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DESIGN AND IMPLEMENTATION OF EFFICIENT ALGORITHMS FOR AUTOMATIC DETERMINATION OF CORRECTED SLANT RANGE

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16. Abstract

This report introduces a systematic approach to the design of algorithms for evaluating the corrected slant range in a radar surveillance system. Applications include air traffic control (ATC) operations requiring real-time continuous computation for a multitude of targets without overtaxing available computational resources. From the point of view of accuracy, utilization of memory, and computational speed, the design technique is capable of providing an algorithm that is superior to the corrected slant range technique presently employed in the National Airspace System (NAS).

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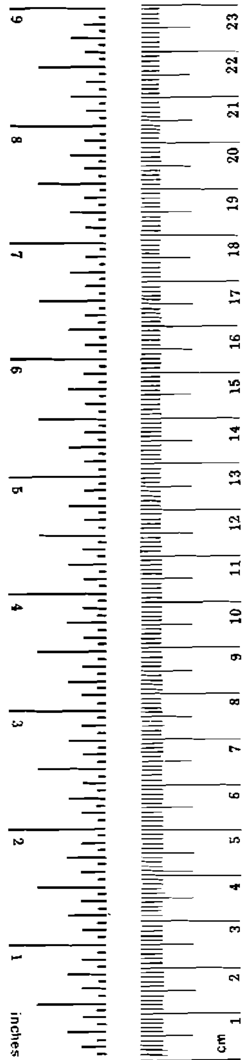
METRIC CONVERSION FACTORS

Approximate Conversions to Metric Measures

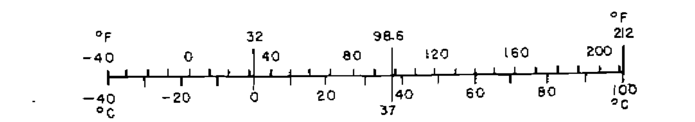
Symbol	When You Know	Multiply by	To Find	Symbol
		LENGTH		
in	inches	*2.5	centimeters	cm
ft	feet	30	centimeters	cm
yd	yards	0.9	meters	m
mi	miles	1.6	kilometers	km
		AREA		
in^2	square inches	6.5	square centimeters	cm^2
ft^2	square feet	0.09	square meters	m^2
yd^2	square yards	0.8	square meters	m^2
mi^2	square miles	2.6	square kilometers	km^2
	acres	0.4	hectares	ha
		MASS (weight)		
oz	ounces	28	grams	g
lb	pounds	0.45	kilograms	kg
	short tons (2000 lb)	0.9	tonnes	t
		VOLUME		
tsp	teaspoons	5	milliliters	ml
Tbsp	tablespoons	15	milliliters	ml
fl oz	fluid ounces	30	milliliters	ml
c	cups	0.24	liters	l
pt	pints	0.47	liters	l
qt	quarts	0.95	liters	l
gal	gallons	3.8	liters	l
ft^3	cubic feet	0.03	cubic meters	m^3
yd^3	cubic yards	0.76	cubic meters	m^3
		TEMPERATURE (exact)		
°F	Fahrenheit temperature	5/9 (after subtracting 32)	Celsius temperature	°C

*1 in = 2.54 (exactly). For other exact conversions and more detailed tables, see NBS Misc. Publ. 286, Units of Weights and Measures, Price $2.25, SD Catalog No. C13.10:286.



Approximate Conversions from Metric Measures

Symbol	When You Know	Multiply by	To Find	Symbol
		LENGTH		
mm	millimeters	0.04	inches	in
cm	centimeters	0.4	inches	in
m	meters	3.3	feet	ft
m	meters	1.1	yards	yd
km	kilometers	0.6	miles	mi
		AREA		
cm^2	square centimeters	0.16	square inches	in^2
m^2	square meters	1.2	square yards	yd^2
km^2	square kilometers	0.4	square miles	mi^2
ha	hectares (10,000 m^2)	2.5	acres	
		MASS (weight)		
g	grams	0.035	ounces	oz
kg	kilograms	2.2	pounds	lb
t	tonnes (1000 kg)	1.1	short tons	
		VOLUME		
ml	milliliters	0.03	fluid ounces	fl oz
l	liters	2.1	pints	pt
l	liters	1.06	quarts	qt
l	liters	0.26	gallons	gal
m^3	cubic meters	35	cubic feet	ft^3
m^3	cubic meters	1.3	cubic yards	yd^3
		TEMPERATURE (exact)		
°C	Celsius temperature	9/5 (then add 32)	Fahrenheit temperature	°F



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1. INTRODUCTION

In the National Airspace System (NAS), the corrected slant range R(S,H) of a target relative to a radar is determined from the measurement S of slant range and the reported altitude H. The computation of R(S,H) involves the calculations of a square root function using an approximation technique. The present approximation was developed at a time when the only use made of radar surveillance data was for display purposes involving movement of the data block on the planned view display The computational accuracy requirements for this purpose were of little significance. Evolution of the air traffic control system towards a greater level of automation (in providing advanced air traffic control functions such as conflict alert) now places substantially increased requirements on the accuracy of the functions which support the radar tracking algorithm. Unfortunately, the need for increased computational accuracy in support of the advanced automation features has been recognized only recently. It is the objective of this report to provide an approximation technique with a specific predetermined error boundary which will guarantee that the computational errors resulting from the function approximation will have absolutely no measurable impact whatsoever on any aspect of system performance. It is important for whatever technique is employed that the algorithm be adaptable to the accuracy of the present radars as well as those in the immediate future.

