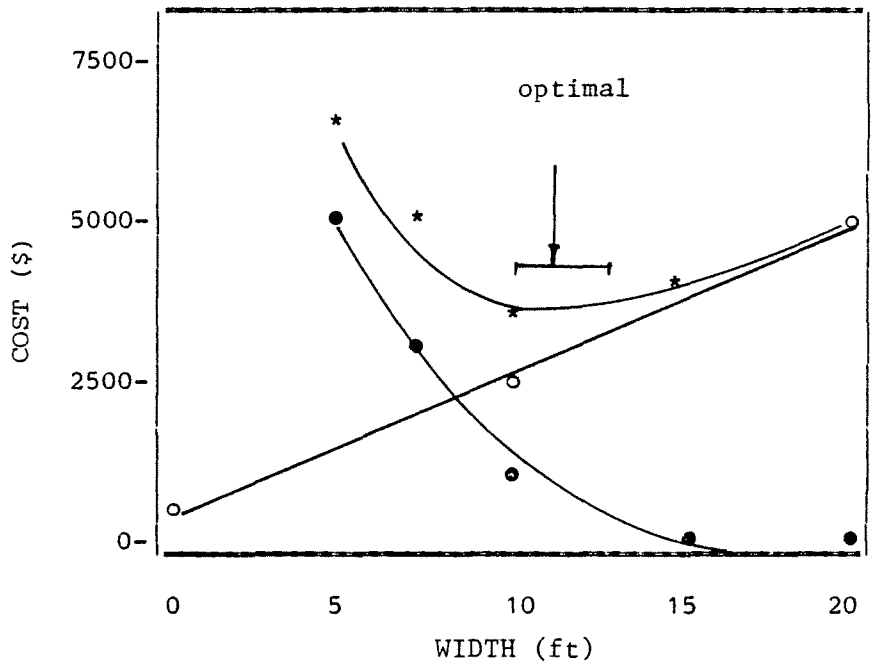


Figure 32 Optimal footing width.

$$C_T = C_c + C_r$$
$$= C_c + P_f \cdot C_f \qquad (50)$$

Construction cost was modelled using a fixed cost of $1000 per footing plus a variable cost of $1000 per foot of width. The total cost accordingly is plotted as a broken line. The minimum at 12 to 14 feet indicates the most efficient footing width.

9. CONCLUSIONS

Risk-based design using means and standard deviations of soil parameters and calculated predictions of performance presents a practical technique for accomodating uncertainty in geotechnical engineering analysis. The technique can be used with the numbers of data typically collected in practice, and does not require extensive computations. The result of this approach is a reliability index β, used to measure the margin of safety in a design in light of the uncertainty in predicted performance.

The central concept of the procedure outlined here is to separately consider the contributions of data scatter and systematic error to predictive uncertainty. The first can be used to assess variability in performance across a site; the second can be used to predict probable error in predictions of average performance.

In making design decisions, risk-based methods allow uncertainty in performance predictions to be explicitly considered, and balanced against cost. In this way, the practice of geotechnical engineering can be made somewhat more objective, and expenditures can be made more efficient.

Appendix A -- REFERENCES

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Appendix B -- SYMBOL LIST

a_i = constant
B = bias correction coefficient
b = footing width
C_f = failure cost
C_r = risk cost
$C_u(L)$ = undrained strength of lacustrine clay
$C_u(m)$ = undrained strength of marine clay
c_u = undrained strength
$C_x(\delta)$ = autocovariance function over separation distance δ
Cx,y = covariance of x and y
D = embedment
D_{crust} = depth of OC crust
D_{till} = depth to till
e = random measurement error
F = factor of safety
FV = field vane
G = derivative matrix with ijth element dy_i/dx_j
g(x) = deterministic function of x
k = undrained strength ratio for NC clay
L = averaging distance, also L_1, L_2, L_3
m = power coefficient for strength increase with OCR
m_x = mean of x
n = number of measurements
N = SPT blow count
P_f = probability of failure
Pr{.} = probability of
q_v = bearing capacity
Δq = applied footing stress
r = correlation coefficient
δ_o = autocorrelation distance, $C_x(\delta_o)=1/e$; also δ_1, δ_2, δ_3 in multiple dimension
R_v = size effect factor
$R_x(\delta)$ = autocorrelation function over separation distance δ
s_x = standard deviation of x
s_{x1} = standard deviation of spatial variation
s_{x2} = standard deviation of measurement noise
s_{x3} = standard deviation of statistical error
s_{x4} = standard deviation of measurement and model bias
t_i = trend of spatial variation at
u_i = residual spatial variation about trend at i
x_i = soil property at location i
y = predicted performance variable
z = observation or measurement, possibly corrupted by noise

β = reliability index
δ = separation distance
ρ = settlement
ρ_o = critical amount of settlement
Σ_x = covariance matrix of $\underline{x}$ = $\{x_1,\ldots,x_n\}$
σ_{vm}' = maximum past pressure
σ_{vo}' = effective vertical stress
τ = undrained strength
Ω_x = coefficient of variation of x

Appendix C -- OTHER METHODS OF UNCERTAINTY ANALYSIS

The approach to propogating uncertainty through an engineering model used here is based on a first-order propogation of variance. This is a common technique and is called many things in the many disciplines to which it finds application. It is sometimes called "first-order second-moment" (FOSM) analysis, and sometimes simply "error analysis." However, there are several other ways to analyze the affect of input uncertainties on output uncertainties. Among the more often encountered of these other methods in civil engineering practice are adjoint methods, simulation, and response surface techniques.

Adjoint techniques evaluate the proportionate effect of a pertubation in input parameter on the resulting purtubation in an output prediction. That is, they lead to an evaluation of the quantity $\{(\Delta y_j/\Delta x_i)\ x_i/y_j\}$, in which y_j is the jth component of the prediction and x_i is the ith input parameter. Adjoint techniques are conveniently applied to large numerical models involving the solution of systems of linear equations. By manipulating the linear algebra of such solutions, adjoint results can be obtained in the course of computations. While adjoint techniques are usually used to obtain sensitivies of a model rather than to perform quantitative uncertainty analysis, the results can be used to numerically obtain derivatives, and thus to provide the means for first-order variance propogation.

Simulation uses many repetitions of deterministic calculations in which values of input parameters are randomly generated from specified probability distributions. The result of simulation is a set of many predictions of each output parameter which are treated as empirical data from which statistical inferences of the means, variances, etc. of output predictions can be made. An advantage of simulation is simplicity. It requires none of the mathematics of variance propogation, adjoint analysis, and related techniques. On the other hand, simulation has three important limitations. It is expensive because the deteministic model must be run many times. For example, at least several hundred trials are typically needed. It requires not only means and variances of input parameters, but entire probability distributions. These may be ambiguous or arbitrary. Finally, the components of uncertainty are lumped together in simulations. Thus, differing effects are hard to unravel. Nevertheless, simulation is an important tool when a model is complicated, involves logical branching, or on other occations when variance propogation and related techniques cannot be used.

Response surface techniques are related both to variance propogation and simulation, finding their most frequent use with models that are numerical, possibly implicit, difficult to analytically propogate variance through, and expensive to run. Response surface techniques are closely related to regression analysis. Multiple runs of the model are made in the vacinity of the mean of the input parameter values and a regression surface of chosen complexity is fit to the output. This regression surface is presumably less complicated than the model function itself, and yet can still be used as an approximation on which variance propogation or other techniques can be used.

At the same time, many fewer runs of the model are made than with simulation, and thus cost is reduced. Response surface approaches are often applied to risk analysis problems associated with nuclear power and waste facilities, and to structural reliability problems.