# A MATHEMATICAL STUDY OF SHEARING STRESSES PRODUCED IN A PAVEMENT BY THE LOCKED WHEELS OF AN AIRPLANE DURING THE WARM-UP OF ITS ENGINES

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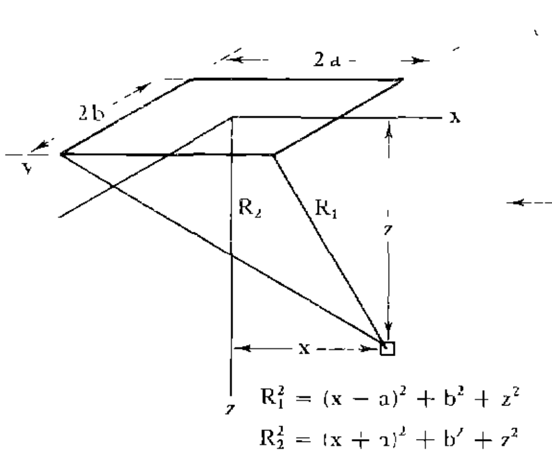
## FOREWORD

The study described in this Note was performed as part of a project on Airport Pavement Performance which is being carried out for the Technical Development Service of the Civil Aeronautics Administration by the Soil Mechanics Laboratory of Princeton University The general scope of this overall program and a partial report on the progress made was presented in a paper "Effect of Vibrations on the Bearing Properties of Soils", Proceedings, Highway Research Board, 1944, pp. 405-425, by G P Tschebotarioff, Director of the Laboratory A final report on this overall program is now in preparation and will include a description of the practical applications of the present theoretical study

CA INFORMATION AND STATISTICS

## CASE 1 STRESSES AT A POINT OF SEMI INFINITE SOLID LOADED OVER A RECTANGULAR AREA WITH A UNIFORM DISTRIBUTION OF PRESSURE NORMAL TO THE SURFACE OF THE SOLID

Formulas are summarized below for the stresses at a point $(x = x, z = z, y = o)$ of a semi-infinite solid loaded over a rectangular area with a uniform distribution of pressure, $p_n$ per unit of area, normal to the surface of the solid The co-ordinates and the dimensions of the loaded area are shown in Figure 1a Positive stresses acting on an element of the solid are shown in Figure 1b.



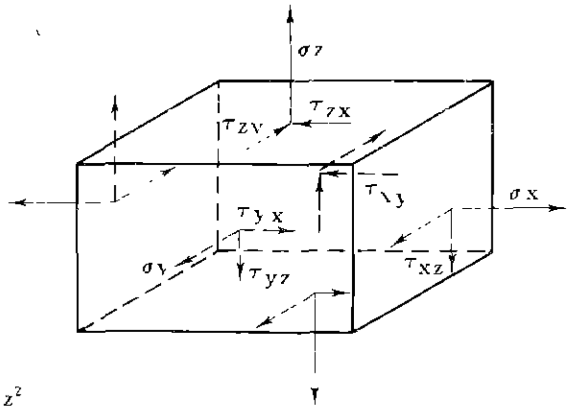




Figure 1 A Semi infinite Solid Loaded Over a Rectangular Area with Dimensions as Shown

The normal stresses, $\sigma$ , and the shear stresses, $\tau$ , at a point$(x=x\ z=z\ ,y=o)$ of the semi-infinite solid may be stated as follows

$$\sigma_{x1} = \frac{p_n}{\pi}\left\{2\mu A_1 - (1 - 2\mu)\,H_1 - zB_1\right\} \quad (1)$$

$$\sigma_{y1} = \frac{p_n}{\pi}\left\{2\mu A_1 - (1 - 2\mu)\,J_1 - zC_1\right\} \quad (2)$$

$$\sigma_{z1} = \frac{p_n}{\pi}\left\{A_1 - zD_1\right\} \quad (3)$$

$$\tau_{yz1} = -\frac{p_n}{\pi}\,zE_1 \quad (4)$$

$$\tau_{zx1} = -\frac{p_n}{\pi}\,zF_1 \quad (5)$$

$$\tau_{xy1} = -\frac{p_n}{\pi}\left\{(1 - 2\mu)\,K_1 + zG_1\right\} \quad (6)$$

At the point $(x = x\ z = z\ y = o)$ the variables $A_1$, $B_1$, $C_1$, $D_1$, $E_1$, $F_1$, $G_1$, $H_1$, $J_1$, and $K_1$ are defined as follows