If the radar site is located at an altitude H_R above mean sea level, then (reference 1)

$$R(S,H) = (S^2 - [H-H_R]^2)^{1/2} \tag{1}$$

Due to the necessity of performing such computations in real time for numerous targets, an approximation of the square root function is employed in NAS that does not severely tax available computational resources. A previous investigation of this approximation (reference 2) demonstrated that errors as large as 0.3 nautical miles (nmi) can be introduced in the corrected slant range. Also, it is shown there that an adjustment of the parameters used in the current approximation algorithm would eliminate errors in excess of 0.125 nmi, the present quantization of slant range measurements. However, this parameter change will not reduce the computation time for slant range correction. Also, because of the structure of the present FAA algorithm it is not easily adaptable to an increased approximation accuracy specification. This report deals with an alternative to the approximation in current use. It can be adapted to meet any reasonable specification of the maximum allowable approximation error.

Briefly stated, our problem is as follows. The altitude H_R of the radar site is a known constant, S is observed at the radar site, and H is observed at the target and then transmitted to the ground in response to an appropriate interrogation signal from the radar. In addition, there is a well-defined region of admissible values for S and H, that is determined by the measurement capabilities of the radar and the altitudes of targets of interest. The objective is to find a simple estimate $R_o(S,H)$ of R(S,H) such that the absolute value of the difference between the two does not exceed a predetermined error bound ϵ over the admissible region.

The region of admissible values for the measurement vector (S,H) is defined in section 2. Basic elements of the algorithm and machine implementations of the algorithm are described in section 3.

Section 4 describes how to choose the parameters of the algorithm so that the approximation error will never exceed the predetermined error bound ϵ over the admissible values of S and H. The algorithm is illustrated by a numerical example in section 5, and comparisons are made with the estimation procedure currently employed in NAS. Section 6 contains possible modifications of the algorithm and section 7 contains conclusions.

2. CONSTRAINTS ON SLANT RANGE AND ALTITUDE

There exist combinations of reported altitude and slant range measurement that are unacceptable in the determination of corrected slant range. For example, admissible values of S are limited by the maximum effective range S_M of the radar as well as interference phenomena encountered at distances less than some minimum range S_m. In addition, due to limitations on altitudes of targets of interest, H can be assumed to be bounded above by some constant K. Moreover, the ratio of $|H-H_R|$ to S is constrained by the so-called cone of silence of the radar to be less than another constant J. In other words, S and H can be viewed as being restricted under practical operating conditions by the following inequalities:

$$0<S_m \leq S \leq S_M \tag{2}$$

$$|H-H_R| \leq K \tag{3}$$

$$|H-H_R| /S<J<1 \tag{4}$$

In a typical NAS air route traffic control center, radar site altitudes are less than 1.646 nmi (10,000 feet), S_M does not exceed 200 nmi, S_m is at least 2 nmi, J is sin 70°, and controlled traffic consists mainly of aircraft at altitudes less than 9.875 nmi (60,000 feet). Thus,

$$K/S_M <J \tag{5}$$

in the case of NAS, and this inequality will be assumed to be satisfied throughout the remaining discussion.

3. ALGORITHMIC STRUCTURE

The reported altitude H and the measured slant range S are hereafter assumed to be constrained by relations (2) - (4) where H_R, S_m, S_M, K, and J are known constants. Subject to these constraints, our objective is to find a simple approximation $R_o(S,H)$ of R(S,H) such that

$$|R_o(S,H) - R(S,H)| \leq \epsilon \qquad (6)$$

where ϵ is some prescribed error bound, say 0.125 nmi. In what follows, it will be shown how this desideratum can be accomplished by an algorithm based upon a partition of the interval between 0 and J^2 into a finite number n of subintervals by boundary points

$$o=y_o<y_1<\ldots<y_n = J^2 \qquad (7)$$

The algorithm is of the following form: if

$$y_{k-1} \leq (|H-H_R| /S)^2<y_k \qquad (8)$$

where k is any one of the integers 1 through n, then set

$$R_o(S,H)=S \left[A_k + B_k \left(|H-H_R| /S\right)^2\right] \qquad (9)$$

The problem is to find n, the n-1 interior boundary points $y_1,\ldots,y_{n-1}$, and the 2n coefficients $A_1,\ldots,A_n$, $B_1,\ldots,B_n$. As will be seen, these depend upon the error bound ϵ. Moreover, it will be shown how these parameters can be selected so that R_o (S,H) is never less than R(S,H). Thus, if instead of R_o (S,H), the corrected slant range is approximated by $R_o(S,H)-\epsilon/2$, then the approximation error will be $\epsilon/2$ rather than ϵ. In fact, it will be shown that if the latter approximation is used as an initial estimate, then after p applications of the Newton-Raphson iteration (reference 3), the corrected slant range can be found to within an error of $(\epsilon/2)^{2^p}$.

Before proceeding with the mechanics of parameter selection for the approximation algorithm, its implementation will be discussed. First of all, we point out that the ratio of $|H-H_R|$ to S is determined in NAS for the purpose of controlling bias errors in radar measurements. Thus, only the square of the ratio is unique to the algorithm. However, we will show that it is possible to eliminate the squaring operation at the expense of memory. On the other hand, as will be seen shortly, the difference $S^2 -(H-H_R)^2$ is required by the Newton-Raphson iteration. Hence, if the approximation algorithim is to be used to supply that iterative technique with an initial estimate of the corrected slant range, then there is little reason to sacrifice memory to eliminate an operation that can be efficiently employed to determine a necessary element involved in the calculation of the final result.

