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# HANDBOOK OF APPLICATIONS OF STATISTICAL CONCEPTS TO THE HIGHWAY CONSTRUCTION INDUSTRY

## *PART II - ACCEPTANCE PLANS*


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16. Abstract This is Part II of a three-part Handbook dealing with practical concepts in the field of statistics and with their applications in the control of materials and procedures in highway construction. The main purpose of this part is to point out and analyze the risk, or expected loss, that may result whenever an inspector or engineer makes a wrong decision by accepting or rejecting a LOT of material or construction. The concepts and procedures described in Part I are applied to the design of statistical acceptance plans and specifications. The concept of good and poor LOTs characterized by assigned average values is introduced. It is shown that the amount of risk associated with the acceptance or rejection of a particular LOT depends on the difference between these averages, on the variability of the measured values, and on the number of tests on which a decision is based. The design of acceptance plans and specifications, the construction of associated operating characteristics curves, and methods of applying reductions in price for materials or construction of marginal quality, are illustrated by fully worked out examples.

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# PREFACE

This book is Part Two of a three-part Handbook dealing with practical methods of mathematical statistics and their applications to the control of highway materials and construction. The main purpose of this Part is to point out and analyze the risk, or expected loss, that may result whenever an inspector or engineer makes a wrong decision by accepting or rejecting a LOT of material or construction. The factors that affect the size of such a loss are explained, and the methods of controlling the risk are discussed.

In Part One, emphasis was placed on the amount of variation to be expected among the results of tests of highway materials and construction. Several ways in which these variations could be measured were presented, and the statistical methods of computation were described. It was said that reliable information could be obtained from tests only if the samples for the tests were taken in a statistically random manner. It was shown that the required number of tests would depend on the variability of the measured values and on the desired accuracy of the test results. In this Part the applications of these principles to the design of statistical acceptance plans and specifications will be covered. The concept of good and poor LOTs of material or construction will be introduced. We shall show how averages can be assigned to the measured values obtained in tests made on samples from these LOTs. We shall also show that the amount of the risk associated with the acceptance or rejection of a particular LOT will depend on the allowable difference between certain averages, on the variability of the measured values, and on the number of tests on which a decision is based.

Part Three of the Handbook deals with more advanced statistical methods that are of interest primarily to persons who are engaged in research or are responsible for the preparation of reports and the presentation of data. The topics covered in that Part include the design of experiments, the analysis of variance, methods of finding relationships between measured values, and practical methods of fitting curves to measured test results.

HANDBOOK OF APPLICATIONS OF STATISTICAL CONCEPTS
TO THE HIGHWAY CONSTRUCTION INDUSTRY

*PART II - ACCEPTANCE PLANS*

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# CHAPTER 6

# BASIC ACCEPTANCE PLANS AND OPERATING CHARACTERISTIC CURVES

## 6.1 RISKS

### 6.1.1 Types of Wrong Decisions

In order that you may be able to understand the principles underlying the design of acceptance plans, you must know something about the theory of risks. This theory not only is very important in applications of statistical principles, but also is involved in every choice of action that we make in everyday life.

Whenever a person has to make a decision of any kind by choosing one of two possible actions, there is a possibility that his decision will be wrong. In other words, the risk of making a wrong decision is always present, whether or not the person realizes this fact. For instance, let us suppose that the weather forecast calls for snow and that a person planning a long trip in his car must decide, before he starts on the trip, whether or not he will need chains. If he puts on chains and it does snow, he will have made the right decision. However, he may make a wrong decision in either of two ways. If he does not put on the chains but it snows, he will have made an error by rejecting the forecast when it was correct. If he puts on chains but it does not snow, he will have made an error by accepting a forecast when it was incorrect. It is the usual practice to call an error of the first kind a *Type I error*, and to call an error of the second kind a *Type II error*.

### 6.1.2 Seller's Risk and Buyer's Risk

An engineer or an inspector on a highway project, as the representative of a State agency or a buyer, must decide many times whether a material or construction complies with the requirements of the specifications governing the project. Since it is not possible to test the entire LOT of material or construction, he must base the acceptance decision on the results of a relatively small number of tests made on samples or made at selected locations. The computed average $\bar{X}$ of the results of a small number of tests will seldom, if ever, be the same as the true average $\bar{X}'$ of the results of all possible tests that could have been made on an entire LOT of material or construction. Also, the computed value of the standard deviation s for the measured values will differ from the true standard deviation $s'$ for all possible results. Because of the nature of the variability of the results of a small number of tests, there is always a chance that a LOT of good material will be rejected or a LOT of poor material will be accepted.

If a decision is made to reject a LOT of material when the material is actually satisfactory, a Type I error will have been made. The risk associated with such an error is called the *alpha* (α) *risk*. (The Greek letter α is called alpha.) In this Handbook, the alpha risk is called the *seller's risk*, because it is the risk taken by a contractor, producer, or manufacturer that acceptable material will be rejected.

If a decision is made to accept a material, but the material is unsatisfactory, a Type II error will have been made. The risk associated with such an error is called the *beta* (β) *risk*. (The Greek letter β is called beta.) In this Handbook the beta risk is called the *buyer's risk*, because it is the risk taken by an agency such as a State Highway Department or Commission, represented by the engineer, that material which does not meet specified requirements will be accepted.

As used here, the term "material" means any kind of item, including a manufactured product or completed construction, to which specified requirements apply.

It is easy to remember that both the word beta and the word buyer start with the same letter, b. Also, the beta risk is related to a Type II error and b is the second letter in the alphabet. The relationship between a type of error and its related risk is shown in Figure 6.1. The definitions are also repeated there.

## 6.2 THE CONCEPT OF LOTS

### 6.2.1 Decision Based on True Average for LOT

The concept that the buyer's risk and the seller's risk are present when an acceptance decision is made is closely related to the concept of LOTs. Although the concept of LOTs has been applied in Part One of this Handbook, some of the basic information will be given again here. A *LOT* is a definite quantity of material or construction produced by the same process. For example, a LOT might be the number of cubic yards of concrete in an entire bridge abutment, the number of tons of hot-mix asphaltic concrete produced in a day, 1000 feet of compacted subbase, or a stockpile of aggregate.

Figure 6.1

# RISKS OF ERROR

| ACTUAL CONDITIONS | ENGINEER'S DECISION | |
|---|---|---|
| | REJECT MATERIAL | ACCEPT MATERIAL |
| MATERIAL ACCEPTABLE | TYPE I ERROR<br>ENGINEER INCORRECT | ENGINEER CORRECT |
| MATERIAL UNACCEPTABLE | ENGINEER CORRECT | TYPE II ERROR<br>ENGINEER INCORRECT |

## TYPE I ERROR

ENGINEER INCORRECTLY REJECTS ACCEPTABLE MATERIAL OR CONSTRUCTION

SELLER'S RISK
OR ALPHA ($\alpha$) RISK

THE RISK THAT ACCEPTABLE MATERIAL OR CONSTRUCTION WILL BE REJECTED, OR A TYPE I ERROR WILL BE MADE

## TYPE II ERROR

ENGINEER INCORRECTLY ACCEPTS UNACCEPTABLE MATERIAL OR CONSTRUCTION

BUYER'S RISK
OR BETA ($\beta$) RISK

THE RISK THAT UNACCEPTABLE MATERIAL OR CONSTRUCTION WILL BE ACCEPTED, OR A TYPE II ERROR WILL BE MADE

Different LOTs of the same kind of material can differ in quality, as indicated by variations in the measured values of some characteristic of the material. Let us suppose, for example, that most of the crushed stone in a particular locality is of such quality that the average loss in the Los Angeles (L.A.) abrasion test is less than 40 percent, and let us suppose that this value is selected as the specification for a certain project. Our acceptance rule will be, if the result of a single L.A. abrasion test is not more than 40 percent, accept the LOT. If more than 40 percent, reject the LOT. We shall assume for the purpose of this explanation that we want to determine the quality of the crushed stone that is obtainable from each of five different sources, and that we select one LOT from each of these sources as a sample for testing. If we could make a test on every possible portion of each sample LOT, we could determine the true average abrasion loss $\overline{X}'$ for each LOT. Let us suppose that these average values for the five LOTs are as follows: LOT from source A, 30 percent; LOT from source B, 35 percent; LOT from source C, 40 percent; LOT from source D, 45 percent; and LOT from source E, 50 percent. We shall assume that the true value of the standard deviation $s'$ for all the measured losses in each LOT is about 7 percent. We shall now consider the seller's risk and the buyer's risk for each source if we accept or reject all the material in a LOT.

Since the value of $\overline{X}'$ for the LOT from source A, which is 30 percent, is much less than the acceptance limit, which is 40 percent, we can conclude that without a doubt the material from this source is of good quality. We would therefore want to have an acceptance procedure that would guarantee a very low seller's risk, or risk of rejecting good material from this source, because of expected variations in the test results due to sampling and testing errors.

The material in the LOT from source B is of fair quality, since 35 percent is well below 40 percent, and we would want a reasonably low risk of rejection for material from this source. The material in the LOT from source C is of acceptable quality because the value of $\overline{X}'$ is not less than the specification acceptance limit of 40 percent. However, because of variations in the measured test results, we would expect that only about half of the material from this source could be accepted and the rest would have to be rejected.

Now let us consider the material from the source D. Although this material is of questionable quality for general use, because of a loss equal to 45 percent which exceeds the acceptance limit, it is possible that such material will be suitable for some purposes on the project. We may therefore decide to purchase material from this source at a reduced price. It would obviously be unfair to the producers of material of good quality for us to pay the same price for material of poor quality. We would want a reasonably low buyer's risk, or risk of accepting unsuitable material from this source.

The material in the LOT from source E, for which the average loss is 50 percent, is of poor quality. Most of the material from this source should be rejected because it would not be suitable for any purpose connected with the project. We would want the acceptance procedure to guarantee a very low buyer's risk.

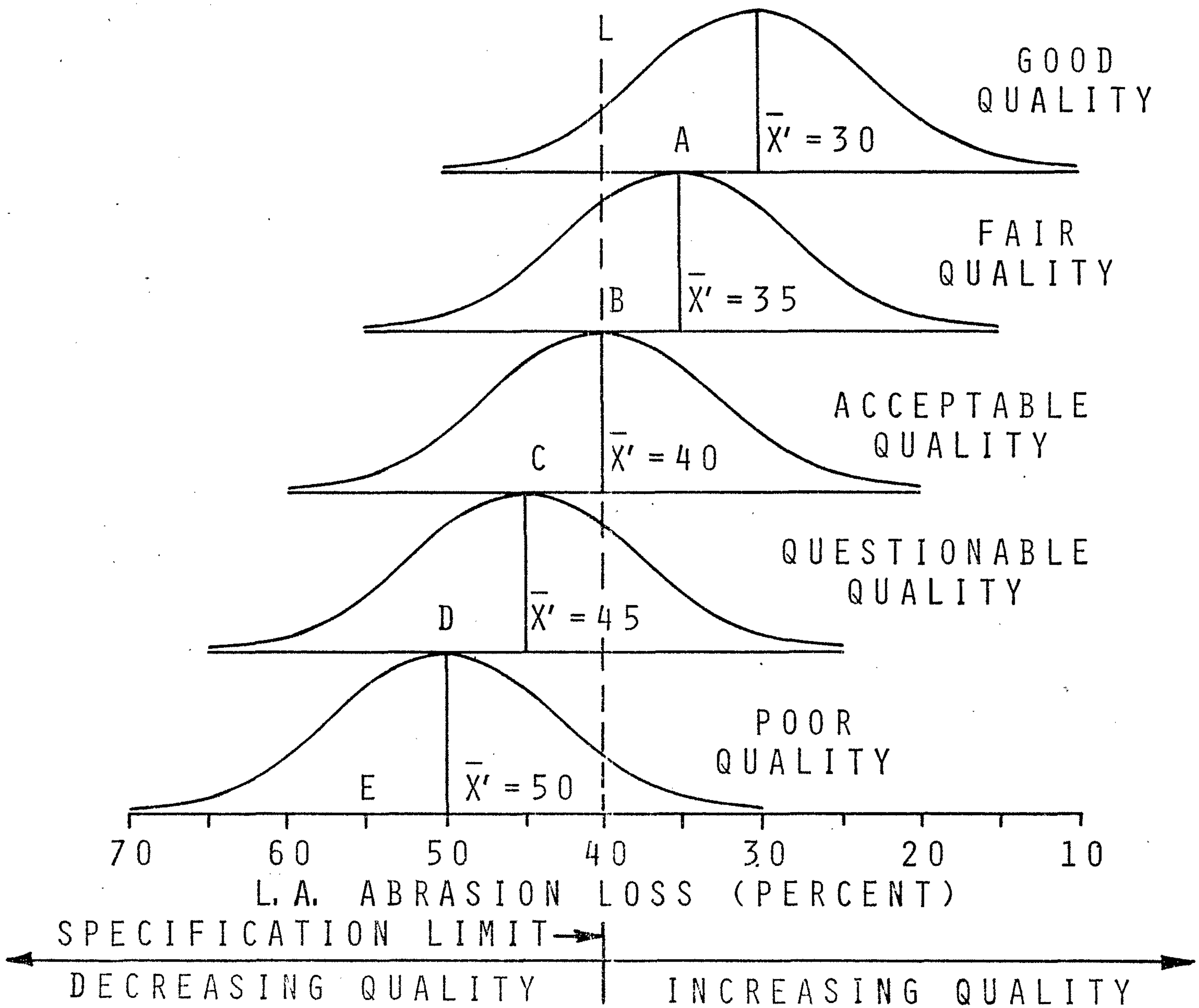
The probability distributions of the measured abrasion losses for the sources of aggregate considered in the preceding example are represented by the normal curves in Figure 6.2. When the value of $s'$ is 7 percent, the range for the individual measured results for a source is relatively large.

### 6.2.2 Decision Based on Actual Average for LOT

In the example in Section 6.2.1, we supposed that tests could be made on every possible portion of a LOT, and that we could calculate the true average $\overline{X}'$ and the true standard deviation $s'$ for each LOT. On an actual highway project, however, we would not know the true values of $\overline{X}'$ and $s'$. Instead, we would have to compute values of the average $\overline{X}$ and the standard deviation s by using the measured results from a relatively small number of tests for each LOT of material. Let us suppose first that we selected only one test portion from each LOT. In accordance with the acceptance rule given in Section 6.2.1, we would accept the entire LOT if the measured abrasion loss is less than 40 percent and we would reject the entire LOT if the measured loss is greater than 40 percent. As indicated by the probability distributions of test results represented by the normal curves in Figure 6.2, there is some chance that the result of a test on some single sample taken from the material in any LOT could be 40 percent or less. However, since most of the test results on material from LOT A would be less than 40 percent, we can assume that the test result for a single sample from that LOT would probably be less than 40 percent and we would probably accept all of the material from that LOT. On the other hand, most of the test results on material from LOT E would be greater than 40 percent, and we would probably reject all of the material in this LOT on the basis of the result of a single test.

Figure 6.2

AVERAGES AND DISTRIBUTIONS OF L. A. ABRASION LOSS FOR DIFFERENT LOTS OF CRUSHED AGGREGATE



If we base our decision to accept or reject a LOT of material on the result of one test, we must consider the risk. We see from Figure 6.2 that some of the measured results of tests on material in a LOT from source A could be greater than 40 percent. So there is some chance that we would reject a LOT of material that is actually of good quality. Also, some of the measured results of tests on material in a LOT from source E could be less than 40 percent, and there is some chance that we would accept a LOT of material that is actually of poor quality. To estimate the relative chances or probabilities of making a wrong decision in regard to the acceptance or rejection of material, we can make use of what we have learned about the characteristics of a normal curve.

You should keep in mind the fact that the area under the normal curve stands for a probability of 1.00 or 100 percent. By the use of a suitable table, we can find the area under the part of the normal curve between the center of the curve and a vertical line drawn through any selected point on the horizontal base. The first step is to compute the horizontal distance z from the center of the curve to the selected point by using the equation

$$z = \frac{L - \overline{X}'}{s'} \qquad \text{or} \qquad z = \frac{\overline{X}' - L}{s'} \tag{6.1}$$

Here L is the percent of abrasion loss at the acceptance limit, $\overline{X}'$ is the true average of the loss, and $s'$ is the true standard deviation. The distance z is expressed in sigma units. If L is greater than $\overline{X}'$, the first relation is used and the distance z is considered positive and lies to the right of the center of the curve. If $\overline{X}'$ is greater than L, the second relation is used and the distance z is considered negative and lies to the left of the center of the curve.

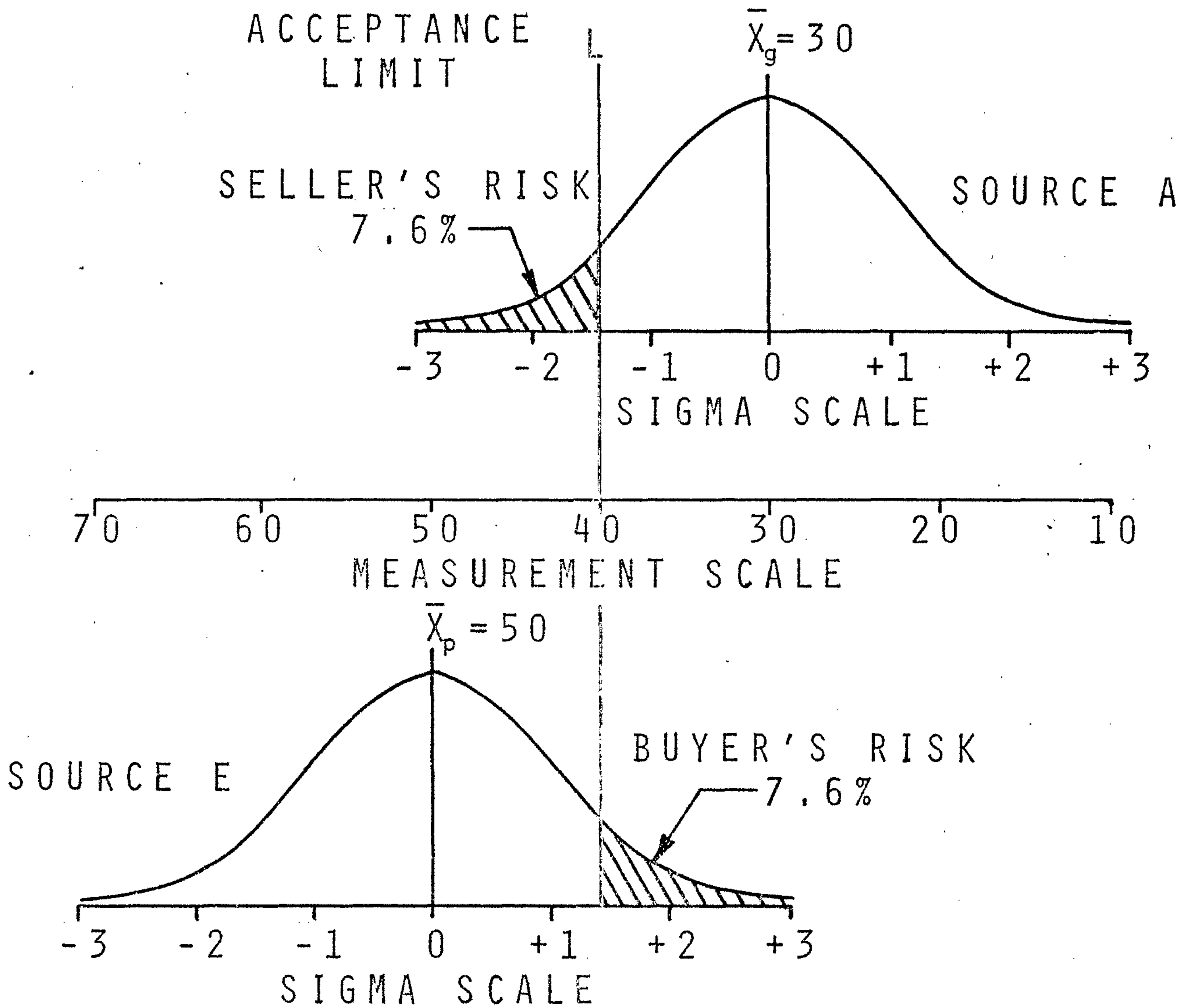
In our example, the acceptance limit L is 40 percent, and we can assume that the value of $s'$ is 7 percent. Then, for the LOT A, for which $\overline{X}'$ = 30 percent,

$$z = \frac{40 - 30}{7} = \frac{10}{7} = 1.43$$

From a one-tailed table of areas for the normal curve, we find that 92.4 percent of the total area under the curve lies to the right of a vertical line located 1.43 sigma units to the left of the center of the curve. The conditions for LOT A are represented in Figure 6.3 by the upper curve. Here, 92.4 percent of the total area under the curve lies to the right of the

Figure 6.3

SELLER'S AND BUYER'S RISKS
BASED ON SINGLE MEASUREMENTS



vertical line for an abrasion loss equal to 40 percent. In other words, 92.4 percent of the measured results of single measurements on portions that are selected in a statistically random manner from a LOT from source A should be less than 40 percent, and the probability that we would accept a LOT from source A is 92.4 percent. Since the total area under a normal curve represents a probability equal to 100 percent, the area under the portion of the curve to the left of a vertical line for an abrasion loss equal to 40 percent would be only 7.6 percent of the total area, and the probability that we would reject a LOT from source A is 7.6 percent.

If we consider the material from source E, for which the true average abrasion loss is 50 percent, we will see that the conditions are reversed. In this case, the distance z is found from the second relation and is equal to $\frac{50 - 40}{7} = 1.43$ units. However, as shown by the position of the lower curve in Figure 6.3, this distance lies to the right of the center of the curve. Since 92.4 percent of the total area under the curve lies to the left of the vertical line for an abrasion loss equal to 40 percent, the probability that a LOT from source E will be rejected is 92.4 percent and the probability that such a LOT will be accepted is only 7.6 percent.

