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GAIN IN FIELD STRENGTH OF FOUR SYMMETRICALLY DISPOSED ANTENNAS COMPARED TO ONE ANTENNA

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GAIN IN FIELD STRENGTH OF FOUR SYMMETRICALLY DISPOSED ANTENNAS COMPARED TO ONE ANTENNA

SUMMARY

An analysis is made of obtainable rms field strength in the horizontal plane with a given radio-frequency power fed into four identical, short vertical antennas which are located in the corners of a square. This field strength is compared to that produced with the same power fed into a single antenna of the same design. The resistance coupled into each antenna by the other three is computed.

A formula giving the gain in field strength for four antennas over that obtained from one antenna is derived, and families of gain curves are plotted.

The theory presented is in essential agreement with measurements of gain in field strength made at the Pittsburgh, Pa. radio range. A gain in field strength of the order of 1.8 / 1 can be realized at the lower frequencies. The gain increases as the coil and ground resistance of the antenna circuit becomes greater, and it also increases as the antenna spacing is reduced.

INTRODUCTION

Experiments conducted at the Pittsburgh radio range on 200 kc and 400 kc have shown that a considerable gain in field strength can be obtained if a given radio-frequency power is fed into four antennas in phase instead of into a single antenna. The antennas are insulated towers 125 feet high, located at the corners of a square whose diagonals measure 600 feet. For range operation, the diagonal towers are excited 180° out of phase with equal currents; whereas, for voice operation, where an essentially circular horizontal field pattern is desired, the antennas are fed in phase with equal currents. This is the method of operation which was used before the simultaneous radio range was developed.

The purpose of this Note is to check the available measurements theoretically, thereby obtaining relations between the several variables occurring in the problem which enable one to predict quantitatively the gain obtainable with similar four-element antenna arrays, and to determine if these principles can be applied to ultra-high-frequency technique.

The procedure followed in the analysis is to find the rms value of the horizontal field pattern for four antennas with unit current in each antenna, and to determine the antenna current with a given radio-frequency power available. This involves calculation of the resistance coupled into each antenna by the other three antennas, besides the radiation resistance of the reference antenna. The evaluation of coupled resistance is given in the literature for antennas whose lengths are multiples of half wavelengths. Values of coupled resistance for antennas shorter than one quarter-wavelength are obtained by multiplying the coupled resistance for the quarter-wave antenna by the square of the ratio of the effective heights. Sinusoidal current distribution in the antennas is assumed, and the earth's surface is taken as a perfectly conducting plane.

The field strength in the horizontal plane obtained with four antennas is compared to that produced by a single antenna of identical height into which the same total power is fed, and the ratio of the two field strengths is plotted versus one-half the diagonal spacing S, and versus the ratio of coil and ground resistance to radiation resistance (R_L/R_r), respectively.

RMS FIELD STRENGTH OF HORIZONTAL FIELD PATTERN OF FOUR ANTENNAS.

Calling the diagonal spacing (in degrees) of the antennas 2S and the length of the sides of the square $S\sqrt{2}$ (figure 1), the equation for the relative field strength in the horizontal plane with in-phase unit current in each antenna is

$$E = 2\left[\cos\left(S\cos\theta\right) + \cos\left(S\sin\theta\right)\right] \qquad (1)$$

where θ is the azimuth angle measured from the diagonals.

The rms field strength in the horizontal plane is defined as

$$E_{rms}^2 = \frac{1}{2}\int_{-\pi}^{\pi} E^2 \, d\theta \qquad (2)$$

Introducing equation 1 into 2 and integrating gives

$$E_{rms} = 2\left[1 + J_o\left(2S\right) + J_o\left(S\sqrt{2}\right)\right]^{1/2} \qquad (3)$$

where J_o designates the Bessel function of the first kind, of order zero. It will be observed that for very close spacing $J_o(2S) = J_o(S\sqrt{2}) \cong 1$, giving $E_{rms} = 4$. This, of course, is the same result as obtained if we let S = 0 in equation (1), and if 4 units of current are fed into a single antenna. As S is increased the pattern becomes constricted, and for $S = \frac{180^\circ}{\sqrt{2}} = 127.2^\circ$, four directions of zero field strength are obtained at 45° off the diagonals. Field strength in the direction of the diagonals and the bisectors, and rms field strengths, are shown in figure 1, plotted as a function of S.

IMPEDANCE OF EACH OF FOUR PARALLEL VERTICAL DIPOLES.

The impedance Z_1 of one dipole in the presence of three other energized dipoles, all four carrying equal currents in-phase, is given by

$$Z_1 = Z_{11} + 2Z_{12} + Z_{13} \tag{4}$$

where Z_{11} = self-impedance of one dipole with the other dipoles removed.

Z_{12} = mutual impedance between adjacent dipoles.

Z_{13} = mutual impedance between diagonally opposite dipoles.