The second implementation item to consider is the obvious need for some procedure to identify the particular integer k for which the relation (8) is satisfied. This can be accomplished by a linear search. Another possibility is to subdivide the continuum from 0 to 1 into intervals of identical length, equal to some negative power of 2, and employ a hashing technique in which higher order bits of a fixed point representation of $(|H-H_R|/S)^2$ are used to make the appropriate identification. For example, suppose m is any integer for which 2^{-m} does not exceed J^2, and let N(m) represent the smallest integer j for which $j2^{-m}$ exceeds or is equal to J^2. Then the sets defined by

$$U_J = \begin{cases} \{t:(j-1)2^{-m} \leq t < j2^{-m}\} & \text{if } j = 1,\ldots, N(m)-1 \\ \{t:[N(m)-1]\ 2^{-m} \leq t < J^2\} & \text{if } j=N(m) \end{cases} \qquad (10)$$

partition the continuum from 0 to J^2 into N(m) disjoint intervals. Suppose that $(|H-H_R|/S)^2$ is a member of U_j. Then the m highest order bits in a fixed point representation of that ratio constitute the binary representation of j-1, and this, in turn, is an automatic signal to the effect that the ratio is indeed a member of U_j. Now choose m to be sufficiently large that each interval contains no more than one of the boundary points $y_0,\ldots,y_n$; e.g., choose m large enough to satisfy the relation

$$2^{-m} \leq \min_{1 \leq k \leq n} (y_k - y_{k-1}) \qquad (11)$$

In the event that U_j does not contain any one of the interior boundary points $y_1,\ldots,y_{n-1}$, our binary signal is tantamount to an automatic identification of the integer k for which (8) is satisfied. Otherwise, a collision occurs in the sense that U_j contains both the ratio and one of the interior boundary points. In this situation, we are confronted with one of the possibilities; i.e., the offending boundary point is identical to the smallest element $(j-1)2^{-m}$ in the interval, or else it exceeds this number. If it exceeds the minimum element, then it must be compared with the ratio in order to make the correct identification of the integer k for which (8) is satisfied. If it is equal to the minimum element, then the relative positions of the ratio and the boundary point are known, and there is no need for a comparison. Needless to say, the likelihood of a collision decreases as m increases. In this way, computational speed can be increased at the expense of memory.

The hashing technique that we have described will never require a comparison if the minimum element of each interval, but the first is an interior boundary point; i.e., n=N(m) and $y_j = j2^{-m}$ for all j=1,...,n-1. As will be shown later, subject to the condition that the minimum elements of the U_j's meet this requirement, there is a way to pick the smallest number m of intervals such that the approximation error satisfies (6). In other words, without sacrificing an undue amount of memory, we will show how to subdivide the continuum from 0 to J^2 in such a way that collisions do not require comparisons.

4. PARAMETER SELECTION

Turning now to the problem of parameter selection, we find it convenient to work with the independent variables S and

$$x = (|H-H_R|/S)^2 \quad (12)$$

rather than S and H. Obviously, the corrected slant range (1) can be expressed as a function of these variables, namely,

$$\bar{R}(S, x) = S(1-x)^{1/2} \quad (13)$$

Likewise, the estimate (9) provided by the algorithm can be represented as a function $\bar{R}_o(S,x)$ of the same variables. Also, the constraints (2) - (4) can be expressed in terms of equivalent restrictions on S and x. In particular, letting $x^{1/2}$ represent the positive square root of x, (3) implies

$$S = |H-H_R|x^{-1/2} \leq Kx^{-1/2} \quad (14)$$

On the other hand, the right side of (14) exceeds S_M only if x falls below $(K/S_M)^2$. From this we conclude that the constraints (2) - (4) are equivalent to the relations

$$0 \leq x \leq J^2 \quad (15)$$

$$0 < S_m \leq S \leq S_M \qquad \text{if } 0 \leq x < (K/S_M)^2 \quad (16)$$

$$0 < S_m \leq S \leq Kx^{-1/2} \qquad \text{if } (K/S_M)^2 \leq x < J^2 \quad (17)$$

It now remains to show how one determines the estimate $\bar{R}_o(S,x)$ of R(S,x) so that the absolute difference between the two does not exceed ε for all vectors (S,x) satisfying relations (15) - (17).

Suppose β is a nonnegative number less than 1. Then the tangent line to the graph of $(1-x)^{1/2}$ at $x=\beta$ can be represented by the function

$$a(x,\beta) = A(\beta) - B(\beta)\,x \quad (18)$$

where

$$A(\beta)=(1-\beta/2)(1-\beta)^{-1/2} \quad (19)$$

and

$$B(\beta)=(1-\beta)^{-1/2}/2 \quad (20)$$

Since the second derivative of $(1-x)^{1/2}$ is negative, it follows that

$$e(x,\beta)=a(x,\beta) - (1-x)^{1/2} \geq 0 \quad (21)$$

for all x in the continuum from 0 to 1. Hence, if corrected slant range (13) is approximated by $Sa(x,\beta)$, then, after multiplying (21) through by S,

$$Se(x.\beta) = S\ a\ (x,\beta) - R(S,x) \tag{22}$$

is the error, and it is always nonnegative. Moreover, it follows from (16) and (17) that

$$0 \leq Se(x,\beta) \leq E\ (x,\beta) \tag{23}$$

where

$$E(x,\beta) = \begin{cases} S_M e(x,\beta) & \text{if } 0 \leq x < (K/S_M)^2 \\ Ke(x,\beta)x^{-1/2} & \text{if } (K/S_M)^2 \leq x < J^2 \end{cases} \tag{24}$$

In fact, it is clear that $E(x,\beta)$ is the maximum error that can be incurred through use of the approximation $Sa(x,\beta)$ so long as S and x satisfy the constraints (15) - (17).