We can define good material as that which will provide satisfactory performance when used for the intended purpose. Accordingly, poor material is that which has reduced quality which could result in increased maintenance cost or reduced safety factor. We do not want to reject good material or to accept poor material. Since the probability of rejecting a LOT from source A is very small, we can describe the quality of the material from source A by saying that there is a seller's risk or alpha risk equal to 7.6 percent. On the other hand, since the probability of accepting a LOT from source E is very small, we can describe the quality of the material from source E by saying that there is a buyer's risk or beta risk equal to 7.6 percent.

## 6.3 CALCULATIONS FOR OPERATING-CHARACTERISTICS CURVE

We can calculate the probabilities of acceptance and rejection for the material from sources B, C, and D in the preceding example by following a procedure like that shown for source A or source E. If we do this and put the results in a table, we would obtain a table like Table 6.1.

Table 6.1

CALCULATIONS FOR OPERATING-CHARACTERISTICS CURVE

| Source | $\overline{X}'$ | $L - \overline{X}'$ | $z = \frac{L - \overline{X}'}{s'}$ | Probability of Accepting (Percent) | Probability of Rejecting (Percent) |
|---|---|---|---|---|---|
| A | 30 | +10 | +1.43 | 92.4 | 7.6 |
| B | 35 | + 5 | +0.71 | 76.1 | 23.9 |
| C | 40 | 0 | 0.00 | 50.0 | 50.0 |
| D | 45 | - 5 | -0.71 | 23.9 | 76.1 |
| E | 50 | -10 | -1.43 | 7.6 | 92.4 |

$L = 40 \qquad s' = 7$

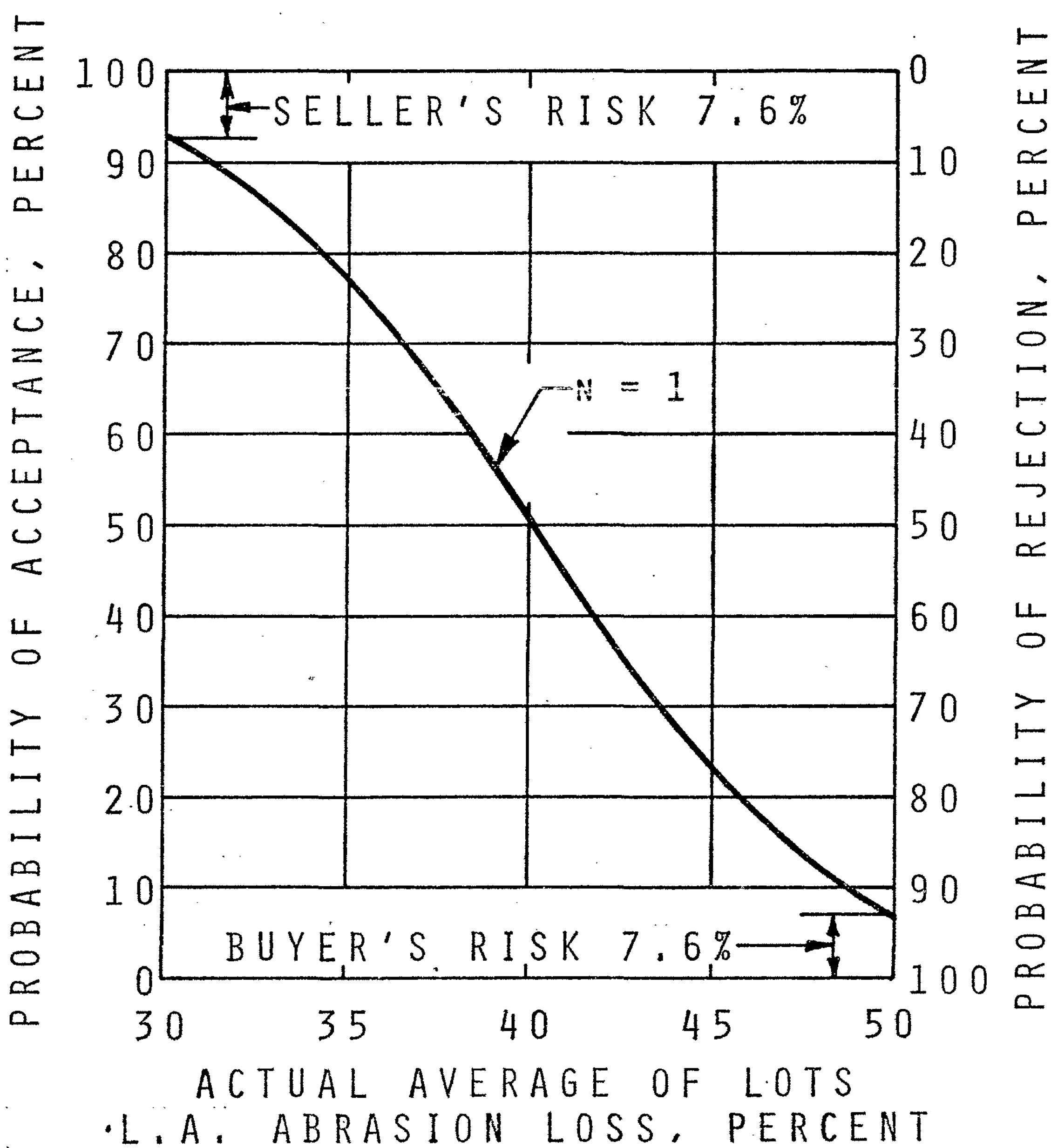
We can use these values to plot a curve showing the probability of acceptance or rejection of any LOT for which the value of $\overline{X}'$ is between 50 and 30 and the value of $s'$ is taken as 7 percent. This curve is called an *operating-characteristics curve* or *OC curve*. The curve for our assumed example is shown in Figure 6.4. We can use this curve to find the probability of acceptance or rejection of any LOT within the range covered. For example, if the average L.A. abrasion loss for a certain LOT is 37 percent, the probability that material from that LOT would be accepted is about 67 percent. For a LOT for which the average abrasion loss is 46 percent, the probability of rejection would be about 80 percent.

## 6.4 REDUCING RISKS

In our assumed example, both the seller's risk for source A and the buyer's risk for source E were 7.6 percent. Therefore, if we based our decision on the result of a test on only one portion from a LOT, about once in 13 times we would make a Type I error and reject a LOT from source A which may be good or we would make a Type II error and accept a LOT from source E which may be poor, as we have defined good and poor material in Section 6.2.2. Let us suppose that we are not satisfied with the size of these risks and we want to reduce them to a much lower value such as 1 percent. You should keep in mind the following fact: When a number of portions $n$ from each LOT are tested and the averages of the measured values for the LOTs are used, the relation between the standard deviation $s'_{\overline{X}}$ for the averages and the standard deviation $s'$ for a single measured value for each LOT can be expressed by equation 3.1. This equation, which is repeated here for convenient reference, is

Figure 6.4

## OPERATING-CHARACTERISTICS CURVE BASED ON SINGLE MEASUREMENTS



$$s'_{\bar{x}} = \frac{s'}{\sqrt{n}} \qquad (3.1)$$

If we reduce the standard deviation, we can obtain more accurate measurements. This increase in accuracy should reduce the size of our risk. Let us suppose that we change our acceptance rule to read as follows: If the average of the results of tests on three portions randomly selected from a LOT is not more than 40 percent, accept the LOT; if such an average is more than 40 percent, reject the LOT. When we measure the abrasion loss for the material in a LOT by using the average of three test results, we may assume that the standard deviation will be

$$s'_{\bar{x}} = \frac{7}{\sqrt{3}} = \frac{7}{1.73} = 4.05, \text{ or say 4 percent}$$

The probability of acceptance of a LOT from source A, for which the average L.A. loss is taken as 30 percent, is now found as follows:

$$z = \frac{L - \bar{X}'}{s'} = \frac{40 - 30}{4} = \frac{10}{4} = 2.5$$

From a one-tailed table of areas for the normal curve, we can find that the portion of the total area to the right of L is 99.4 percent, as shown in Figure 6.5. Hence, the probability of acceptance of a LOT from source A is 99.4 percent and the probability of rejection, or the seller's risk, is 0.6 percent. We can calculate the probabilities of acceptance and rejection for each source and put them in a table. The results are shown in Table 6.2. By using these values we can plot a new operating-characteristics curve, as shown in Figure 6.6. If we compare this curve with the one in Figure 6.4, we see that we not only have reduced the buyer's and seller's risks but we also have made the acceptance plan more effective for accepting material of fair quality and for rejecting material of questionable quality. For example, under the previous plan the probability of accepting material for which the abrasion loss is 38 percent was about 62 percent. Under the new plan such material would be accepted about 75 percent of the time. This increased power of discrimination is shown by the steeper slope of the central portion of the operating charatteristics curve. In general, the more nearly vertical this portion of the curve is, the greater will be the power of discrimination.

Figure 6.5

SELLER'S AND BUYER'S RISKS
BASED ON AVERAGES OF THREE MEASUREMENTS

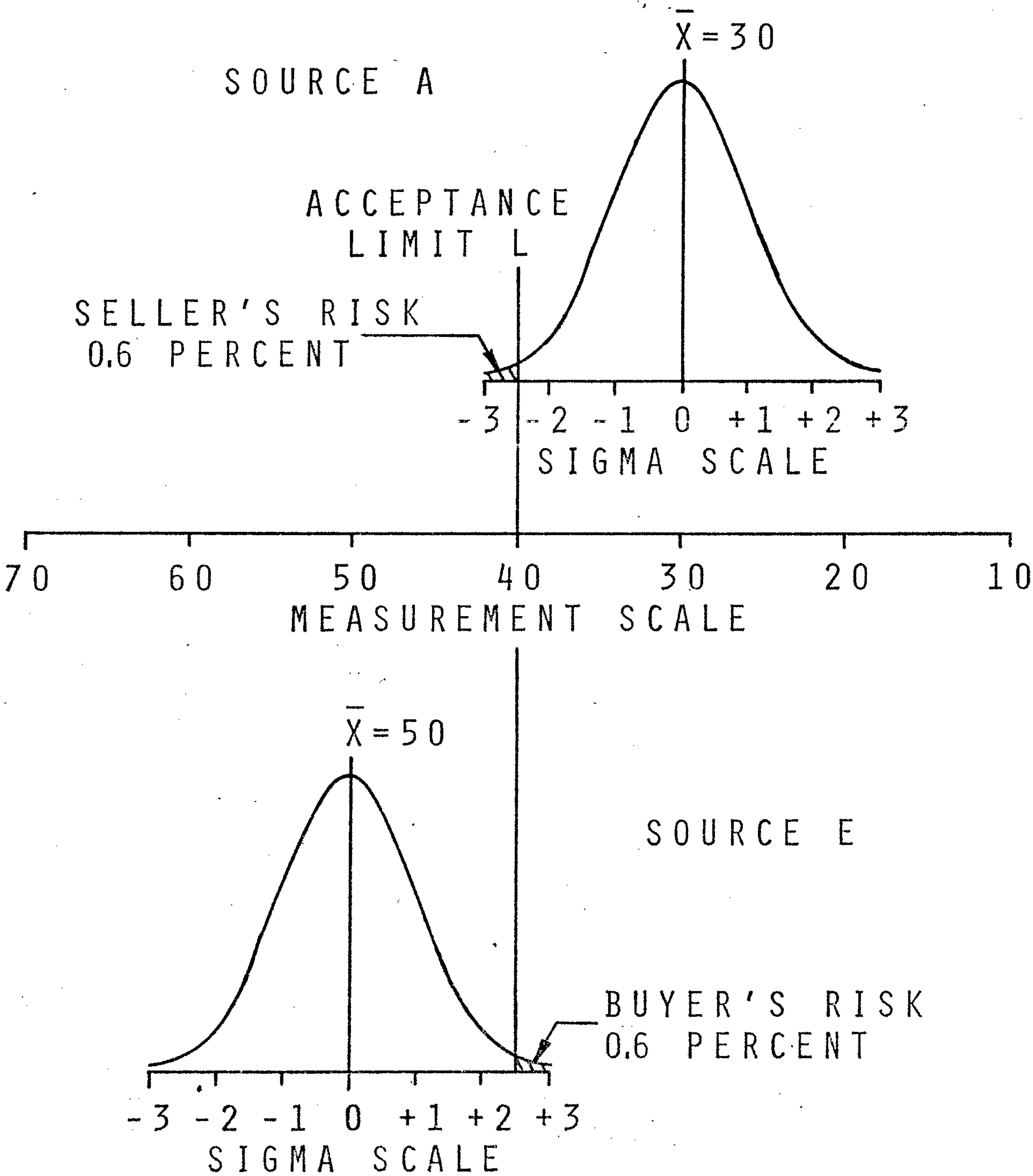


Figure 6.6

OPERATING-CHARACTERISTICS CURVE
BASED ON AVERAGES OF THREE MEASUREMENTS

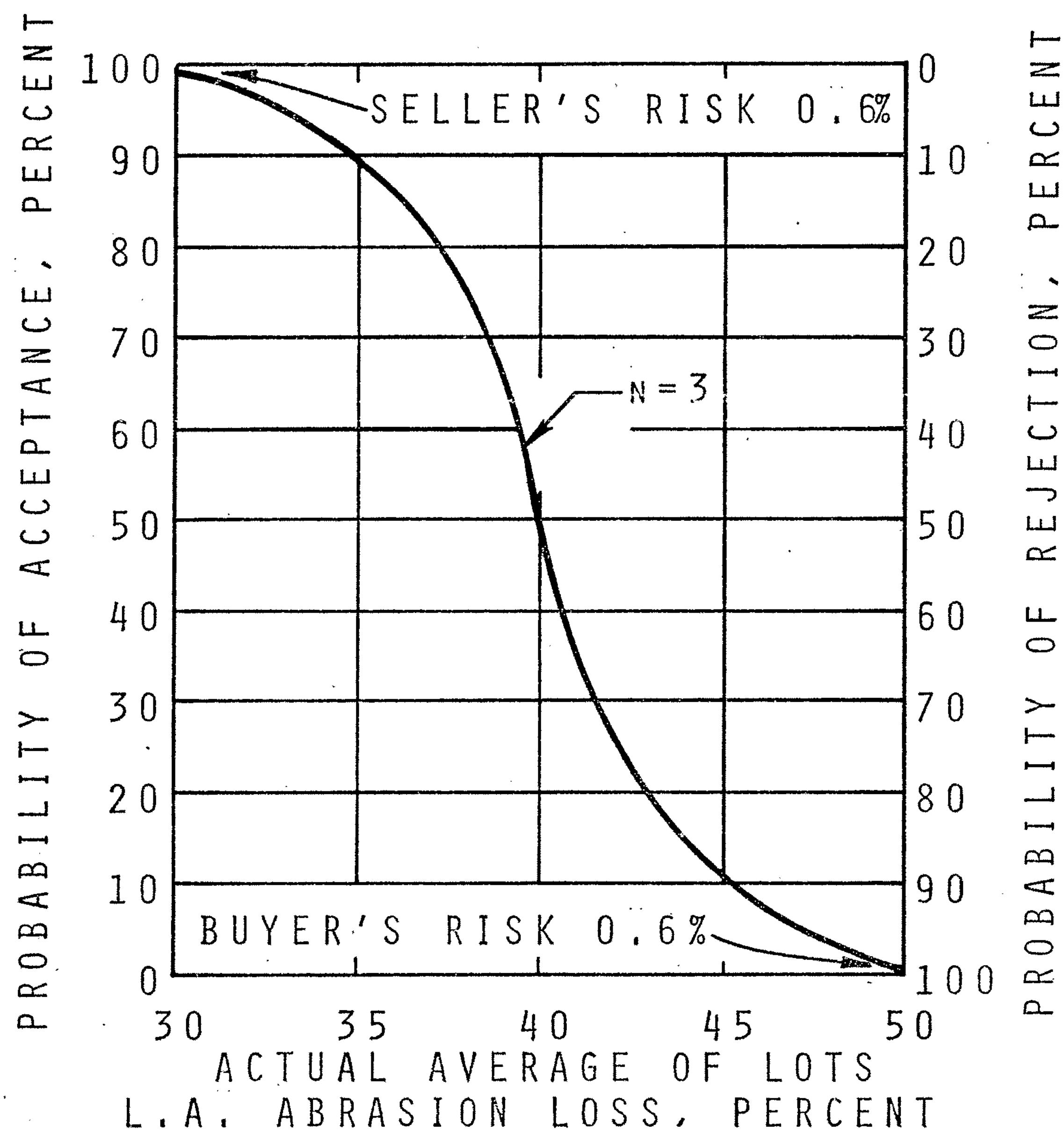


Table 6.2

CALCULATIONS FOR OPERATING-CHARACTERISTICS CURVE

| Source | $\bar{X}'$ | $L - \bar{X}'$ | $z = \frac{L - \bar{X}'}{s'_{\bar{X}}}$ | Probability of Accepting (Percent) | Probability of Rejecting (Percent) |
|---|---|---|---|---|---|
| A | 30 | +10 | +2.50 | 99.4 | 0.6 |
| B | 35 | + 5 | +1.25 | 89.4 | 10.6 |
| C | 40 | 0 | 0.00 | 50.0 | 50.0 |
| D | 45 | - 5 | -1.25 | 10.6 | 89.4 |
| E | 50 | -10 | -2.50 | 0.6 | 99.4 |

$L = 40 \qquad s' = 4$

## 6.5 EFFECT OF POSITION OF AN ACCEPTANCE LIMIT

In the preceding example, the seller's risk for source A and the buyer's risk for source E were equal, and both risks were reduced by using the average of a number of measurements. When we have only a single acceptance limit, we might attempt to reduce the size of the buyer's risk by moving our acceptance limit closer to the average value for good material. Of course, this change would increase the seller's risk. However, we might say that such a change would require the seller to produce a more uniform material of higher quality in order to avoid frequent rejections or a reduction in price.

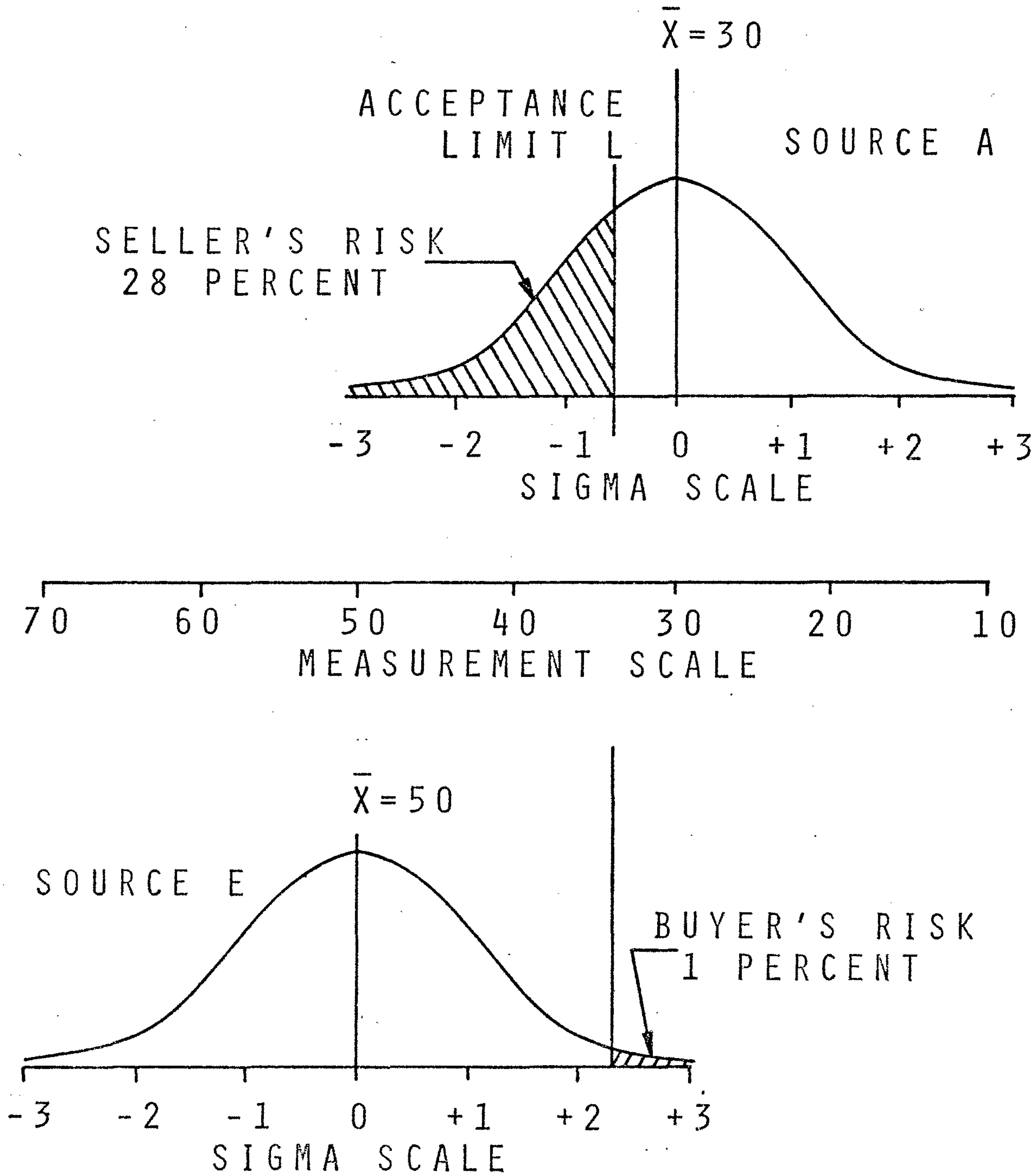
Let us suppose that we decide to accept or reject a LOT of material on the basis of a single test result, but we want to limit the buyer's risk to about 1 percent. To make the buyer's risk about 1 percent, we shall set our acceptance limit 2.33 sigma units from the average for poor material. Since the standard deviation $s'$ is taken as 7 percent, the corresponding distance in measurement units will be $2.33(7) = 16.3 \approx 16$ percent. Our acceptance rule will be as follows: Take one random test portion from a LOT. If the L.A. abrasion loss is not more than 34 percent, accept the LOT. If the loss is more than 34 percent, reject the LOT.

