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Derivation of Global Positioning System (GPS) User Errors with Representative Examples

by

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## ABSTRACT

Expressions for Global Positioning System (GPS) user errors in position and time are derived for the situation of the receiver utilizing data either from four satellites or from three satellites plus an altimeter. The derivation assumes that the satellite pseudorange/altimeter measurement errors are independent, but not necessarily equal. The predicted user behavior is given in this report for representative measurement errors and for the special case of a user located at the FAA Technical Center, Atlantic City Airport (ACY), New Jersey, when NAVSTAR satellites 3 through 6 are visible.



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Key Words

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GPS
User Errors
Geometric Dilution of Precision
System Bias

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# INTRODUCTION

## PURPOSE

The objectives of this report are (1) to develop the matrix equations which predict NAVSTAR Global Positioning System (GPS) user position/time errors (bias and standard deviation) for a given constellation of four satellites or of three satellites plus encoded altimeter data, and (2) to illustrate typical behavior of the user error parameters when the pseudorange/altimeter variance measurements are independent, but not equal.

## BACKGROUND

NAVSTAR GPS is a prototype Department of Defense (DOD) satellite system which will provide a suitably equipped user with precise position, velocity, and time information.[1] The Federal Aviation Administration (FAA) is currently evaluating GPS as a potential navigation system for civil aviation operations. A GPS receiver measures the propagation delay of the transmitted signals from four or more different satellites (that is, the effective satellite to receiver distances known as pseudoranges) and solves for its position and its clock offset with respect to GPS time. The errors in the pseudorange measurements are dependent upon uncertainties in the satellite ephemeris and clock, atmospheric delay, multipath, receiver noise and resolution, user dynamics, etc. The relative orientation of the satellites to the user also determines the accuracy that the user can determine his position and time. Under certain simplifying assumptions, the geometry effects can be separated from the pseudorange errors through the dilution of precision parameters:[2]

GDOP = geometric dilution of precision (four dimensions),
PDOP = position dilution of precision (three dimensions),
HDOP = horizontal dilution of precision (two dimensions), (1)
VDOP = vertical dilution of precision (one dimension),
TDOP = time dilution of precision (one dimension),

where

$$GDOP = [ (PDOP)^2 + (TDOP)^2]^{1/2}$$

and

$$PDOP = [ (HDOP)^2 + (VDOP)^2]^{1/2}. \qquad (2)$$

For example, the standard deviation in the horizontal position determination of the user is given by the satellite pseudorange error, one sigma value, multiplied by HDOP.

## SCOPE

The matrix equations for GPS user position errors are solved for a four satellite constellation and for a three satellite constellation and encoded altimeter. For the former case, the user standard deviation in position and time is shown to be proportional to GDOP when the ranging measurement errors from each satellite are equal and independent. In general, these errors are not equal, and the satellite clock and ephemeris data have biases. However, the general assumption is made throughout this report that the pseudorange errors (from satellites or encoded altimeter) are independent. The resultant expressions are useful not only for determining the adequacy of a given satellite configuration for user location, but also for correcting the system bias in differential GPS schemes for attaining better user accuracy.

Representative user error parameters (biases, single and multiple dimensional standard deviations, and correlation coefficients) are shown for a receiver processing signals from NAVSTAR satellite numbers 3 through 6 when (1) the measured residuals and one sigma errors from one satellite/altimeter are ten times larger than those from the other three satellites, and (2) the measured biases and variances are respectively averaged. The user location is chosen to be at the FAA Technical Center, Atlantic City Airport (ACY), New Jersey.

## DERIVATION OF USER ERRORS

### USER POSITION DETERMINATION - FOUR SATELLITES

Consider the vector diagram given in figure 1 for a user located at ($x_u$, $y_u$, $z_u$) and the $i^{th}$ satellite at ($x_i$,$y_i$,$z_i$), with both positions given in an arbitrary earth-centered coordinate system[2,3]. The vector displacement from the user to the satellite is given by

$$\vec{D}_i = \vec{R}_i - \vec{R}_u. \tag{3}$$

The distance, $D_i$, between the user and satellite is obtained from the unit vector $\hat{e}_i$ along $\vec{D}_i$ by the dot product

$$D_i = \hat{e}_i \cdot \vec{D}_i = \hat{e}_i \cdot (\vec{R}_i - \vec{R}_u) = [(x_i-x_u)^2 + (y_i-y_u)^2 + (z_i-z_u)^2]^{1/2}. \tag{4}$$

The unit vector $\hat{e}_i$ is expressed in terms of its components by

$$\hat{e}_i = e_{ix}\,\hat{x} + e_{iy}\,\hat{y} + e_{iz}\,\hat{z} = \frac{(x_i-x_u)}{D_i}\,\hat{x} + \frac{(y_i-y_u)}{D_i}\,\hat{y} + \frac{(z_i-z_u)}{D_i}\,\hat{z}, \tag{5}$$

where $e_{ix}$, $e_{iy}$, and $e_{iz}$ are the user to satellite direction cosines with respect to the unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$ of the coordinate system.

The GPS satellites broadcast navigation messages containing Keplerian-type position parameters, time, system correction terms, and system health. A four dimensional position fix, with user time being the fourth variable, requires measurements of the signal transit times from four different satellites to the user. The pseudorange measurement from the $i^{th}$ satellite is then

$$f_i = D_i + ct_i = D_i + B_u + B_i, \tag{6}$$

where c is the velocity of the rf signal and $t_i$ is the time offset between the clocks of the user and $i^{th}$ satellite. The quantities $B_u$ and $B_i$ are respectively the user clock offset and satellite time offset, both scaled by c with respect to GPS time.

The user position may be rewritten in terms of the pseudorange from the above equations as

$$\hat{e}_i \cdot \vec{R}_u - B_u = \hat{e}_i \cdot \vec{R}_i + B_i - f_i = p_i. \tag{7}$$

Consider the case for four satellites where the satellite and system parameters are known and pseudoranges are measured. Recasting the left side of equation (7) in terms of user to satellite direction cosines, one obtains four non-linear simultaneous equations which permit the solution of user position and time:

$$
\begin{aligned}
e_{1x}\,x_u+e_{1y}\,y_u+e_{1z}\,z_u-B_u &= p_1 \\
e_{2x}\,x_u+e_{2y}\,y_u+e_{2z}\,z_u-B_u &= p_2 \\
e_{3x}\,x_u+e_{3y}\,y_u+e_{3z}\,z_u-B_u &= p_3 \\
e_{4x}\,x_u+e_{4y}\,y_u+e_{4z}\,z_u-B_u &= p_4.
\end{aligned} \tag{8}
$$

Equation (8) may be condensed in matrix form as

$$GX_u = P, \tag{9}$$

where

$$G_{(4x4)} = \begin{bmatrix} e_{1x} & e_{1y} & e_{1z} & 1 \\ e_{2x} & e_{2y} & e_{2z} & 1 \\ e_{3x} & e_{3y} & e_{3z} & 1 \\ e_{4x} & e_{4y} & e_{4z} & 1 \end{bmatrix} = \begin{bmatrix} E_1^T, & 1 \\ E_2^T, & 1 \\ E_3^T, & 1 \\ E_4^T, & 1 \end{bmatrix},$$

$$(E_i^T)_{(1x3)} = (e_{ix}\ e_{iy}\ e_{iz}),$$

$$(X_u)_{(4x1)} = \begin{bmatrix} x_u \\ y_u \\ z_u \\ -B_u \end{bmatrix},$$

$$P_{(4x1)} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} E_1^T,-1 & 0\ 0\ 0\ 0 & 0\ 0\ 0\ 0 & 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 & E_2^T,-1 & 0\ 0\ 0\ 0 & 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 & 0\ 0\ 0\ 0 & E_3^T,-1 & 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 & 0\ 0\ 0\ 0 & 0\ 0\ 0\ 0 & E_4^T,-1 \end{bmatrix}_{(4x16)} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}_{(16x1)} - \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix},$$

$$(X_i)_{(4x1)} = \begin{bmatrix} x_i \\ y_i \\ z_i \\ -B_i \end{bmatrix}, \tag{10}$$

and the superscript T denotes the transpose matrix where element $(A_{ij})^T = A_{ji}$.

If G is nonsingular, the user position matrix is given by

$$X_u = G^{-1}P. \tag{11}$$

When measurements are made from more than four satellites, G is no longer a square matrix and does not have an inverse. For such a case, both sides of equation (9) should be multiplied by $G^T$ before solving for the overdetermined user position:

$$X_u = (G^TG)^{-1} G^TP, \tag{12}$$

where $X_u$ is solved by the least squares method and the pseudoinverse matrix is $(G^TG)^{-1}G^T$ for $(G^TG)$ nonsingular.

CHOICE OF COORDINATE SYSTEM

Phase 1 user equipment, such as the Magnavox Z-set currently being tested by the FAA, performs all primary navigation functions in earth-centered-earth-fixed (ECEF) coordinates.[4] The positive direction of the z-axis is along the earth's spin axis, that is, from the center of the earth to the north pole. The x-axis is chosen to lie in the equatorial plane such that the x-z plane is coincident with the Greenwich meridian plane. The y-axis is fixed in the equatorial plane such that the x, y, z axes form a right-handed coordinate system. The geodetic parameters of latitude, longitude and altitude are computed by the set and displayed with respect to the 1972 World Geodetic Survey (WGS-72) reference ellipsoid.

User corrections are conveniently described in terms of an east-north-up (ENU) coordinate system; that is, in a horizontal plane (east-north) and vertical direction(up-down) referenced to the user. If the position solution equations were initially given in ECEF coordinates, then transformation to four dimensional ENU coordinates is accomplished by

$$X_{(ENU)} = SX_{(ECEF)} = SG^{-1}{}_{(ECEF)}P = G^{-1}{}_{(ENU)}P. \tag{13}$$

The rotation matrix S is defined by

$$S = \begin{bmatrix} -\sin N & \cos N & 0 & 0 \\ -\cos N \sin M & -\sin N \sin M & \cos M & 0 \\ \cos N \cos M & \sin N \cos M & \sin M & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{14}$$

where M and N are the geodetic latitude and longitude, respectively. Since S is an orthogonal matrix,

$$S^T = S^{-1}$$

and (15)

$$SS^T = SS^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1_{(4x4)},$$

where $1_{(4x4)}$ is the identity matrix. All subsequent position equations are given in ENU coordinates.

TYPICAL RECEIVER SOLUTION FOR POSITION

A GPS receiver demodulates the information on the Keplerian parameters for each satellite from either the almanac or ephemeris data blocks and computes satellite position. When in the navigation mode, the receiver performs an iterative solution based on the a priori estimate of $X_u$ in equation (11). Although $G^{-1}$ is a non-linear function of the user position and time, most receivers employ a linearized model for the basic ranging equations. The above equations may be rewritten as

$$h_i = f_i - B_i = [(x_i-x_u)^2+(y_i-y_u)^2+(z_i-z_u)^2]^{1/2}+B_u. \qquad (16)$$

Let $(H)_E$ be the estimated pseudorange value (offset by the satellite bias $-B_i$) and $(X_u)_E$ the user position/time matrix determined from $(H)_E$; then the difference between the measured and estimated values is

$$\Delta' H = H_M-(H)_E. \qquad (17)$$

The iterative change in the user state becomes

$$\Delta' X_u = (X_u)_M - (X_u)_E \qquad (18)$$

as the receiver solution, $(X_u)_M$, converges to the mathematical solution, $X_u$.

If $h_i$ is expanded about its estimated value, $\Delta' h_i$ to first order becomes[5]

$$\Delta' h_i = \frac{-[(x_i-(x_u)_E)\Delta' x_u+(y_i-(y_u)_E)\Delta' y_u+(z_i-(z_u)_E)\Delta' z_u]+\Delta' B_u}{[(x_i-(x_u)_E)^2+(y_i-(y_u)_E))^2+(z_i-(z_u)_E)^2]^{1/2}}. \qquad (19)$$

In matrix notation, equation (19) may be rewritten for four satellites as

$$\Delta' H = -G_E \Delta' X_u$$
or
$$\Delta' X_u = -G_E^{-1} \Delta' H = G_E^{-1} \Delta' P. \qquad (20)$$

Thus, the incremental pseudorange measurement (difference between the actually measured value and that predicted by the user computer) is linearly related to $\Delta' X_u$, which is the correction that the user equipment will make to update its last estimate of position and time. The same result may have been obtained from equation (11) by taking differentials of both sides and keeping $G^{-1}$ (inverse of the direction cosine matrix) constant.

DILUTION OF PRECISION PARAMETERS

The accuracy of the GPS navigation solution is usually characterized by the uncorrelated errors in the pseudorange measurements[3,6]. One sigma error budgets are assigned to the space segment (satellite ephermeris and clock), the propagation link (atmospheric delay and multipath), and the user segment (receiver processing and dynamics). Each contribution is assumed to come from either a random process and/or a bias error. The latter is viewed as a one sigma value for error budget purposes and is comprised of slowly varying functions of time whose ensemble average is zero. The uncorrelated system error is defined as the user equivalent range error which is the root sum square of the error components.