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$$A_1 = -\left\{\pi - \cos^{-1}\frac{b\,(a-x)}{\sqrt{\{(a-x)^2+z^2\}\{b^2+z^2\}}} - \cos^{-1}\frac{b\,(a+x)}{\sqrt{\{(a+x)^2+z^2\}\{b^2+z^2\}}}\right\} \qquad (7)$$

$$B_1 = -\left\{\frac{(a-x)}{(a-x)^2+z^2}\ \frac{b}{R_1} + \frac{(a+x)}{(a+x)^2+z^2}\ \frac{b}{R_2}\right\} \qquad (8)$$

$$C_1 = -\left\{\frac{b}{b^2+z^2}\ \left(\frac{a-x}{R_1}+\frac{a+x}{R_2}\right)\right\} \qquad (9)$$

$$D_1 = -\,(B_1 + C_1) \qquad (10)$$

$$E_1 = 0 \qquad (11)$$

$$F_1 = \left\{\frac{z}{(a-x)^2+z^2}\ \frac{b}{R_1} - \frac{z}{(a+x)^2+z^2}\ \frac{b}{R_2}\right\} \qquad (12)$$

$$G_1 = 0 \qquad (13)$$

$$H_1 = \left\{\tan^{-1}\frac{b}{a-x} + \tan^{-1}\frac{b}{a+x} - \tan^{-1}\frac{bz}{(a-x)\,R_1} - \tan^{-1}\frac{bz}{(a+x)\,R_2}\right\} \qquad (14)$$

$$J_1 = \left\{\tan^{-1}\frac{a-x}{b} + \tan^{-1}\frac{a+x}{b} - \tan^{-1}\frac{z\,(a-x)}{bR_1} - \tan^{-1}\frac{z\,(a+x)}{bR_2}\right\} \qquad (15)$$

$$K_1 = 0 \qquad (16)$$

In substituting numerical values into equations 7, 14, and 15, it is well to remember the following

(a) In equation 7 all the inverse cosines are taken to have values between 0 and $\pi$.

(b) In equations 14 and 15 all the inverse tangents have values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

(c) Signs of inverse functions in equations 7, 14, and 15 are to be observed as written

An inspection of equations 4, 6, 11, 13, and 16 shows that $\tau_{yz1}$ and $\tau_{xy1}$ are zero at any point $x = x$ $z = z$ $y = o$ for a load symmetrical about the x axis as in Figure 1 Thus, the normal stress $\sigma_{y1}$ at every point $x = x$ $z = z, y = o$, is a principal stress by definition of principal stresses

To determine the other two principal stresses and the maximum shear at every point $x = x$ $z = z, y = o$ the following order of computations is suggested

(1) Compute by use of equations 7-16 the values of $A_1$ $B_1$ $C_1$, $D_1$, $F_1$ $H_1$, $G_1$, $H_1$ $J_1$ and $K_1$ at every point $x = x,$ $z = z,$ and $y = o$

(2) Compute by use of equations 1-6 the stresses $\sigma_{x1}$ $\sigma_{y1}$ $\sigma_{z1}$ and $\tau_{xz1}$ at every point $x = x,$ $z = z,$ $y = o$

(3) Since $\tau_{xy1} = \tau_{yz1} = o$ and $\sigma_{y1}$ is a principal stress at every point $x = x$ $z = z, y = o$ the computations for the other two principal stresses, $\sigma_{xm1}$ and $\sigma_{zm1}$, at each point may be made in the same way as in the case of a plane state of stress where $\sigma_y = o$, $\tau_{xy} = o$, and $\tau_{yz} = o$ Substitute the values of $\sigma_{x1}$, $\sigma_{z1}$, and $\tau_{xz1}$ at each point, (computed

in (2)), in the construction of Mohr's circle, the Mohr-Land dyadic circle, or in equation 17

$$\left.\begin{matrix}\sigma_{xm1} \\ \sigma_{zm1}\end{matrix}\right\} = \frac{\sigma_{x1} + \sigma_{z1}}{2} \pm \sqrt{\left(\frac{\sigma_{x1} - \sigma_{z1}}{2}\right)^2 + \tau_{xz1}{}^2} \tag{17}$$