For the derivation of the self and mutual impedance terms in equation (4), the reader is referred to a paper by P. S. Carter, "Circuit Relations in Radiating Systems and Applications to Antenna Problems," Proc. I.R.E. 20, 1004, June, 1932. For convenience the equations for Z_{11}, Z_{12}, and Z_{13} are reproduced below.

$$Z_{11} = 30\left[\gamma + \log_e 2\beta l - \mathrm{Ci}\,(2\beta l) + j\,\mathrm{Si}\,(2\beta l)\right] \text{ ohms} \tag{5}$$

$\gamma = 0.577$ = Euler's constant

$$\beta = \frac{2\pi}{\lambda}$$

l = length of dipole

$$Ci\ (u) = \int_{\infty}^{u} \frac{\cos x}{x}\, dx = \text{cosine integral}$$

$$Si\ (u) = \int_{0}^{u} \frac{\sin x}{x}\, dx = \text{sine integral}$$

$Z_{11} = 73.2 + j\ 42.5$ ohms

$$Z_{12} \underset{\text{(ohms)}}{=} 30 \left[\begin{array}{l} 2\, Ci\ (S\sqrt{2}) \qquad\qquad - j\ 2\ Si\ (S\sqrt{2}) \\ - Ci\left(\sqrt{2S^2 + \frac{\lambda^2}{4}} + \frac{\lambda}{2}\right) + j\ Si\left(\sqrt{2S^2 + \frac{\lambda^2}{4}} + \frac{\lambda}{2}\right) \\ - Ci\left(\sqrt{2S^2 + \frac{\lambda^2}{4}} - \frac{\lambda}{2}\right) + j\ Si\left(\sqrt{2S^2 + \frac{\lambda^2}{4}} - \frac{\lambda}{2}\right) \end{array} \right] \qquad (6)$$

$$Z_{13} \underset{\text{(ohms)}}{=} 30 \left[\begin{array}{l} 2\, Ci\ (2\ S) \qquad\qquad - j\ 2\ Si\ (2\ S) \\ - Ci\left(\sqrt{4S^2 + \frac{\lambda^2}{4}} + \frac{\lambda}{2}\right) + j\ Si\left(\sqrt{4S^2 + \frac{\lambda^2}{4}} + \frac{\lambda}{2}\right) \\ - Ci\left(\sqrt{4S^2 + \frac{\lambda^2}{4}} - \frac{\lambda}{2}\right) + j\ Si\left(\sqrt{4S^2 + \frac{\lambda^2}{4}} - \frac{\lambda}{2}\right) \end{array} \right] \qquad (7)$$

In the present problem the reactive terms in equations (6) and (7) need not be calculated as the reactance is tuned out.

Referring to figure 2, the resistance looking into a quarter-wave radiator at the current maximum in the presence of three other energized quarter-wave radiators is shown as a function of the antenna spacing.

As the antennas are assumed to be perfect conductors, this input resistance (1/2 R_1) is equal to the radiation resistance of each antenna. The relation of R_1 to the self-resistance and coupled resistance of each antenna is

$$R_1 = R_{11} + 2R_{12} + R_{13}$$

where R_{11} = radiation resistance of a half-wave dipole

R_{12} = coupled resistance between adjacent dipoles

R_{13} = coupled resistance between diagonal dipoles

Self-resistance and coupled resistance for quarter-wave antennas over a perfectly conducting plane are one-half of the corresponding resistances of half-wave antennas.

For comparison, the input resistance of a single quarter-wave radiator is shown (1/2 R_{11} = 36.6 ohms) with the other radiators removed. It will be observed that for small spacings the coupled resistance is positive and increases the radiation resistance. For S = 0

$$R_1 = 4\ R_{11}$$

For S = 95°; 227°; 393° the coupled resistance is zero, giving $R_1 = R_{11}$. Between 95° and 227° the coupled resistance is negative. The minimum total resistance is found at S = 135° and is 8.2 ohms.

EXTENSION OF ANALYSIS TO RADIATORS SHORTER THAN ONE-QUARTER WAVELENGTH.

The radiation resistance R_r of a thin vertical wire of effective height h whose lower end is near the surface of a perfectly conducting horizontal plane is given by the equation

$$R_r = 1579\left(\frac{h}{\lambda}\right)^2 \text{ ohms} \qquad (8)$$

$$= 40 \tan^2 \left(180^\circ \frac{H}{\lambda}\right)$$

This expression is accurate for antennas much shorter than one-quarter wavelength. For antennas one-quarter wave long the above formula gives a value of radiation resistance 10 percent too high. The effective height h is

$$h = \frac{\lambda}{2\pi} \tan \pi \frac{H}{\lambda} \qquad (9)$$

where H = physical height of antenna.

The coupled resistance is also assumed to vary as the square of the effective height of the antennas. This may be understood by the following reasoning

$$P_r = I_1^2 R_r = K_1 I_1^2 h^2 = K_2 E_2^2$$

$$E_2 = K_3 h$$

where P_r = radiated power of a single antenna

I_1 = rms value of maximum current in one antenna

K_1 K_2 K_3 K_4 = constants

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E_2 = axial field strength at radiator 2 due to current in radiator 1

The total induced voltage is $V_2 = h\ E_2 = K_3\ h^2$

$$V_2/I_1 = Z_{12} = K_4\ h^2$$

While it is realized that the values of coupled resistance computed on this basis are not as accurate as desirable, it is believed that they hold within a few percent.

In order to check the above procedure of calculating coupled resistance for short antennas, the following approximate formula was derived:

$$\frac{R_c}{R_r} = 2\left[J_o^2\ (S/\sqrt{2}) - J_1^2\ (S/\sqrt{2})\right] + J_o^2\ (S) - J_1^2(S) \tag{10}$$

where J_o, J_1 = Bessel functions of first kind of order 0 and 1.

R_c = resistance coupled into each antenna by the other three.

R_r = radiation resistance of a single antenna with the others removed.

The equation is based on four vertical doublet antennas located in the corners of a square on a perfectly conducting horizontal plane. The antennas are assumed to have equal currents in phase, the currents increasing linearly from zero at the top to the maximum value at the ground plane, and the vertical field pattern of each antenna is assumed to vary with the cosine of the angle of elevation.

The table below gives a comparison of values of R_c/R_r calculated by cosine integrals and Bessel functions, respectively.