Let $(X_u)_M$ be the receiver determined solution which is a minimum variance (unbiased), consistent estimate to $X_u$[2,3]. For a linear estimator, one may take differentials of equation (11) and solve for the user error, $\delta X_u$, to first order, where

$$\delta X_u = (X_u)_M - X_u = G^{-1}\,\delta P, \tag{21}$$

where $G^{-1}$ is assumed to be constant for small variations about $X_u$. The covariance of the user error in the estimate of $X_u$ is defined as

$$\operatorname{cov}(\delta X_u) = \langle \delta\tilde{X}_u\, \delta\tilde{X}_u^T\rangle = G^{-1}\langle \delta\tilde{P}\, \delta\tilde{P}^T\rangle\, (G^{-1})^T = G^{-1}\operatorname{cov}(\delta P)(G^T)^{-1}, \tag{22}$$

where the symbol $\langle\ \rangle$ denotes the expected value of the ensemble parameter for a constant user-satellite geometry, and

$$\delta\tilde{X}_u = X_u - \langle X_u\rangle \tag{23}$$

and

$$\delta\tilde{P} = P - \langle P\rangle.$$

Note that $\langle\delta\tilde{X}\rangle$ and $\langle\delta\tilde{P}\rangle$ are identically zero. Furthermore, $\langle\delta X\rangle$ and $\langle\delta P\rangle$ are zero from equation (21) since $(X_u)_M$ is unbiased. However, on any given day, the receiver determined solution may indeed have an instantaneous error or time averaged bias with respect to its surveyed position (which may be offset from the mathematical model solution).

The quantity $\operatorname{cov}(\delta P)$ represents the receiver precision in making pseudorange measurements while the $G^{-1}$ and $(G^T)^{-1}$ matrices determine the influence of the satellite geometry on the user solution. The diagonal terms in $\operatorname{cov}(\delta P)$ are the variances (squares of the one sigma values), and the off diagonal terms express the correlations; namely

$$\operatorname{cov}(\delta P) = \begin{bmatrix} \langle(\delta\tilde{p}_1)^2\rangle & \langle\delta\tilde{p}_1\,\delta\tilde{p}_2\rangle & \langle\delta\tilde{p}_1\,\delta\tilde{p}_3\rangle & \langle\delta\tilde{p}_1\,\delta\tilde{p}_4\rangle \\ \langle\delta\tilde{p}_2\,\delta\tilde{p}_1\rangle & \langle(\delta\tilde{p}_2)^2\rangle & \langle\delta\tilde{p}_2\,\delta\tilde{p}_3\rangle & \langle\delta\tilde{p}_2\,\delta\tilde{p}_4\rangle \\ \langle\delta\tilde{p}_3\,\delta\tilde{p}_1\rangle & \langle\delta\tilde{p}_3\,\delta\tilde{p}_2\rangle & \langle(\delta\tilde{p}_3)^2\rangle & \langle\delta\tilde{p}_3\,\delta\tilde{p}_4\rangle \\ \langle\delta\tilde{p}_4\,\delta\tilde{p}_1\rangle & \langle\delta\tilde{p}_4\,\delta\tilde{p}_2\rangle & \langle\delta\tilde{p}_4\,\delta\tilde{p}_3\rangle & \langle(\delta\tilde{p}_4)^2\rangle \end{bmatrix}. \tag{24}$$

If the satellite pseudorange errors are independent of each other, then for $i \neq j$,

$$\langle\delta\tilde{p}_i\,\delta\tilde{p}_j\rangle = \langle\delta\tilde{p}_i\rangle\langle\delta\tilde{p}_j\rangle = 0, \tag{25}$$

(which holds even for a general receiver solution). Then, $\operatorname{cov}(\delta P)$ is a diagonal matrix containing the variance terms $\langle(\delta\tilde{p}_i)^2\rangle$, whose values are normally chosen as the square of the user equivalent range errors from each satellite. These errors set a nominal upper bound on the variance of the measured errors for a given receiver class.

For the special case where the pseudorange errors are independent and have identical variances equal to $a^2$, cov( $\delta P$) reduces to the 4 x 4 identity matrix multiplied by the constant $a^2$. The covariance matrix of the user error,

$$\mathrm{cov}(\delta X_u) = a^2\, G^{-1}(G^T)^{-1} = a^2(G^TG)^{-1}, \tag{26}$$

then becomes a product of the pseudorange measurement error multiplied by a function of the line of sight direction cosines. The matrix $(G^TG)^{-1}$ may be written in terms of user-satellite position and time elements as

$$(G^TG)^{-1} = \begin{bmatrix} g_{xx} & g_{xy} & g_{xz} & g_{xt} \\ g_{yx} & g_{yy} & g_{yz} & g_{yt} \\ g_{zx} & g_{zy} & g_{zz} & g_{zt} \\ g_{tx} & g_{ty} & g_{tz} & g_{tt} \end{bmatrix}, \tag{27}$$

where $(G^TG)^{-1}$ is a symmetric matrix with elements

$$g_{ij} = g_{ji}. \tag{28}$$

When equation (26) holds, the variances in the user position and time are proportional to the square of the dilution of precision parameters. The quantity GDOP is defined as

$$\mathrm{GDOP} = [\mathrm{trace}(G^TG)^{-1}]^{1/2} = [g_{xx}+g_{yy}+g_{zz}+g_{tt}]^{1/2}, \tag{29}$$

which relates the user equivalent range error or the measurement error (one sigma value) into the uncertainty in the user position and time.

The quantity PDOP is defined as

$$\mathrm{PDOP} = [g_{xx}+g_{yy}+g_{zz}]^{1/2}, \tag{30}$$

which is proportional to the standard deviation in the user position. It can be shown from the rotation matrix S in equation (14) that both GDOP and PDOP are invariant to either an ECEF or an ENU coordinate system.

The HDOP parameter,

$$\mathrm{HDOP} = [g_{xx}+g_{yy}]^{1/2}, \tag{31}$$

multiplied by a , gives the user one sigma error in the horizontal plane of the user (ENU coordinates). The quantities VDOP and TDOP, respectively, are proportional to the one sigma error for the user altitude and clock bias,

$$\mathrm{VDOP} = (g_{zz})^{1/2}$$

and (32)

$$\mathrm{TDOP} = (g_{tt})^{1/2}.$$

The correlation coefficient, $w_{ij}$, between components in $X_u$ is given by

$$w_{ij} = \frac{g_{ij}}{[g_{ii}g_{jj}]^{1/2}} , \tag{33}$$

where $i \neq j$. The error in the $i^{th}$ component is a linear function of the $j^{th}$ component when $w_{ij} = \pm 1$ (the upper/lower bounds). When $w_{ij}=0$, the user component errors are uncorrelated, but not necessarily independent. Hence, the linear dependence between the error components in $X_u$ is a function of the satellite geometry.

GENERAL SOLUTION FOR USER ERRORS

In general, when the satellite pseudorange measurement errors are independent of each other, but their variances are not equal, the user covariance matrix of the user error becomes

$$\text{cov}(\delta X_u) = \sum_{i,j,k=1}^{4} (G^{-1})_{ik} \langle (\delta \tilde{P})^2 \rangle_{kk} [(G^T)^{-1}]_{kj} = \begin{bmatrix} v_{xx} & v_{xy} & v_{xz} & v_{xt} \\ v_{yx} & v_{yy} & v_{yz} & v_{yt} \\ v_{zx} & v_{zy} & v_{zz} & v_{zt} \\ v_{tx} & v_{ty} & v_{tz} & v_{tt} \end{bmatrix} , \tag{34}$$

where the off-diagonal terms are related by

$$v_{ij} = v_{ji} . \tag{35}$$

The standard deviations in $x_u$, $y_u$, $z_u$, and $-B_u$ are respectively given as

$$s_{xx} = (v_{xx})^{1/2},$$

$$s_{yy} = (v_{yy})^{1/2},$$

$$s_{zz} = (v_{zz})^{1/2},$$

and (36)

$$s_{tt} = (v_{tt})^{1/2}.$$

In a similar manner, the user standard deviations in two dimensions (the horizontal plane), three dimensions, and four dimensions are respectively defined as

$$d_2 = (v_{xx} + v_{yy})^{1/2},$$

$$d_3 = (v_{xx} + v_{yy} + v_{zz})^{1/2},$$

and (37)

$$d_4 = (v_{xx} + v_{yy} + v_{zz} + v_{tt})^{1/2}.$$

For the general case, the dilution of precision parameters may only be used to estimate the adequacy of the satellite configuration to provide accurate measurements since the effects of geometry and pseudorange errors are not separated as in equation (26) . In the Results section, user errors for independent, but not equal, measurement errors are compared to those computed for the pooled pseudorange covariance matrix $cov(\delta P)_p$ with equal diagonal elements $v_p$ given by

$$v_p = (1/4)\ [\langle(\delta\tilde{p}_1)^2\rangle + \langle(\delta\tilde{p}_2)^2\rangle + \langle(\delta\tilde{p}_3)^2 + \langle(\delta\tilde{p}_4)^2\rangle]. \tag{38}$$

The correlation coefficient for the general case becomes

$$w_{ij} = \frac{v_{ij}}{[v_{ii}\ v_{jj}]^{1/2}}, \tag{39}$$

which reduces to equation (33) when the pseudorange measurement variances are equal.

SYSTEM BIAS

In this report, the term system bias error for the user refers to the difference between the receiver measurement/solution and that corresponding to a reference one. For example, the GPS test range facility at Yuma Proving Ground has a fiducial GPS receiver driven by an atomic clock at a surveyed position. Yuma records differences between the receiver determined, $(X_u)_M$, and reference (surveyed), $(X_u)_s$, positions and times, and computes the measurement residuals from each satellite. The system bias error for the user, $\Delta X_u$, to first order is obtained by taking differentials of equation (11), where

$$\Delta X_u = (X_u)_M - (X_u)_s = -G^{-1}\,\Delta H = G^{-1}\,\Delta P, \tag{40}$$

$G^{-1}$ is assumed to be constant for small variations about $(X_u)_s$, and $\Delta H$ is the bias in the pseudorange measurement. Note the user error discussed here results in a correction-$\Delta B_u$ to the receiver clock offset-$B_u$ previously discussed.

On the other hand, the GPS master control station at Vandenberg Air Force Base computes every 15 minutes the difference between the filtered pseudorange estimate obtained from widely separated monitoring stations during the satellite visibility period and the pseudorange value predicted by the ephemeris and clock parameters in the satellite's message.[7] This computation is essentially similiar to an inverted range operation (corrected for known biases) in which the position of each satellite is calculated as if it were the user and the monitoring stations (four or more) the space vehicles. The pseudorange residual in this case is the sum of the ephemeris contributions, which are functions of the user-satellite direction cosines, and the satellite clock contributions, which are independent of user geometry.

The system residuals predicted by the Vandenberg filter algorithms and those determined by the Yuma ground truth receiver are within specification[8] when the system is operating properly. Information either from the master control station or from fiducial receivers may be utilized to provide near real-time corrections to aircraft via diffential GPS techniques and/or to apprise pilots of overall GPS system accuracy for satellites visible in that area.

SOLUTION FOR THREE SATELLITES PLUS ALTITUDE

Consider the case when three satellite positioning data plus user altitude information are available to the receiver navigation computer. Let the subscripts 1 to 3 in equation (8) hold for the three satellites. The user altitude enters into the navigation solution as a distance referenced from the center of the earth. The fourth line of equation (8) then becomes

$$-z_u = p_4, \qquad (41)$$

which corresponds to a satellite at the center of the earth synchronized to the receiver clock. The G matrix is given by

$$G = \begin{bmatrix} e_{1x} & e_{1y} & e_{1z} & 1 \\ e_{2x} & e_{2y} & e_{2z} & 1 \\ e_{3x} & e_{3y} & e_{3z} & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \qquad (42)$$

and the covariance matrix and user bias are solved as above.

# RESULTS

## GENERAL DISCUSSION

The results presented in this report are modeled for a GPS receiver located at the FAA Technical Center (ACY) with coordinates 39°26'58.4" North and 74°34'00.4" West and altitude 14.1 meters above the WGS-72 ellipsoid. The receiver is assumed to track either four satellites (NAVSTAR 3 through 6) during the primary visibility period or any three of these satellites. In the latter case, the receiver utitizes encoded barometric altitude information in its navigation solution. The satellite coordinates are determined from GPS almanac data as a function of time. Figure 2 shows the computed satellite elevation and azimuth angles for ACY on February 26, 1982. The date is chosen merely to establish a convenient reference time in order to compare results in the error analysis.

In order to illustrate the dependence of the receiver determined position and clock offset on the measurement errors from the satellites/altimeter, we vary the measurement errors in the following manner:

1. For the four satellite case, pseudorange errors from three satellites are assumed to have an equal bias of 1 meter and an equal standard deviation of 1 meter. The measurements from the fourth satellite are chosen to exhibit a bias of 10 meters and standard deviation of 10 meters (see table 1). The larger errors from the last satellite do not exceed GPS system specifications[6] for projected GPS standard positioning service without ionospheric corrections and degradation (denial of accuracy). The errors are then permuted among satellites in the pseudorange matrix and are independent. Finally, an averaged bias value and a pooled variance value (for the permuted examples) are assigned for the pseudorange errors.

2. The same is done with the three satellites plus altitude case, with the stipulation that the larger errors are always assigned to the altimeter measurement except for the averaged (equal) case. According to the Federal Radionavigation Plan[8], aircraft altimetry system errors can easily exceed the 10 meter bias and/or standard deviation and still be within specification.