The conditions for materials from sources A and E are represented in Figure 6.7. The probability of acceptance of a LOT from source A, for which the average L.A. loss is 30 percent, is

$$z = \frac{L - \bar{X}'}{s'} = \frac{34 - 30}{7} = \frac{4}{7} = 0.57$$

Figure 6.7

SELLER'S AND BUYER'S RISKS
BASED ON SINGLE MEASUREMENTS



From a one-tailed table of areas for the normal curve, we see that the probability of acceptance of a LOT from source A is now about 72 percent. So the probability of rejection, or the seller's risk, is about 28 percent. If we calculate the probabilities of acceptance and rejection for the other sources and we construct a table, we obtain Table 6.3.

Table 6.3

CALCULATIONS FOR OPERATING-CHARACTERISTICS CURVE

| Source | $\overline{X}'$ | $L - \overline{X}'$ | $z = \frac{L - \overline{X}'}{s'}$ | Probability of Accepting (Percent) | Probability of Rejecting (Percent) |
|---|---|---|---|---|---|
| A | 30 | + 4 | +0.57 | 71.6 | 28.4 |
| B | 35 | - 1 | -0.14 | 44.4 | 55.6 |
| C | 40 | - 6 | -0.86 | 19.5 | 80.5 |
| D | 45 | -11 | -1.57 | 5.8 | 94.2 |
| E | 50 | -16 | -2.29 | 1.1 | 98.9 |

$L = 34 \qquad s' = 7$

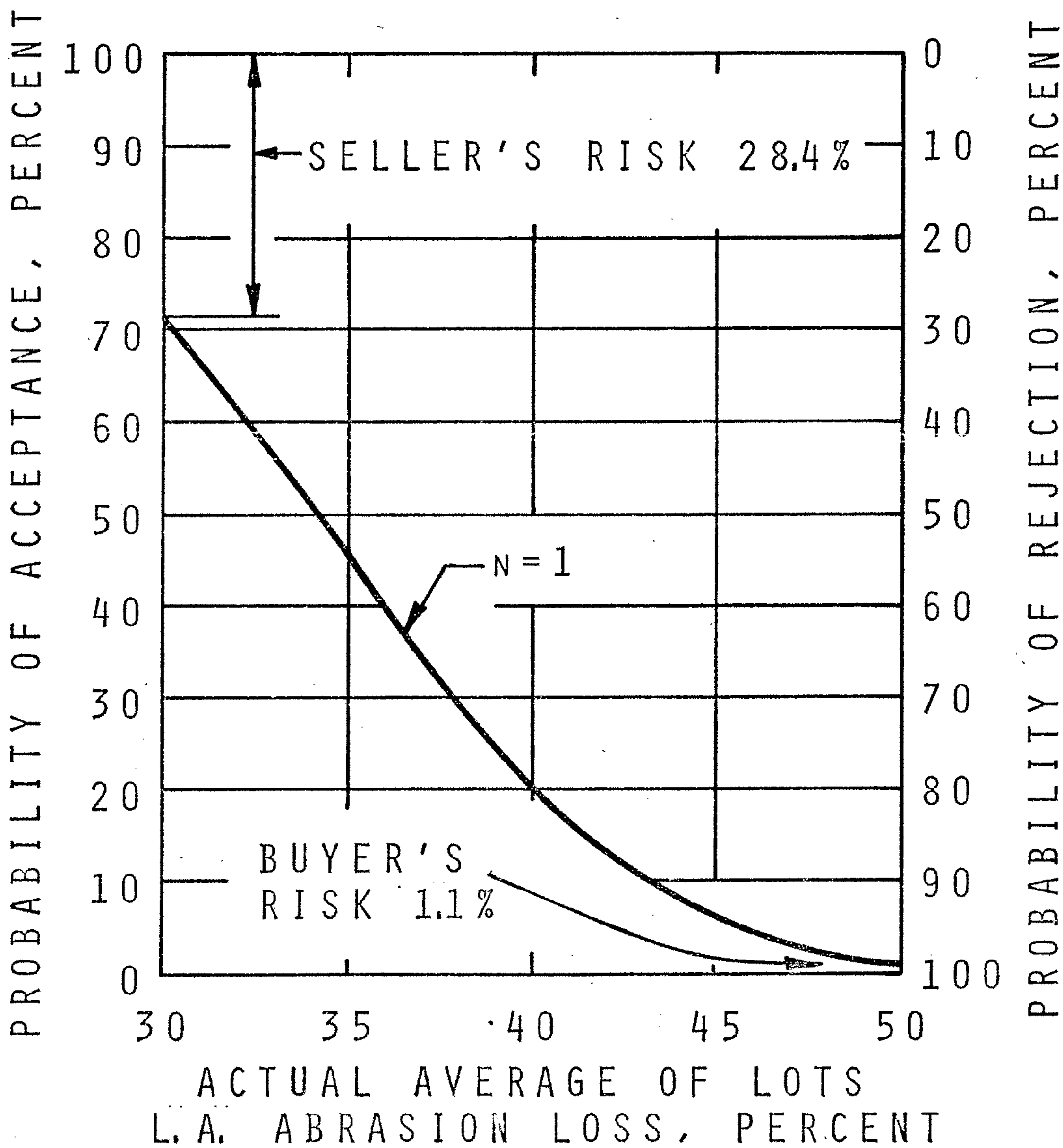
By using these values, we can construct the operating-characteristics curve shown in Figure 6.8.

Actually, this would be a very bad acceptance plan. If it were strictly enforced, about one out of four LOTs of good material would be rejected, and about half the LOTs of material of fair quality would also be rejected. The result would probably be a large increase in the price of the material.

In this example, the variability of the measured results has been exaggerated in order to show clearly the effects of basing a decision on a single measurement. The example also shows the effect of arbitrarily setting an acceptance limit that is too close to the average value of some characteristic for economically available good material.

Figure 6.8

OPERATING-CHARACTERISTICS CURVE
BASED ON SINGLE MEASUREMENTS



## 6.6 EFFECT OF POSITION OF DOUBLE ACCEPTANCE LIMIT

### 6.6.1 General Procedure

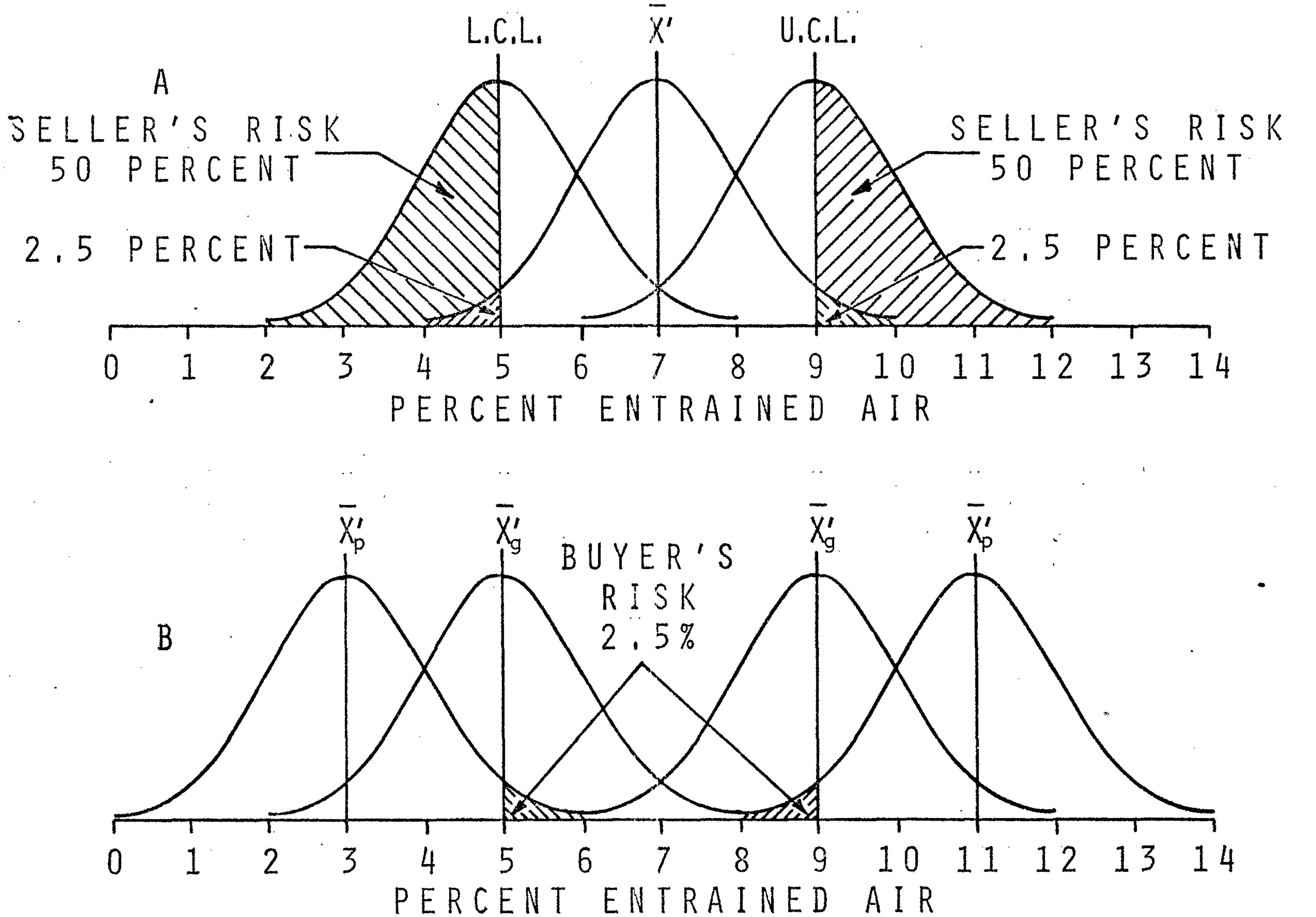
When we have a double-limit specification, such as upper and lower limits for the asphalt content in a mixture, the percentage of aggregate passing a given sieve, or the percentage of entrained air in concrete, the design of an acceptance plan is a little more difficult. For example, Section 501.03 of the AASHO Interim Specifications recommends that the percent of entrained air in pavement concrete be specified as 7 plus or minus 2 percent. A large number of designed experiments have shown that the standard deviation for single measured values of air content varies from about 0.8 to about 1.5 percent, and that the average is about 1 percent. About half of the variation is caused by sampling and testing errors.

The design of a realistic acceptance plan to meet the preceding requirements presents some problems. First we shall suppose that the conditions are as shown in Figure 6.9 by each of the three curves in the set designated A. If the actual average air content is maintained at the target value, or 7 percent, as indicated by the middle curve, about 95 percent of the results of single tests would be between the specified lower control limit (L.C.L.), or 5 percent, and the upper control limit (U.C.L.), or 9 percent. If the actual average air content is either 5 percent or 9 percent, as indicated by one of the other curves, only 50 percent of the measured results would be between the specified limits. This means that for these three possible conditions, the seller's risk of having concrete with an actual air content between 7 and 2 percent rejected would be between 5 percent and 50 percent.

In order to estimate the buyer's risk for the conditions in Figure 6.9, we must define poor material. For this purpose, we shall assume that $s' = 1$ and that material will be considered good if the air content indicated by a single measurement is anywhere between 5 percent and 9 percent, which are The specified values for the L.C.L. and the U.C.L. Also, we shall arbitrarily set the average of single measurements for poor material at a distance of 2 sigma units from the specified values for the L.C.L. and the U.C.L. The conditions for poor material are shown in Figure 6.9 by the curves in the set designated B. Here, the notation $\overline{X}'_g$ is used to denote the average air content for good material and $\overline{X}'_p$ is used for the average for poor material. Since $5 - 2 = 3$ and $9 + 2 = 11$, the values for $\overline{X}'_p$ are 3 percent and 11 percent.

Figure 6.9

SELLER'S AND BUYER'S RISKS FOR PERCENT ENTRAINED AIR IN PAVEMENT CONCRETE BASED ON SINGLE MEASUREMENTS



There is some justification for the preceding assumptions, because concrete in which the average air content is only 3 percent would have poor durability whereas concrete in which the average air content is 11 percent would probably have significantly reduced strength. On the basis of the new assumptions, the buyer's risk is about 2.5 percent and the seller's risk is from 5 to 50 percent.

For the conditions represented by the set of curves B in Figure 6.9, we can reduce the buyer's risk by basing acceptance on the average of the results of a number of tests. However, the seller's risk will always approach 50 percent when the actual average air content approaches either the U.C.L. or the L.C.L., no matter how many results we average. Let us suppose that we make a slight concession and place the U.C.L. and the L.C.L. at 2.5 percent below and above the target value of 7 percent. Also, we shall base the acceptance decision on the average of four test results.

The acceptance rule will be as follows: Take a test portion from each of four batches selected at random from each LOT. If the average of the four test results is between 4.5 and 9.5, accept the LOT. If the average is not between these limits, reject the LOT. The conditions are indicated by the curves shown in Figure 6.10.

The standard deviation for the new conditions may be taken as

$$s'_{\bar{x}} = \frac{s'}{\sqrt{n}} = \frac{1.0}{\sqrt{4}} = \frac{1.0}{2} = 0.5$$

The buyer's risk is zero for all practical purposes, because the distance from either the L.C.L. or the U.C.L. to the average $\bar{X}'_p$ is equal to $3s'_{\bar{x}}$. The seller's risk is found as follows:

$$z = \frac{\text{U.C.L.} - \bar{X}'_g}{s'_{\bar{x}}} = \frac{9.5 - 9.0}{0.5} = 1.0$$

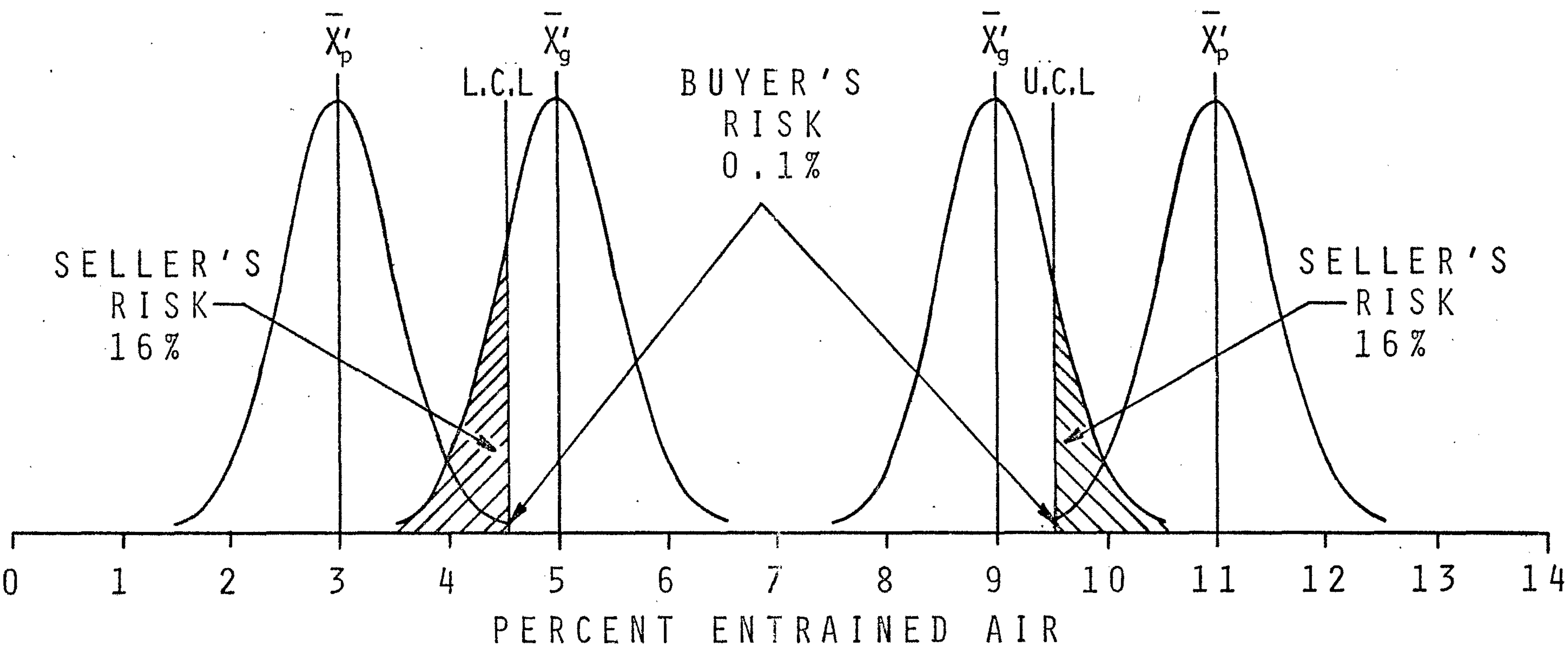
From the one-tailed table of areas for the normal curve, we see that when z = 1.0 the portion of the area to the left of the U.C.L. is 84.1 percent. So the seller's risk, or the area to the right of the U.C.L., is about 16 percent.

The operating-characteristics curve for this acceptance plan is shown in Figure 6.11.

You should note that it is only necessary to calculate areas for locating points on one half of the plan, since the corresponding areas are the same on both sides of the central value.

Figure 6.10

# SELLER'S AND BUYER'S RISKS FOR PERCENT ENTRAINED AIR IN PAVEMENT CONCRETE BASED ON FOUR MEASUREMENTS

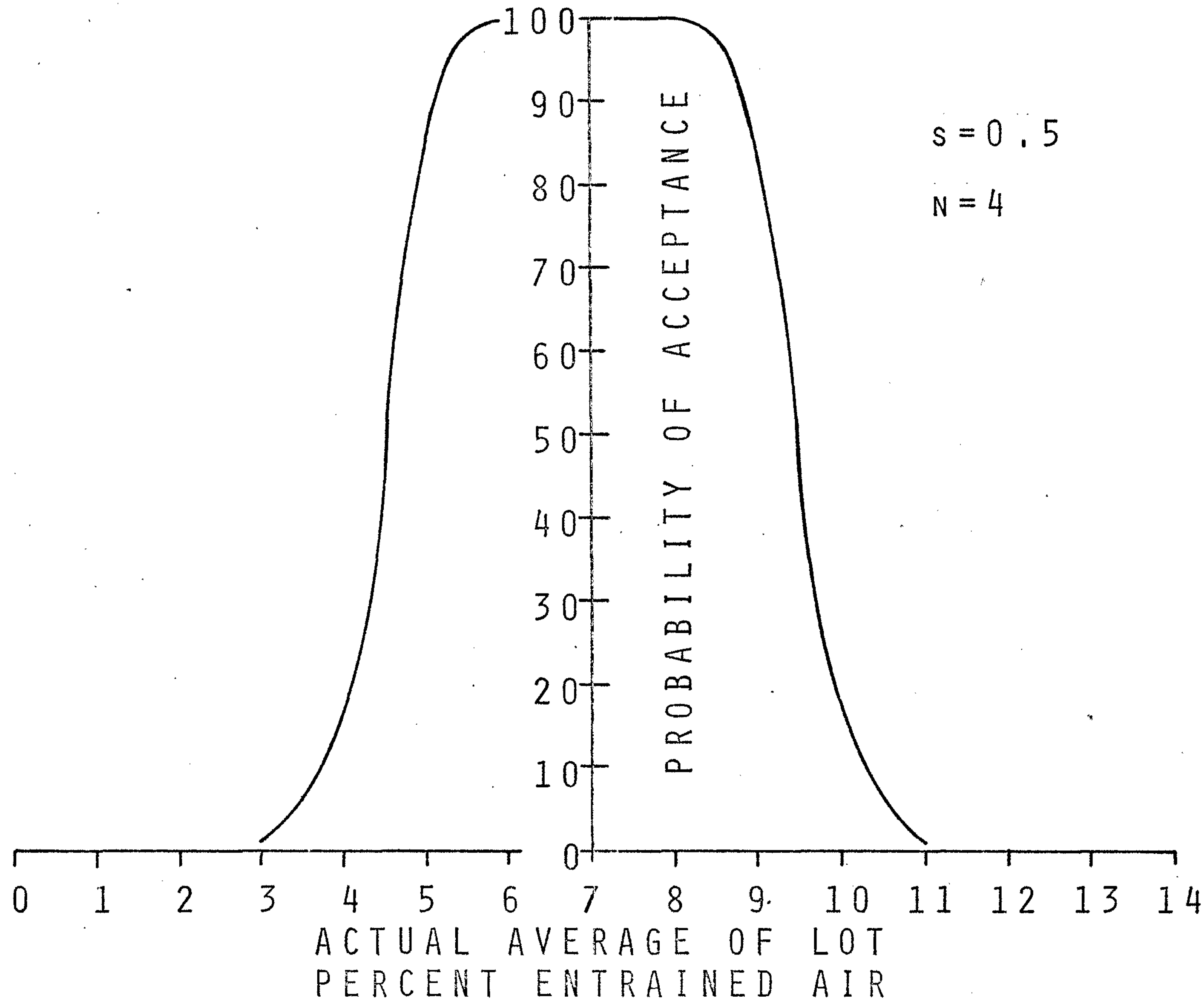



L.C.L. = 4.5 U.C.L. = 9.5

$s_{\bar{X}} = 0.5$

Figure 6.11

OPERATING-CHARACTERISTICS CURVE FOR PERCENT ENTRAINED AIR IN PAVEMENT CONCRETE BASED ON FOUR MEASUREMENTS



### 6.6.2 Simplified Design of Acceptance Plan When Standard Deviation is Known

If we have a good estimate of the value of the standard deviation for measured results of tests made on the output of a process that is in statistical control, we can design an acceptance plan similar to one of those previously discussed by applying a simplified method. The general equation is

$$n = \left(\frac{(z_1 + z_2)s'}{D}\right)^2 \qquad (6.2)$$

$n$ = number of tests

$s'$ = standard deviation for the measured results

D = difference, in measurement units, between the average $\overline{X}'_g$ for good material and the average $\overline{X}'_p$ for poor material

$z_1$ = a constant, expressed in sigma units, corresponding to the probability of accepting good material

$z_2$ = a constant, expressed in sigma units, corresponding to the probability of rejecting poor material.