Table of R_c/R_r

f(kc)	S^o	R_c/R_r Calculated by Cosine Integrals	R_c/R_r Calculated by Bessel Functions
200	22	2.76	2.78
300	33	2.48	2.527
400	44	2.16	2.20

It will be observed that for small spacings the values of R_c/R_r agree within 2 percent; for greater spacing, they differ more. For $S = 360^o$ the difference is about 10 percent. It requires considerably less time to compute R_c/R_r by equation (10) than using equations (6) and (7).

It can be readily shown that, for a given frequency and input power, the field strength produced is nearly independent of the antenna height, provided the antenna height is smaller than one-quarter wavelength, and coil and ground resistance are negligible, and impedance matching is perfect. In practice these conditions cannot be realized.

GAIN IN FIELD STRENGTH DUE TO FOUR RADIATORS OVER THAT WITH ONE RADIATOR FOR CONSTANT TOTAL INPUT POWER.

For a single antenna the field strength E_1 produced with an input power P_1 is

$$E_1 = K \sqrt{\frac{P_1}{R_r + R_L}} \qquad (11)$$

For four antennas the rms field strength E_4 is

$$E_4 = K \frac{E_{rms}}{2} \sqrt{\frac{P_1}{R_r + R_L + R_c}} \tag{12}$$

where R_c and R_r are as defined for equation (10) and

R_L = coil and ground resistance for a single antenna

K = constant.

E_{rms} = rms value of field strength for unit current in each antenna (by equation 3).

The gain will then be the ratio of equations (11) and (12).

$$\frac{E_4}{E_1} = \frac{E_{rms}}{2} \sqrt{\frac{R_r + R_L}{R_r + R_L + R_c}} \tag{13}$$

$$= \frac{E_{rms}}{2} \sqrt{\frac{1 + \eta}{1 + \eta + \frac{R_c}{R_r}}} \tag{14}$$

where $\eta = \frac{R_L}{R_r}$

Under the above assumption that the coupled resistance R_c and radiation resistance R_r both are proportional to h^2, a plot of E_4/E_1 versus η holds for any antenna heights equal or smaller than one-quarter wave. It must be remembered, however, that η varies with h in any practical case.

Equation (14) has been used to calculate the curves of figure 3 showing the gain in field strength (E_4/E_1) versus $\eta = \frac{R_L}{R_r}$ with one-half diagonal spacing S as parameter.

It is observed that for close spacings and a ratio R_L/R_r of approximately five or over, a considerable gain in field strength can be obtained if the r-f power is fed into four antennas instead of a single antenna. The power gain is proportional to the square of the field strength gain, and can be made of the order of 3·1.

In figure 4 the gain in field strength calculated by equation (14), is shown as a function of one-half diagonal antenna spacing for a number of values of η. For values of S greater than about 80° little or no gain can be obtained for any value of η. For a spacing $S = 95^\circ$ all the curves intersect. This is because at this spacing the coupled resistance is zero, and the ratio E_4/E_1 equals $1/2\ E_{rms}$. It is apparent that the greatest gain in field strength is obtained with highest coil and ground resistance and for smallest spacings. Spacings S in excess of approximately 45° should not be chosen if maximum gain in field strength is desired. These conditions can readily be fulfilled in the present low-frequency radio ranges, as will be brought out below.

In order to check the deviation from the desired circular field pattern, calculated field patterns (equation 1) are shown in the table below for several spacings. Due to symmetry the field patterns need only be calculated between 0° and 45°.

Relative Horizontal Field Strength

°	S = 44°	S = 85°	S = 135°	S = 270°
0	3.438	2.174	0.586	2.0
10	3.438	2.15	0.570	1.229
20	3.434	2.10	0.190	-0.648
30	3.428	2.038	-0.139	-2.589
40	3.426	2.00	-0.351	-3.77
45	3.426	1.994	-0.376	-3.926
26.0			0	
16.9				0

MEASUREMENTS

In October, 1934, measurements of field strengths obtained by energizing one and four antennas, respectively, were made by Mr. R. P. Battle at the Pittsburgh, Pennsylvania, radio range. Measurements were made at 200 kc and 400 kc at a location 0.82 miles north of the range station, in the direction 3° off the bisector of the diagonals. The r-f resistance (radiation, coil and ground resistance) of individual towers was measured with the remaining three towers grounded. Thus the effect of coupled resistance was not present. The measured resistances for the four towers at 400 kc were:

R_{SE} = 6.45 ohms R_{NE} = 6.6 ohms R_{NW} = 5.95 ohms R_{SW} = 7.2 ohms

With the southeast tower energized alone (the other towers grounded) r-f current was 7.0 amperes, giving a field strength of 43,500 μV/m.

Energizing all four towers simultaneously the currents were:

I_{SE} = 2.0 amperes I_{NE} = 2.0 amperes I_{NW} = 1.8 amperes I_{SW} = 1.2 amperes giving a field strength of 41,250 μV/m.

For diagonal tower spacing 2S = 600 feet, we obtain, at 400 kc,

$$S = 44^\circ$$

The height H of the towers (125 feet) gives a radiation resistance of 1.04 ohm. The curve of R_r versus H/λ shown in figure 5 is a plot of equation (8) and may be used for finding R_r.