The detailed figures, which follow, illustrate user position/clock standard deviations in single and multiple dimensions, the user correlation coefficients, and the user biases versus time during the satellite visibility period on February 26, 1982 at ACY. The user errors are computed from the preceding equations (equation (36) for $s_{ii}$, equation (37) for $d_i$, and equation (40) for $\Delta X_u$) and are given in meters. Note that $s_{ii}$ is functionally independent of the measurement bias errors, and $\Delta X_u$ is functionally independent of the measurement standard deviations. The correlation coefficients, computed from equation (39), are dimensionless and vary from -1 to +1. The Greenwich mean time (GMT) time is given in 15 minute increments. Another set of figures compares the user horizontal position errors, $d_2$, for the permuted satellite measurement errors and the pooled variance case.

The predicted user errors are functions of the resultant user-satellite direction cosine matrix and the relevant pseudorange measurement errors. It is reemphasized that the curves are computed for one user location (ACY) during the primary satellite viewing period for NAVSTAR 3 through 6 (approximately two hours and fifteen minutes for a 10° mimimum elevation angle). The results typify the representative behavior in the user errors that might be expected for different pseudorange measurement errors. All computations and plots[9] were accomplished on a Hewlett Packard 1000 model 45 computer system.

FOUR SATELLITES CASE

Figures 3 through 6 illustrate the wide diversity of GPS user errors when the pseudorange measurement from one satellite has a bias and standard deviation 10 times larger than the others. The error data are outlined in table 1. The pseudorange biases and variances are respectively averaged for the results given in figure 7. Note that the solution of equation (40) reveals that the user position biases are zero when the pseudorange residuals are equal. However, the error in the estimated user clock offset, $-\Delta B_u$, is equal to -3.75 meters, the negative value of the averaged pseudorange bias. The dilution of precision parameters are obtained by dividing the user multiple dimensional errors $d_i$ in figure 7 by the pseudorange standard deviation (5.074 meters in this example).

The two dimensional user errors, $d_2$, for the above cases are replotted in figure 8. Curve 7 illustrates the behavior of $d_2$ computed from the pooled pseudorange variance $v_p$. The HDOP value rapidly increases (above 20) as NAVSTAR 3 and 4 approach elevation angles below 10° (at approximately 13:00 GMT time). For one sigma error ratios between pseudorange measurements of the order of 10 to 1 or less, the $d_i$ values for the pooled case give a reasonable approximation of the expected multiple dimensional user errors. Thus, one may justify the treatment often found in the literature that leads to the introduction of the dilution of precision parameters, namely, the individual satellite pseudorange measurement variances are assumed equal.

THREE SATELLITES PLUS ALTITUDE CASE

Figures 9 through 12 show the predicted user error curves when measurements are made from three satellites and the aircraft altimeter. Table 1 displays the satellite permutations and pseudorange errors, where the altimeter errors are 10 times larger. Note the quantity $s_{zz}$ is constant (10 meters for these examples) when the G matrix in equation (42) is substituted into equation (34). The other predicted user errors for the altitude-aided receiver are very dependent upon the direction cosine matrix, as was previously seen for the unaided receiver utilizing four satellite inputs. The $d_i$ parameters may exhibit very large extreme values, may exhibit relatively constant behavior (figure 11) throughout the satellite visibility period, and/or may be actually lower (figure 11) than results derived from four satellites. The user bias in the vertical direction, $\Delta z_u$ is equal to the altimeter bias, 10 meters.

A wide diversity in the user error behavior is also observed in figures 13 through 16 when the biases and variances in the satellite and altimeter measurements are respectively averaged (see table 1). Note that $s_{zz}$ and $\Delta z$ are constant and equal to 5.074 meters (square root of pooled variance) and 3.75 meters (averaged bias), respectively. Again, the dilution of precision parameters are found by dividing $d_i$ by the one sigma measurement error.

Figures 17 and 18 compare the behavior of $d_2$ on a condensed scale for the results previously illustrated in figures 9 through 16. The magnitudes of the curves in figure 17 - (altimeter variance 100 times larger than the permuted satellite measurement errors) are less than the corresponding ones in figure 18 - (pooled variance for altimeter and permuted satellite measurement errors). The behavior exhibited in figure 17 represents the more realistic case encountered by an altitude-aided receiver since the expected altimeter errors will normally exceed the satellite measurement errors (perhaps with one sigma ratios even greater than 10 to 1). The wide variations in the two dimensional user standard deviations indicate that altitude aiding should be employed only as an emergency back-up navigation mode unless the HDOP difference from the four satellite constellation is accurately known beforehand. For example, future GPS receivers should compute and display HDOP as a measure of the adequacy of the satellite/altimeter geometry.

## SUMMARY AND CONCLUSIONS

Equations for GPS user position and time errors are derived in a matrix formulation when the satellite pseudorange measurements errors are assumed to be independent, but not necessarily equal. The resulting expressions reveal that the user errors are dependent upon satellite geometry and the magnitudes of each pseudorange residual and its variance. The well-known dilution of precision parameters (geometric scaling factors) enter into the solution only when the pseudorange variances are equal and independent. The case for an altitude-aided receiver tracking three satellites is also treated.

A number of figures are presented to illustrate representative user error behavior in position and time (biases, single and multiple dimensional standard deviations, and correlation coefficients). The selected example is for a receiver located at the FAA Technical Center, Atlantic City Airport (ACY), New Jersey, when (1) NAVSTAR satellites 3 through 6 are in view, (2) the measurement errors from one satellite/altimeter are 10 times larger than those from the other three satellites, and (3) the measured biases and variances are respectively averaged. The results indicate a wide diversity in the predicted user errors depending upon satellite geometry, ranging errors, and altitude aiding. The dilution of precision parameters give a reasonable approximation of expected user errors for the four satellite case when the ratio of pseudorange standard deviation is less than 10 to 1. The altitude-aided receiver exhibits extremely large user errors under unfavorable geometry conditions which are governed by the user-satellite direction cosine matrix.

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3. Milliken, R.J. and Zoller, C.J., "Principle of Operation of NAVSTAR and System Characteristics", Navigation, Vol. 25, pages 95-106, 1978.

4. "User's Manual (Computer Program) for User Equipment Set Z of the NAVSTAR Global Positioning System", Magnavox Report Number CDRL Item AOOX.

5. Jorgensen, P.S., "NAVSTAR/Global Positioning System 18-Satellite Constellation", Navigation Vol. 27, pages 89-100, 1980.

6. Martin, E.H., "GPS User Equipment Error Models", Navigation, Vol. 25, pages 201-210, 1978.

7. Russell, S.S. and Schaibly, J.H., "Control Segment and User Performance", Navigation, Vol. 25 pages 166-172, 1978.

8. Federal Radionavigation Plan, Department of Defense and Department of Transportation, Report Numbers DOD-4650.4P and DOT-TSC-RSPA-81-12, Vol. 2, page 11, March 1982.

9. Buchberger, E., "A Program to Compute the User Errors of the Global Positioning System", FAA Report Number CT-82-112LR.

$\vec{R}_i$ = vector from center of earth to $i^{th}$ satellite

$\vec{R}_u$ = vector from center of earth to user

$\vec{D}_i$ = vector from user position to $i^{th}$ satellite

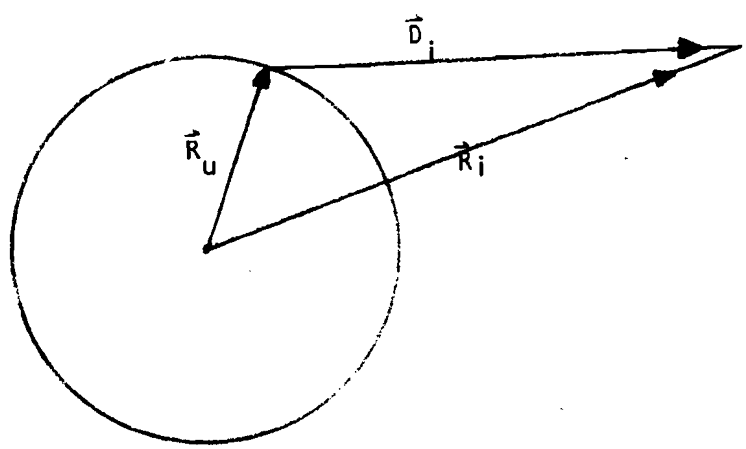


FIGURE 1. USER-SATELLITE GEOMETRY.

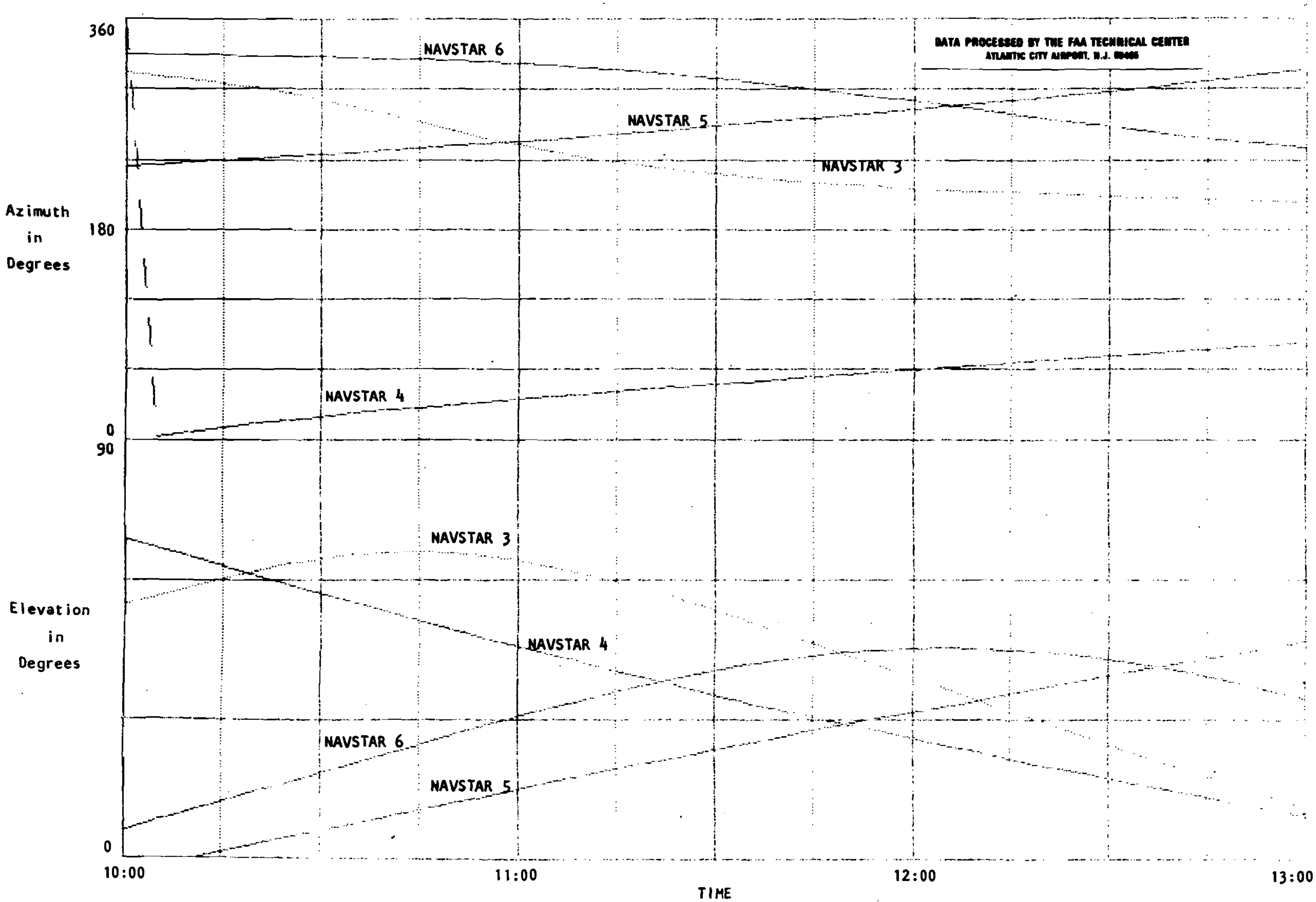


FIGURE 2. NAVSTAR 3 THROUGH 6 AZIMUTH AND ELEVATION ANGLES VERSUS TIME DURING PRIME VISIBILITY PERIOD FOR USER AT ACY ON FEBRUARY 26, 1982.