(4) Compute the maximum shear at each point $x = x$ $z = z, y = o$. The maximum shear at each point is equal to one-half the algebraic difference between the largest principal stress and the smallest principal stress The principal stresses at each point are $\sigma_{y1}$, $\sigma_{xm1}$, $\sigma_{zm1}$

CASE 2 STRESSES AT A POINT OF A SEMI INFINITE SOLID LOADED OVER A RECTANGULAR AREA WITH A UNIFORM DISTRIBUTION OF FORCES TANGENTIAL TO THE SURFACE OF THE SOLID

Formulas are summarized below for the stresses at a point $(x = x, z = z, y' = o)$ of a semi-infinite solid loaded over a rectangular area with a uniform distribution of forces, $p_t$ per unit of area, tangential to the surface of the solid The coordinates and dimensions of the loaded area are shown in Figure 1a The forces, $p_t$ per unit of area, are in the direction of the x axis Positive stresses acting on an element of the solid are shown in Figure 1b

The normal stresses, $\sigma$ , and the shear stresses, $\tau$ , at a point $(x = x, z = z, y = o)$ of the semi-infinite solid may be stated as follows

$$\sigma_{x2} = \frac{p_t}{\pi}\left\{2A_2 + B_2 - (1 - 2\mu) C_2 + (1 - 2\mu) D_2\right\} \tag{18}$$

$$\sigma_{y2} = \frac{p_t}{\pi}\left\{2\mu A_2 - E_2 + (1 - 2\mu) F_2\right\} \tag{19}$$

$$\sigma_{z2} = \frac{p_t}{\pi} G_2 \tag{20}$$

$$\tau_{yz2} = 0 \tag{21}$$

$$\tau_{xz2} = -\frac{p_t}{\pi}\left\{H_2 - J_2\right\} \tag{22}$$

$$\tau_{xy2} = 0 \tag{23}$$

At the point $x = x, z = z, y = o$ the variables $A_2$, $B_2$ $C_2$, $D_2$, $E_2$, $F_2$, $G_2$, $H_2$, and $J_2$ are defined as follows

$$A_2 = \left\{\tan h^{-1}\frac{b}{R_2} - \tan h^{-1}\frac{b}{R_1}\right\} \tag{24}$$

$$B_2 = \left\{\frac{b}{R_2}\ \frac{(x + a)^2}{\{(x + a)^2 + z^2\}} - \frac{b}{R_1}\ \frac{(x - a)^2}{\{(x - a)^2 + z^2\}}\right\} \tag{25}$$

$$C_2 = \left\{\frac{R_2}{b} - \frac{R_1}{b}\right\} \tag{26}$$

$$D_2 = \left\{\frac{x+a}{b}\ \frac{x+a}{(R_2+z)} - \frac{x-a}{b}\ \frac{x-a}{(R_1+z)}\right\} \tag{27}$$

$$E_2 = \left\{\frac{b}{R_2} - \frac{b}{R_1}\right\} \tag{28}$$

$$F_2 = \left\{\frac{b}{R_2+z} - \frac{b}{R_1+z}\right\} \tag{29}$$

$$G_2 = \left\{\frac{b}{R_2}\ \frac{z^2}{\{(x+a)^2+z^2\}} - \frac{b}{R_1}\ \frac{z^2}{\{(x-a)^2+z^2\}}\right\} \tag{30}$$

$$H_2 = \left\{\tan^{-1}\frac{x+a}{z}\ \frac{b}{R_2} - \tan^{-1}\frac{x-a}{z}\ \frac{b}{R_1}\right\} \tag{31}$$

$$J_2 = \left\{\frac{b}{R_2}\ \frac{(x+a)\,z}{\{(x+a)^2+z^2\}} - \frac{b}{R_1}\ \frac{(x-a)\,z}{\{(x-a)^2+z^2\}}\right\} \tag{32}$$

In substituting numerical values into equations 24 and 31, it is wel_ to remember the following.