Suitable values of $z_1$ and $z_2$ corresponding to various selected probabilities $p$ for acceptance of good material and rejection of poor material are given in Table 6.4.

Table 6.4

VALUES OF $z$ FOR SELECTED PROBABILITIES

| $p$ | $z$ |
|---|---|
| 0.80 | 0.842 |
| 0.85 | 1.037 |
| 0.90 | 1.282 |
| 0.95 | 1.645 |
| 0.975 | 1.960 |
| 0.99 | 2.326 |
| 0.995 | 2.576 |
| 0.999 | 3.090 |

We must select numerical values, in measurement units, for the averages $\overline{X}'_g$ and $\overline{X}'_p$. In the case of a double-limit specification, the difference between the target average $\overline{X}$ and the average $\overline{X}'_g$ must be the same as the difference between $\overline{X}$ and $\overline{X}'_p$. Also, the difference D between $\overline{X}'_g$ and $\overline{X}'_p$ cannot be less than the value computed by the following equation:

$$D = \frac{2(z_1 + z_2)s'}{\sqrt{n}} \qquad (6.3)$$

There is another requirement in every case. The value of D must be so related to the value of $(z_1 + z_2)$ that the required number of tests $n$ computed by applying equation 6.2 will not be too large. The values, in measurement units, for the upper acceptance limit U.L. and the lower acceptance limit L.L. are $\overline{X} + 0.5D$ and $\overline{X} - 0.5D$.

For a single-limit specification, the smallest allowable difference D between the averages $\overline{X}'_g$ and $\overline{X}'_p$ can be found by applying the equation

$$D = \frac{(z_1 + z_2)s'}{\sqrt{n}} \qquad (6.4)$$

The values, in measurement units, for the upper and lower acceptance limits are

$$\overline{X}'_g + \frac{z_1 s'}{\sqrt{n}} \qquad \text{and} \qquad \overline{X}'_g - \frac{z_1 s'}{\sqrt{n}}$$

where $z_1$ is the constant for the seller's risk. In some cases, only the lower limit is used; in other cases, only the upper limit is used; and in still other cases, both limits are used. Precalculated constants for different buyer's and seller's risks are shown in Table 6.5. Here the symbol ± stands for *plus or minus*.

Table 6.5

CONSTANTS FOR ACCEPTANCE PLANS

| Criticality | Buyer's Risk (Percent) | Seller's Risk (Percent) | No. of Measurements ($n$) | D | U.L. and L.L. for Sample Averages |
|---|---|---|---|---|---|
| Critical | 0.5 | 5.0 | 6 | $1.72s'$ | $\overline{X}'_g \pm 0.67s'$ |
| Major | 5.0 | 1.0 | 5 | $1.78s'$ | $X'_g \pm 1.04s'$ |
| Minor | 10.0 | 0.5 | 4 | $1.93s'$ | $X'_g \pm 1.29s'$ |
| Contractual | 20.0 | 0.1 | 3 | $2.27s'$ | $X'_g \pm 1.78s'$ |

## 6.7 EXAMPLES OF TYPICAL SPECIFICATIONS

The following examples are given to show how the principles described here are applied in writing the acceptance clauses of three typical specifications. You must remember that these principles apply only in cases where it can be assumed that the standard deviation for the measured results is known.

### 6.7.1 Clause for Only Lower Acceptance Limit

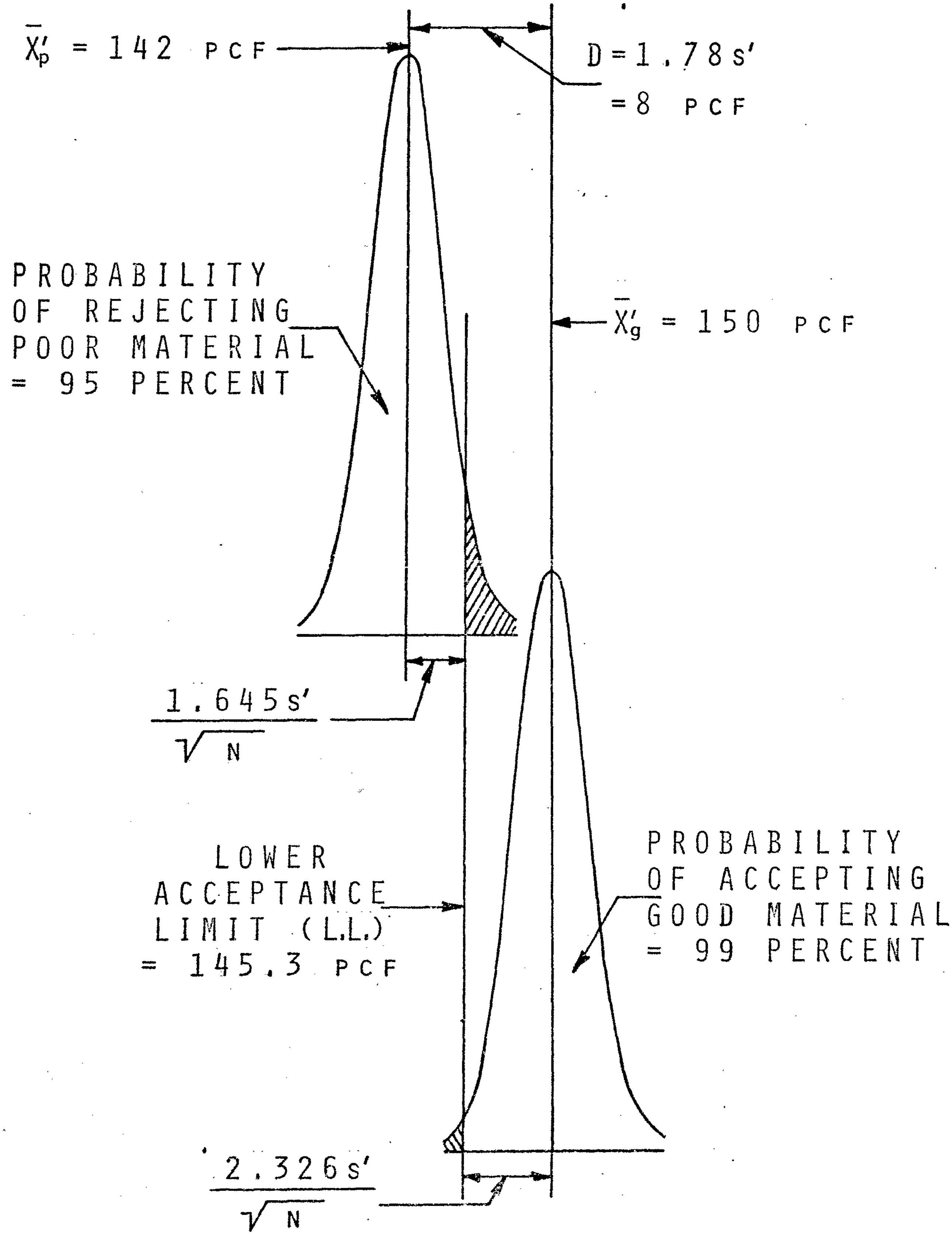
We shall suppose that tests on a large number of samples selected at random indicate that a crushed-stone base of a certain specified type is of good quality when the average density is 150 pcf (pounds per cubic foot). We shall suppose also that the standard deviation for measured results may be taken as 4.5 pcf, and that adequate density of the base is considered to be a *major requirement*. In this case, we assume that $\overline{X}'_g$ is 150 pcf. Also, from Table 6.5, we find that we should take five test samples for each LOT and that the lower acceptance limit should be $\overline{X}'_g - 1.04s' = 150 - 1.04(4.5) = 145.3$ pcf.

The acceptance clause of the specification should be as follows: The desired minimum density of the crushed-stone base is 150 pounds per cubic foot. Acceptable base shall have an average density of not less than 145.3 pounds per cubic foot based on the results of five density tests made in accordance with AASHO Designation T-147 at random locations within an area designated by the Engineer.

The operating characteristics for this acceptance clause are shown graphically in Figure 6.12. The lower curve at the right represents the distribution of the averages of the results of tests on five samples from a LOT

Figure 6.12

# OPERATING CHARACTERISTICS FOR LOWER ACCEPTANCE LIMIT ONLY



with respect to the desired average $\overline{X}'_g$, which is 150 pcf; while the upper curve at the left represents the distribution with respect to the acceptable average $\overline{X}'_p$, which is equal to 150 - D = 142 pcf. The lower acceptance limit L.L. is 145.3 pcf. A vertical line representing this value cuts the lower distribution curve at such a place that 99 percent of the area of that curve is to the right of the lower acceptance limit. Hence, if the average density for a LOT is 150 pcf and the standard deviation is 4.5 pcf, the probability of acceptance of the LOT would be 0.99. A small shaded portion of the curve, representing 1 percent of the total area, lies to the left of the lower acceptance limit. In this case the probability of rejecting the LOT would be 0.01.

If the average density of a LOT were slightly higher than 150 pcf, say 151.5 pcf, the distribution curve would lie to the right of the lower curve in Figure 6.12, and only a very small part of such a curve would lie to the left of the lower acceptance limit. As a matter of fact, the probability that the material would be rejected would be only 0.001, or the risk would be one chance in a thousand.

On the other hand, if the average density of a LOT were less than 150 pcf, the distribution curve would lie to the left of the lower curve in Figure 6.12, and more of the area would lie to the left of the lower acceptance limit. So the probability of rejecting the LOT would be greater than 0.01. When the average density is at 147.9 pcf, this probability is 0.10; and for 145.3 pcf, it is 0.50. In general, when the average density of the LOT is the same as the lower acceptance limit, there is an even (50/50) chance that the LOT will be rejected or accepted. If the average density falls below the acceptance limit, the area of the portion of the distribution curve to the left of the lower acceptance limit is greater than 50 percent of the total area, and the probability of rejecting the LOT would be greater than 0.50. If the actual average density were only equal to the minimum acceptable average $\overline{X}'_p$, which in this case is 142 pcf, the probability of rejection would be 0.95. Also if the average were 140.6 pcf, or about 6 percent less than $\overline{X}'_p$, the probability of rejection would be 0.99.

The facts developed in the foregoing discussion can be summarized as follows: If the average of the test results for a LOT is only about 1 percent above the desired average, the LOT will be accepted 999 times out of 1000; whereas the LOT for which the average of the test results is about 6.3 percent below the desired average will be accepted about 10 times out of 1000.

Obviously, there is considerable incentive for a contractor to try to keep the average value high. Variations in the standard deviation have somewhat the same effect as variations in the average values. In other words, a decrease in uniformity results in an increased risk of rejection.

### 6.7.2 Clause for Only Upper Acceptance Limit

Let us suppose now that the coarse aggregate available in a certain area has a good record of satisfactory service as a component of bituminous concrete. A large amount of available data indicates that the average Los Angeles abrasion loss is 35 percent and that the standard deviation is about 5 percent. In this application the abrasion loss may be classified as a *minor requirement*. Accordingly, $\overline{X}'_g$ = 35 percent and, from Table 6.5, we find that we should take four test samples from each LOT and that the upper acceptance limit should be $\overline{X}'_g + 1.29s' = 35 + 1.29(5) = 41.4$ percent.

The acceptance clause of the specification should be as follows: The desired maximum average percentage of wear of the aggregate is 35 percent. Acceptable aggregate shall have an average percent of wear of not more than 41.4 percent based on the results of four tests made in accordance with AASHO T-96 on random samples taken from the prepared and stockpiled aggregate.

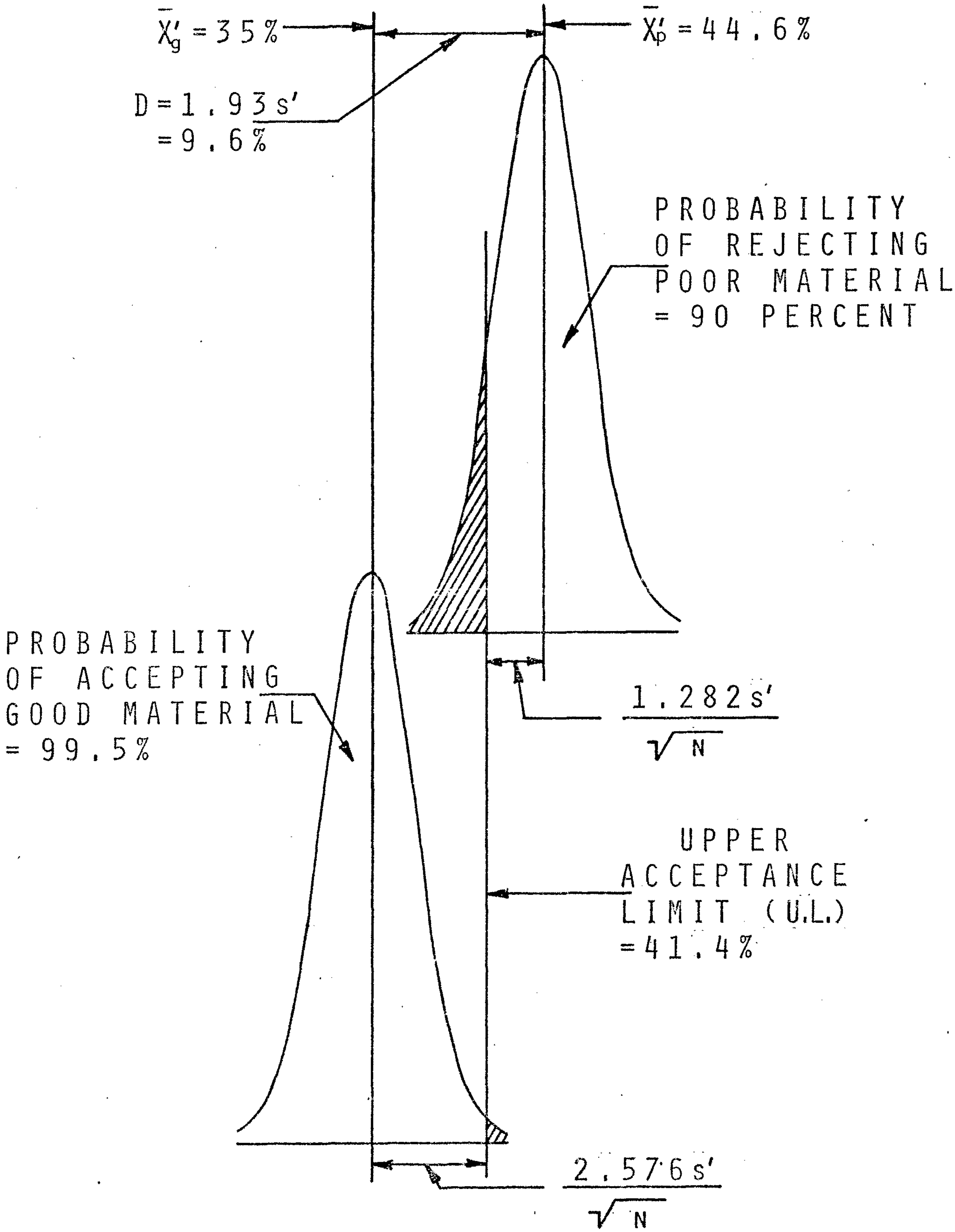
The operating characteristics for this acceptance clause are shown in Figure 6.13. The lower curve at the left represents the distribution of the averages of the results of four tests with respect to the desired average $\overline{X}'_g$, which is 35 percent. The upper acceptance limit is 41.4 percent. In this case, 99.5 percent of the area of the lower curve lies to the left of a vertical line representing the upper acceptance limit. It is easy to see that a slight decrease in the abrasion loss below 35 percent will almost certainly insure acceptance of the LOT. The upper distribution curve at the right is so positioned that 90 percent of its area lies to the right of the vertical line representing the upper acceptance limit. A small excess in the average abrasion loss above 44.6 percent would greatly increase the risk of rejection of the LOT above 90 percent.

### 6.7.3 Clause for Upper and Lower Acceptance Limits

Let us suppose that a study of bituminous-concrete surface mixture indicates that the best performance is obtained when the percent of the total aggregate passing the No. 8 sieve is maintained at 40 percent. Also, tests

Figure 6.13

# OPERATING CHARACTERISTICS FOR UPPER ACCEPTANCE LIMIT ONLY



made on samples taken from batches in trucks indicate that the standard deviation is about 2.5 percent at most plants, and the grading requirement is classified as minor. In this case, $\overline{X}'_g$ = 40 percent. From Table 6.5, we find that we should take four test samples from each LOT. Also, L.L. = $\overline{X}'_g - 1.29s'$ = 40 - 3.2 = 36.8 percent and U.L. = $\overline{X}'_g + 1.29s'$ = 40 + 3.2 = 43.2 percent.

The acceptance clause of the specification would be as follows: The desired average value of the percent of the total aggregate passing the No. 8 sieve is 40 percent. An acceptable mixture shall have an average percent passing that is not less than 36.8 percent and not more than 43.2 percent based on the results of four mechanical analyses made in accordance with AASHO T-30. Samples on which the tests are made shall be taken from the center of the batch, during discharge from the mixer and at randomly selected times, by use of the sampling method directed by the engineer.

The operating characteristics for this acceptance clause may be visualized by reference to Figure 6.14. Here the upper curve represents the distribution of the averages of the results of four tests with respect to the desired average $\overline{X}'_g$, which is 40 percent. The upper and lower acceptance limits are 36.8 and 43.2 percent. In theory, the probability of accepting a LOT for which the average abrasion loss is 40 percent would be only about 98 percent instead of 99.5 percent, because the distribution curve extends across both vertical lines representing the upper and lower acceptance limits. However, this condition has little practical significance. If the average loss for a LOT is about 37.5 percent, the probability of the rejection of the LOT would be 5 percent, which is the same as if there were only a lower limit. If the average value for a LOT is much larger or much smaller than the desired average, the probability of its rejection is relatively high. A LOT for which the average value is either 36.8 percent or 43.2 percent has only a 50/50 chance of being accepted. There is a strong incentive for the producer to install some method of quality control.

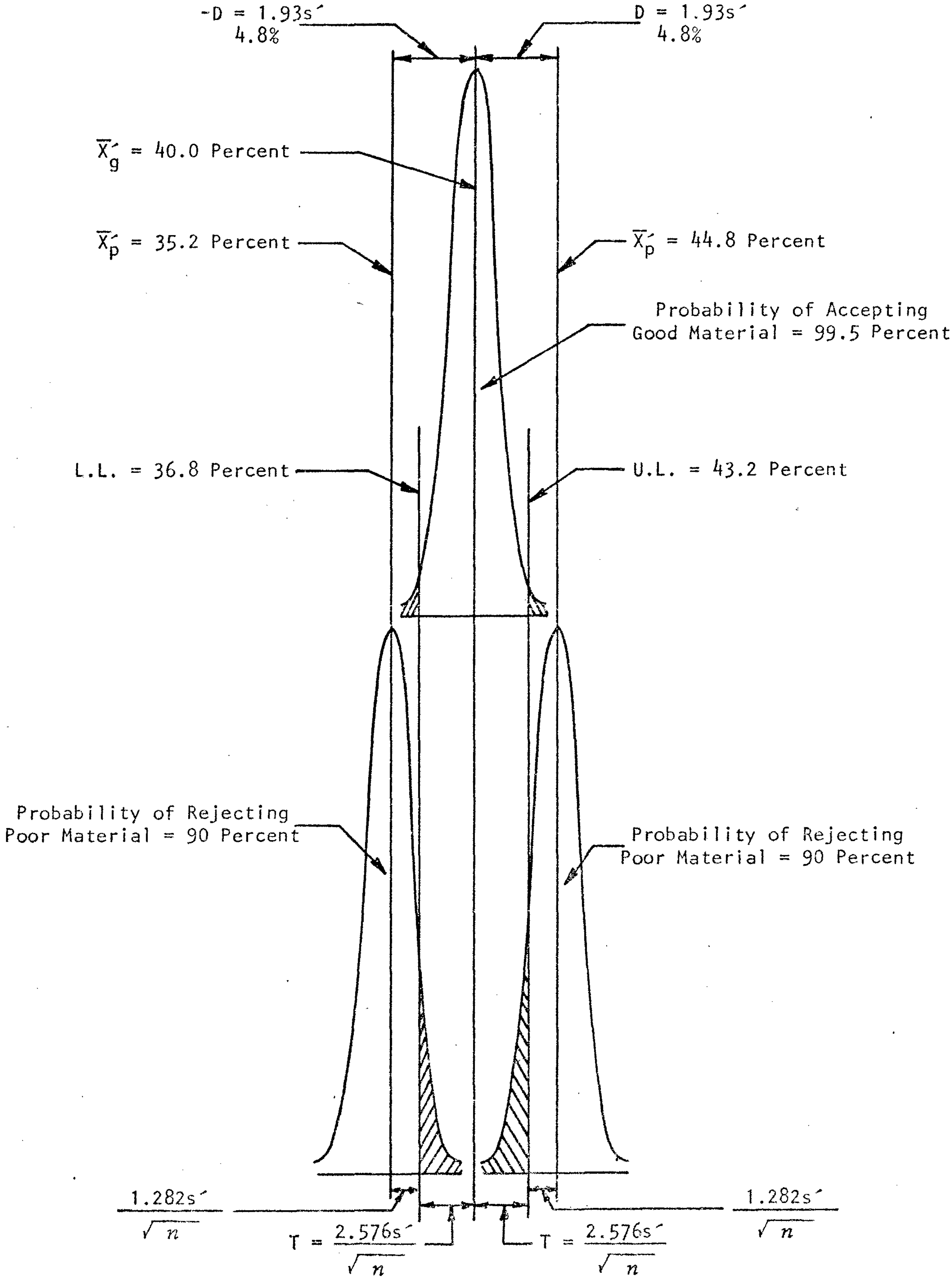
## 6.8 VISUALIZING ACCEPTANCE SITUATIONS

### 6.8.1 Use of Piles of Items

When an acceptance situation is being considered, it may be helpful to visualize the LOTs and to use graphical methods to show the relationship of buyer's and seller's risks, the position of the acceptance limit, and the number of measurements.