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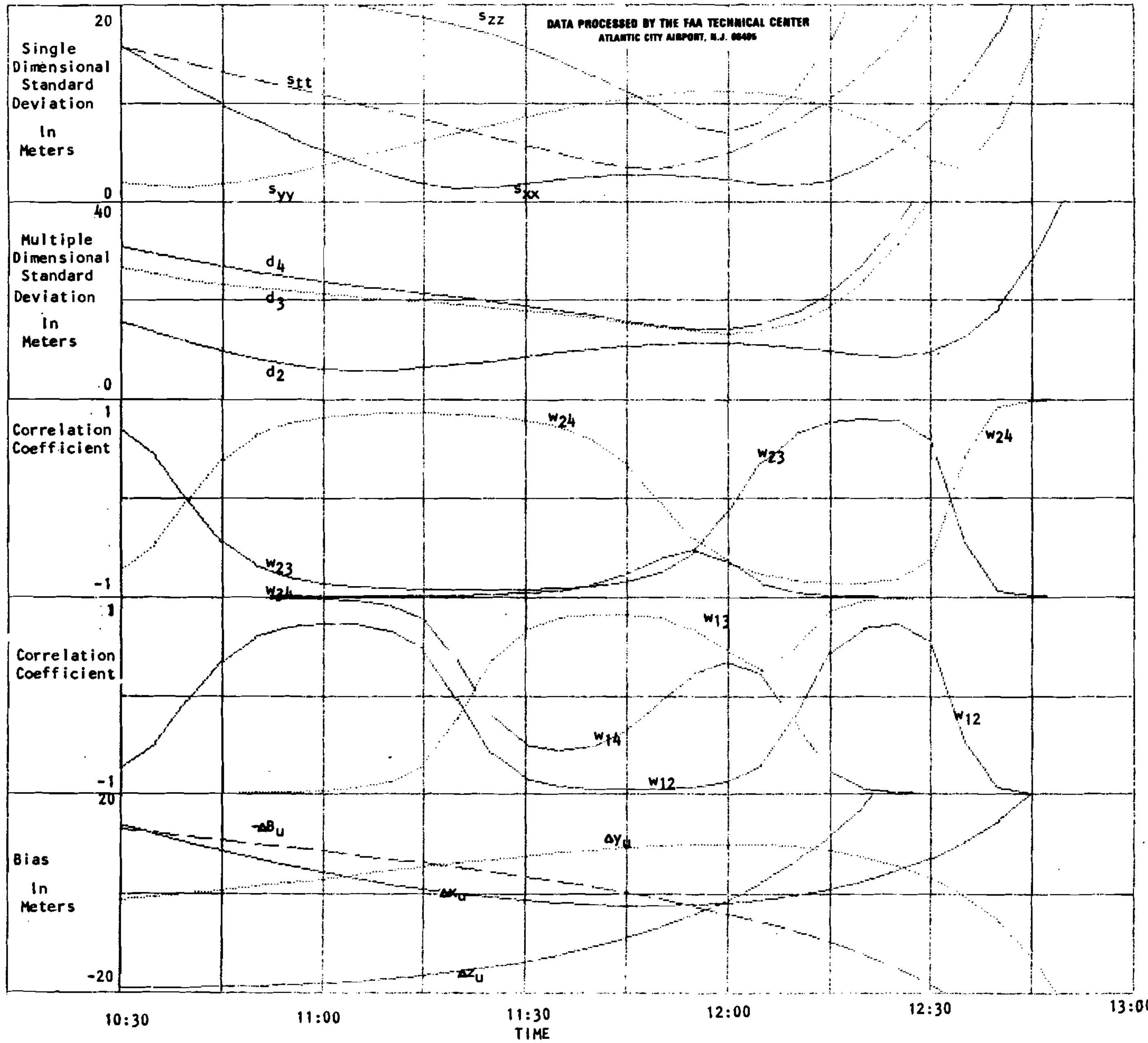


FIGURE 3. PREDICTED GPS USER ERRORS AT ACY WHEN THE MEASURED BIASES AND STANDARD DEVIATIONS ARE BOTH 10 METERS FROM NAVSTAR 3 AND 1 METER FROM NAVSTAR 4, 5, and 6.

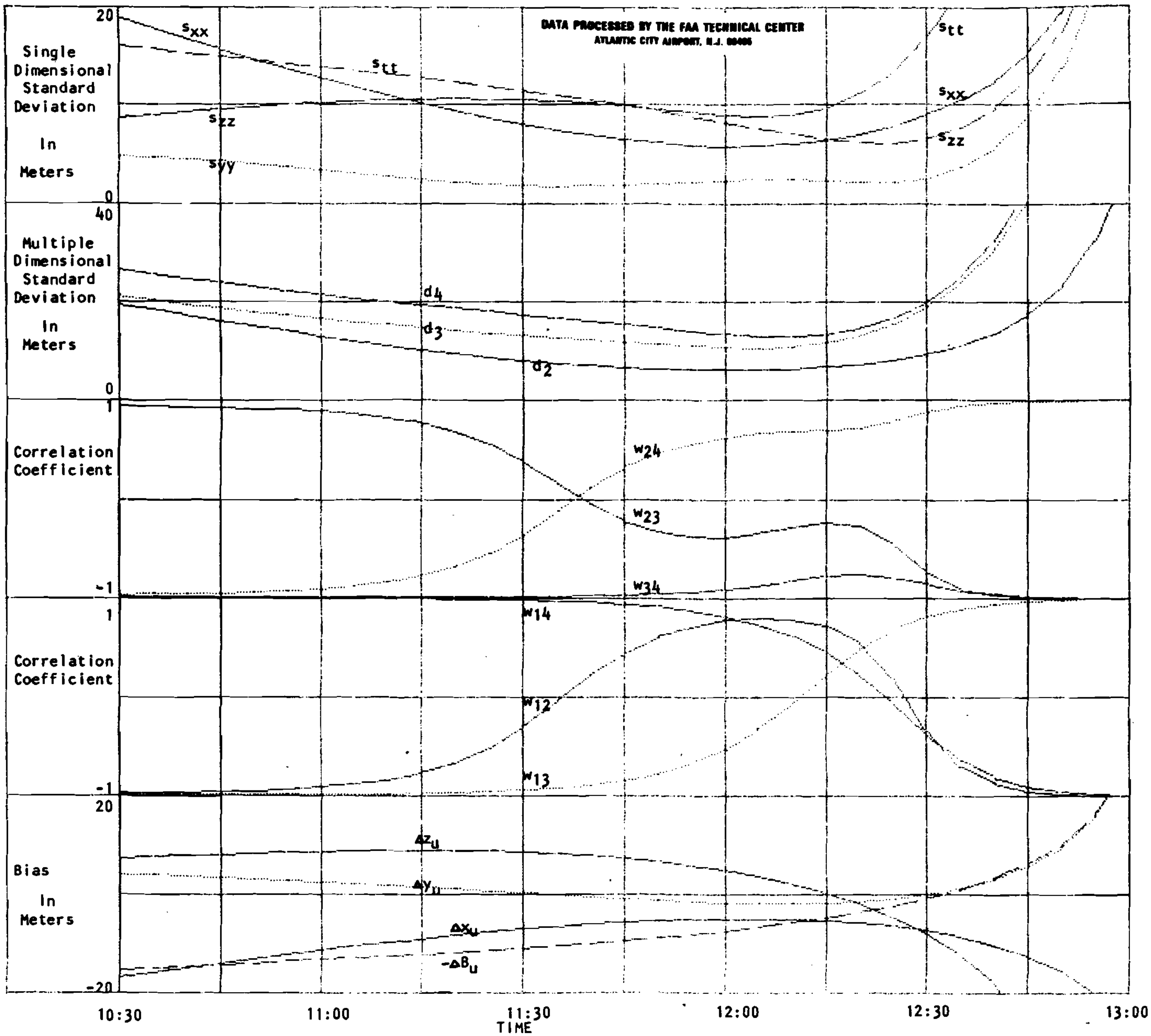


FIGURE 4. PREDICTED GPS USER ERRORS AT ACY WHEN THE MEASURED BIASES AND STANDARD DEVIATIONS ARE BOTH 10 METERS FROM NAVSTAR 4 AND 1 METER FROM NAVSTAR 3, 5, and 6.

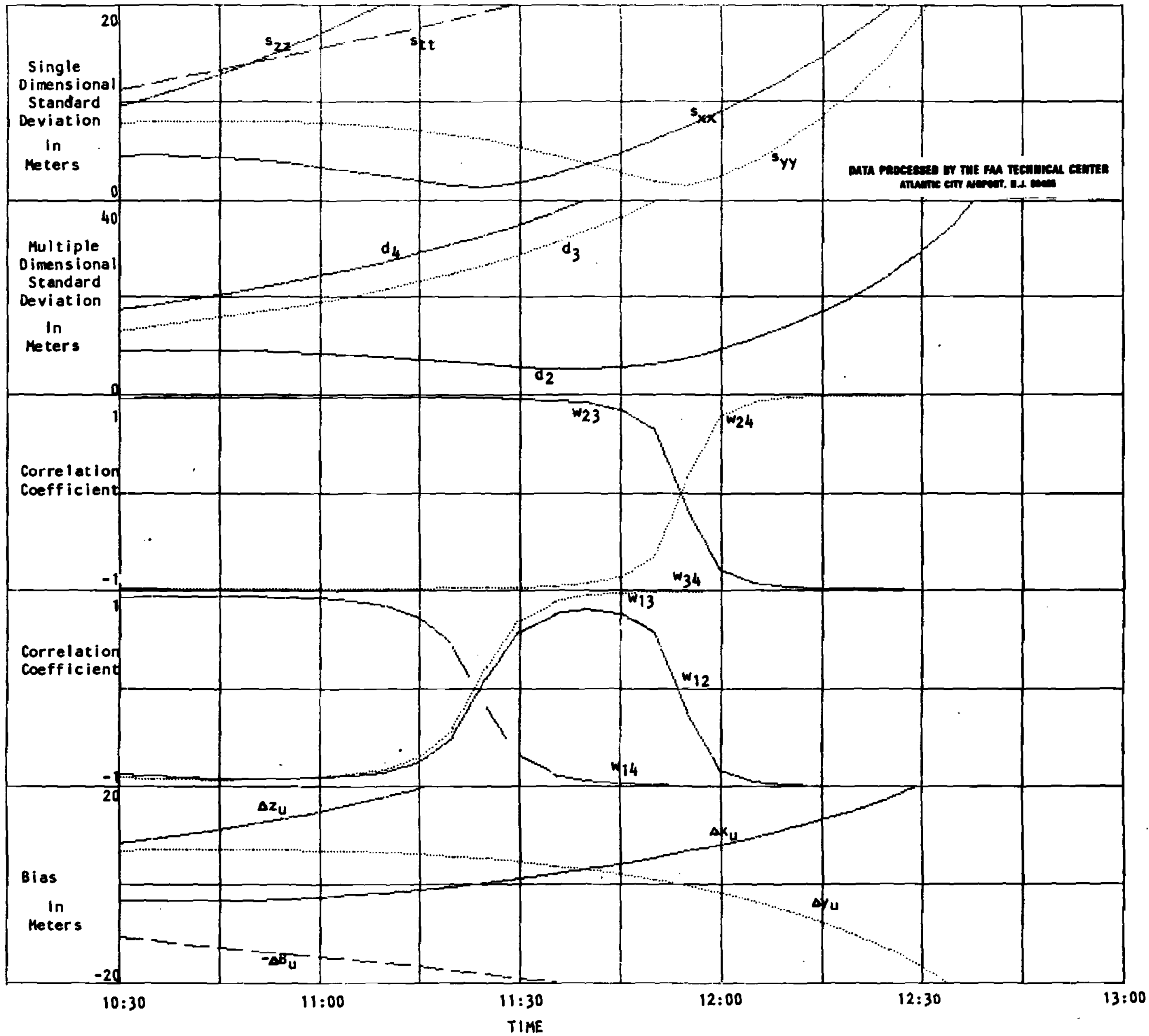


FIGURE 5. PREDICTED GPS USER ERRORS AT ACY WHEN THE MEASURED BIASES AND STANDARD DEVIATIONS ARE BOTH 10 METERS FROM NAVSTAR 5 AND 1 METER FROM NAVSTAR 3, 4, and 6.