It can be verified by differentiation that when $E(x,\beta)$ is viewed as a function of x alone, it decreases monotonically as x increases from 0 to β, it vanishes at $x=\beta$, and it increases monotonically as x increases from β to 1. Thus, corresponding to any positive value that one might care to assign to ϵ, there must exist numbers $x_1(\beta)$ and $x_2(\beta)$ such that

$$0 \leq x_1\ (\beta) \leq \beta < x_2\ (\beta) \leq 1 \tag{25}$$

and

$$E(x,\beta) \leq \epsilon \tag{26}$$

so long as

$$X_1(\beta) \leq x < X_2\ (\beta) \tag{27}$$

The situation is illustrated in figure 1. In other words, if the restriction (27) is added to the constraints (15) - (17), then $S\ a(x,\beta)$ approximates the corrected slant range to within an error ϵ. Thus, from figure 1, it appears that by picking several different β's between 0 and 1, we should be able to develop an algorithm along the lines of (7) - (9) that will estimate the slant range correction to within an error ϵ for all possible values of x between 0 and J^2.

The assertion at the end of the preceding paragraph can be established in a rigorous fashion. In particular, let us consider the case where the number n of boundary points is identical to N(m); i.e., the smallest interger by which 2^{-m} can be multiplied to yield a number greater than or equal to J^2, and the interior boundary points are given by

$$y_j = j2^{-m};\ j = 1,\ldots,n-1 \tag{28}$$

It is shown in appendix A that for each $k=1,\ldots,n$ there is one and only one number β_k between 0 and J^2 such that $E(y_{k-1}, \beta_k)$ and $E(y_k,\beta_k)$ are the same. Moreover, it is shown that there exists a positive integer m_o such that

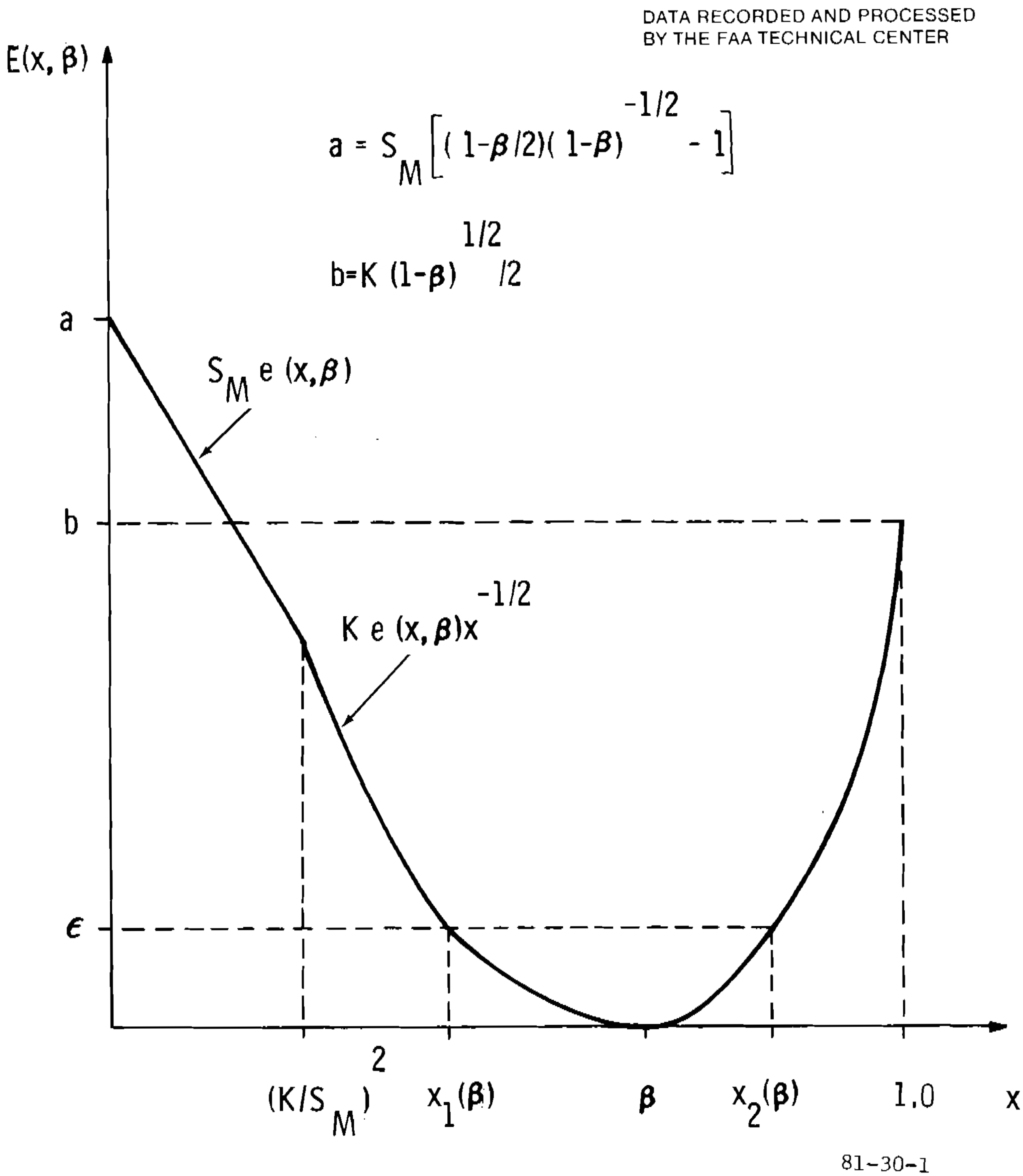

FIGURE 1. ERROR DUE TO THE APPROXIMATION $a(x;\beta)$

$$E\ (y_k\ ,\ \beta_k) \leq \epsilon \text{ for all } k=1,\ldots,n \tag{29}$$

if $m=m_\circ$, and in this case

$$E\ (x,\beta_k) \leq\ \epsilon \text{ whenever } y_{k-1} \leq x < y_k \tag{30}$$