Figure 6.14

OPERATING CHARACTERISTICS - LOWER AND UPPER ACCEPTANCE LIMITS



For example, let us suppose that we have the task of designing an acceptance plan for structural concrete having a nominal 28-day compressive strength of 3000 psi (pounds per square inch). We know from previous experience that the standard deviation for 28-day compressive strengths is about 460 psi. We decide that, with this expected variability, concrete having an average strength of 4200 psi will be good and acceptable without doubt, and that concrete having an average strength of 3200 psi will be too poor to be acceptable. We want to design an acceptance plan that will provide a low buyer's risk of accepting concrete of poor quality for which the average strength is below 3200 psi and at the same time will not have a high seller's risk of rejecting good concrete for which the average strength is at least 4200 psi.

First, we can visualize a LOT of good or poor concrete as a pile of test cylinders. In Figure 6.15 the strength of each cylinder in hundreds of pounds per square inch is indicated by the number on the end of the cylinder. The LOT A represents good concrete, because the numbers are high. The LOT B represents poor concrete, because the numbers are relatively low. The cylinders in each pile are arranged so that, in general, the strength increases from left to right. If we did not have numbers on the ends of the cylinders and we selected a single cylinder at random from one of the LOTs and tested it, we might have great difficulty in deciding whether the concrete in the LOT is good or poor. If, for example, the strength of the cylinder we tested was 3900 psi, it could have come from either LOT, and there would be a high risk of making either a Type I error or a Type II error if we based a decision on this single test.

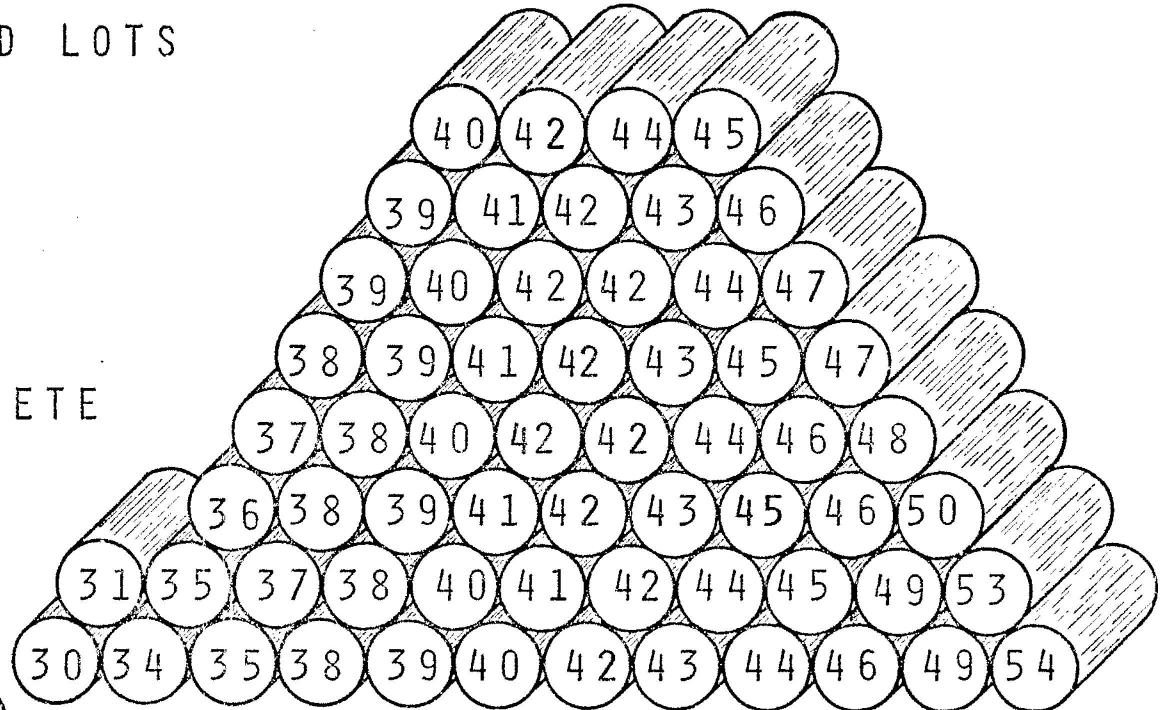
Let us suppose that as a first step in designing an acceptance plan, we decide that we want to take only a 1-percent risk of accepting a LOT of poor concrete and that we will base our acceptance decision on the average of the results of tests on two cylinders. Since we will be dealing with the standard deviation for this average, we divide the assumed deviation for individual test results by the square root of 2 and get

$$s_{\hat{\bar{x}}} = \frac{460}{\sqrt{2}} = 325 \text{ psi}$$

Since we want to limit to 1 percent the risk of making a Type II error by accepting poor material, we shall set our acceptance limit so as to include only 99 percent of the distribution of the poor LOT B in Figure 6.15. From a

Figure 6.15

# POOR AND GOOD LOTS

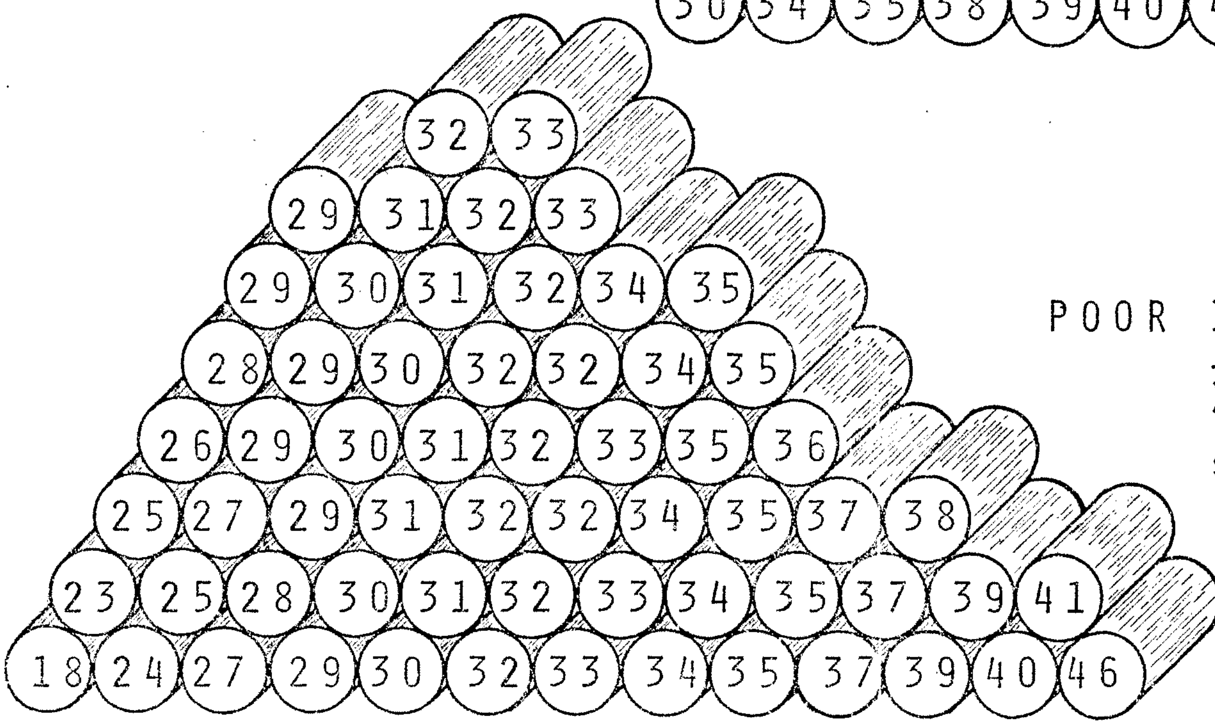



LOT A

GOOD 3000-PSI CONCRETE

$\overline{X}' = 4200$ PSI

$s' = 460$ PSI




LOT B

POOR 3000-PSI CONCRETE

$\overline{X}' = 3200$ PSI

$s' = 460$ PSI

single-tailed table of areas for the normal curve, we find that for 99 percent of the area under the curve the distance z should be 2.33 sigma units. So we set our lower acceptance limit at 3200 + 2.33(325) = 3200 + 757 = 3957 psi.

If we consider only Type II errors, our acceptance rule could be as follows: Make and test two cylinders from each LOT of concrete. If the average of the two test results is more than 3957 psi, accept the LOT.

We are now protected from making a Type II error more than once in 100 times, but we have not considered the probability of making a Type I error. Our acceptance limit, which is 3957 psi, is 4200 - 3957 = 243 psi below the average for the good material in LOT A in Figure 6.15. From a single-tailed table of areas for the normal curve, we find that for $z = \frac{243}{325} = 0.75$, the percent of the area that is included is 77. To get the percent of the area below the acceptance limit, we subtract 77 from 100 and obtain 23 percent. This means that we will be making a Type I error and rejecting good material once in about four times. Since the trial plan is not very satisfactory, at least as far as the seller of the concrete is concerned, we should make some change that will reduce the seller's risk.

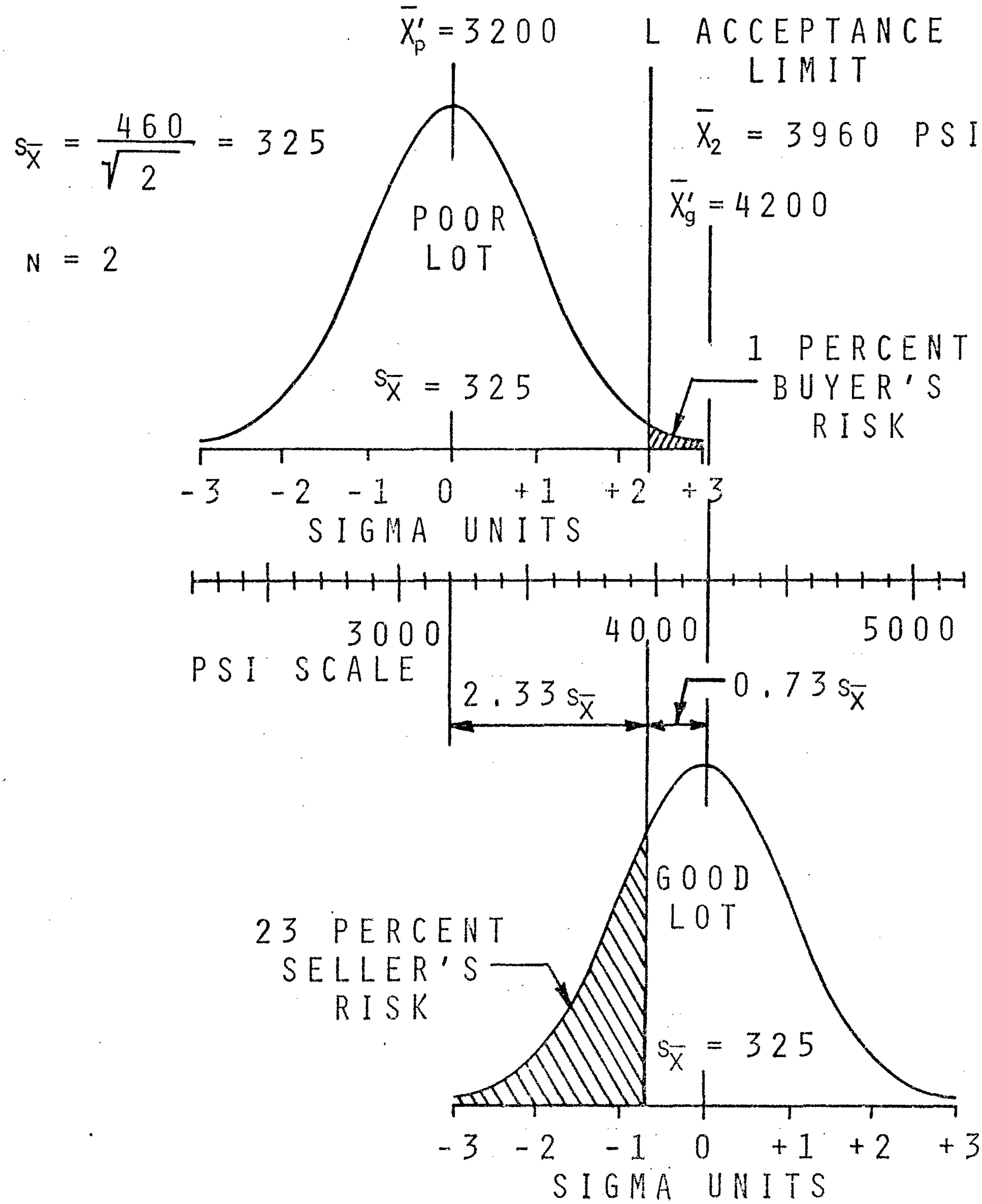
6.8.2 Use of Normal Curves

We can visualize an acceptance plan in either of two ways. One of these is to use sketches of the normal curve, as in Figure 6.16. The first step in drafting Figure 6.16 is to lay out the measurement scale representing pounds per square inch. For this purpose, it is convenient to use a No. 60 measuring scale that is graduated so that there are 6 main divisions to the inch and each main division contains ten subdivisions. Since the difference between the averages $\overline{X}_g'$ and $\overline{X}_p'$ is 4200 - 3200 = 1000 psi and since each curve will extend a distance equal to $3s_{\overline{x}}'$ on each side of the corresponding average, the length of the measurement scale should cover at least 1000 + 6(325) = 2950 psi. If it is assumed that each of the main divisions on the measuring scale is equivalent to 100 psi, it will be necessary to mark off 30 main divisions along a horizontal line located as shown in Figure 6.16. These division marks are numbered so that the values for $\overline{X}_p'$ and $\overline{X}_g'$ will be near the center of the measurement scale, and vertical lines are drawn at these two values, which are 3200 and 4200 psi.

The next step is to draw the normal curves for a poor LOT and a good LOT. The horizontal base line for each of these curves should extend a

Figure 6.16

# GOOD AND POOR LOTS OF CONCRETE



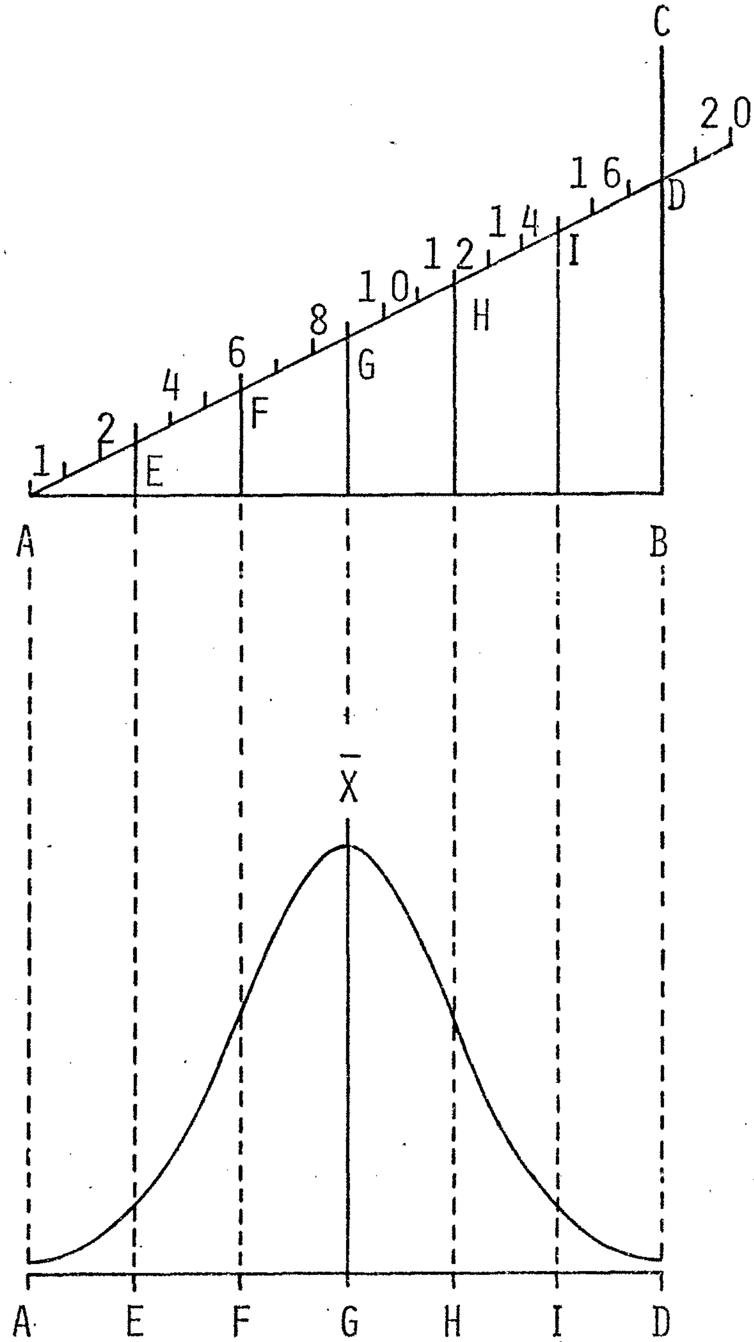
distance equal to $3s'_{\overline{X}}$ on each side of the centerline of the curve. So the total length of each base line should be 6(325) = 1950 psi or 19.5 main divisions on the measuring scale. However, since this length should be divided into six equal parts representing sigma units, the division marks should be located in the manner indicated in Figure 6.17. Here, the line AB having a length of 19.5 units on the No. 60 scale represents the base line. A vertical line BC is drawn through the right-hand end B. Then a No. 50 measuring scale graduated with 5 main divisions to the inch is placed in an inclined position so that it passes through the point A and intersects the vertical line BC at a point D at a main division that is some multiple of 6. In Figure 6.17, the eighteenth division is used. Since one-sixth of 18 is 3, intermediate points are marked along the inclined line at intervals of 3 divisions on the scale. These points are designated E, F, G, H, and I in Figure 6.17. To locate the division marks on the horizontal base line, vertical lines are drawn through the division points on the inclined line. The normal curve can now be sketched in. For sketching purposes, the height of the curve should be about 0.7 times the base width at the centerline or $\overline{X}'$. The relative heights at plus and minus $1s'$ should be about 0.4 and the heights at plus and minus $2s'$ should be about 0.1. The curve does not quite touch the base line at plus and minus $3s'$. After the normal curves have been drawn above the base lines, as shown in Figure 6.16, a vertical line is drawn at the acceptance limit, which in this case is 3960 psi. The area of the normal curve for a poor LOT that represents the buyer's risk and the area of the curve for a good LOT that represents the seller's risk are shown cross-hatched.