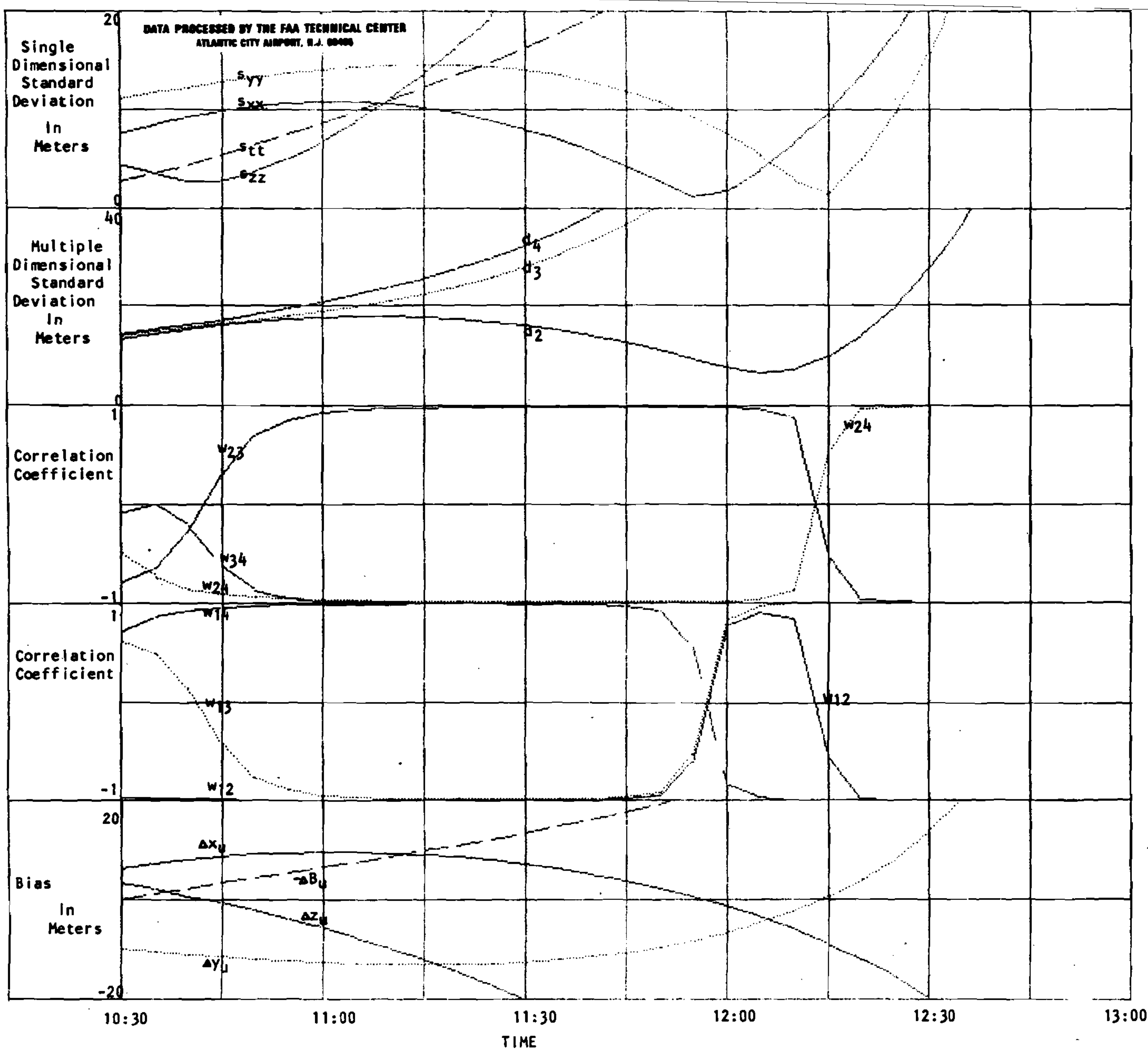


FIGURE 6. PREDICTED GPS USER ERRORS AT ACY WHEN THE MEASURED BIASES AND STANDARD DEVIATIONS ARE BOTH 10 METERS FROM NAVSTAR 6 AND 1 METER FROM NAVSTAR 3, 4, AND 5.

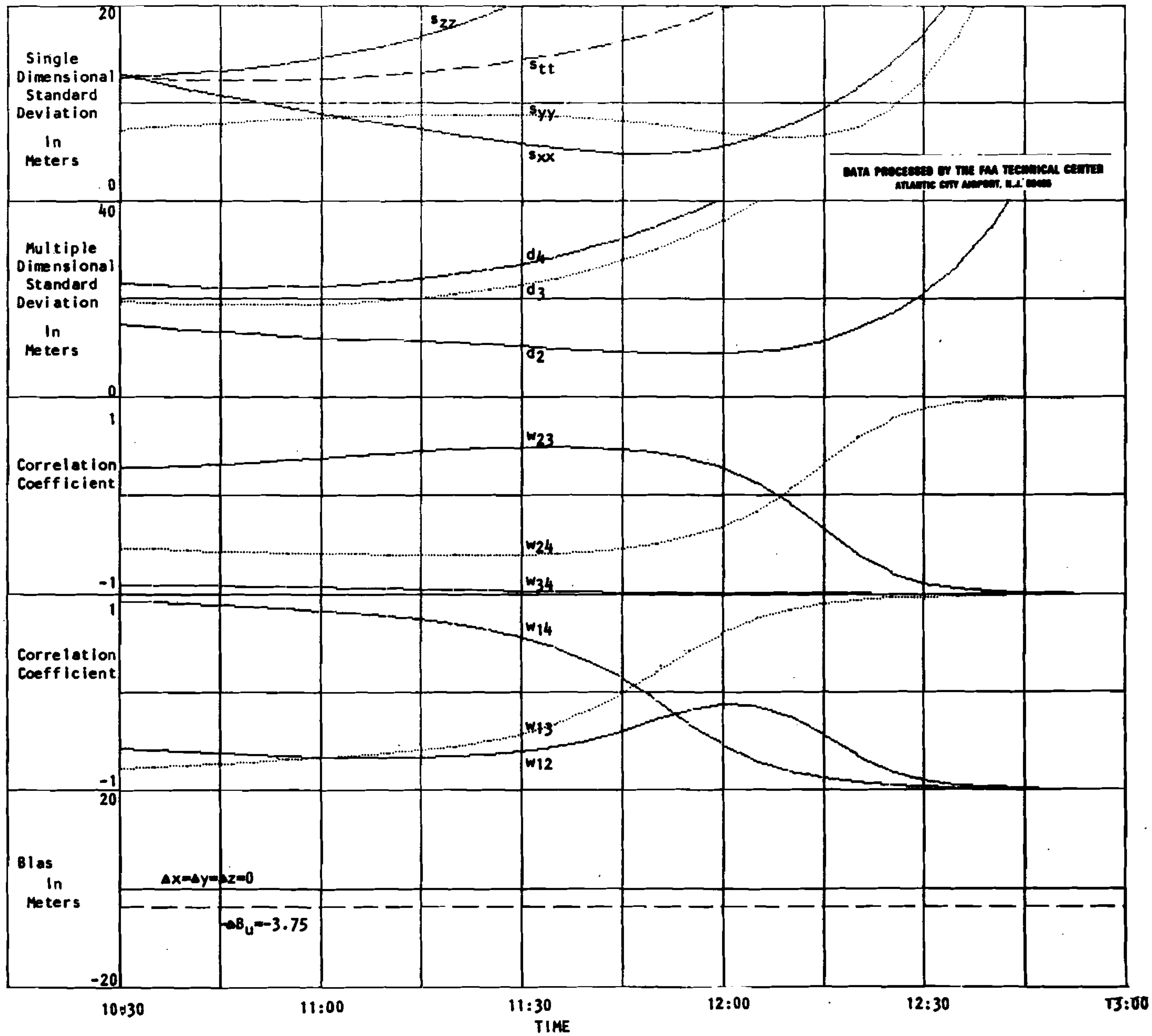


FIGURE 7. PREDICTED GPS USER ERRORS AT ACY WHEN THE AVERAGED BIAS IS 3.75 METERS AND THE POOLED VARIANCE IS (5.074 METERS)$^2$ FROM NAVSTAR 3, 4, 5 and 6.

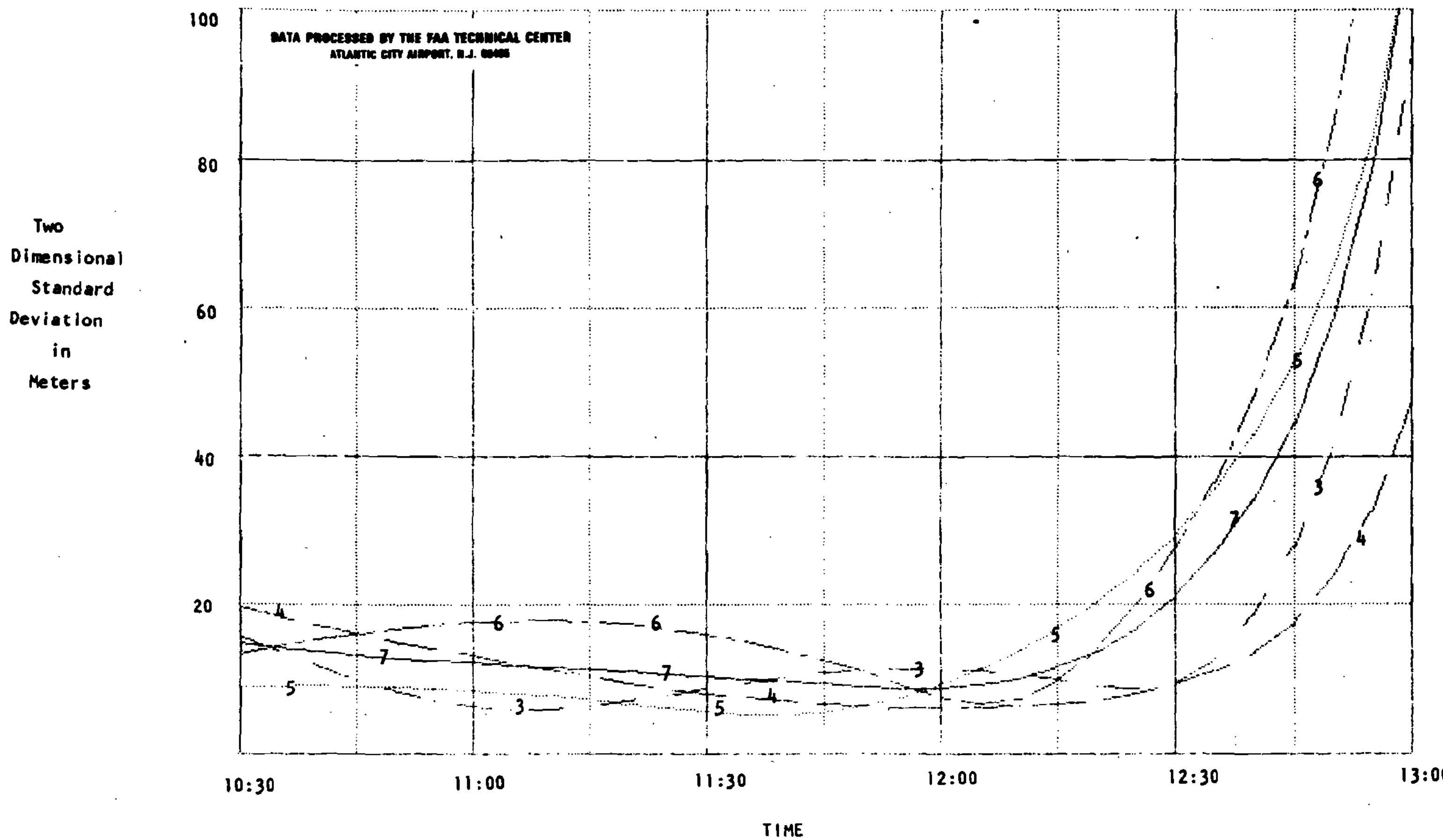


FIGURE 8. TWO DIMENSIONAL USER STANDARD DEVIATIONS FOR DATA PRESENTED IN FIGURES 3 THROUGH 7.

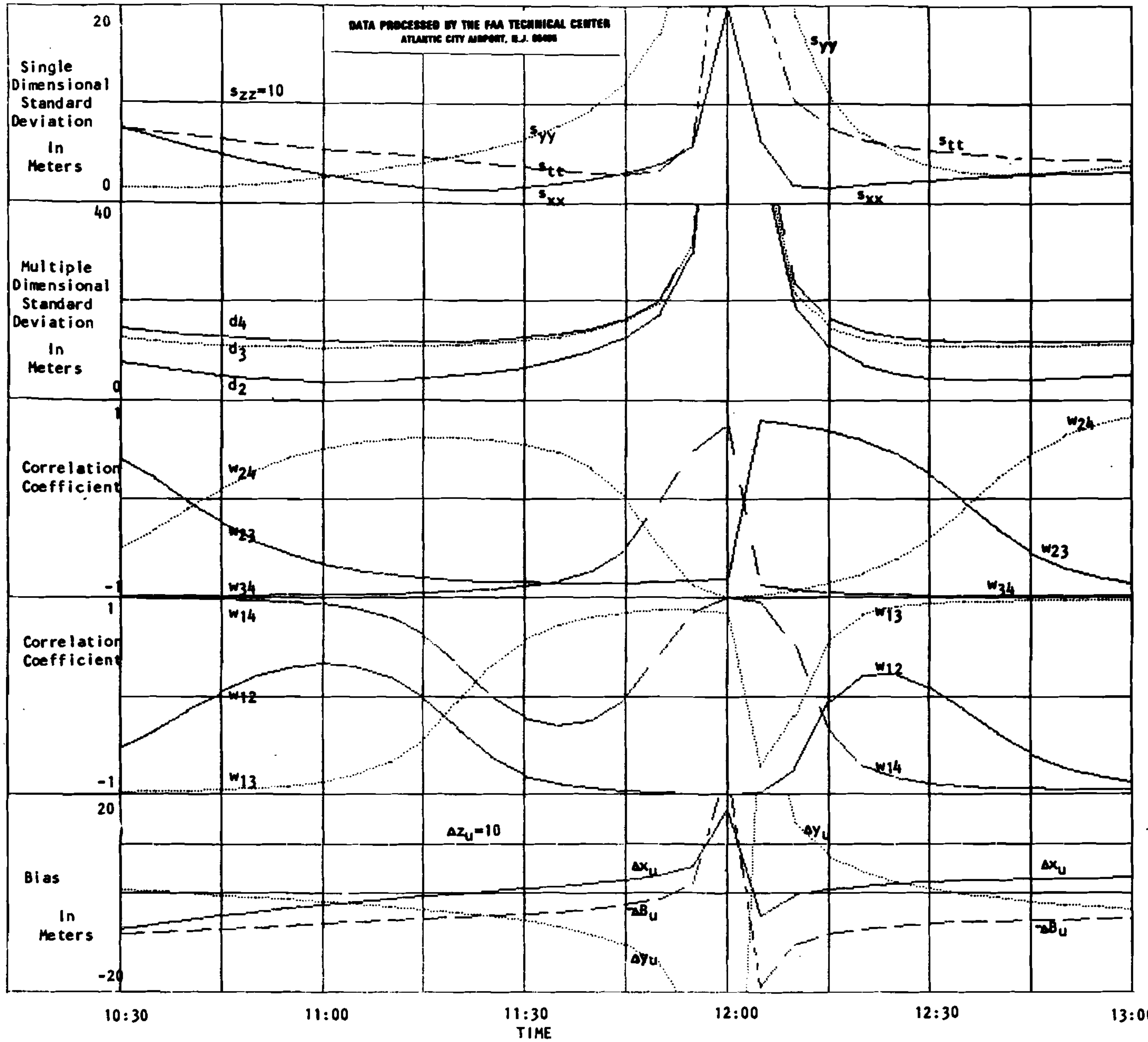


FIGURE 9. PREDICTED GPS USER ERRORS AT ACY WHEN THE MEASURED BIASES AND STANDARD DEVIATIONS ARE BOTH 10 METERS FROM AN ALTIMETER AND 1 METER FROM NAVSTAR 4, 5, and 6.