(a) In equation 31 all the inverse tangents have values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

(b) Signs of inverse functions in equations 24 and 31 are to be observed as written

Any inspection of equations 21 and 23 shows that $\tau_{yz2}$ and $\tau_{xy2}$ are zero at any point $x = x, z = z, y = o$ Thus, the normal stresses $\sigma_{y2}$ at every point $x = x, z = z, y = o$ is a principal stress by definition of principal stresses

### CASE 3 STRESSES AT A POINT OF A SEMI INFINITE SOLID LOADED OVER A RECTANGULAR AREA WITH A UNIFORM DISTRIBUTION OF NORMAL PRESSURES AND TANGENTIAL FORCES

The stresses at a point $(x = x, z = z, y = o)$ of a semi-infinite solid loaded over a rectangular area with a uniform distribution of pressure, $p_n$ per unit of areas, normal to the surface of the solid and also with a uniform distribution of forces, $p_t$ per unit of area, tangential to the surface of the solid are as follows

$$\sigma_{x3} = \sigma_{x1} + c\,\sigma_{x2} \tag{33}$$

$$\sigma_{y3} = \sigma_{y1} + c\,\sigma_{y2} \tag{34}$$

$$\sigma_{z3} = \sigma_{z1} + c\,\sigma_{z2} \tag{35}$$

$$\tau_{yz3} = \tau_{yz1} + c\,\tau_{yz2} = 0 \tag{36}$$

$$\tau_{zx3} = \tau_{zx1} + c\,\tau_{zx2} \tag{37}$$

$$\tau_{xy3} = \tau_{xy1} + c\,\tau_{xy2} = 0 \tag{38}$$

where $p_t = c\ \ p_n$ (39)

The normal stress $\sigma_{y3}$ is a principal stress To determine the other two principal stresses and the maximum shear at every point $x = x$ $z = z$ $y = o$ the following order of computations is suggested

(1) Compute by use of equations 1–16 the stresses $\sigma_{x1}$, $\sigma_{y1}$ $\sigma_{z1}$, and $\tau_{xz1}$ at every point $x = x$ $z = z$ $y = o$

(2) Compute by use of equations 18–32 the stresses $\sigma_{x2}$, $\sigma_{y2}$, $\sigma_{z2}$, and $\tau_{xz2}$ at every point $x = x$ $z = z$ $y = o$

(3) Compute by use of equations 33–39 the stresses $\sigma_{x3}$ $\sigma_{y3}$, $\sigma_{z3}$, and $\tau_{xz3}$ at every point $x = x$, $z = z$ $y = o$

(4) Since $\tau_{xy3} = \tau_{yz3}$ and $\sigma_{y3}$ is a principal stress at every point $x = x$ $z = z$ $y = o$ the computations for the other two principal stresses, $\sigma_{xm3}$ and $\sigma_{zm3}$, at each point may be made in the same way as previously namely,

$$\left.\begin{matrix}\sigma_{xm3}\\ \sigma_{zm3}\end{matrix}\right\} = \frac{\sigma_{x3} + \sigma_{z3}}{2} \pm \sqrt{\left(\frac{\sigma_{x3} - \sigma_{z3}}{2}\right)^2 + \tau_{xz3}^2} \tag{40}$$

(5) Compute the maximum shear at each point $x = x$ $z = z$ $y = o$ The maximum shear at each point is equal to one half the algebraic difference between the largest principal stress and the smallest principal stress The principal stresses at each point are $\sigma_{xm3}$, $\sigma_{zm3}$, and $\sigma_{y3}$

## NUMERICAL EXAMPLES

Case 1 The solid is loaded over a rectangular area with a uniform distribution of pressure, $p_n$ per unit of area, normal to the surface of the solid Figure 2 shows the distribution of maximum shears for different ratios of $\frac{x}{a}$ and $\frac{z}{a}$ The ratio $\frac{a}{b}$ was taken as equal to 1 50 Poisson's ratio, $\mu$, was taken as equal to 0 25

Case 3 The solid is loaded over a rectangular area with a uniform distribution of pressure, $p_n$ per unit of area, normal to the surface of the solid, and also with a uniform distribution of forces, $p_t$ per unit of area, tangential to the surface of the solid Figure 3 shows the distribution of maximum shears for different ratios of $\frac{x}{a}$ and $\frac{z}{a}$ The following values were taken for the ratios $\frac{a}{b}$, $\frac{p_t}{p_n}$, and $\mu$

$$\frac{a}{b} = 1\ 50 \qquad \frac{p_t}{p_n} = \frac{1}{3} \qquad \mu = 0\ 25$$

Figure 4 compares the biggest maximum shears of Case 1 and Case 3 for different ratios of $\frac{z}{a}$