In order to find the total input power with four towers energized, the individual measured currents squared have to be multipled by the sum of their measured and coupled resistances. The coupled resistance R_c is found by multiplying the ratio (R_c/R_r) (as calculated for quarter-wave radiators) by the radiation resistance of one 125-foot tower. To save the work of evaluating the real parts of equations (6) and (7), a curve of (R_c/R_r) versus S was drawn through a few calculated points (figure 6). From the curve we find for $S = 44^\circ$

$$\frac{R_c}{R_r} = 2.21$$

The coupled resistance for each 125-foot tower is

$$R_c = 2.21 \times 1.04 = 2.3 \text{ ohms}$$

under the assumption of equal currents in all the towers.

The total input power into the four antennas is

$$
\begin{aligned}
& 2.0^2\ (6.45 + 2.3) = 35.0 \text{ watts} \\
+\ & 1.8^2\ (5.95 + 2.3) = 26.75 \text{ watts} \\
+\ & 2.0^2\ (6.6 + 2.3) = 35.6 \text{ watts} \\
+\ & 1.2^2\ (7.2 + 2.3) = \underline{13.68} \text{ watts} \\
& \qquad\qquad\qquad\quad 111.03 \text{ watts}
\end{aligned}
$$

The field strength for this condition was measured and found to be 41,250 μV/m at 0.82 miles.

The power input with the southeast tower energized alone is

$$7.0^2 \times 6.45 = 316 \text{ watts}$$

The field strength for this condition was measured and found to be 43,500 μV/m at 0.82 miles.

Adjusting the field strength with four towers energized to the same input power as with one tower energized gives

$$41{,}250 \sqrt{\frac{316}{111}} = 69{,}600 \ \mu\text{V/m}$$

$$\underline{\text{Measured gain}} = \frac{69{,}600}{43{,}500} = \underline{1.60}$$

The calculated gain is found by equation (14). For S = 44°

$$E_{rms} = 3.42$$

$$\eta = \frac{6.45 + 6.6 + 5.95 + 7.2 - 4 \times 1.04}{4 \times 1.04}$$

$$\eta = 5.3$$

$$R_c/R_r = 2.21$$

$$\frac{E_4}{E_1} = \frac{3.42}{2} \sqrt{\frac{1 + 5.3}{1 + 5.3 + 2.21}}$$

<u>Calculated gain = 1.47</u>

The same measurements were repeated at 200 kc.

$S = 22^\circ$ $R_r = 0.25$ ohm $\frac{R_c}{R_r} = 2.8$

$R_{SE} = 5.63$ ohms $\quad I_{SE} = 2.2$ amps.

$R_{NE} = 6.07$ ohms $\quad I_{NE} = 2.1$ amps.

$R_{NW} = 5.97$ ohms $\quad I_{NW} = 2.2$ amps.

$R_{SW} = 5.7$ ohms $\quad I_{SW} = 2.0$ amps.

One tower energized:

$I_{SE} = 7.05$ amps. yielding $E = 21,150\ \mu V/m$

$P_i = 280$ watts

Four towers energized:

$P_i = 118.3$ watts yielding $E = 28,500\ \mu V/m$

Adjusting the field strength with four towers energized to same input power as with one tower energized gives

$$\sqrt{\frac{280}{118.3}} \times 28,500 = 43,750\ \mu V/m$$

$$\underline{\text{Measured gain}} = \frac{43,750}{21,150} = \underline{2.07}$$

The calculated gain is obtained as follows:

$$E_{rms} = 3.85$$

$$\eta = \frac{R_L}{R_r} = 22.37$$

$$\frac{E_4}{E_1} = \frac{3.85}{2}\sqrt{\frac{23.37}{23.37 + 2.8}} = 1.82$$

<u>Calculated gain = 1.82</u>

It is observed that at 200 kc the measured and calculated gains differ by 12 percent.

At 400 kc the agreement is better, the difference being about 8 percent. In view of the several assumptions made in the analysis, the above agreements between measured and calculated gain are considered satisfactory.

CONCLUSIONS

With four vertical antennas shorter than one-quarter wavelength, located in the corners of a square, a considerable gain in field strength can be obtained over that produced by a similar single antenna into which the same total r-f power is fed.

The gain obtainable is higher the larger the loss resistance (coil and ground resistance) and the smaller the spacing. No gain will be realized when the loss resistance is zero.

The spacing between diagonal towers (2S) must be smaller than approximately 170° to realize any gain and to keep a substantially circular horizontal field pattern.

From the above, it follows that with ultra-high frequencies no gain is realized by feeding a given power in-phase into several antennas located symmetrically on a circle, over that of a single antenna. This is because the radiators are one-half or one-quarter wavelength long, requiring no loss resistance for tuning and grounding.