### 6.8.3 Use of Arithmetic Probability Paper

Another way of showing an acceptance plan visually is by the use of a special kind of graph paper, called arithmetic probability paper. Graduation lines on such paper are located as shown in Figure 6.18. The horizontal graduation lines are spaced at uniform distances and are not identified by printed numbers. Suitable numbers representing measurement units can be inserted by the person using the paper. The vertical graduation lines are located in accordance with a certain system and are numbered as shown to represent probabilities. There are two sets of numbers (one at the top and one at the bottom), which increase in opposite directions, and these sets are labeled seller's risk and buyer's risk. An important advantage of using arithmetic

Figure 6.17

# DIVISION OF LINE INTO EQUAL PARTS

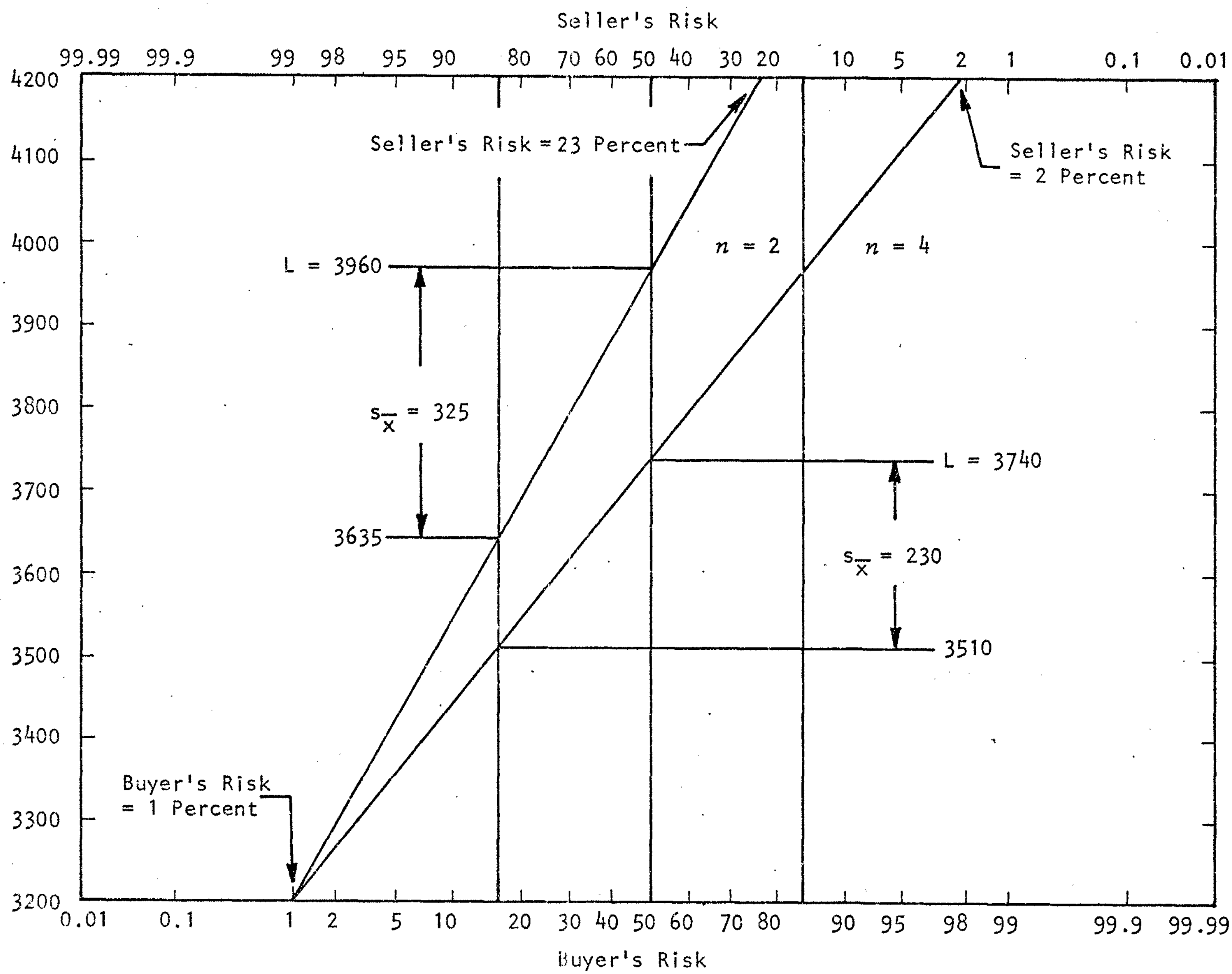



AD = 18
ON 50 SCALE

AB = 19.5
ON 60 SCALE

Figure 6.18

GRAPHICAL REPRESENTATION OF ACCEPTANCE PLAN



probability paper is that points which would lie on an operating characteristics curve on ordinary paper lie on a straight line on the special paper. There are several sources of arithmetic probability paper. Types available from two of these sources are Codex paper No. 3227, prepared by the Codex Book Company, Inc., which has 100 vertical divisions, and K & E paper No. 46 8000, prepared by the Keuffel and Esser Company, which has 90 vertical divisions.

To prepare the paper for use, the vertical scale is numbered to correspond to the number of measurement units between $\overline{X}_p'$ and $\overline{X}_g'$. Vertical lines are drawn across the sheet at the 16, 50, and 84 points of the probability scales. One of these scales is labeled as seller's risk and the other as the buyer's risk.

A complete acceptance plan can be shown on arithmetic probability paper by drawing a single straight line. One point on this line is located where the value of the measurement scale is equal to $\overline{X}_p'$ and the value on the probability scale is equal to the buyer's risk. Another point on the line is located where the value of the measurement scale is equal to $\overline{X}_g'$ and the value on the probability scale is equal to the seller's risk. For the values used in the example considered here, one point is at 3200 psi and 1 percent, and the other point is at 4200 psi and 23 percent. The line drawn through these points in Figure 6.18 is the line for $n$ = 2. This line crosses the vertical line through 50 on the probability scale at the value on the measurement scale equal to the acceptance limit L, or 3960 psi. It also crosses the vertical line through 16 on the scale for buyer's risk at a point indicating a value on the measurement scale equal to 3635 psi. The difference between these two values, or 3960 - 3635 = 325 psi, is the standard deviation $s_{\overline{X}}'$ of the average of two measurements.

We could have drawn the inclined line for the acceptance plan without knowing the seller's risk. It would only have been necessary to set a straightedge through the point at 3200 psi on the measurement scale and at 1 percent on the scale for the buyer's risk. We could then swing the straightedge to a position for which the difference between the values on the measurement scale at the vertical line through 16 and 50 on the scale for the buyer's risk would be 325 psi. Thus, 3960 - 3635 = 325.

Since the acceptance plan designed in Section 6.8.1 would have a high seller's risk, it is not very satisfactory. One way to reduce the risk is to increase the number of tests that are made to determine the average measured result. If we decide to take the result of a test as the average of the strengths of three cylinders (instead of two, as previously decided), the standard deviation would be

$$s'_{\bar{x}} = \frac{460}{\sqrt{3}} = 266 \text{ psi}$$

If we want to keep the buyer's risk at 1 percent, we can determine the corresponding seller's risk by drawing a line in Figure 6.18 in the following position. A straightedge is set so as to pass through the point at 3200 psi on the measurement scale and 1 on the scale for buyer's risk. The straightedge is swung so that the difference between the values on the measurement scale at the vertical lines through 16 and 50 on the scale for buyer's risk would be 266 psi. The straightedge would then cut the horizontal line representing 4200 psi on the measurement scale at a point representing 7 percent on the scale for seller's risk. Also the corresponding acceptance limit, or the value on the measurement scale where the inclined line cuts the vertical line through 50, would be 3820 psi. If we based our acceptance of a LOT on the average of three test results, we might, on the average, reject a good LOT one time in 14.

We can reduce the seller's risk still further by basing our acceptance of a LOT on the average of four test results. In this case, the standard deviation would be

$$s'_{\bar{x}} = \frac{460}{\sqrt{4}} = 230 \text{ psi}$$

The acceptance plan would then be represented in Figure 6.18 by the inclined line for $n = 4$. In this case, the seller's risk would be about 2.3 percent and the acceptance limit would be 3740 psi.

Our acceptance rule would be as follows: Make and test four cylinders from each LOT of concrete. If the average of the four test results is more than 3740 psi, accept the LOT. If the average is less, reject the LOT.

If we apply this rule to the acceptance of every LOT of concrete, we will, on the average, accept poor LOTs one time in 100 and reject good LOTs about one time in 44.

### 6.8.4 Replotting Acceptance Plan

After we have designed a satisfactory acceptance plan by using arithmetic probability paper, we may want to be able to visualize the acceptance situation better. To do this we can sketch normal curves for the good and poor distributions and show the acceptance limit, as was done in Figure 6.16 for $n = 2$. We may also want to plot an operating-characteristics curve on ordinary paper. No computations need be made for this purpose, because the straight lines for $n = 2$ and $n = 4$ in Figure 6.18 really are equivalent to operating-characteristics curves. To replot an OC curve on ordinary paper, it is only necessary to note the values on the probability scale corresponding to the selected values on the measurement scale. The required values for the line for $n = 4$ in Figure 6.18 are listed in Table 6.6.

Table 6.6

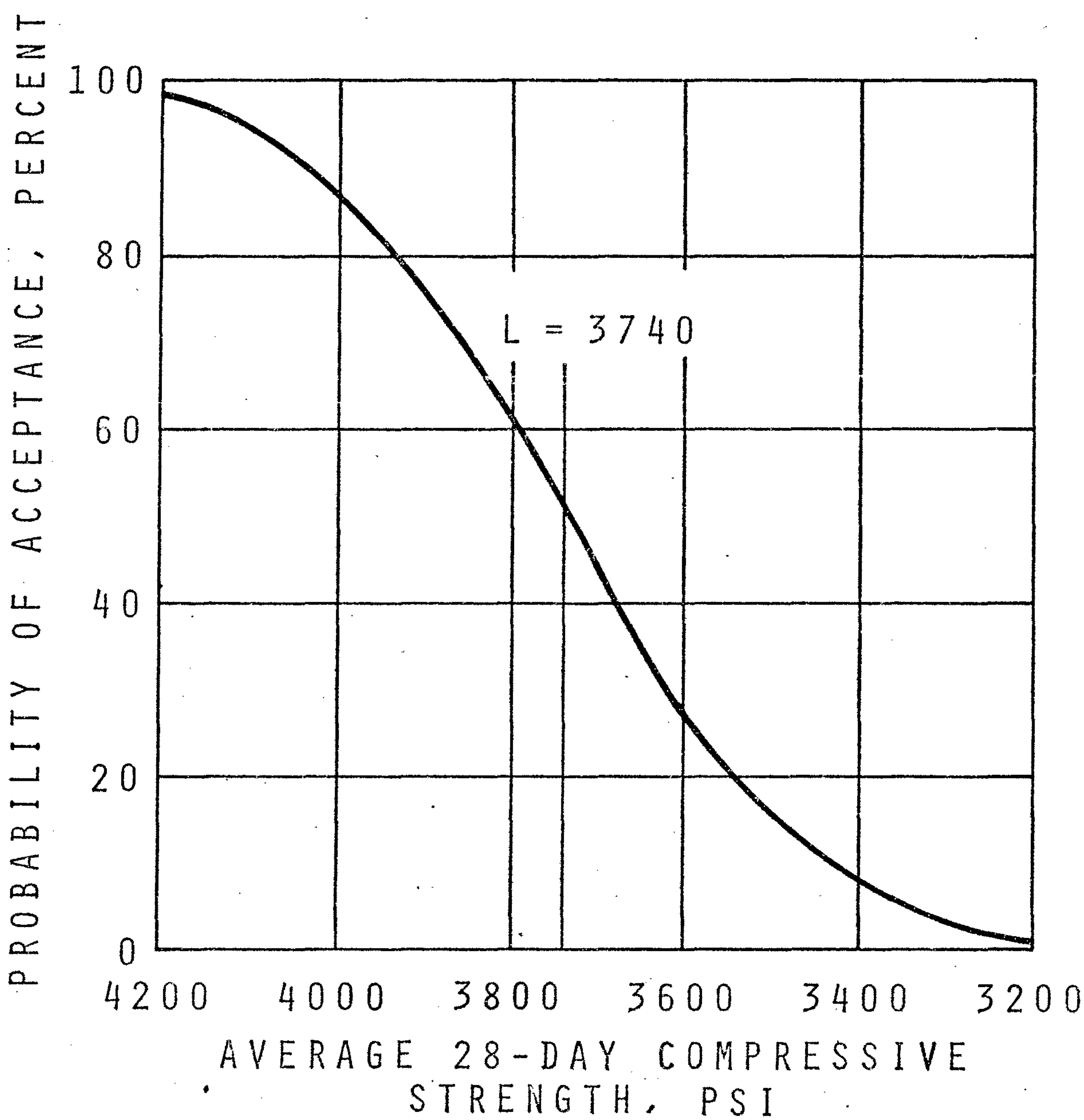
VALUES FOR OPERATING-CHARACTERISTICS CURVE IN FIGURE 6.19

| LOT Average | Probability of Acceptance |
|---|---|
| 3200 | 1 |
| 3300 | 3 |
| 3400 | 7 |
| 3500 | 15 |
| 3600 | 28 |
| 3700 | 44 |
| 3800 | 60 |
| 3900 | 76 |
| 4000 | 87 |
| 4100 | 94 |
| 4200 | 98 |

The operating-characteristics curve, as it would be plotted on ordinary paper, is shown in Figure 6.19. It should be understood that these probabilities would be correct if the processes for producing concrete, sampling, and testing remain in statistical control, and the standard deviation is very nearly 460 psi.

Figure 6.19

# OPERATING-CHARACTERISTICS CURVE FOR ACCEPTANCE OF LOTS OF STRUCTURAL CONCRETE WHEN s = 460 AND n = 4



## 6.9 SEQUENTIAL SAMPLING PLAN

### 6.9.1 Risks with Fixed Number of Measurements

As stated in Section 6.6.2, when we can assume that we know the standard deviation $s'$ of measurements made under certain conditions, we can calculate a fixed number of required tests by the use of equation 6.2. When we do this for a particular plan, we fix both the seller's ($\alpha$) and buyer's ($\beta$) risks. We would prefer to have a low value for both of these risks, but if the standard deviation of the measurements is large, the number of measurements required for an acceptance decision may be more than is desirable or economical. Suppose, for example we decide that the compacted density of the asphaltic concrete wearing course is critical on a particular project. Our target value $\overline{X}'_g$ is 94 percent of the maximum theoretical density and we consider that a density of 92 percent ($\overline{X}'_p$) would not be acceptable. We would like to have only a one percent $\beta$ risk of accepting pavement having an average density of $\overline{X}'_p$ and would prefer that the $\alpha$ risk of rejecting pavement having an average density of $\overline{X}'_g$ be limited to five percent. From a large number of tests on cores taken from similar pavement we have found that the standard deviation $s'$ of the measurements is 1.6 percent. To find the required number of cores to be taken from each LOT consisting of one day's construction we use equation 6.2

$$n = \left(\frac{(z_1 + z_2)s'}{|\overline{X}'_g - \overline{X}'_p|}\right)^2$$

$$= \left(\frac{(1.645 + 2.326)1.6}{|94 - 92|}\right)^2$$

$$= \left(\frac{(3.971)1.6}{2}\right)^2$$

$$= \left(\frac{6.354}{2}\right)^2$$

$$= 3.177^2$$

$$\approx 10$$

Our lower acceptance limit will be

$$L = \overline{X}'_p + \frac{z_2 s'}{\sqrt{n}}$$

$$= 92 + \frac{2.326(1.6)}{\sqrt{10}}$$

$$= 92 + \frac{3.722}{3.162}$$

$$= 92 + 1.18$$

$$= 93.18 \text{ percent of theoretical density}$$

Our acceptance rule will be: Take 10 cores at random locations from each LOT. If the average density of the 10 cores equals or exceeds 93.18 percent of theoretical density accept the LOT. If the 10-core average is less than 93.18 percent of theoretical density reject the LOT and require remedial measures.

### 6.9.2 Risks with Sequential Sampling Single Acceptance Limit

The procedure of Section 6.9.1 results in a perfectly good acceptance plan except for one thing. We may not consider it practical to expend the cost and effort required in the taking and measuring 10 cores for each day's construction and may also be reluctant to cut and patch 10 holes in an otherwise unblemished pavement. On the other hand, we may not believe it advisable to take a greater risk of accepting poor construction. A way of avoiding these difficulties is to start out with a small number of measurements but keep our buyer's ($\beta$) risk constant. When we do this the seller's ($\alpha$) risk will increase, but if the quality is sufficiently better than our target value, we may find it possible to accept a LOT on the basis of a greatly reduced number of samples.

If we follow this plan we will have a different acceptance limit L and a different seller's ($\alpha$) risk for each number of measurements. For example, suppose that we take three cores initially. Then

$$L = \bar{X}'_p + \frac{z_2 s'}{\sqrt{n}}$$

$$= 92 + \frac{2.326(1.6)}{\sqrt{3}}$$

$$= 92 + \frac{3.722}{1.732}$$

$$= 92 + 2.15$$

$$= 94.15$$

To find the seller's ($\alpha$) risk we use the equation

$$z = \frac{(\bar{X}'_g - L)\sqrt{n}}{s}$$

$$= \frac{(-0.15)\sqrt{3}}{1.6}$$

$$= \frac{-0.26}{1.6}$$

$$= -0.1625$$

From a one-tailed table of areas of the normal curve we find that a z value of -0.1625 is equivalent to an $\alpha$ risk of about 56 percent. Using this procedure we can construct a table, similar to Table 6.7, showing the acceptance limits and seller's risks for different numbers of measurements.

Using this plan, only a minimum number of measurements are required if the construction is expected to be of higher quality than the target value. The acceptance procedure would be: Estimate the actual density of the compacted pavement and take the corresponding number of cores as indicated in Table 6.7. If the average of the measured densities is equal to or exceeds the acceptance number in the table, accept the LOT. If the average density is less than the acceptance number, take additional cores, up to a limit of 10, until a decision is made to accept or reject the LOT on the basis of the average of the total number of cores.

Table 6.7

ACCEPTANCE LIMITS FOR DENSITY OF COMPACTED ASPHALT WEARING COURSE

$\overline{X}'_g$ = 94 Percent $\overline{X}'_p$ = 92 Percent $s'$ = 1.6 Percent

Buyer's Risk ($\beta$) = 1 Percent

| Number of Measurements ($n$) | Acceptance Limit (L) (Average Percent of Theoretical Density) | Seller's Risk ($\alpha$) (Percent) |
|---|---|---|
| 2 | 94.63 | 71 |
| 3 | 94.15 | 56 |
| 4 | 93.86 | 43 |
| 5 | 93.66 | 32 |
| 6 | 93.52 | 23 |
| 7 | 93.41 | 16 |
| 8 | 93.32 | 12 |
| 9 | 93.24 | 8 |
| 10 | 93.18 | 5 |

It should be noted that the seller's ($\alpha$) risks in Table 6.7 apply only when the actual process average of the compacted density is equal to the target value $\overline{X}'_g = 94$. If the actual process average was 96 percent of the theoretical density, and acceptance was based on the average of three cores, only about one out of 50 of these averages would be below the acceptance number of 94.15.

This sequential sampling approach is applicable in cases where material or construction remains available for repeated sampling. It can also be used by taking the maximum number of random samples and testing only that number required to reach an acceptance decision.

# CHAPTER 7

# COMPUTATIONS ON DATA FROM SMALL SAMPLES

## 7.1 THE t DISTRIBUTION

### 7.1.1 Need for t Distribution

In the previous discussions, we have generally assumed that we had a good estimate of the true standard deviation or s′. In actual highway construction, this is seldom the case. In order that an estimated value of s′ may be reliable, it must be based on a large number of measurements pertaining to the output of a process that is in statistical control. We can then assume that the variability of the measurements has remained constant over a suitable period of time, and that the variability of future measurements will remain the same.

Since so many factors affect the variability of measurements pertaining to highway materials and construction, we can seldom know the exact value of the true standard deviation s′ of all the possible measurements for a LOT. Sometimes we may have a good general idea in regard to the probably value of s′ for a certain type of measurement on a particular material. However, when we make an acceptance decision for a LOT, we usually have to base our decision on measurements made on a few samples taken from the LOT. When this is the case, and we do not know the true value of s′, we must use as an estimate of the standard deviation s′ the value s that we calculate from the measurements on the samples. It has been found that, under these conditions, we obtain more exact estimates of risks and percentages within limits if we base our calculations on what is called the t distribution.

---

Note: The t distribution was originally developed by an amateur statistician, named W. S. Gosset, in 1908. He was a chemist who worked for a brewery. Because the management of the brewery did not approve of publication of a technical paper by an employee, he used the signature "Student" on the paper that described the concept. For this reason, reference is often made to "Student's t Distribution".

### 7.1.2 Comparison of Normal Distribution and t Distribution

The t distribution is very much like the normal distribution in some ways. However, the shape of the curve representing the normal distribution is determined entirely by the size of the standard deviation. The shape of the curve representing the t distribution depends on the number of *degrees of freedom.* For our purposes in this part of the Handbook, the number of degrees of freedom is simply one less than the number of measurements. If the notation d.f. is used to stand for degrees of freedom and $n$ denotes the number of measurements, d.f. = $n$ - 1. When there is only a small number of measurements, the curve for the t distribution is flatter than the curve for the normal distribution and its tails are more spread out. In Figure 7.1 are shown points on the normal curve and points on the curve for the t distribution when $n$ = 3 and d.f. = 2. When there is a large number of measurements, the t distribution is the same as the normal distribution.

In Table 7.1 are given relative factors for determining the vertical distances from the horizontal reference axis to points on the curve for the t distribution. The values in each horizontal row in this table apply to a particular number $n$. You will see that the values for $n = \infty$ in the bottom row are the values for the normal distribution.

Since the curve for the t distribution based on a small number of measurements is flatter and wider than the curve for the normal distribution, more of the area is in the tails. As a result, if a vertical line is drawn at a certain number of sigma units from the centerline of the curve for such a t distribution, the percent of the area under the curve between the centerline and the vertical line would be less than the corresponding percent for the curve for the normal distribution.