for all $k = 1,\ldots,n$. Now let us choose the coefficients $A_1,\ldots, A_n, B_1,\ldots, B_n$ of the algorithm (7) - (9) in accord with the relations

$$A_k = A\ (\beta_k) \text{ and } B_k = B(\beta_k) \tag{31}$$

where $A(\beta)$ and $B(\beta)$ are defined by (19) and (20). Then, when $(|H-H_R|/S)^2$ is substituted for x, the estimate $R_\circ(S,H)$ provided by the algorithm is just

$$\tilde{R}_\circ\ (S,x)= \sum_{k=1}^{n} Sa(x,\beta_k)\ I(x,\beta_k) \tag{32}$$

where $a(x,\beta)$ is given by (18) and

$$I(x,\beta_k)= \begin{cases} 1 & \text{if } y_{k-1} \leq x < y_k \\ 0 & \text{otherwise} \end{cases} \tag{33}$$

It remains to show that (32) approximates the corrected slant range (13) to within an error ϵ.

Using (21) - (23) and (32), it follows that the difference $\tilde{R}_\circ(S,x) - \bar{R}(s,n)$ between the estimate and the corrected slant range must satisfy the inequality

$$0 \leq \tilde{R}_\circ\ (S,x) - \bar{R}(s,x) \leq E_\circ(x) \tag{34}$$

where

$$E_\circ\ (x) = \sum_{k=1}^{n} E(x,\beta_k)\ I(x,\beta_k) \tag{35}$$

Moreover, since (23) holds for all S and x satisfying (15) - (17), the same is true for (34). Needless to say, if $m=m_\circ$, then (30), (33), and (35) imply that $E_\circ(x)$ cannot exceed the error bound ϵ for all x in the continuum from 0 to J^2. In other words, when $m=m_\circ$, the estimate (32) provided by the algorithm cannot be less than the corrected slant range, nor can it exceed the corrected slant range by an amount greater than ϵ. Thus, the estimate $R_0(S,x) - \epsilon/2$ must approximate the corrected slant range to within an error $\epsilon/2$.

By now the reader should be cognizant of a systematic approach to the design of an algorithm for estimating corrected slant range subject to a prescribed limitation on the estimation error. In particular, starting with m=1, one determines for each $k=1,\ldots,N(m)-1$ the number β_k for which $E([k-1]2^{-m},\beta_k)$ and $E(k2^{-m},\beta_k)$ are equal as well as the number $\beta_{N(m)}$ for which $E([N(m)-1]2^{-m},\beta_{N(m)})$ and $E(J^2,\beta_{N(m)})$ are the same. If $E([k-1]2^{-m},\beta_k)$ does not exceed ϵ for each $k=1,\ldots,N(m)$, then m is identical to m_0. In this case, one sets $n=N(m)$, chooses the coefficients $A_1,\ldots,A_n,B_1,\ldots,B_n$ in accord with (31), and selects the interior boundary point y_k to be $k2^{-m}$ for each $k=1,\ldots,n-1$. Otherwise, m is increased by 1, and the entire procedure is repeated.

5. EXAMPLE

We point out that the integer m_0 is dependent on the error bound ϵ. In fact, as ϵ decreases toward 0, m_0 increases without bound. Thus, the cost in terms of memory of implementing the algorithm becomes prohibitive as ϵ approaches 0. However, this does not mean that the algorithm has no practical value. For example, in current NAS operations, slant range measurements are quantized into units of 1/8 nmi, and altitude is reported in units of 0.0165 nmi (100 feet). Suppose we choose ϵ to be 1/8 nmi. Then, under the assumption that S_m = 2 nmi, S_M = 200 nmi, J = sin 70°, and K = 9.875 nmi in accord with the operation of a typical air route traffic control center, it turns out that $m_0 = 2$ and

$$\beta_1 = 0.0630, \quad \beta_2 = 0.3705, \quad \beta_3 = 0.6294, \quad \beta_4 = 0.8214$$

The coefficients of the algorithm can be determined from (19), (20), and (31). From (34), the minimum error incurred through use of the estimate (32) provided by the algorithm is 0, and the maximum error is $E_0(x)$. As shown in figure 2, $E_0(x)$ never exceeds the prescribed error bound of 1/8 nmi. Hence, by subtracting 1/16 nmi from the estimate supplied by the algorithm, it is possible to determine the corrected slant range to within an error of no more than 1/16 nmi.

A bimodal algorithm is currently employed by NAS to determine corrected slant range. In the case where the ratio $|H-H_R|/S$ is greater than or equal to 6/10, the algorithm computes S^2 and $(H-H_R)^2$, and then, by means of a linear search procedure, it applies a continuous approximation consisting of six straight line segments to the difference. When the ratio is less than 6/10, the algorithm approximates the corrected slant range by the expression $S^2 - (H-H_R)^2/2S$ which is equivalent to the product of S and the tangent to the graph of $(1-x)^{1/2}$ at x = 0. Obviously, both modes of the algorithm require division and squaring operations. Moreover, using the values assigned to S_m, S_M, J, and K in our example, it can be shown that errors in excess of 1/8 nmi are possible within the constraints (2) - (4).