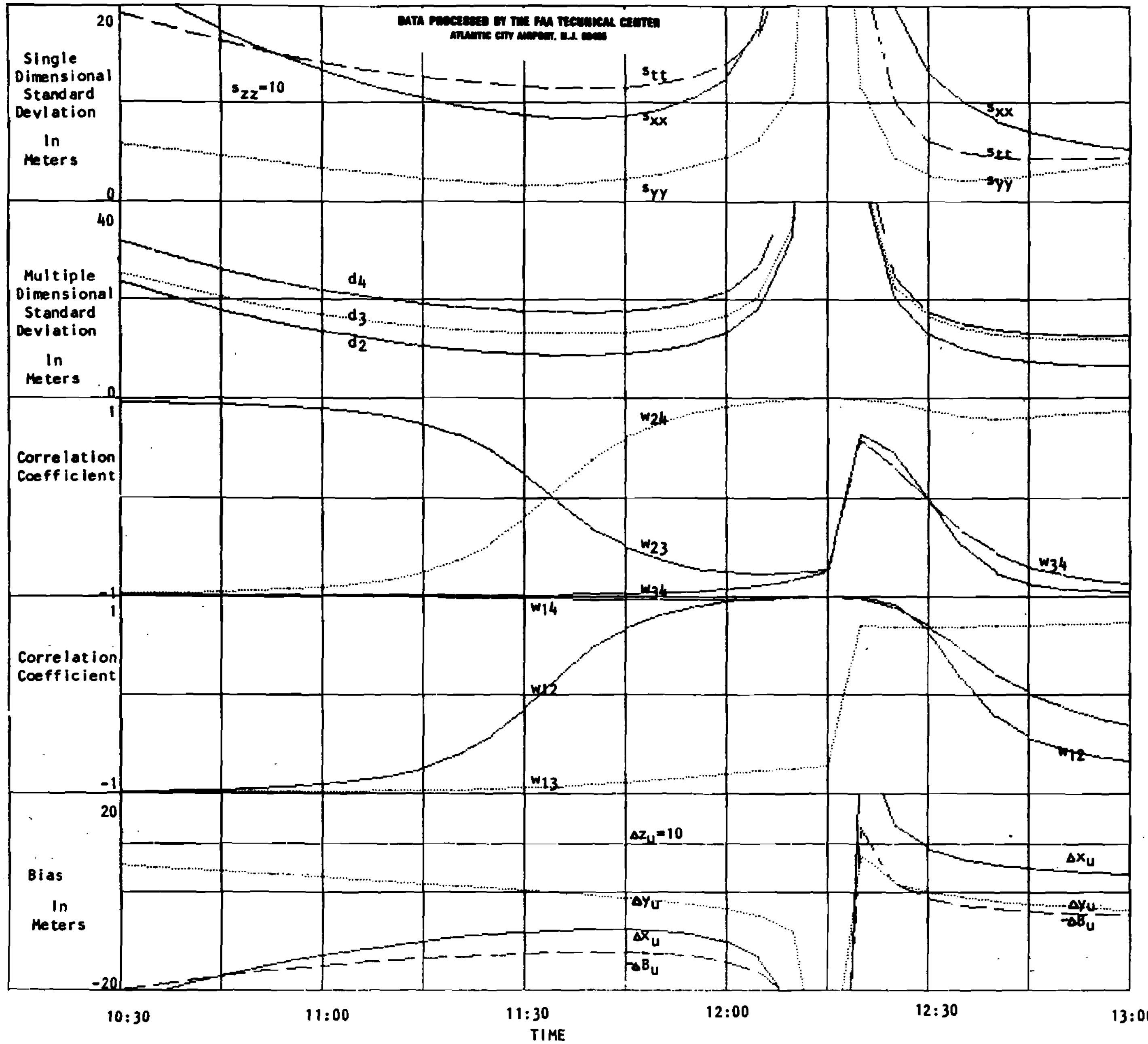


FIGURE 10. PREDICTED GPS USER ERRORS AT ACY WHEN THE MEASURED BIASES AND STANDARD DEVIATIONS ARE BOTH 10 METERS FROM AN ALTIMETER AND 1 METER FROM NAVSTAR 3, 5, and 6.

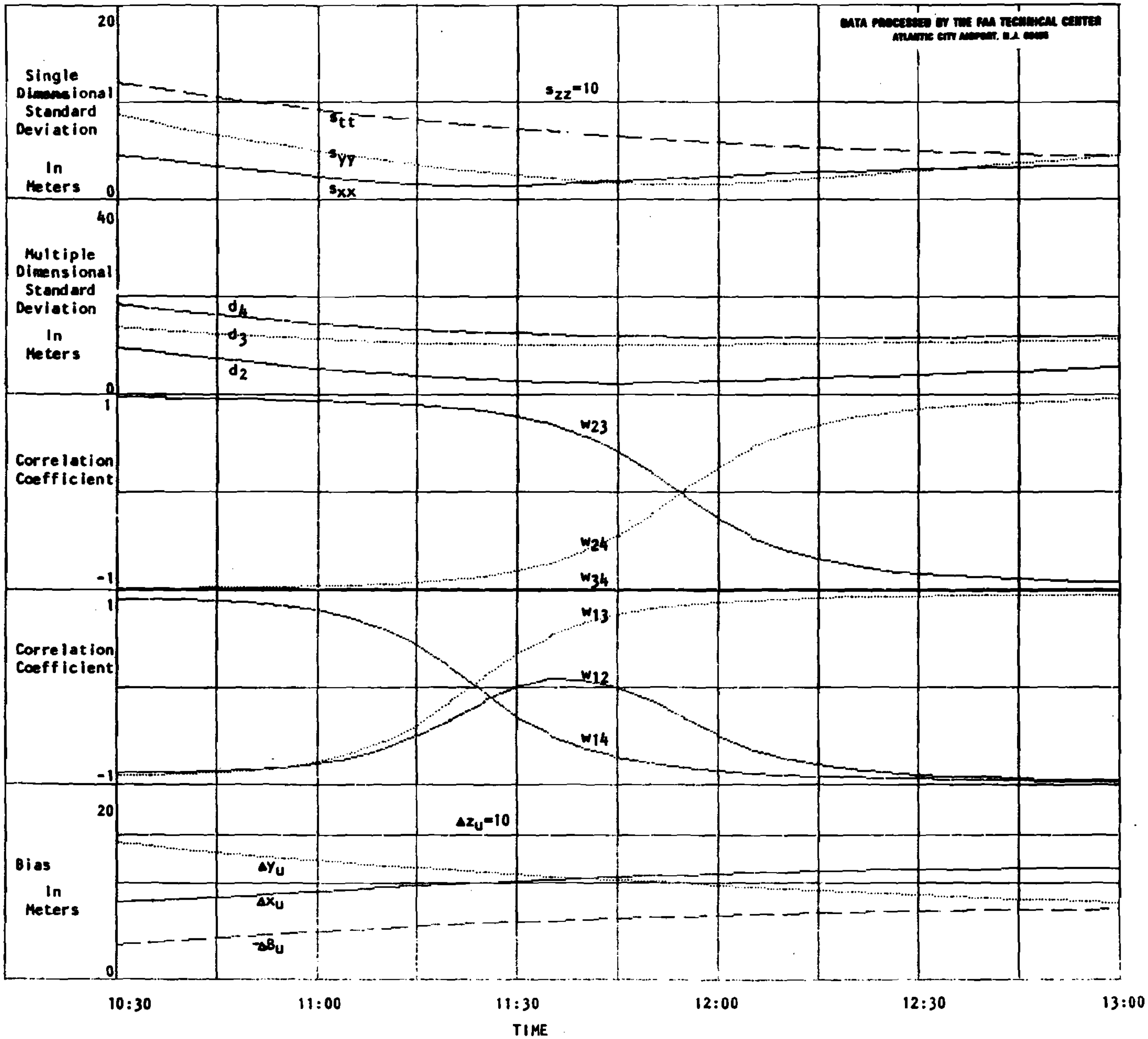


FIGURE 11. PREDICTED GPS USER ERRORS AT ACY WHEN THE MEASURED BIASES AND STANDARD DEVIATIONS ARE BOTH 10 METERS FROM AN ALTIMETER AND 1 METER FROM NAVSTAR 3, 4, and 6.

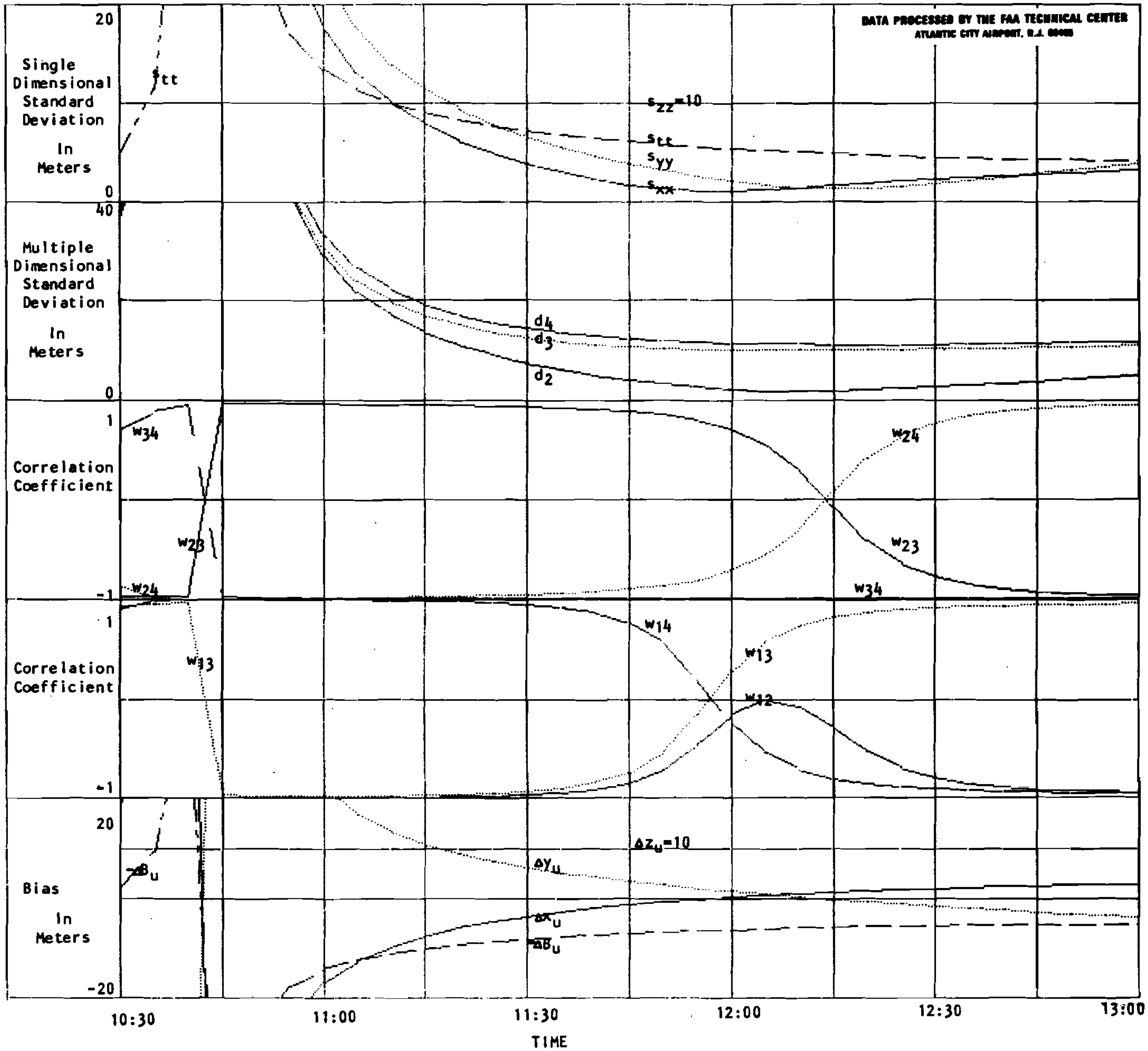


FIGURE 12. PREDICTED GPS USER ERRORS AT ACY WHEN THE MEASURED BIASES AND STANDARD DEVIATIONS ARE BOTH 10 METERS FROM AN ALTIMETER AND 1 METER FROM NAVSTAR 3, 4, and 5.