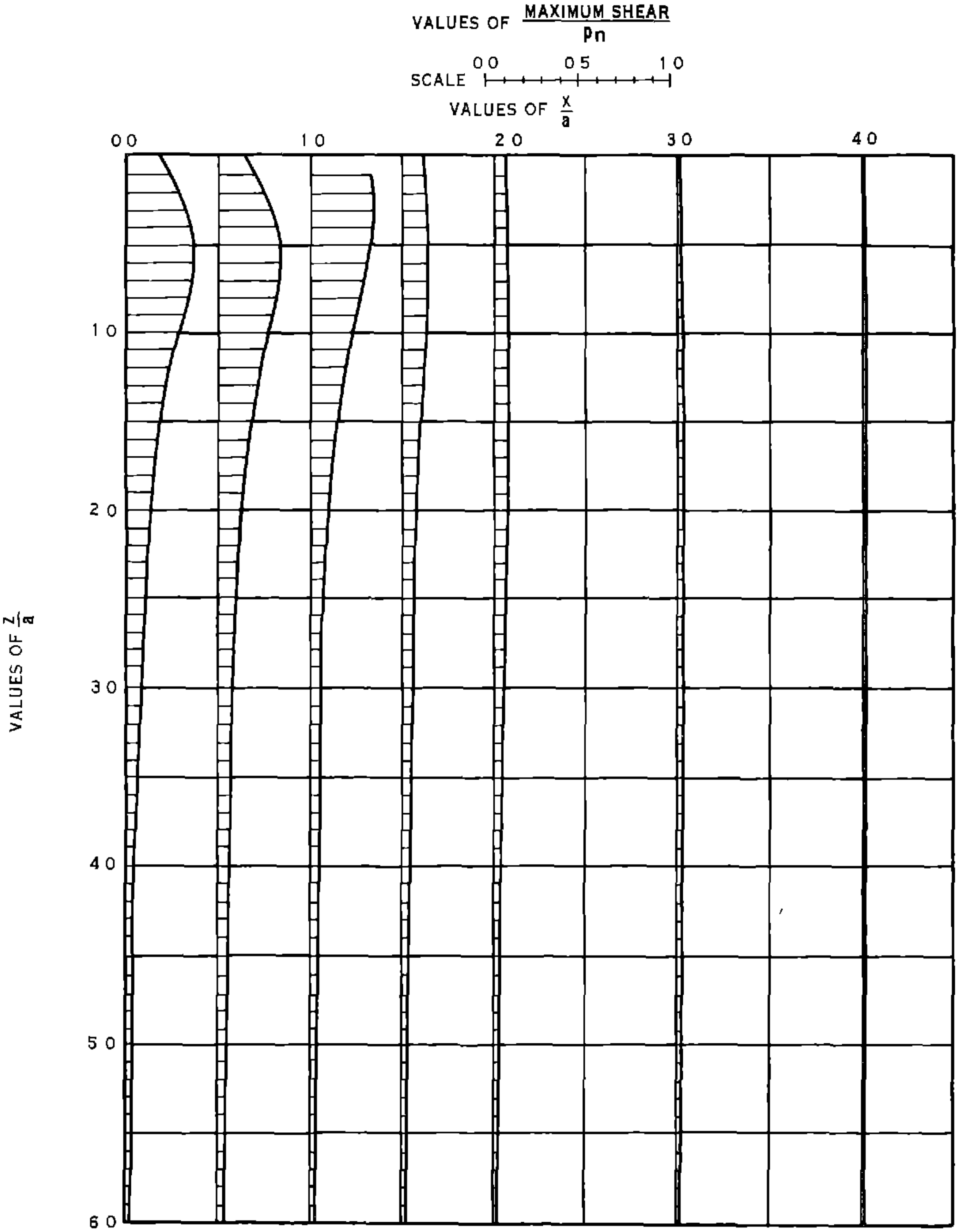



Figure 2 Maximum Shears for Different Values of $\frac{z}{a}$ and Semi infinite Solid Loaded over a Rectangular Area with a Uniform Distribution of Normal Pressures, $p_n$ Per Unit of Area $\frac{a}{b} = 1\ 50$, $\mu = 0\ 25$