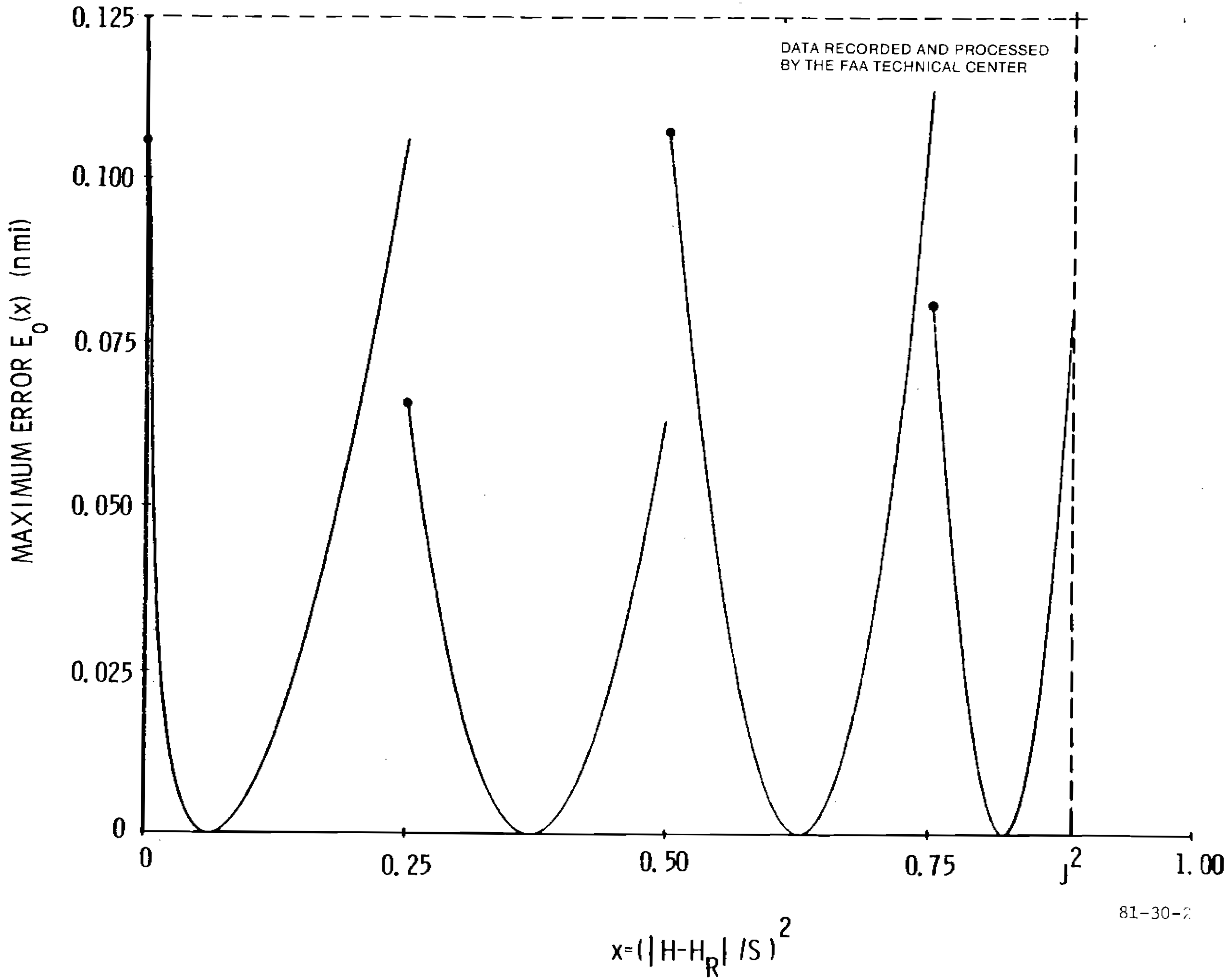

FIGURE 2. ERROR DUE TO THE APPROXIMATION $\bar{R}_o$ (S,x)

Although the bimodal algorithm has met the needs of the air traffic control system for many years, it is evident that the algorithm of our example offers some very definite advantages in terms of accuracy, utilization of memory, and computational speed.

6. MODIFICATIONS

As already indicated, the amount of memory required by machine realizations of the approximation (32) increases indefinitely as the error bound ε decreases toward 0.

To be more specific, in our example, it has been shown that if $\varepsilon = 1/8$ nmi, then a partition consisting of four intervals is sufficient to maintain the approximation error incurred through use of the estimate

$$Z_\circ = \bar{R}_\circ (S,x) - \varepsilon/2 \qquad (36)$$

to within $\varepsilon/2$ nmi. However, if ε is reduced to 1/16 nmi, then the reader can verify that a partition of similar structure must have at least 8 intervals in order to guarantee that the absolute difference between (36) and the corrected slant range (13) will not exceed $\varepsilon/2$ nmi. On the other hand, for a given ε, we can always use (36) as an initial estimate of the corrected slant range. This, in turn, can be used to find a better estimate by means of some iterative technique.

For instance, let us use α to denote the corrected slant range $\bar{R}(S,x)$. This is just the positive solution to the equations obtained by setting w of

$$w = z^2 - \alpha^2 \qquad (37)$$

equal to 0. According to the Newton-Raphson iteration, if z_k is the kth ($K \geq 0$) estimate of the solution, then the (k+1)th estimate is just the intersection of the z-axis with the tangent to the graph of (37) at $z = z_k$, i.e.,

$$z_{k+1} = (z_k + \alpha^2/z_k)\ /2 \qquad (38)$$

In appendix B it is shown that if the minimum range S_m of the radar is at least 2 nmi and J is sin 70°, as is the case in NAS, then, after $p(p \geq 1)$ applications of (38) starting with the initial estimate $z_\circ$ the error satisfies the relation

$$|z_p - \alpha| \leq (\varepsilon/2)^{2^p} \qquad (39)$$

provided that the error bound ε does not exceed 0.368 nmi. In our numerical example, ε is 1/8 nmi, and so the estimate $z_\circ$ obtained by way of the algorithm

(7) - (9) must be within 1/16 nmi of α. Thus, after one application of the Newton-Raphson iteration, we are assured that the absolute value of the difference between z_1 and the corrected slant range will be no more than 1/256 nmi.