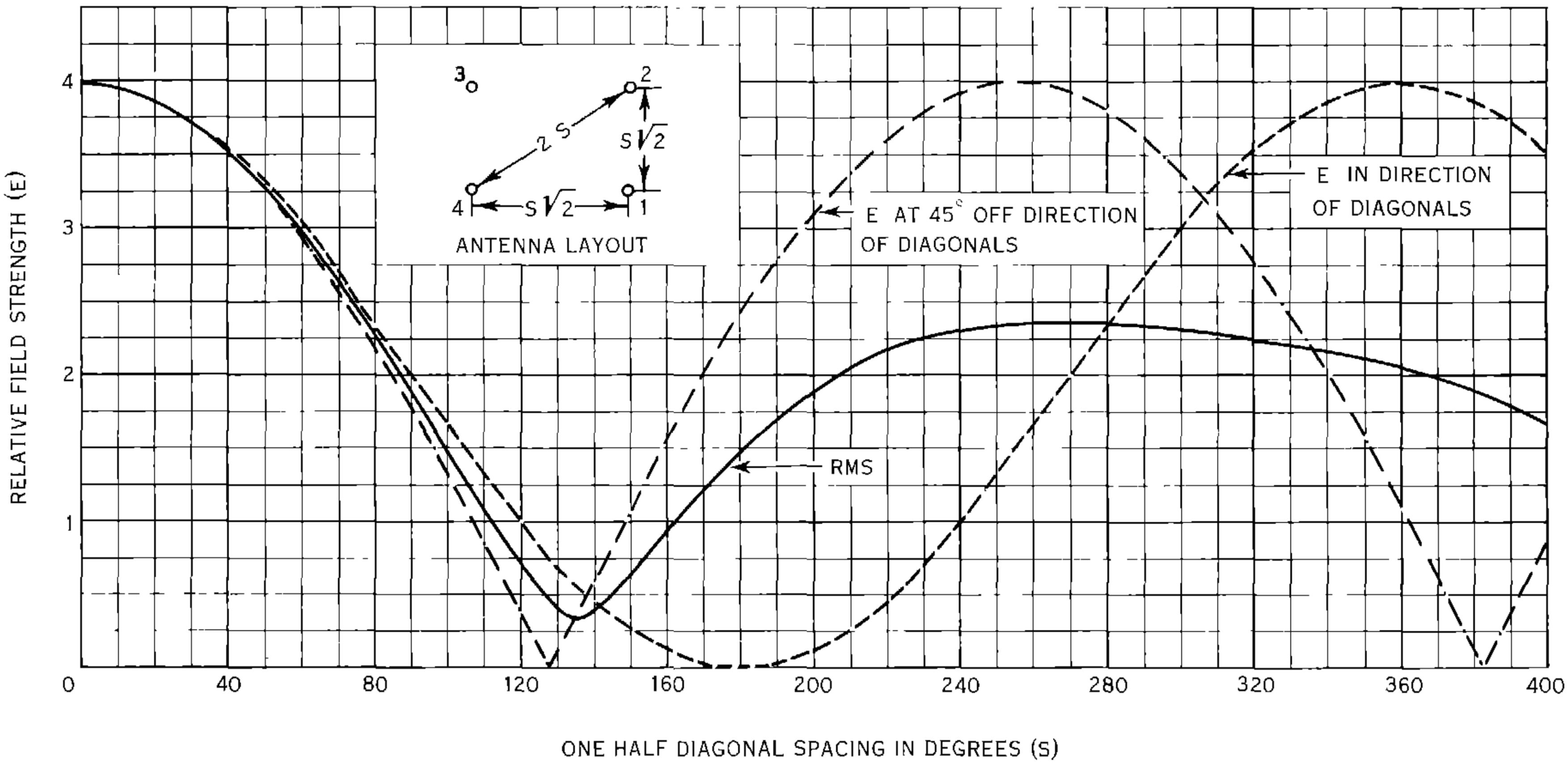

FIGURE 1. Horizontal Field Strength Versus Antenna Spacing. Antenna Currents Equal and in Phase RMS Value of Field Strength Taken in all Directions of Horizontal Plane