### 7.1.3 Table of Areas for t Distribution

Percents of areas under the t-distribution curve are shown in Table 7.2. Since we are usually interested only in the higher percents of the total area for the t distribution, the table does not give values for less than 80 percent of a symmetric area like that shown cross-hatched in the left-hand sketch or for less than 90 percent of an unsymmetric area like that shown cross-hatched in the right-hand sketch. Also, the one-tailed table and the two-tailed table are combined, and the proper column to use is shown by the sketches and the values in the rows above the body of the table designated $P_1$

Figure 7.1

## NORMAL CURVE AND *t* DISTRIBUTION CURVE
## FOR N = 3 AND D. F. = 2

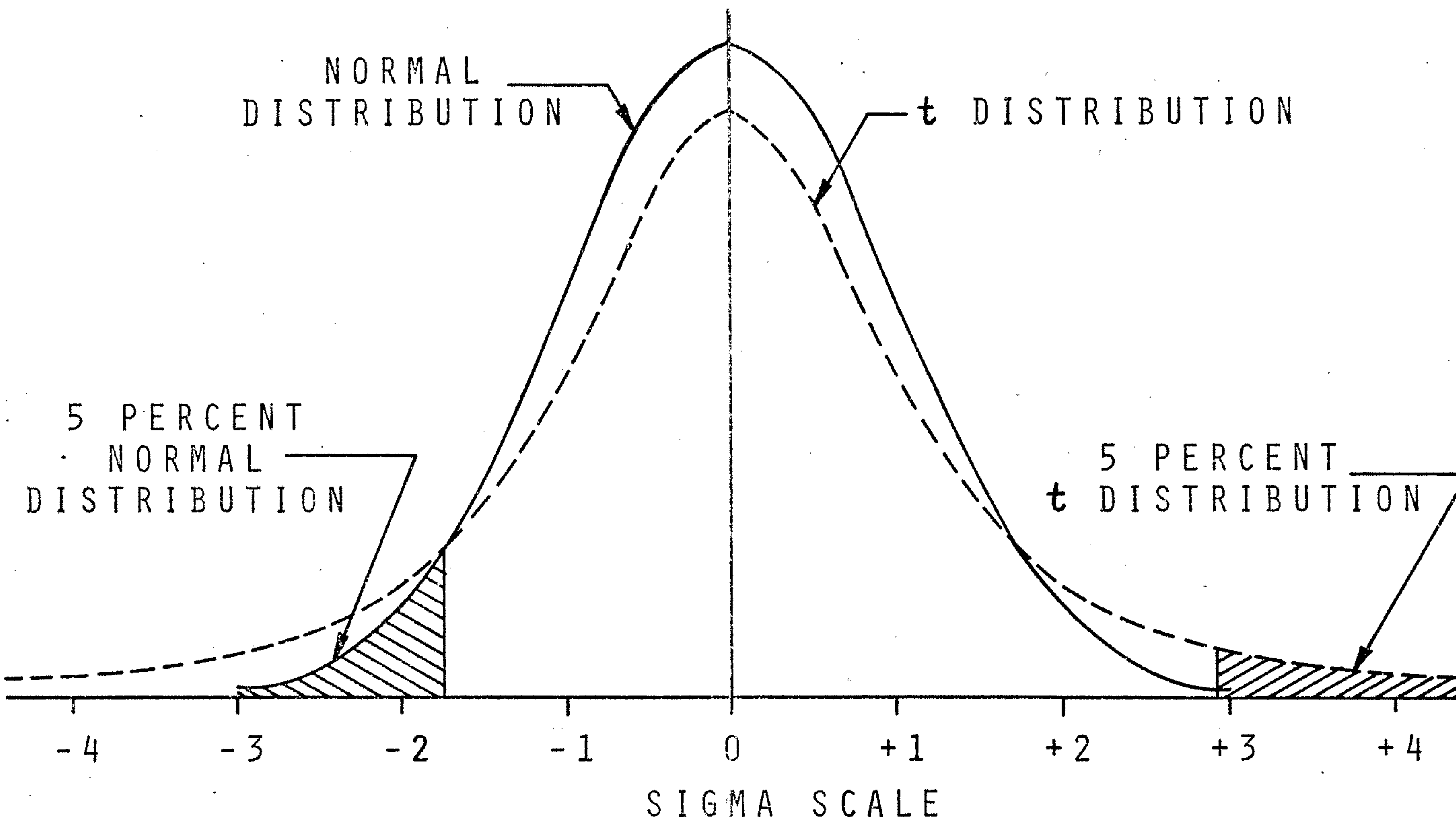


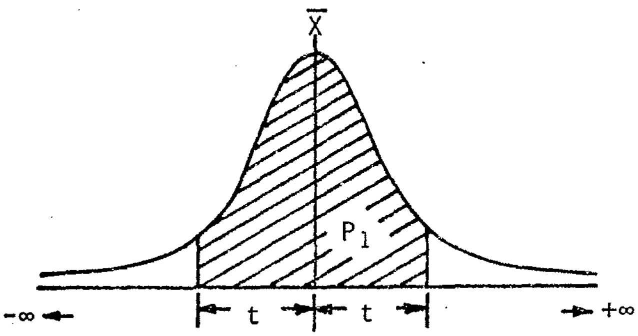
Table 7.1

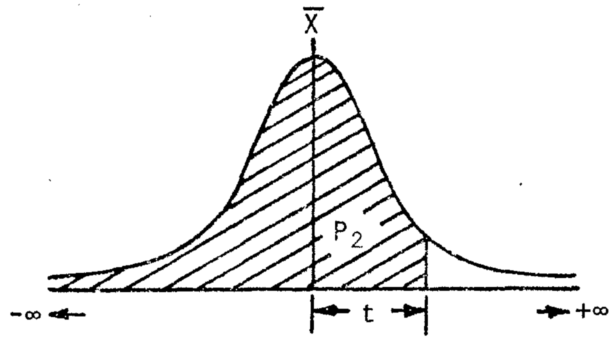
RELATIVE FACTORS FOR PLOTTING t-DISTRIBUTION CURVES

| | Plus or Minus Sigma Distance from $\overline{X}$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 0.00 | 0.20 | 0.40 | 0.60 | 0.80 | 1.00 | 1.20 | 1.40 | 1.60 | 2.00 | 3.00 | 4.00 |
| 2 | 0.32 | 0.31 | 0.28 | 0.23 | 0.19 | 0.16 | 0.13 | 0.11 | 0.09 | 0.07 | 0.03 | 0.02 |
| 3 | 0.35 | 0.34 | 0.32 | 0.28 | 0.23 | 0.19 | 0.16 | 0.13 | 0.10 | 0.07 | 0.03 | 0.01 |
| 4 | 0.37 | 0.36 | 0.33 | 0.29 | 0.25 | 0.21 | 0.17 | 0.13 | 0.11 | 0.07 | 0.02 | 0.01 |
| 5 | 0.38 | 0.37 | 0.34 | 0.30 | 0.26 | 0.21 | 0.17 | 0.14 | 0.11 | 0.07 | 0.02 | 0.01 |
| 7 | 0.38 | 0.37 | 0.35 | 0.31 | 0.27 | 0.22 | 0.18 | 0.14 | 0.11 | 0.06 | 0.02 | ---- |
| 10 | 0.39 | 0.38 | 0.36 | 0.32 | 0.28 | 0.23 | 0.18 | 0.15 | 0.11 | 0.06 | 0.01 | ---- |
| $\infty$ | 0.40 | 0.39 | 0.37 | 0.33 | 0.29 | 0.24 | 0.19 | 0.15 | 0.11 | 0.05 | ---- | |

Table 7.2

AREAS OF THE t-DISTRIBUTION CURVE




| Percent of Area | $P_1$ | 80.0 | 90.0 | 95.0 | 98.0 | 99.0 |
|---|---|---|---|---|---|---|
| | $P_2$ | 90.0 | 95.0 | 97.5 | 99.0 | 99.5 |
| | d.f. | Values of t in Sigma Units | | | | |
| | 1 | 3.078 | 6.314 | 12.706 | 31.821 | 63.657 |
| | 2 | 1.886 | 2.920 | 4.303 | 6.965 | 9.925 |
| | 3 | 1.638 | 2.353 | 3.182 | 4.541 | 5.841 |
| | 4 | 1.533 | 2.132 | 2.776 | 3.747 | 4.604 |
| | 5 | 1.476 | 2.015 | 2.571 | 3.365 | 4.032 |
| | 6 | 1.440 | 1.943 | 2.447 | 3.143 | 3.707 |
| | 7 | 1.415 | 1.895 | 2.365 | 2.998 | 3.499 |
| | 8 | 1.397 | 1.860 | 2.306 | 2.896 | 3.355 |
| | 9 | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 |
| | 10 | 1.372 | 1.812 | 2.228 | 2.764 | 3.169 |
| | 11 | 1.363 | 1.796 | 2.201 | 2.718 | 3.106 |
| | 12 | 1.356 | 1.782 | 2.179 | 2.681 | 3.055 |
| | 13 | 1.350 | 1.771 | 2.160 | 2.650 | 3.012 |
| | 14 | 1.345 | 1.761 | 2.145 | 2.624 | 2.977 |
| | 15 | 1.341 | 1.753 | 2.131 | 2.602 | 2.947 |
| | 16 | 1.337 | 1.746 | 2.120 | 2.583 | 2.921 |
| | 17 | 1.333 | 1.740 | 2.110 | 2.567 | 2.898 |
| | 18 | 1.330 | 1.734 | 2.101 | 2.552 | 2.878 |
| | 19 | 1.328 | 1.729 | 2.093 | 2.539 | 2.861 |
| | 20 | 1.325 | 1.725 | 2.086 | 2.528 | 2.845 |
| | 21 | 1.323 | 1.721 | 2.080 | 2.518 | 2.831 |
| | 22 | 1.321 | 1.717 | 2.074 | 2.508 | 2.819 |
| | 23 | 1.319 | 1.714 | 2.069 | 2.500 | 2.807 |
| | 24 | 1.318 | 1.711 | 2.064 | 2.492 | 2.797 |
| | 25 | 1.316 | 1.708 | 2.060 | 2.485 | 2.787 |
| | 26 | 1.315 | 1.706 | 2.056 | 2.479 | 2.779 |
| | 27 | 1.314 | 1.703 | 2.052 | 2.473 | 2.771 |
| | 28 | 1.313 | 1.701 | 2.048 | 2.467 | 2.763 |
| | 29 | 1.311 | 1.699 | 2.045 | 2.462 | 2.756 |
| | 30 | 1.310 | 1.697 | 2.042 | 2.457 | 2.750 |
| | 40 | 1.303 | 1.684 | 2.021 | 2.423 | 2.704 |
| | 60 | 1.296 | 1.671 | 2.000 | 2.390 | 2.660 |
| | 120 | 1.289 | 1.658 | 1.980 | 2.358 | 2.617 |
| | ∞ | 1.282 | 1.645 | 1.960 | 2.326 | 2.576 |

and $P_2$. The distance t is expressed in sigma units. The symbol ∞ stands for *infinity*.

When we have a certain specified symmetric area and we want to find the distance t from the centerline of the curve to each vertical line bounding the area, we would use the sketch for $P_1$ and the value of $P_1$ in the upper of the two rows of numbers above the table. If we want to include 95 percent of the total area under the curve in a symmetric portion, we would use the values of t in the column below the number 95.0 in the row for $P_1$. For a small number of measurements, the curve would be flat and wide, and the value of t would be large. For instance, if $n = 2$, the number of degrees of freedom would be 2 - 1 = 1 and the value of t would be 12.7 sigma units. As the number of measurements is increased, the value of t becomes smaller. You should note that the last value in the column is 1.960, which is the same as the value of z in a table for the normal curve.

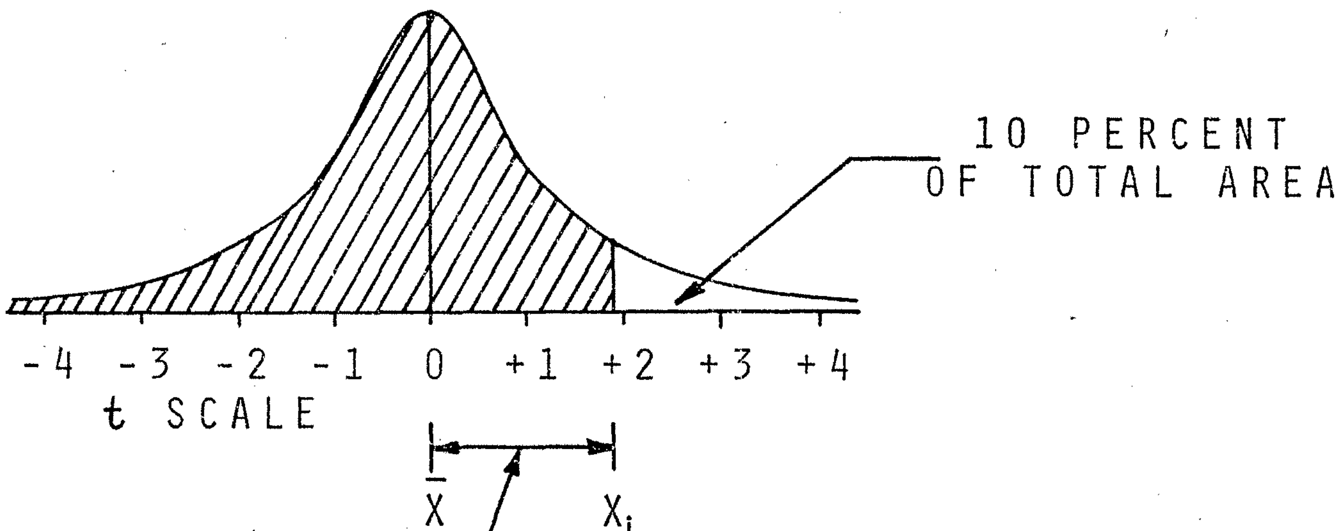
Now let us suppose that we have an unsymmetric area and the required percent $P_2$ is 95 percent. In this case, we would use the values of t in the column under 95.0 in the row for $P_2$. If the number of degrees of freedom is 1, the value of t would be about 6.3 sigma units. If the number of degrees of freedom is extremely large, the value of t would be 1.645, which is the same as the value of z in a table for the normal curve.

You may be able to obtain a clearer picture of the relationship of the value of t to the percent of the total area under the curve and the number of degrees of freedom by examining Figure 7.2. The curve in the sketch represents the t distribution. If the number of degrees of freedom was 2, as assumed for the curve shown, 90 percent of the total area would be to the left of a vertical line located at a distance of 1.886 sigma units to the right of the centerline. However, if the number of degrees of freedom was 10, 90 percent of the total area would be to the left of a vertical line located at a distance of 1.372 sigma units to the right of the centerline.

The important thing to remember is that when we must compute the standard deviation s from measurements on a sample, we use the t distribution to get the right answer. We use the normal distribution only when we have made such a large number of measurements that we can assume that we know the true value of s´.

Figure 7.2

# RELATIONSHIP OF $t$ TO PERCENT OF TOTAL AREA UNDER CURVE AND D.F.




$$t = \frac{X_i - \bar{X}}{s} = 1.886 \text{ FOR D.F.} = 2$$

1.372 FOR D.F. = 10

1.296 FOR D.F. = 60

1.282 FOR D.F. = $\infty$

## 7.2 USES OF THE t DISTRIBUTION

### 7.2.1 Known and Unknown Standard Deviation

Let us suppose that we want to estimate the average compressive strength of a group of four cylinders made from a certain LOT of concrete that will be exceeded 95 percent of the time. We shall consider two possible conditions. In one case, we shall suppose that we know the true standard deviation $s'$.

In the other case, we shall suppose that we have no information in regard to the true average $\overline{X}'$ or the true standard deviation $s'$.

Example 7.1. Let us suppose that we have made tests on over 200 cylinders from a LOT of concrete and have found from the results of those tests that the average strength is 4000 psi and the standard deviation is 500 psi. What is the average strength of four cylinders from the LOT that will be exceeded 95 percent of the time?

Solution. Since over 200 measurements have been made, we can assume that $\overline{X}'$ = 4000 psi and that $s'$ for a single test result is equal to 500 psi, and we can base our calculations for an acceptance plan on the normal distribution. From the normal table for one-tailed areas, we see that a vertical line located at 1.645 sigma units from the centerline of the curve will cut off 95 percent of the area.

The value of $s'$ for the average of the results of tests on four cylinders will be

$$s'_{\overline{X}} = \frac{500}{\sqrt{4}} = \frac{500}{2} = 250 \text{ psi}$$

The average strength L of four cylinders that we estimate will be exceeded 95 percent of the time can be found by the equation

$$L = \overline{X}' - zs'_{\overline{X}} \qquad (7.1)$$

In this example,

$$L = 4000 - 1.645(250)$$

$$= 4000 - 411$$

$$= 3589 \approx 3590 \text{ psi}$$

We can therefore say that we are 95 percent confident that the average of the compressive strengths of a group of four cylinders sampled from the same LOT will be more than about 3590 psi.

<u>Example 7.2</u>. We shall suppose that we have no information in regard to the true average $\overline{X}'$ or the true standard deviation $s'$ for the compressive strength of cylinders sampled from the LOT in Example 7.1 and that we test four cylinders sampled from random locations in the LOT. By using the test results on these cylinders, we find that the computed average $\overline{X}$ is 4000 psi and the computed standard deviation s is 500 psi. We want to estimate the average strength of four future test cylinders that will be exceeded 95 percent of the time.

<u>Solution</u>. Since we do not know the true standard deviation $s'$, we must base our calculations on the t distribution. In this case, we want to use the values in Table 7.2 for a one-tailed, or $P_2$, situation. So we shall find our value of t in the column under the number 95.0 in the row for $P_2$. Since we have four measurements, the number of degrees of freedom is 4 - 1 = 3. In the column of the table for $P_2 = 95.0$ and the horizontal line for d.f. = 3, we find that t = 2.353. Then the average strength L of four cylinders that we estimate will be exceeded 95 percent of the time can be found by the equation

$$U = \overline{X} + ts\sqrt{\frac{1}{m} + \frac{1}{n}} \quad \text{or} \quad L = \overline{X} - ts\sqrt{\frac{1}{m} + \frac{1}{n}} \qquad (7.2)$$

Where:

m = number of measurements in future sample

n = number of measurements in present sample

Then

$$L = 4000 + (-2.353)(500)\sqrt{\frac{1}{4} + \frac{1}{4}}$$

$$= 4000 - 1176.5\sqrt{0.50}$$

$$= 4000 - 1176.5(0.707)$$

$$= 4000 - 832$$

$$= 3168 \approx 3170$$

Under these conditions, we can say that the average of the compressive strengths of the next group of four cylinders sampled from the same LOT will be more than about 3170 psi with a 95 percent probability.

You should note that the estimate in Example 7.2 is more cautious than the one we made in Example 7.1 when we assumed that we knew the true standard deviation s′ and the true average $\overline{X}'$. This practice makes good sense, since we have less information on which to base our estimate. In general, when we do not know s′ and $\overline{X}'$ and must use s and the t distribution, we will always obtain wider confidence limits and a greater required number of tests than we would if we based our acceptance plan on the normal distribution. This is the price we pay for not having enough information.

### 7.2.2 Calculating Confidence Limits

In section 2.10.2 we discussed the procedure for placing confidence limits on an average by use of the range when we take the average of only a small number of measured results. We can apply a similar procedure for any number of measured results by use of the table for the t distribution.

Example 7.3. Suppose that we have made 16 stability tests on samples from a LOT of hot-mix bituminous concrete, and that the average of the measured results is 1600 pounds and the standard deviation is 200 pounds. We want to decide on limits between which the true average $\overline{X}'$ should lie 95 percent of the time.

Solution. In this case we use the equations

$$\text{U.C.L.} = \overline{X} + \frac{1}{\sqrt{n}}(s) \quad \text{and} \quad \text{U.C.L.} = \overline{X} - \frac{1}{\sqrt{n}}(s) \qquad (7.3)$$

To find the value of t from Table 7.2, we consider the value for $P_1$. In the column of the table under 95.0 in the top row and on the horizontal line for d.f. = 15, we find that the value of t is 2.131 sigma units. Hence,

$$U.C.L. = 1600 + \frac{2.131}{\sqrt{16}}(200)$$

$$= 1600 + \frac{426.2}{4}$$

$$= 1600 + 107$$

$$= 1707 \approx 1710 \text{ pounds}$$

and

$$L.C.L. = 1600 - 107$$

$$= 1493 \approx 1490 \text{ pounds}$$

We can therefore say that we are 95 percent confident that the average of all possible stability measurements on a LOT of material would be between about 1490 and 1710 pounds.

### 7.2.3 Calculating Statistical Tolerance Limits

Another use for the table for the t distribution is to compute tolerance limits. Confidence limits define the interval within which we expect the true average $\overline{X}'$ for a certain characteristic to fall some specified number of times in 100. Statistical tolerance limits define the interval within which we expect some specified percent of future measured results to lie on the average or with some specified probability.

In many situations a statement pertaining to the confidence limits for the average of measured results is not very useful. It would be much better to have an estimate of the limits between which some percent of future measured results will lie some number of times in 100. If we know the true standard deviation, we can say that we would expect about 95 percent of future measured results to be included between $\overline{X}' + 2s'$ and $\overline{X}' - 2s'$. This statement is no longer true when we calculate the standard deviation from the measured results of tests on samples. However, we can calculate the upper tolerance limit U.T.L. and the lower tolerance limit L.T.L. which will include 95 percent of future measured results, on the average, by use of the equations

$$U.T.L. = \overline{X} + t\sqrt{\frac{n+1}{n}}(s) \quad \text{and} \quad L.T.L. = \overline{X} - t\sqrt{\frac{n+1}{n}}(s) \qquad (7.4)$$

In Example 7.3 in Section 7.2.2, we calculated the 95-percent confidence limits to be 1710 and 1490 pounds. The corresponding tolerance limits between which 95 percent of the test results would lie on the average would be as follows:

$$\text{U.T.L.} = \overline{X} + t\sqrt{\frac{n+1}{n}}(s)$$

$$= 1600 + 2.131\sqrt{\frac{17}{16}}(s)$$

$$= 1600 + 439$$

$$= 2039 \approx 2040 \text{ pounds}$$

and

$$\text{L.T.L.} = 1600 - 439$$

$$= 1161 \approx 1160 \text{ pounds}$$

Therefore, on the average, we would expect about 95 percent of future measured results to be between 1160 and 2040 pounds.

If we wanted to calculate a single tolerance limit above which, on the average, 95 percent of the measured results would lie, we would use the value of t for $P_2$ = 95 percent. The equation would be

$$\text{L.T.L.} = \overline{X} - t\sqrt{\frac{n+1}{n}}(s)$$

$$= 1600 - 1.753\sqrt{\frac{17}{16}}(s)$$

$$= 1600 - 360$$

$$= 1240 \text{ pounds}$$

### 7.2.4 Calculating Required Number of Measurements

7.2.4.1 General Procedure. One purpose for which the t distribution is often used is to help in answering the following question: How many measurements should be made to find an average value with a desired precision? The procedure is very much like that for finding confidence limits for an average, but the basic equation is Equation 5.1, which is repeated here. It is

$$n = \left( \frac{ts}{\Delta} \right)^2 \qquad (7.5)$$

Here, $n$ is the required number of measured results, each for a randomly sampled test portion or test location; t is the value found from Table 7.2 for the selected value of $P_1$ and $(n - 1)$ measured results; s is the standard deviation computed from the $n$ measured results and expressed in measurement units; and $\Delta$ is the allowable value of $\overline{X} - \overline{X}'$ or $\overline{X}' - \overline{X}$ expressed in measurement units. As discussed in Section 5.6.1, it is not easy to use this equation. Before we start sampling, we do not know the value of $n$ to be used for finding the proper value of t. Also, we usually do not have a good estimated value for s. For 95-percent confidence limits, we can use the nomograph in Figure 5.4 to arrive at an approximate number of required measurements. If we did not have the nomograph, we would have to find $n$ by the method of repeated trials, also called *iteration*.