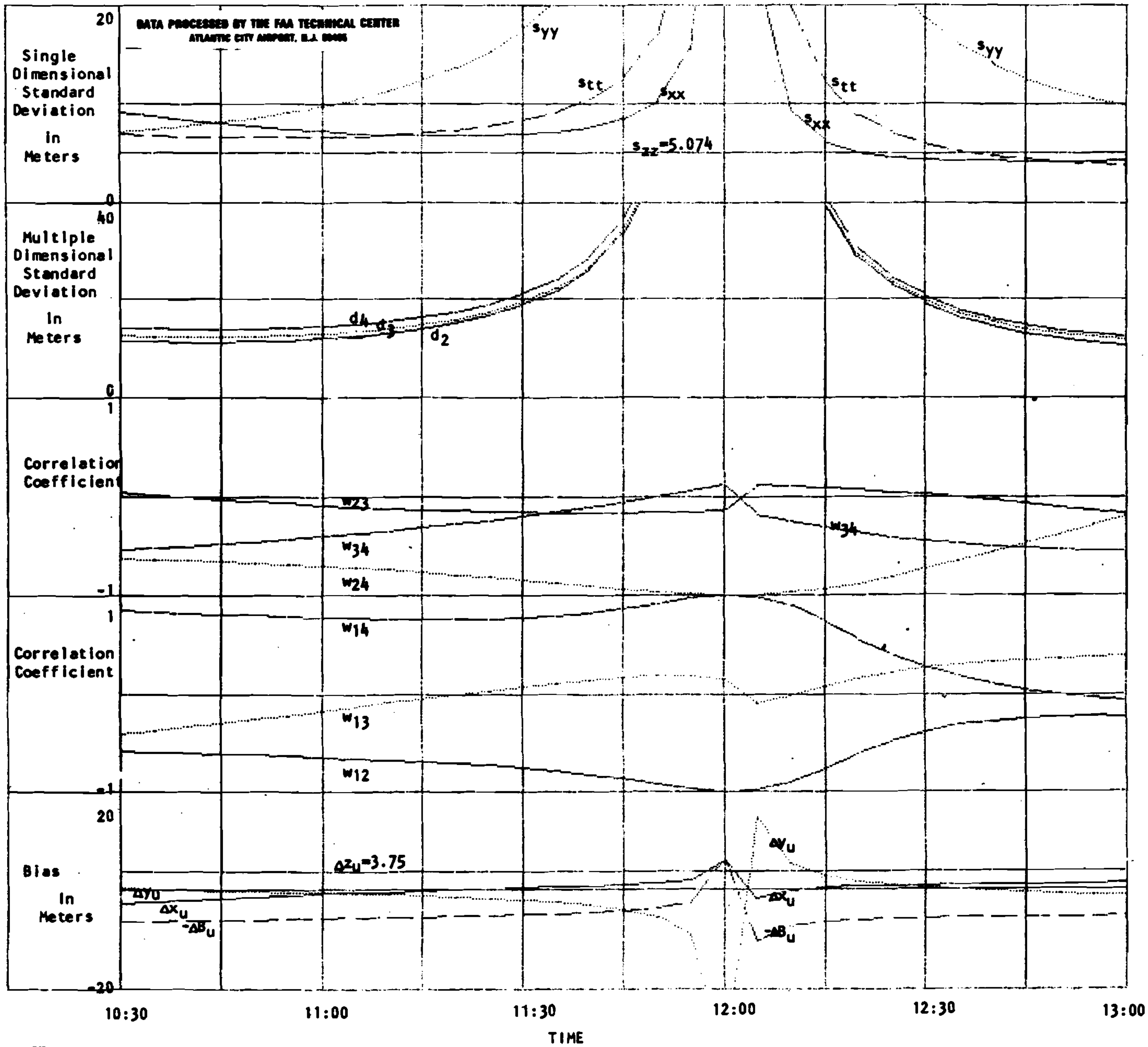


FIGURE 13. PREDICTED GPS USER ERRORS AT ACY WHEN THE AVERAGED BIAS IS 3.75 METERS AND THE POOLED VARIANCE IS (5.074 METERS)$^2$ FROM AN ALTIMETER AND NAVSTAR 4, 5, and 6.

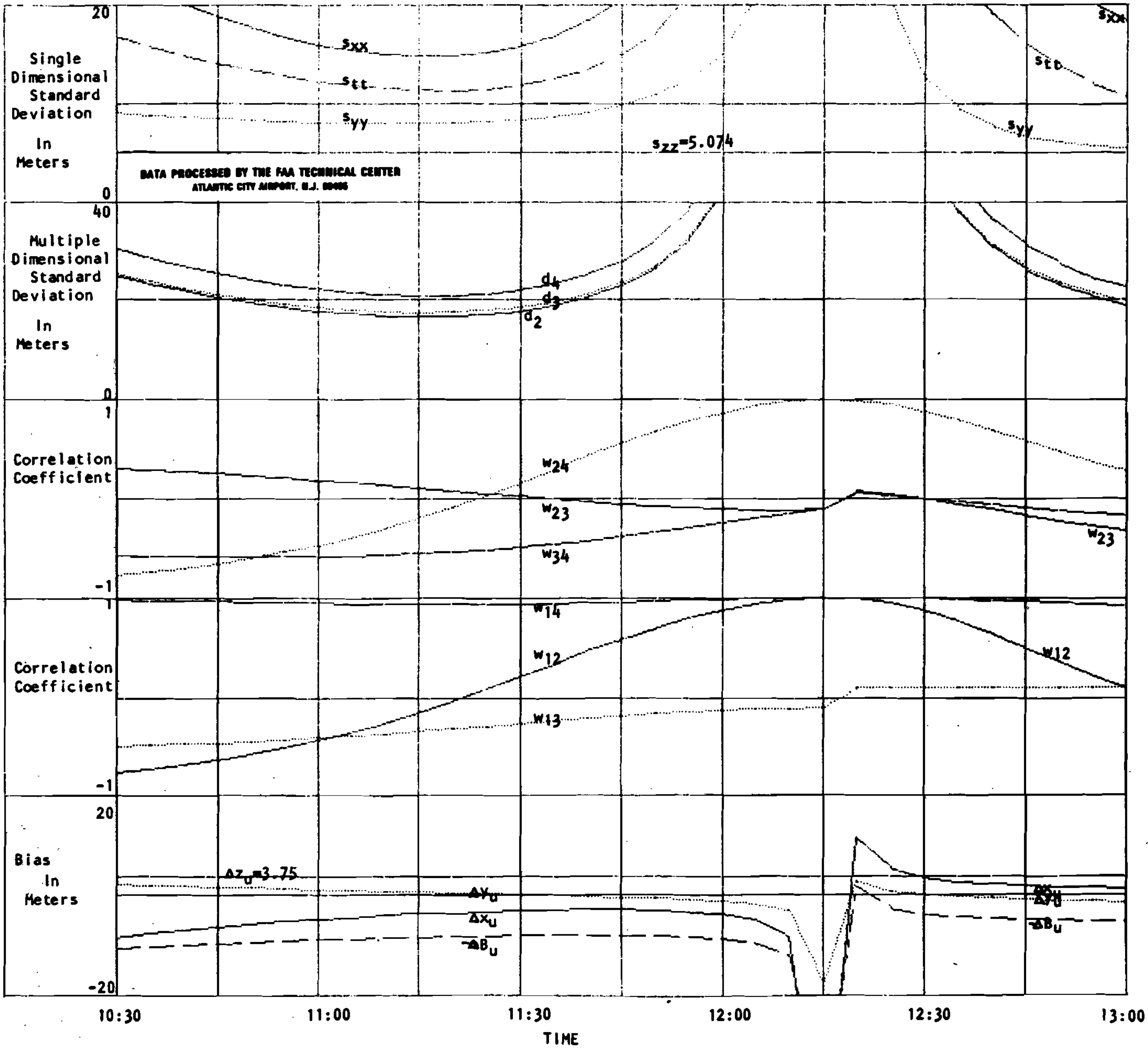


FIGURE 14. PREDICTED GPS USER ERRORS AT ACY WHEN THE AVERAGED BIAS IS 3.75 METERS AND THE POOLED VARIANCE IS (5.074 METERS)$^2$ FROM AN ALTIMETER AND NAVSTAR 3, 5 and 6.

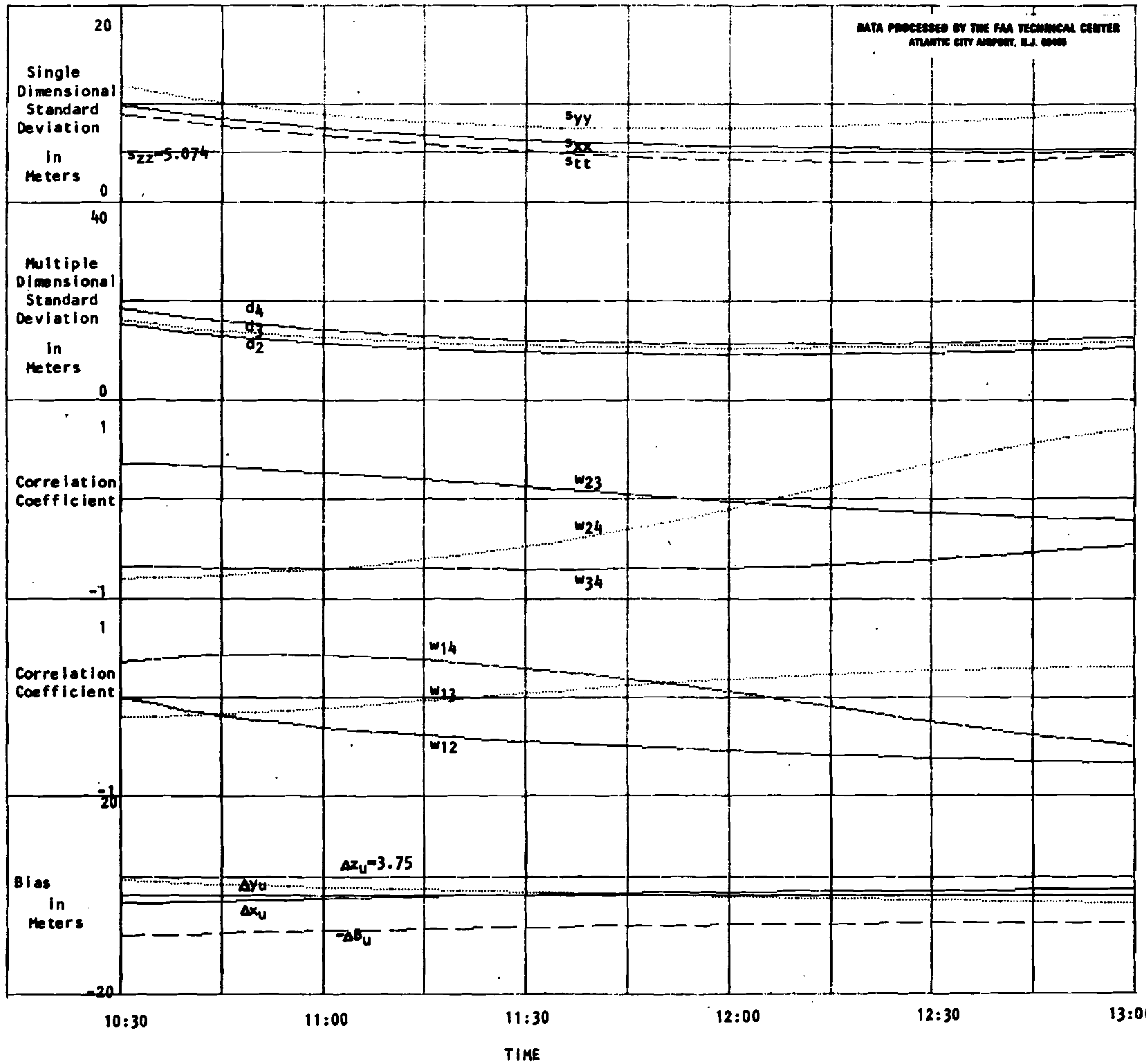


FIGURE 15. PREDICTED GPS USER ERRORS AT ACY WHEN THE AVERAGED BIAS IS 3.75 METERS AND THE POOLED VARIANCE IS (5.074 METERS)$^2$ FROM AN ALTIMETER AND NAVSTAR 3, 4, and 6.