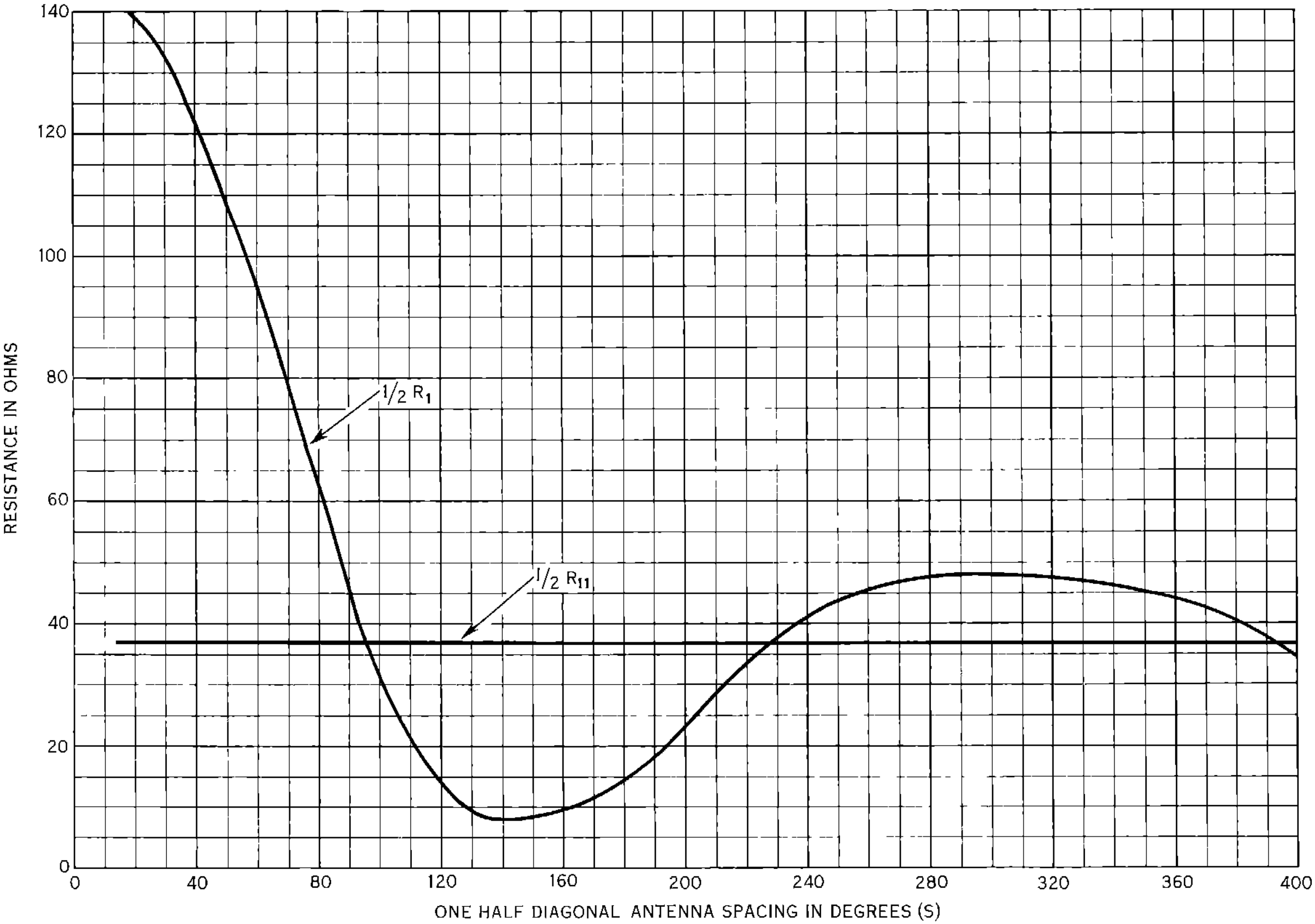

FIGURE 2. Input Resistance ($\frac{1}{2}R_1$) of Each of Four One-Quarter-Wave Antennas Carrying Equal Currents in Phase, Versus S
R = Radiation Resistance of a Half-Wave Dipole.

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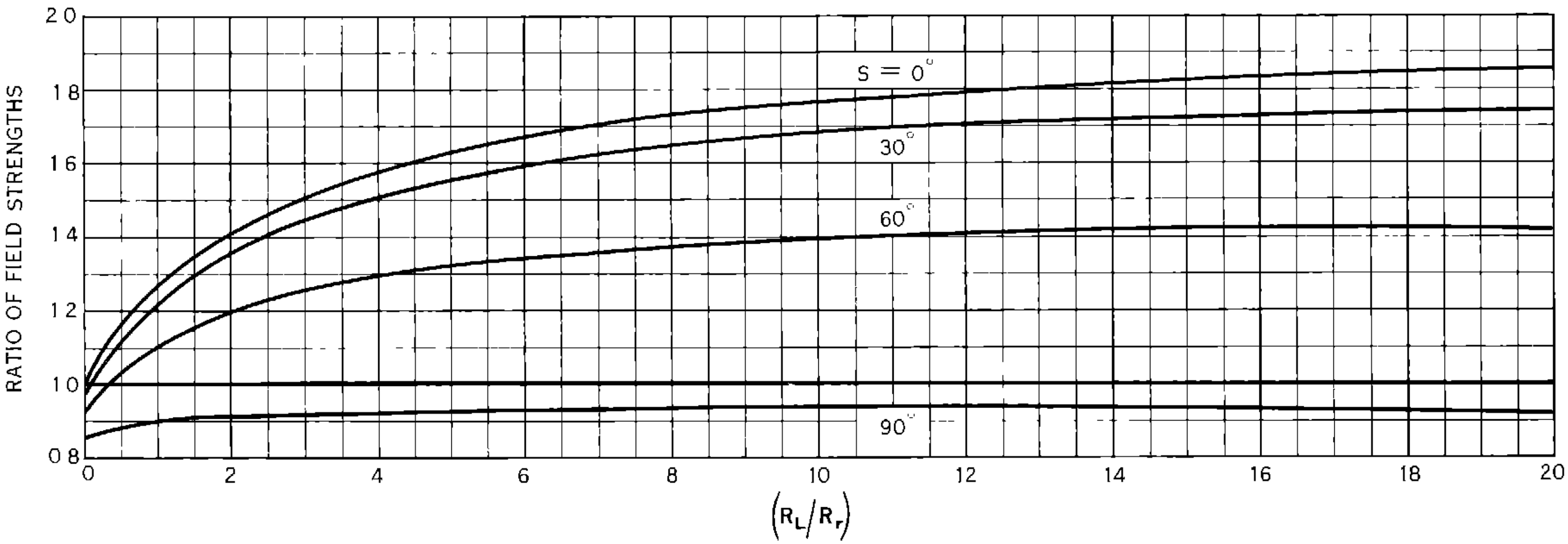

FIGURE 3 Calculated Gain in Field Strength Obtainable by Feeding a Given Power Into Four Antennas Instead of a Single Antenna Versus R_L/R_r, With S as Parameter