Our development of the algorithm (7) - (9) is based upon the idea that one can use a tangent line to approximate a function to within a prescribed error bound ϵ over a small subinterval of the total range of the independent variable. The square of the ratio

$$y = |H-H_R|/S \tag{40}$$

appears in (9) due to the fact that we have selected the function to be $(1-x)^{1/2}$ where, of course, x is equivalent to y^2. We point out that if one chooses the function to be $(1-y^2)^{1/2}$ and proceeds with a development similar to that of the preceding paragraphs, then the result will be an algorithm along the lines of (7) - (9) with the exception that $|H-H_R|/S$ replaces its square in (9) and $y_n = J$ in (7). However, elimination of the squaring operation exacts a price in terms of memory. In particular, the curvature of the graph of the function $(1-y^2)^{1/2}$ is greater than that of $(1-x)^{1/2}$ on the interval from 0 to 1. Consequently, for a given error bound ϵ, one can expect that a greater number of straight line segments will be required to approximate the former function. For instance, if is 1/8 nmi, then, as shown in figure 2, the algorithm (7) - (9) requires a partition of the interval from 0 to J^2 consisting of four subintervals. On the other hand, if the squaring operation is eliminated, then it can be shown that this error bound can be satisfied by a similar subdivision of the interval from 0 to J provided that the number of subintervals is at least 16.

7. CONCLUDING REMARKS

We have introduced a systematic approach to the design of algorithms for estimating corrected slant range, subject to prescribed limitations on the estimation error. In terms of accuracy, utilization of memory, and computational speed, the approach is capable of providing an algorithm that is superior to the method for determining corrected slant range that is currently employed in NAS. It also provides a vehicle for meeting more stringent error requirements in the future that are consistent with the needs of advanced air traffic control functions such as Conflict Alert.

8. REFERENCES

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3. Wilf, H. W., Mathematics for the Physical Sciences, John Wiley, New York, 1962.

APPENDIX A

PROPERTIES OF THE FUNCTION $E(u,\beta)$

Suppose u is a member of the continuum from 0 to J^2, and let us consider $E(u,\beta)$ as a function of β alone. From (18) - (21) and (24), it is clear that the function vanishes at $\beta = u$. Other characteristics of the function can be determined by examining the partial derivative with respect to β. First, it can be shown that the function monotonically decreases as β increases from 0 to u, and thereafter it monotonically increases as β moves from u toward J^2. Moreover, if $u_1 < u_2$, then $E(u_1,\beta)$ is less than $E(u_2,\beta)$ when $\beta < u_1$, and the reverse is true when $\beta > u_2$. As a result, the functions $E(u_1,\beta)$ and $E(u_2,\beta)$ intersect at one and only one point in the continuum from 0 to J^2 and this is a number β^* that is greater than u_1 and less than u_2. The situation is illustrated in figure A-1.

We now point out that when the function $E(u,\beta)$ is viewed as a function of the vector (u,β) on the set of values for which

$$0 \leq u \leq J^2 \text{ and } 0 \leq \beta \leq J^2 \tag{A-1}$$

it is continuous. Moreover, because the set is closed, it is uniformly continuous (see reference 3); i.e., to each $\epsilon > 0$ there corresponds some $\epsilon > 0$ such that

$$|E(v_1,w_1) - E(v_2,w_2)| \leq \epsilon \tag{A-2}$$

whenever the Euclidean distance between (v_1,w_1) and (v_2,w_2) is less than α. Thus, recognizing that ...,. $E(u,\beta)$ vanishes at $\beta = u$, it follows directly from figure A-1 that to any error-bound $\epsilon > 0$ there corresponds a positive integer m_o such that

$$E(u_1,\beta^*) = E(u_2,\beta^*) \tag{A-3}$$

whenever $m \geq m_o$ and

$$0 \leq u_2 - u_1 < 2^{-m} \tag{A-4}$$

Suppose now that u lies between u_1 and u_2 of figure A-1. Then, from the preceding paragraph, we know that

$$E(u,\beta^*) \leq \begin{cases} E(u_1,\beta^*) & \text{if} \quad u \leq \beta^* \\ E(u_2,\beta^*) & \text{if} \quad u > \beta^* \end{cases} \tag{A-5}$$

Consequently, $E(u,\beta^*)$ cannot exceed ϵ whenever $m > m_o$ and u_1 and u_2 satisfy (A-4). The statements made in connection with (29) and (30) follow directly.