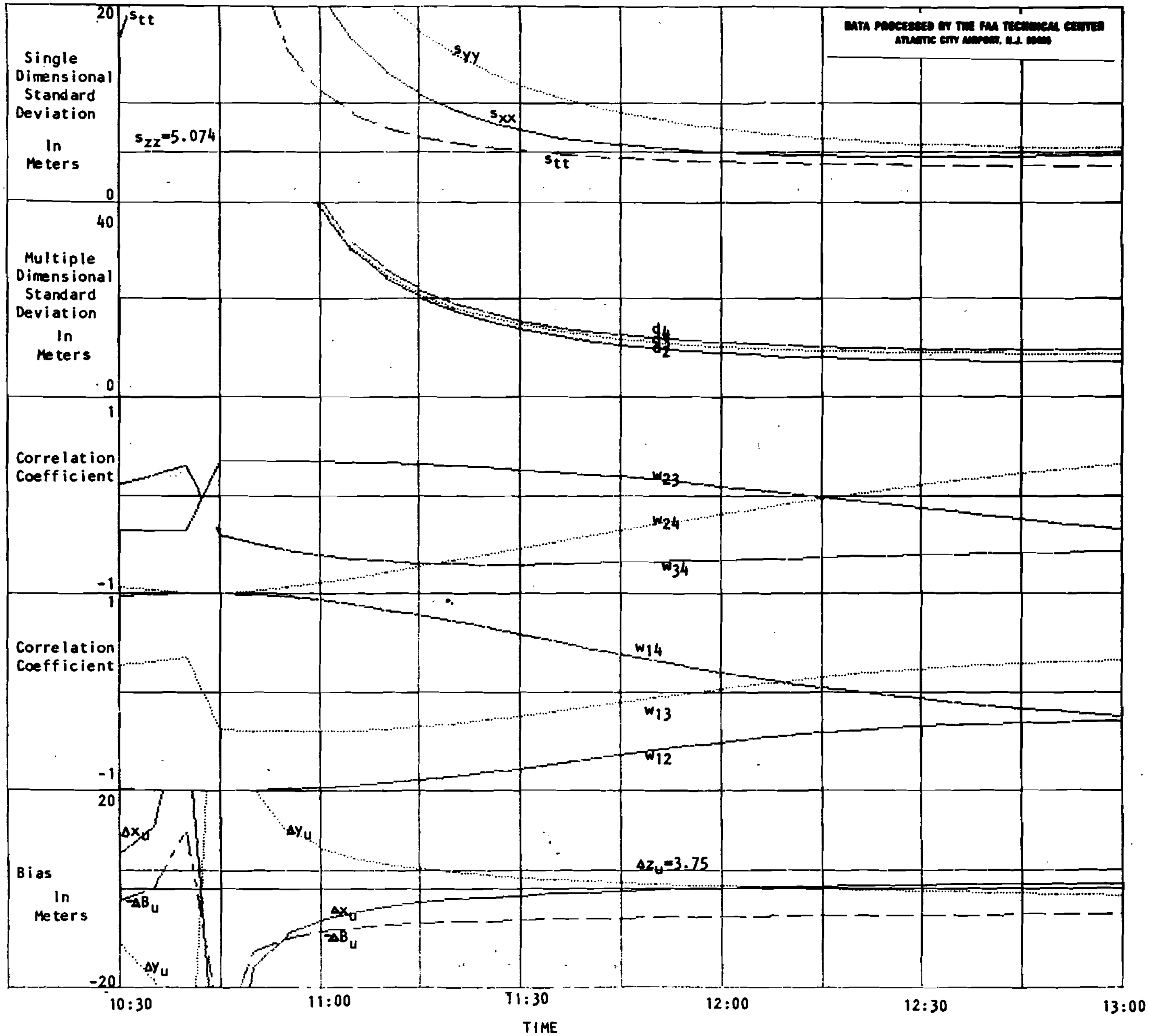


FIGURE 16. PREDICTED GPS USER ERRORS AT ACY WHEN THE AVERAGED BIAS IS 3.75 METERS AND THE POOLED VARIANCE IS (5.074 METERS)$^2$ FROM AN ALTIMETER AND NAVSTAR 3, 4, and 5.

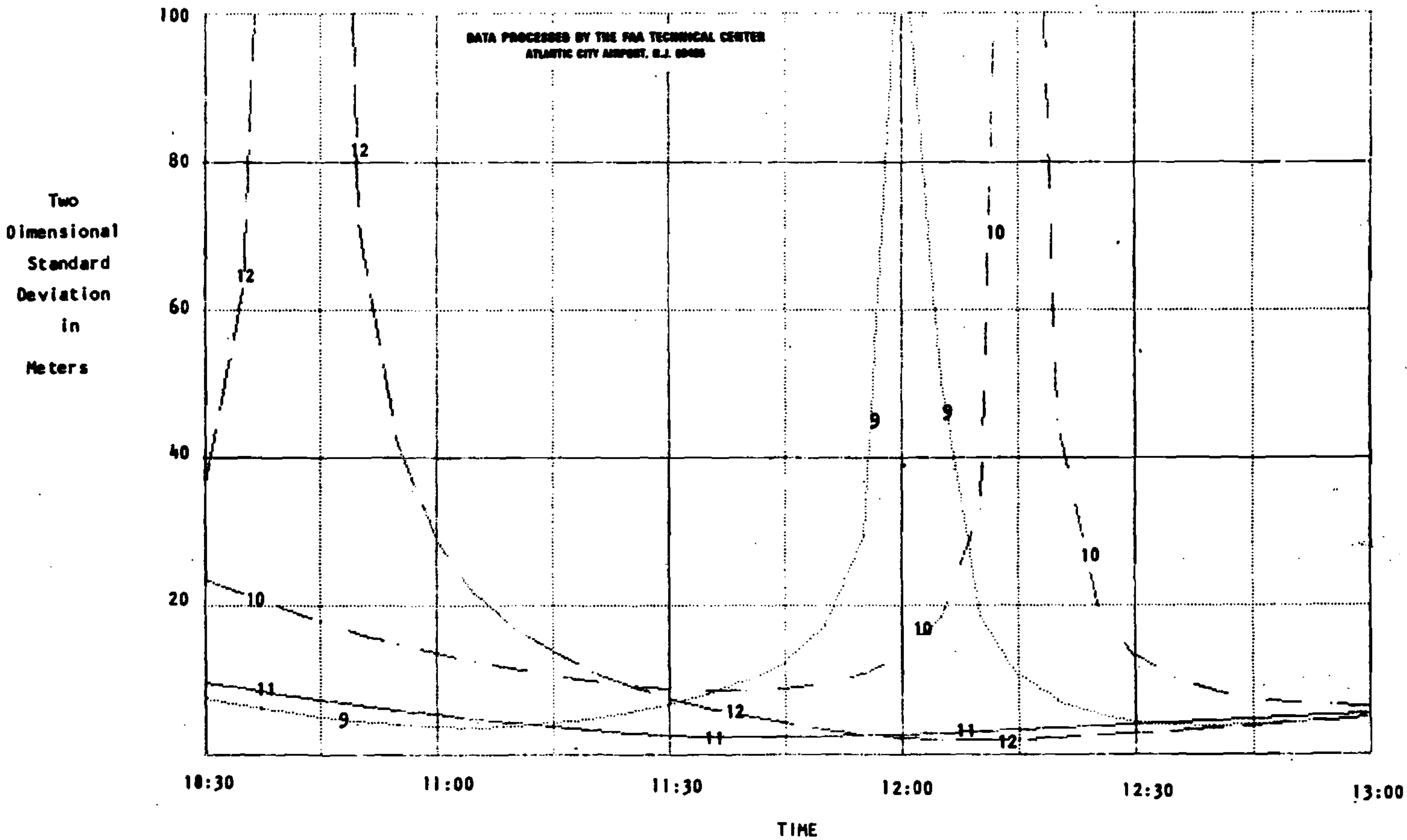


FIGURE 17. TWO DIMENSIONAL USER STANDARD DEVIATIONS FOR DATA PRESENTED IN FIGURES 9 THROUGH 12.

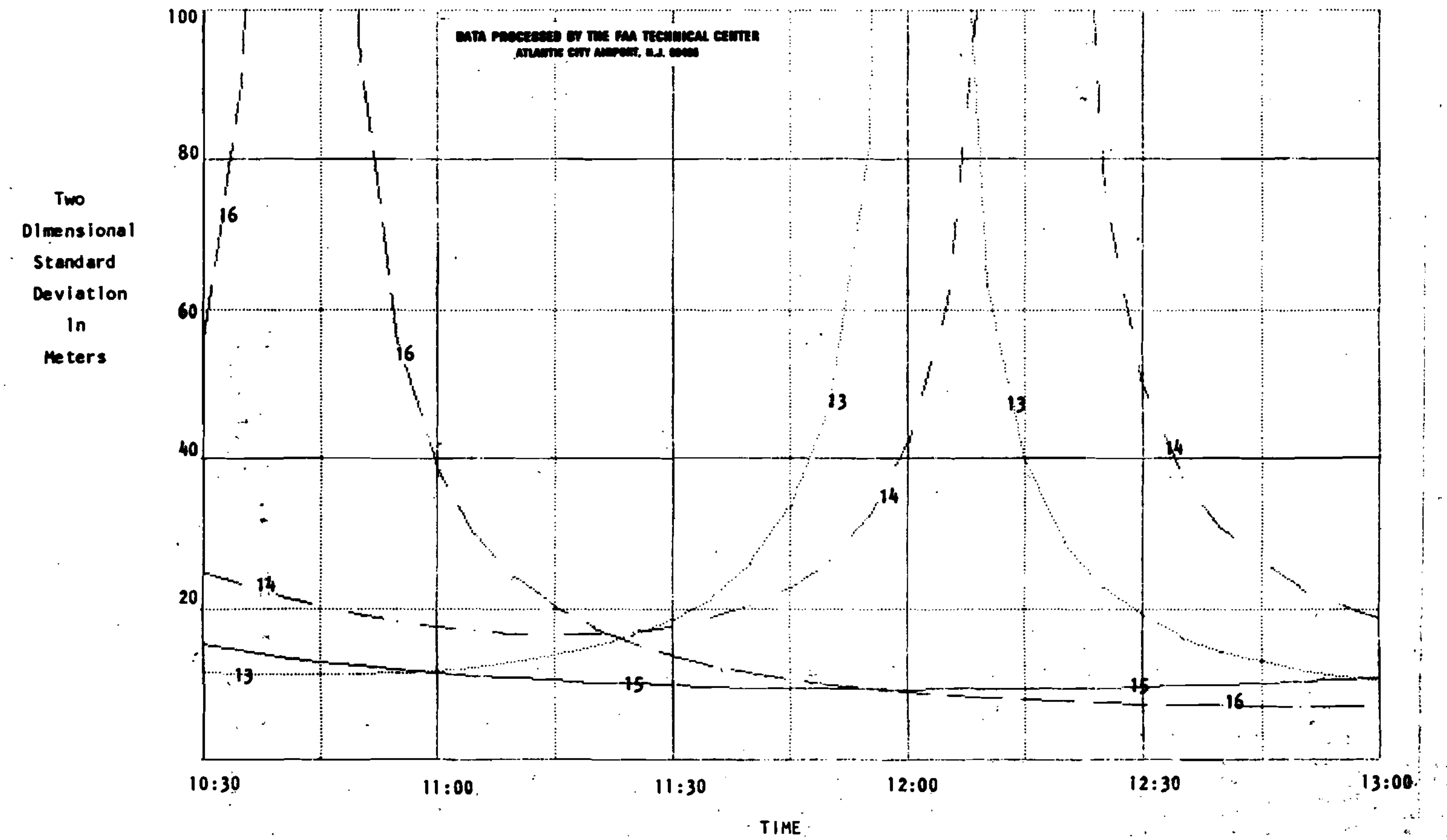


FIGURE 18. TWO DIMENSIONAL USER STANDARD DEVIATIONS FOR DATA PRESENTED IN FIGURES 13 THROUGH 16.

| Figure | Source | Bias Error | One Sigma Error |
|---|---|---|---|
| 3 | NAVSTAR 3 | 10 | 10 |
| | NAVSTAR 4,5,6 | 1 | 1 |
| 4 | NAVSTAR 4 | 10 | 10 |
| | NAVSTAR 3,5,6 | 1 | 1 |
| 5 | NAVSTAR 5 | 10 | 10 |
| | NAVSTAR 3,4,6 | 1 | 1 |
| 6 | NAVSTAR 6 | 10 | 10 |
| | NAVSTAR 3,4,5 | 1 | 1 |
| 7 | NAVSTAR 3,4,5,6 | 3.75 | 5.074 |
| 9 | Altimeter | 10 | 10 |
| | NAVSTAR 4,5,6 | 1 | 1 |
| 10 | Altimeter | 10 | 10 |
| | NAVSTAR 3,5,6 | 1 | 1 |
| 11 | Altimeter | 10 | 10 |
| | NAVSTAR 3,4,6 | 1 | 1 |
| 12 | Altimeter | 10 | 10 |
| | NAVSTAR 3,4,5 | 1 | 1 |
| 13 | Altimeter | 3.75 | 5.074 |
| | NAVSTAR 4,5,6 | 3.75 | 5.074 |
| 14 | Altimeter | 3.75 | 5.074 |
| | NAVSTAR 3,5,6 | 3.75 | 5.074 |
| 15 | Altimeter | 3.75 | 5.074 |
| | NAVSTAR 3,4,6 | 3.75 | 5.074 |
| 16 | Altimeter | 3.75 | 5.074 |
| | NAVSTAR 3,4,5 | 3.75 | 5.074 |

Table (1) Satellite and altimeter measurement errors in meters for results given in specified figures.