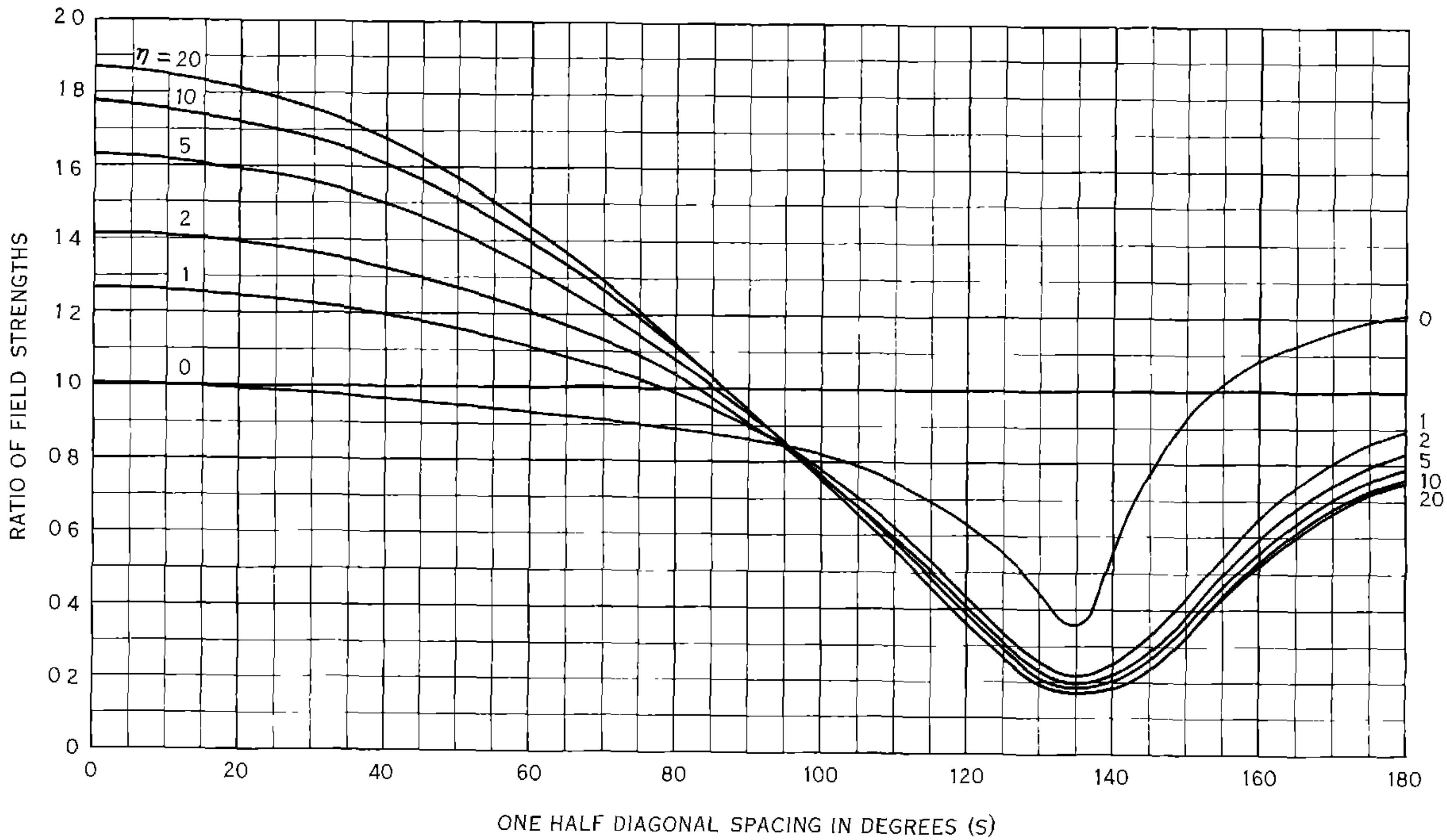

FIGURE 4. Calculated Gain in Field Strength Obtainable by Feeding a Given Power Into Four Antennas Instead of a Single Antenna Versus S , With $\eta = \frac{R_L}{R_r}$ as Parameter