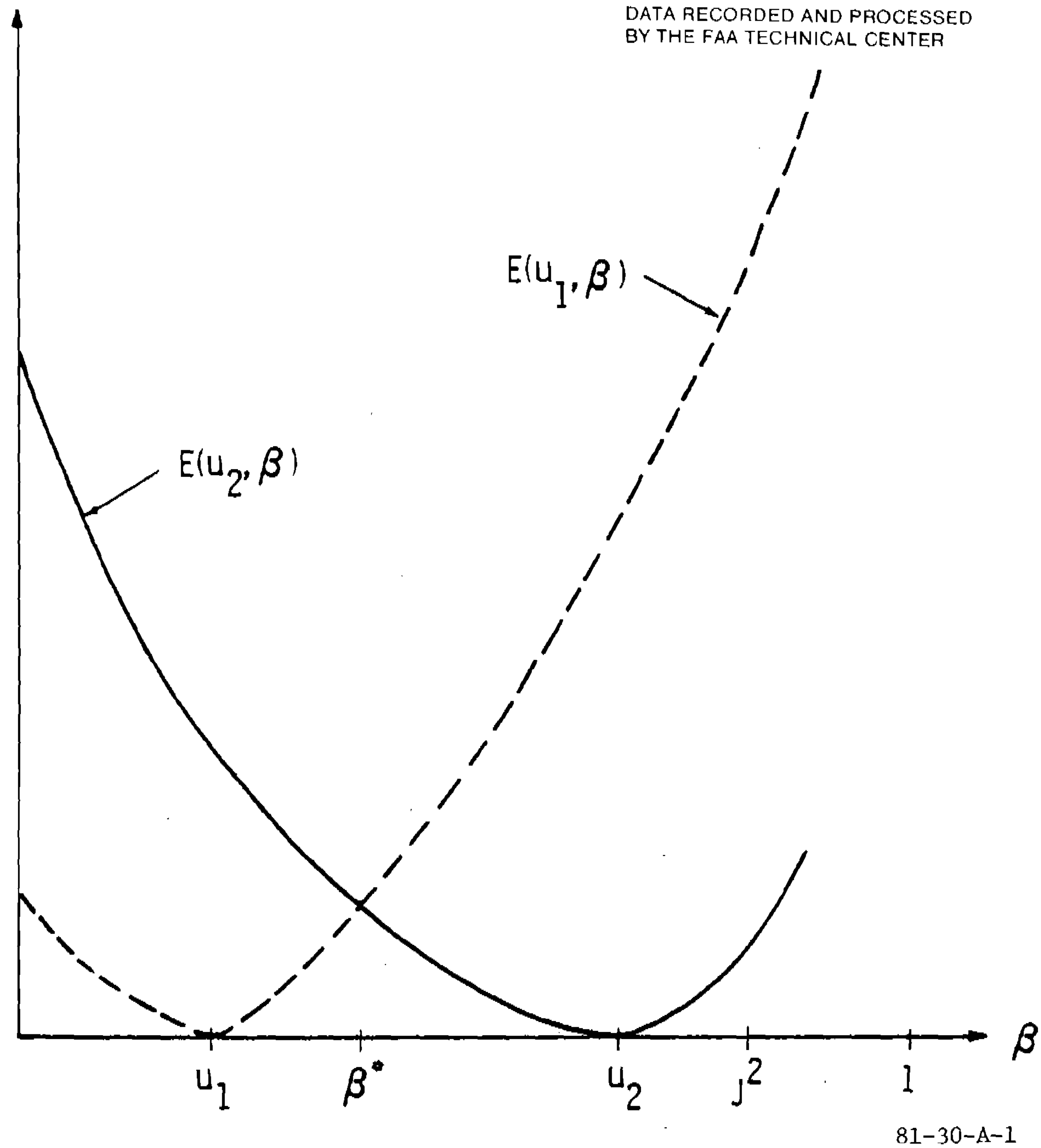

FIGURE A-1. LOCATION OF THE INTERSECTION OF THE FUNCTIONS $E(u_1, \beta)$ AND $E(u_2, \beta)$

APPENDIX B

ERROR BOUND AFTER p NEWTON-RAPHSON ITERATION

Using (38), it can be shown by direct calculation that

$$z_{k+1} = \alpha = (z_k - \alpha)^2 \; / 2z_k \qquad \text{(B-1)}$$

Moreover, since the first and second derivatives of the function (37) with respect to z are positive for all $z>0$, it follows that the tangent to the graph of the function at any point $z_k>0$ must intersect the horizontal or z-axis at a point z_{k+1} greater than or equal to α. In other words, starting with an initial estimate $z_\circ>0$ of corrected slant range, the estimates $z_1, z_2, \ldots$ provided by the Newton-Raphson iteration will all be at least as large as α. Consequently, if α is known to be greater than 1/2 nmi, 2 $z_\circ$ is greater than 1 nmi, and $|z_\circ - \alpha|$ does not exceed $\epsilon/2$ nmi, then (39) follows directly from (B-1) for all integers $p \geq 1$. As will be shown next, all three of these conditions can be met under typical NAS operations.

Relations (13) and (15) - (17) imply

$$\alpha = R(S,x) \geq S_m \, (1-J^2)^{1/2} \qquad \text{(B-2)}$$

Also, since $E_\circ$ (x) does not exceed ϵ for the estimate (32) provided by the algorithm (7) - (9), it follows from (34) and (36) that the absolute value of the difference between $z_\circ$ and α cannot be greater than $\epsilon/2$. As a result, $z_\circ$ will never be less than $\alpha - \epsilon/2$. Thus, from (B-2), we conclude that

$$2z_\circ \geq 2\alpha - \epsilon \geq 2S_m \, (1-J^2)^{1/2} - \epsilon \qquad \text{(B-3)}$$

But the right side of (B-3) is greater than or equal to 1 if

$$\epsilon \leq 2 \, S_m \, (1-J^2)^{1/2} - 1 \qquad \text{(B-4)}$$

Hence, when S_m is 2 nmi and J is sin 70°, (B-2) implies α is greater than 1/2, and, from (B-4), it follows that $2z_\circ$ must exceed 1 if the error bound ϵ is less than 4 cos 70° -1 (0.368).