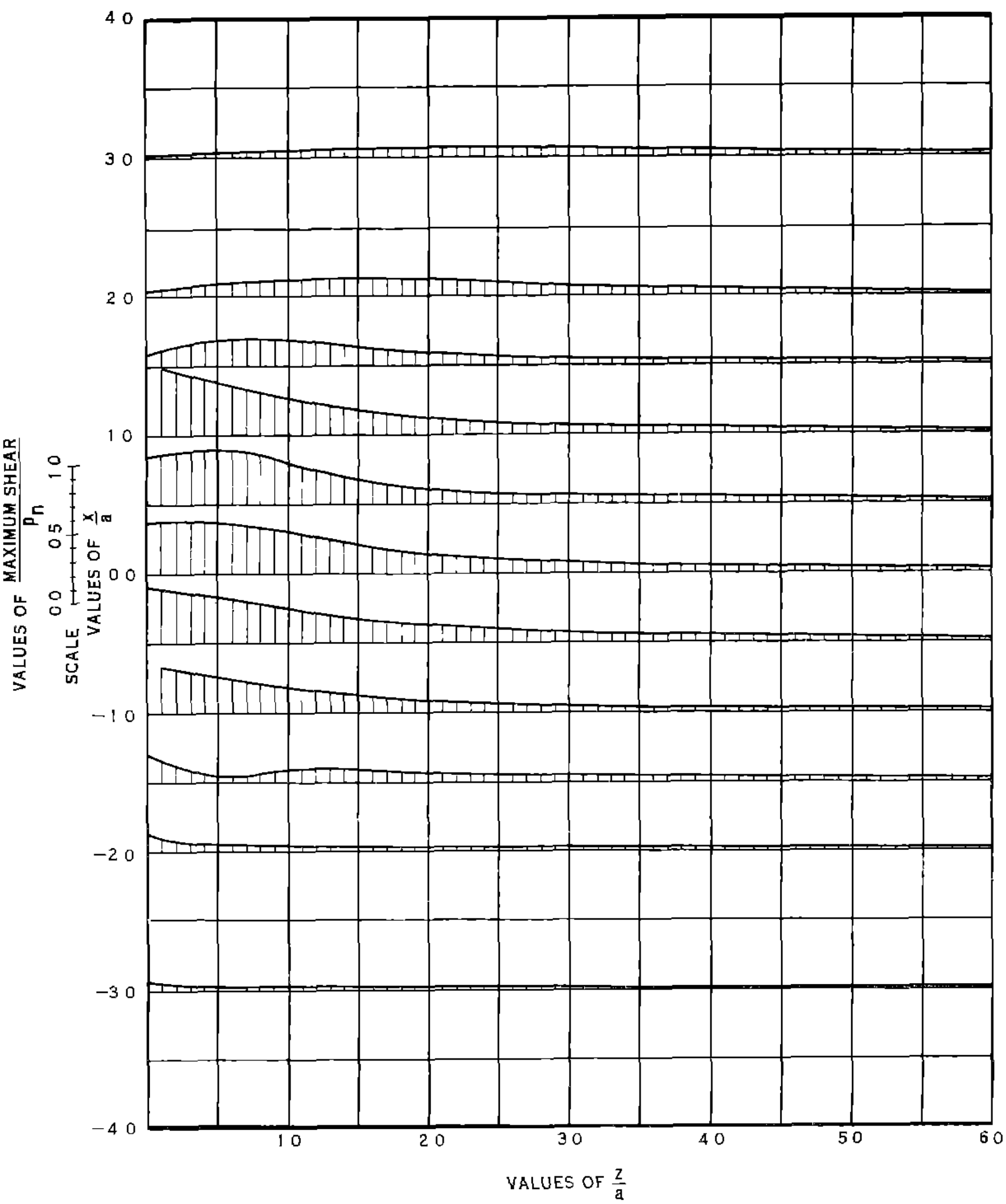
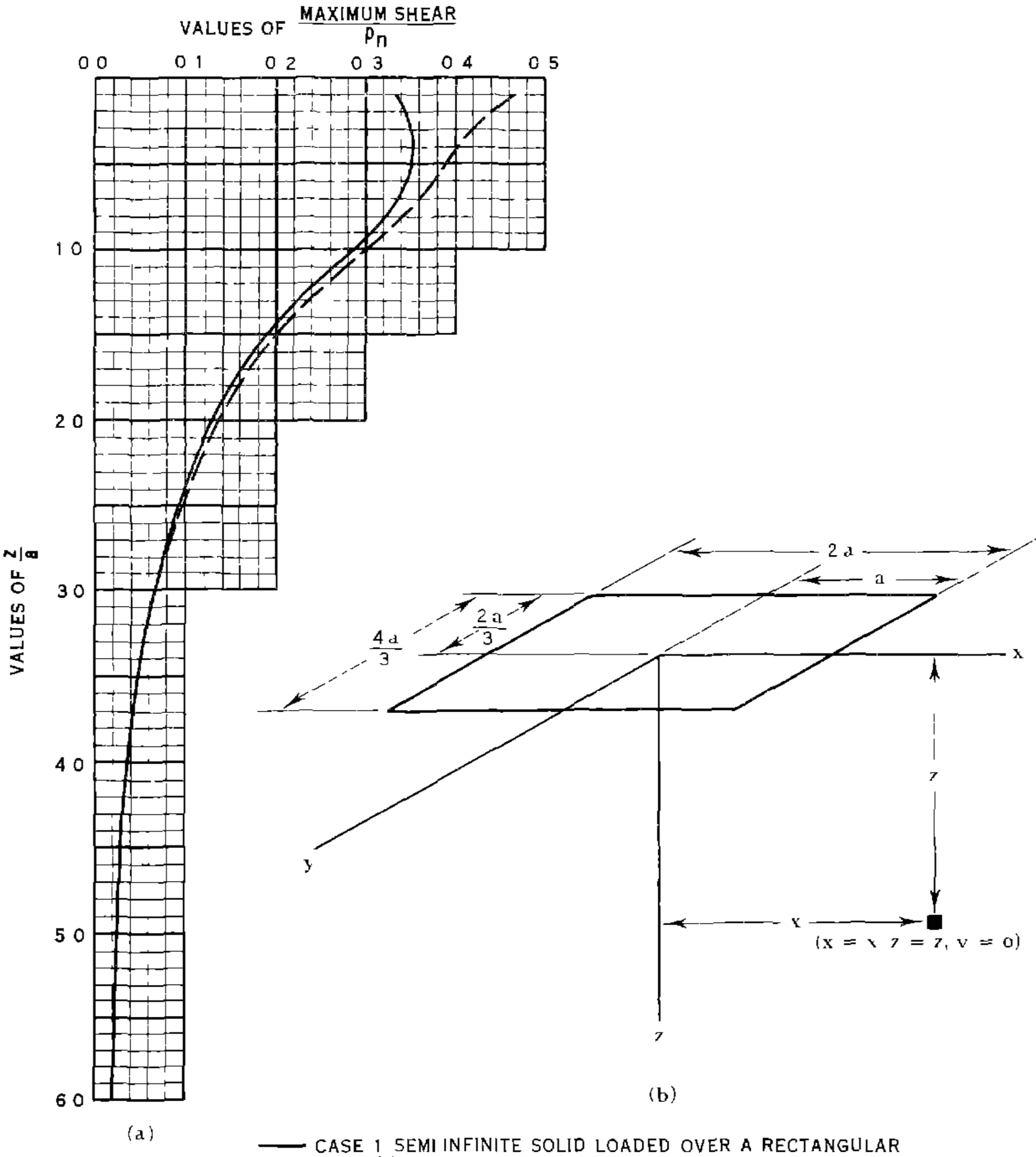



Figure 3 Maximum Shears for Different Values of $\frac{z}{a}$ and $\frac{x}{a}$ Semi infinite Solid Loaded Over a Rectangular Area with a Uniform Distribution of Normal Pressures $p_n$ Per Unit of Area, and with Tangential Forces, $p_t$ Per Unit of Area $\frac{a}{b} = 1\ 50$, $\mu = 0\ 25$, $p_t = \frac{1}{3}\ p_n$




—— CASE 1 SEMI INFINITE SOLID LOADED OVER A RECTANGULAR AREA AS IN (b) WITH A UNIFORM DISTRIBUTION OF NORMAL PRESSURES $p_n$ PER UNIT OF AREA

— — CASE 3 SEMI INFINITE SOLID LOADED AS IN CASE 1 BUT ALSO WITH A UNIFORM DISTRIBUTION OF TANGENTIAL FORCES, $p_t$ PER UNIT OF AREA, IN THE DIRECTION OF THE x AXIS $p_t$ WAS TAKEN AS $\frac{1}{3} p_n$

Figure 4 Comparison of the Biggest Maximum Shears of Case 1 and Case 3 for Different Ratios of $\frac{z}{a}$ Poisson's Ratio was taken as 0 25