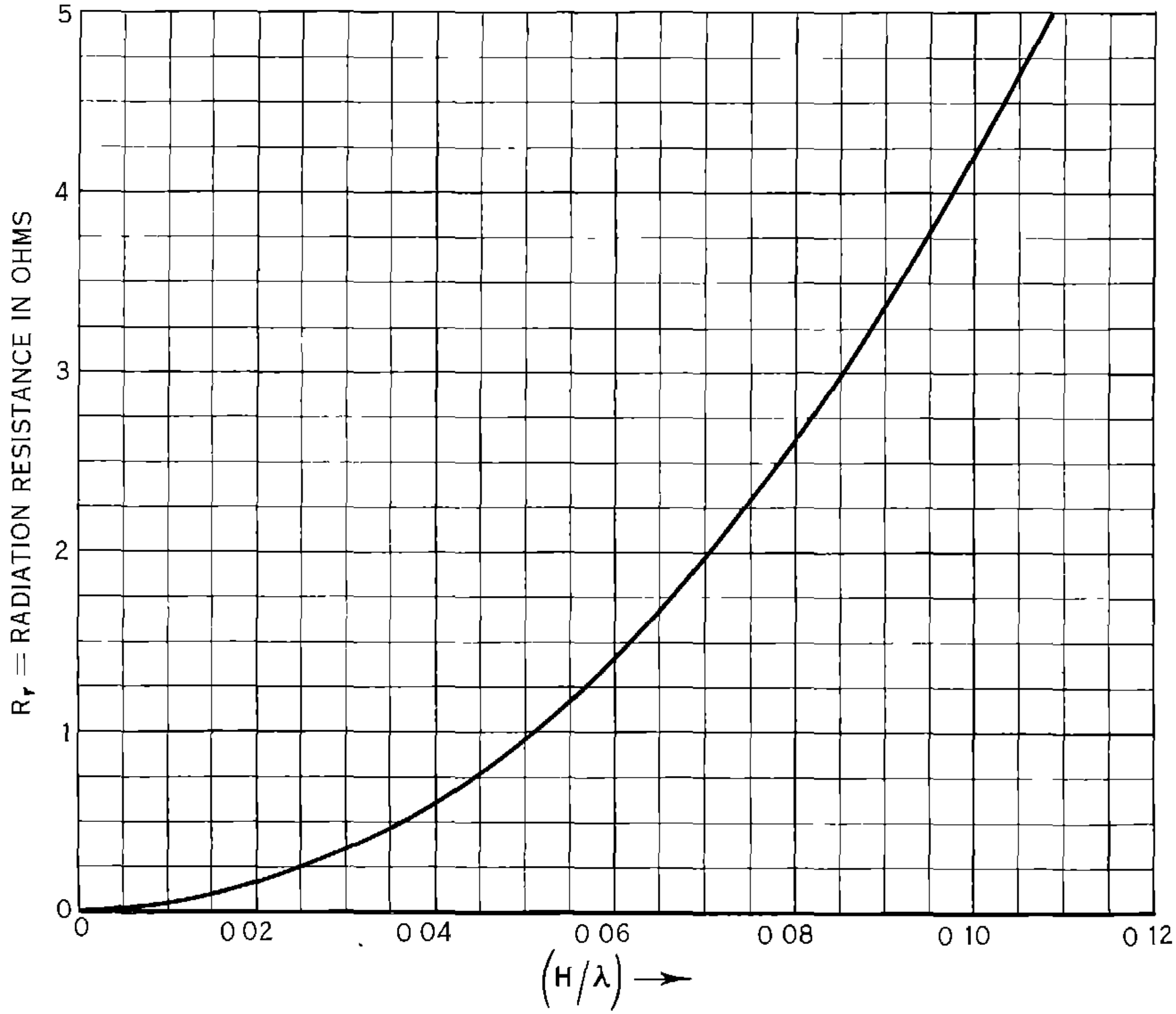

FIGURE 5. Radiation Resistance of Thin Vertical Wire, the Lower End of Which is Near a Perfectly Conducting Horizontal Plane, Plotted Versus Ratio of Physical Height to Wavelength.

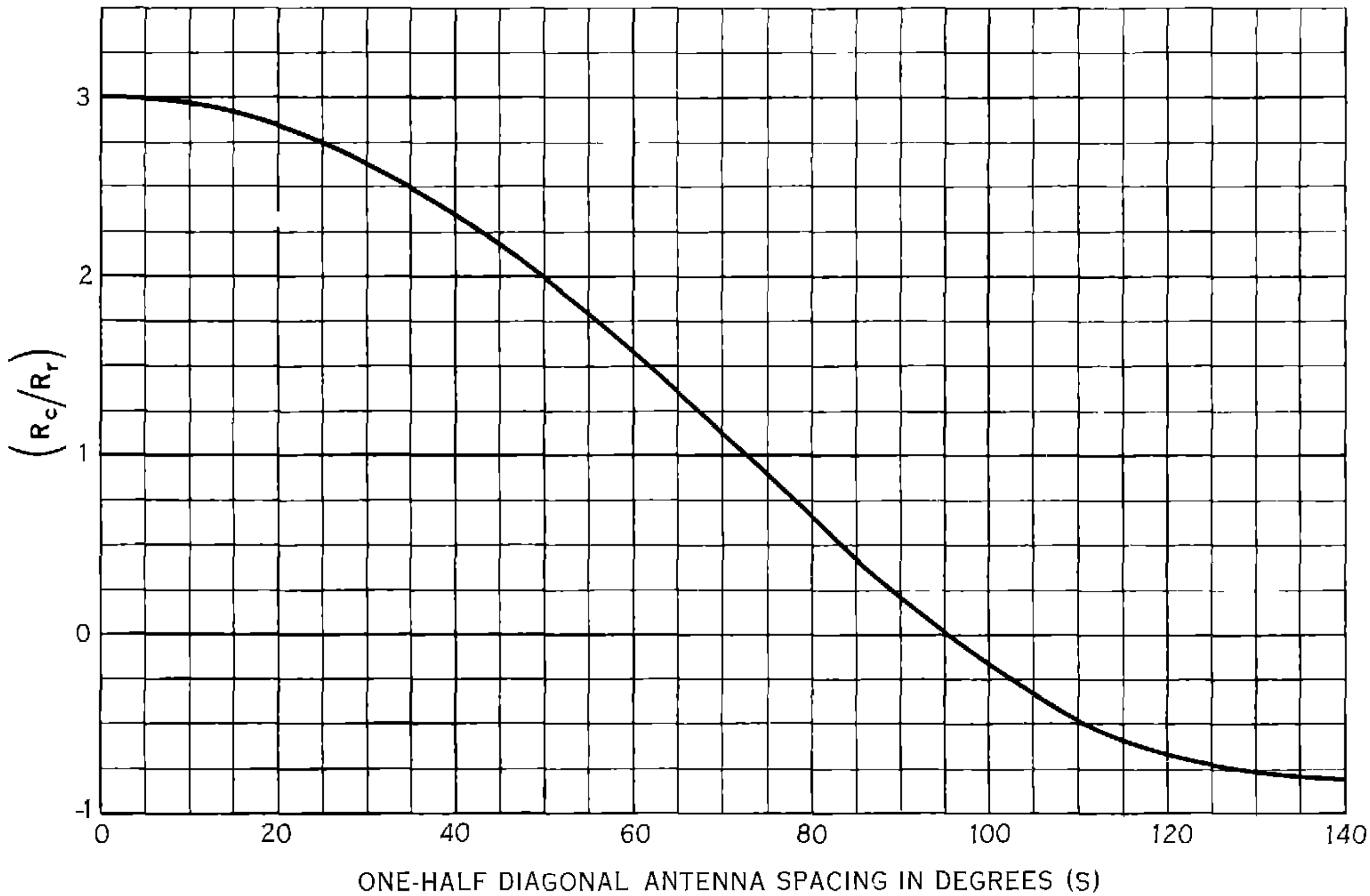

FIGURE 6. Ratio of Coupled Resistance R_c to Radiation Resistance R_r, Versus S.