Example 7.4. We shall suppose that we want to estimate the true average density of a LOT of crushed-stone base. We are willing to take a chance that we will be wrong one time out of 20. Also, from previous work, we can assume that the standard deviation will be about 4.5 pcf. We would like to compute a sample size so that the length of the 95-percent confidence interval will be 6 pcf or less 50 percent of the time. This makes $\Delta$ = 3 pcf.

Solution. We will have to find the proper value of t and the corresponding value of $n$ by the method of repeated trials. To start, we can substitute z for t in Equation 7.6. We can find z from our double-ended normal table. In order that the probability of our being right will be 95 percent, the value of z must be 1.96. The required number of measurements would then be

$$n = \left( \frac{zs}{\Delta} \right)^2$$

$$= \left( \frac{1.96(4.5)}{3} \right)^2$$

$$\approx 9$$

However, from Table 7.2 the value of t in the column under $P_1$ = 95.0 and on the horizontal line for d.f. = 8 is 2.31. Since this value is much larger than 1.96, we must assume a larger value for t in Equation 7.5. If we try the average of 1.96 and 2.31, which is 2.13, we get

$$n = \left( \frac{2.13(4.5)}{3} \right)^2$$

$$\approx 10$$

From Table 7.2, the value of t for d.f. = 9 is 2.26. If we use the average of 2.13 and 2.26, which is about 2.2, we get

$$n = \left( \frac{2.2(4.5)}{3} \right)^2$$

$$\approx 11$$

For d.f. = 10, we find that t = 2.23. So we can conclude that the exact value of $n$ is between 11 and 12, and we would decide to make 12 density tests.

7.2.4.2 Modified Procedure. The use of the method just described for computing the number of measurements which must be averaged to obtain a desired degree of accuracy sometimes gives an excessive result because unrealistic tolerance limits are specified for a characteristic. For example, the job-mix tolerance for the asphalt content of a bituminous mixture is commonly specified as 0.4 percent, and the corresponding range between the tolerance limits is 0.8 percent. The standard deviation for the results of extraction tests made to determine asphalt content is about 0.3 percent. A rule of thumb is that test results should be accurate to within about one-tenth of the specified range. In this case enough test results should be averaged to make Δ less than 0.08 percent. If we again are willing to choose 95 percent for the probability of being right, and we substitute z for t in Equation 7.6 to compute a trial value for $n$, we get

$$n = \left( \frac{1.96(0.3)}{0.08} \right)^2$$

$$= 54$$

The final value for t should be about 2.01, and

$$n = \left( \frac{2.01(0.3)}{0.08} \right)^2$$

$$= 57$$

It would not be practical to make this number of tests. The trouble is not with the statistical approach. The large value of $n$ is obtained because the arbitrarily specified tolerance for asphalt content is too small with respect to the variability of the measurements.

We cannot change a specified tolerance. Nor can we reduce greatly the variability of the measured results. If we want to reduce the required number of tests, we will have to accept a lesser degree of accuracy. Let us suppose that we decide to use the average of four tests to determine the asphalt content of a LOT of hot-mix bituminous concrete. We can determine the corresponding value of $\Delta$ by first writing equation 7.5 in the form

$$\sqrt{n} = \frac{ts}{\Delta}$$

and then getting

$$\Delta = \frac{ts}{\sqrt{n}} \qquad (7.6)$$

From Table 7.2 we find that the value of t for $P_1$ = 95.0 percent and d.f. = 3 is 3.18. Hence,

$$\Delta = \frac{3.18(0.3)}{\sqrt{4}}$$

$$= 0.477$$

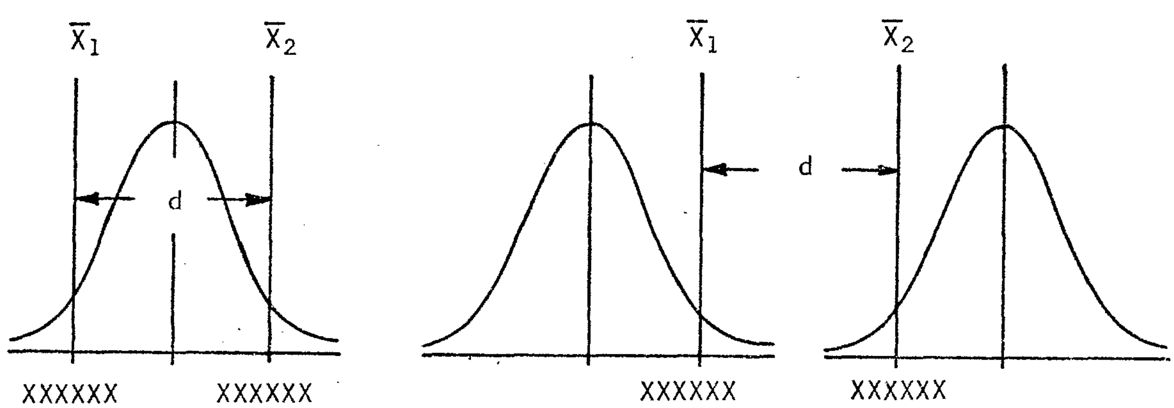
This calculation shows that if we take the average of four test results, we can expect the estimated asphalt content of a LOT to be within about one-half percent of the true average for the LOT, 95 times in 100.

### 7.2.5 Tests for Significant Differences Between Two Sample Averages

7.2.5.1 Paired Data. The tests that we are going to discuss now are uses of the t distribution to determine whether the difference between the averages of two sets of measurements is great enough to be due in part to an assignable cause. If we make a very large number of similar measurements on one LOT of a certain material, we will get a distribution of values for which the average may be designated as $\overline{X}_1'$. If we make another similar set of measurements on another LOT of the same material, we will get another distribution of values for which the average is $\overline{X}_2'$. If there is an appreciable difference between $\overline{X}_1'$ and $\overline{X}_2'$, we can be reasonably sure that there is a real difference between the two LOTs of the material. However, when we make only a small number of measurements on the samples from each LOT, we cannot be sure that a difference between the two averages is real. In this case, the difference may have been due to chance.

To take an extreme example, it would be possible to have the same difference d between $\overline{X}_1$ and $\overline{X}_2$ if these sample averages came from opposite ends of the same distribution or adjacent ends of widely different distributions as shown in Figure 7.3.

Figure 7.3
DIFFERENCE BETWEEN SAMPLE AVERAGES




When there is a small number of measurements, statistical methods make is possible to judge, with some degree of probability, whether differences between the averages of two sets of data are significant. For example, let us suppose that we want to decide whether or not two methods will produce the same result. If the test is not destructive, a possible procedure would be to test the same test portion by each of the two methods. For example, we may first use a nuclear gage for asphalt content and take readings on six different test portions. After these measurements have been made, the same portions can be tested by using the reflux method. If the proper calibration curve is used with the nuclear gage and if we make a large number of measurements on paired test portions, the average asphalt content should be the same for each method. However, if we used only six paired test results, we may get an appreciable difference between the average asphalt contents indicated by the two methods. Part of this difference may be due to assignable causes, or the entire difference could be due to chance. In other words, we have not proved whether or not there is a real difference between the results obtained by the two methods.

To determine whether or not there is a real difference, we must first find the averages of the results for the two methods. We shall designate them as $\overline{X}_1$ and $\overline{X}_2$. We then find the following values: the individual difference d between each pair of results; the square of each of these differences, denoted by $d^2$; the sum of these squares, denoted by $\Sigma d^2$; and the average of the difference, denoted by $\overline{d}$. If we let $n$ denote the number of measurements, we can calculate the value of t by using equation

$$t = \frac{\overline{X}_1 - \overline{X}_2}{\sqrt{\dfrac{\Sigma d^2 - \overline{d}^2}{n(n-1)}}} \qquad (7.7)$$

There are three possible situations. If we have reason to believe that the measurements represented by $\overline{X}_1$ should be larger than those represented by $\overline{X}_2$ we would decide that there was a significant difference if the computed t value was larger than a $P_2$ (one-tailed) t value of a chosen probability. If we were of the opinion that $\overline{X}_2$ should be larger than $\overline{X}_1$ we would decide that there was a significant difference if the $P_2$ value of t was larger than the computed value. However, if we are unsure as to whether $\overline{X}_1$ or $\overline{X}_2$ should be the larger, we would decide that there was a significant difference if the computed value of t was larger than the chosen $P_1$ (two-tailed) value from Table 7.2.

Example 7.5. Six pairs of measured values of asphalt content are shown in Table 7.3. The values in the first column, headed NU, are those obtained by the nuclear method, and the values in the second column, headed RF, are those obtained by the reflux method. We want to find out whether or not the two methods give different results.

Table 7.3

DIFFERENCES BETWEEN ASPHALT CONTENTS

MEASURED BY THE NUCLEAR (NU) AND REFLUX (RF) METHODS

| $X_1$ NU | $X_2$ RF | $(X_1 - X_2) = d$ | $(X_1 - X_2)^2 = d^2$ |
|---|---|---|---|
| 8.05 | 8.2 | -0.15 | 0.0225 |
| 8.05 | 8.4 | -0.35 | 0.1225 |
| 8.62 | 8.6 | +0.02 | 0.0004 |
| 8.27 | 8.6 | -0.33 | 0.1089 |
| 8.46 | 8.3 | +0.16 | 0.0256 |
| 8.10 | 8.6 | -0.50 | 0.2500 |
| $\Sigma X_1 =$ 49.55 | $\Sigma X_2 =$ 50.7 | $\Sigma d =$ -1.15 | $\Sigma d^2 =$ 0.5299 |
| $\overline{X}_1 =$ 8.2583 | $\overline{X}_2 =$ 8.4500 | $\overline{d} =$ -0.1917 | |

By equation 7.7,

$$t = \frac{8.2583 - 8.4500}{\sqrt{\frac{0.5299 - 0.1917^2(6)}{6(5)}}}$$

$$= \frac{-0.1917}{\sqrt{\frac{0.5299 - 0.0367(6)}{30}}}$$

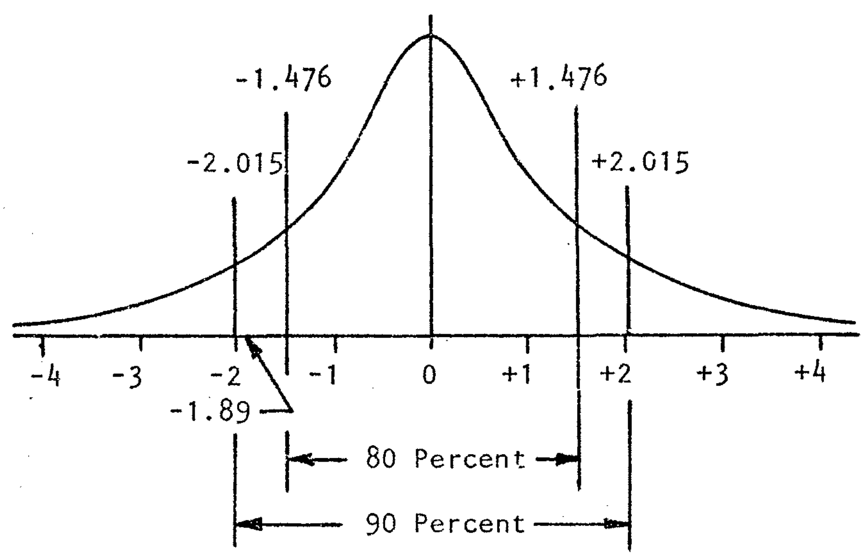
$$= \frac{-0.1917}{\sqrt{\frac{0.5299 - 0.2202}{30}}}$$

$$= \frac{-0.1917}{\sqrt{0.0103}}$$

$$= -1.89$$

In this case we have no reason to believe that either $\overline{X}_1$ or $\overline{X}_2$ is larger so we will use the two-tailed or $P_1$ t distribution to decide whether there is a significant difference. The value -1.89 in Table 7.2 on the horizontal line for d.f. = 5 would be between 1.476 and 2.015, as shown in Figure 7.4. The corresponding values in the row for $P_1$ are 80 and 90 percent. A probability of 90 percent would indicate that an average difference as large as a negative value 0.1917 could be expected by chance once in about 10 trials. Therefore, we cannot be sure that the two methods do give different results. In order that we could say with 95 percent confidence that a real difference existed, the computed value of t based on six pairs of measured test results would have to be about 2.6.

Figure 7.4
PROBABILITY OF SIGNIFICANT DIFFERENCE




7.2.5.2 Unpaired Measurements. In most cases we do not have pairs of measured results on the same test portion, and we may not even have the same number of results in the two sets of measured values which we wish to test for a real difference. First, we shall assume that the true standard deviation s′ is the same for both sets of measured results.

Example 7.6. We shall suppose that a contractor has used Brand "A" cement for the first part of a project and then has switched to Brand "B" for the rest of the project. The average 28-day compressive strength of the concrete obtained with Brand "B" is apparently greater, as shown by the test results in Table 7.4.

Table 7.4

RESULTS OF STRENGTH TESTS ON CONCRETE WITH DIFFERENT BRANDS OF CEMENT

| | Brand of Cement | |
|---|---|---|
| | A | B |
| Number of tests ($n$) | 16 | 9 |
| Average strength $f'_c$ ($\overline{X}$) | 3850 psi | 4350 psi |
| Standard deviation (s) | 560 psi | 630 psi |

We can assume that the true standard deviation is the same for both brands of cement. We want to analyze the difference in strength indicated by the test results.

Solution. The first step is to compute the standard deviation for all the test results by using the equation

$$s = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}} \qquad (7.8)$$

In this case, $n_1 = 16$, $s_1 = 560$ psi, $n_2 = 9$, and $s_2 = 630$. Hence,

$$s = \sqrt{\frac{(16 - 1)(560)^2 + (9 - 1)(630)^2}{16 + 9 - 2}}$$

$$= \sqrt{\frac{15(313,600 + 8(396,900)}{23}}$$

$$= \sqrt{\frac{4,704,000 + 3,175,200}{23}}$$

$$= \sqrt{342,573.9130}$$

$$= 585.2981$$

We then compute the value of t for all the test results by using the equation

$$t = \frac{|\overline{X}_1 - \overline{X}_2|}{s\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} \qquad (7.9)$$

Since $|\overline{X}_1 - \overline{X}_2| = 4350 - 3850 = 500$ psi,

$$t = \frac{500}{585.2981\sqrt{\dfrac{1}{16} + \dfrac{1}{9}}}$$

$$= \frac{500}{585.2981\sqrt{0.0625 + 0.1111}}$$

$$= \frac{500}{585.2981\sqrt{0.1736}}$$

$$= \frac{500}{243.8937}$$

$$= 2.05$$

The number of degrees of freedom for all the tests is (16 - 1) + (9 - 1) = 23. From Table 7.2, we see that the value of t for $P_1 = 95.0$ and d.f. = 23 is 2.07. Since 2.05 is less than 2.07 we conclude that the change in the brand of cement does not produce a significant change in the quality of the concrete if the probability level is taken as 95 percent.

We also consider the confidence limits for the difference in strength of the concrete. If we assume that $P_1 = 95$ percent and $t = 2.07$, we get the following limits:

$$\text{U.C.L.} = |\overline{X}_1 - \overline{X}_2| + ts\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$= 500 + 2.07(585.3)\sqrt{\frac{1}{16} + \frac{1}{9}}$$

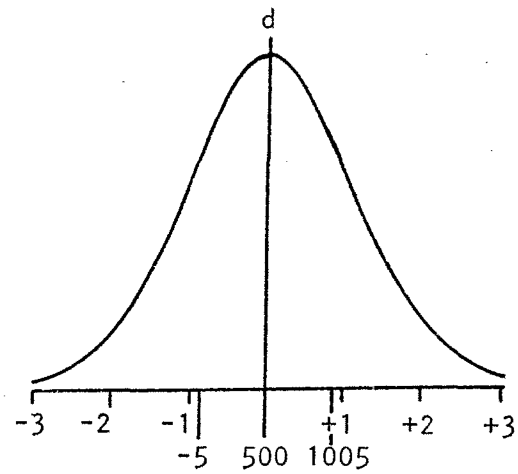
$$= 500 + 505 = 1005 \text{ psi}$$

$$\text{L.C.L.} = 500 - 505 = -5 \text{ psi}$$

These limits indicate that the difference in strength might have been as low as -5 or as high as 1005 psi. Since, as shown in Figure 7.5, zero lies between -5 and 1005, the conclusion based on these confidence limits would be the same as that arrived at by the preceding analysis.

Figure 7.5

POSITION OF CONFIDENCE LIMITS




In Example 7.7, we assumed that the true standard deviation s´ was the same for the two sets of measured results. If we want to see whether or not there is a real difference between the averages of two sets of measured results for which the values of s´ are different, much more arithmetic is involved. The value of t can be calculated by the equation

$$t = \frac{|\overline{X}_1 - \overline{X}_2|}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} \qquad (7.10)$$

Also, the number of degrees of freedom for this condition can be calculated by the equation

$$\text{d.f.} = \frac{\left(\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}\right)^2}{\dfrac{\left(\dfrac{s_1^2}{n_1}\right)^2}{n_1 + 1} + \dfrac{\left(\dfrac{s_2^2}{n_2}\right)^2}{n_2 + 1}} - 2 \qquad (7.11)$$

Example 7.8. We shall suppose that we have made two sets of unpaired measurements for determining the air content in the concrete in Example 7.7 by using the Roll-a-Meter for one set and the Chace Air Indicator for the other set. The results are shown in Table 7.5. We want to analyze the difference in air content indicated by the test results.

Table 7.5

RESULTS OF AIR CONTENT TESTS ON CONCRETE BY DIFFERENT METHODS

| | Roll-a-Meter | Chace |
|---|---|---|
| Number of tests ($n$) | 20 | 30 |
| Average air content ($\overline{X}$) | 5.0 % | 5.6 % |
| Standard deviation (s) | 0.76 % | 1.38 % |

Solution. In this example, $|\overline{X}_1 - \overline{X}_2| = 5.6 - 5.0 = 0.6$. By Equation 7.10,

$$t = \frac{0.6}{\sqrt{\frac{0.76^2}{20} + \frac{1.38^2}{30}}}$$

$$= \frac{0.6}{\sqrt{\frac{0.5776}{20} + \frac{1.9044}{30}}}$$

$$= \frac{0.6}{\sqrt{0.02888 + 0.06348}}$$

$$= \frac{0.6}{\sqrt{0.09236}}$$

$$= \frac{0.6}{0.3039}$$

$$= 1.97$$

Also, by Equation 7.11,

$$\text{d.f.} = \frac{\left(\frac{0.76^2}{20} + \frac{1.38^2}{30}\right)^2}{\frac{\left(\frac{0.76^2}{20}\right)^2}{20 + 1} + \frac{\left(\frac{1.38^2}{30}\right)^2}{30 + 1}} - 2$$

$$= \frac{\left(\frac{0.5776}{20} + \frac{1.9044}{30}\right)^2}{\frac{\left(\frac{0.5776}{20}\right)^2}{21} + \frac{\left(\frac{1.9044}{30}\right)^2}{31}} - 2$$

$$= \frac{(0.02888 + 0.06348)^2}{\frac{0.02888^2}{21} + \frac{0.06348^2}{31}} - 2$$

$$= \frac{0.09236^2}{\frac{0.000834}{21} + \frac{0.004030}{31}} - 2$$

$$= \frac{0.0085304}{0.0000397 + 0.0001300} - 2$$

$$= \frac{0.0085304}{0.0001697} - 2$$

$$= 50.27 - 2$$

$$= 48.27 \approx 48$$

By looking at the values in the row in Table 7.2 for d.f. = 48, we can see that the difference between 1.97 and the value of t in the table is small for $P_1$ = 95 percent but is quite large for $P_1$ = 90 percent. We decide that the difference between the air contents found by the two test methods could be as large as 0.6 percent or larger by chance about once in 17 trials. We can conclude that there is probably a real difference between the average values of air content indicated by the two methods. However, in order that we could be 99 percent confident, the computed value of t would have to be 2.7.

# CHAPTER 8

# DESIGN OF ACCEPTANCE PLANS AND SPECIFICATIONS

## 8.1 PRACTICAL ACCEPTANCE PLANS

In the discussion of basic acceptance plans in Chapter 6, it was assumed that the true standard deviation $s'$ for the measured results could be estimated accurately and that the true average $\overline{X}'_g$ for good material and the true average $\overline{X}'_p$ for poor material were known or could be estimated. In actual practice, however, we may find that the standard deviation for measured results varies widely from one LOT to another. For example, the standard deviation associated with aggregate gradation (measured percentages passing a sieve of a certain size) depends so much on the methods of forming and reclaiming stockpiles that measured values for samples from different stockpiles could not be expected to have the same standard deviation. Also, it is often difficult to assign suitable values for $\overline{X}'_g$ and $\overline{X}'_p$. Yet, if we can choose a suitable arbitrary value for the average $\overline{X}'_p$ for poor material, we can design a simple acceptance plan without making any assumptions in regard to the actual standard deviation for the measured results.

The general procedure is nearly the same as that described in Chapter 6. But when the standard deviation is not known, we can fix only one risk. We can fix either the buyer's risk or the seller's risk. Since a Highway Department is a buyer, we shall fix the buyer's risk. This risk will be calculated by the use of the t distribution.

## 8.2 DESIGN OF SINGLE-LIMIT ACCEPTANCE PLAN

### 8.2.1 General Procedure

A single-limit acceptance plan can be stated in either of two ways. One way is to say that we shall take only a certain risk of accepting a LOT for which the average $\overline{X}$ of the measured values is too close to our arbitrary average $\overline{X}'_p$ for poor material. Another way is to say that we shall reject a LOT for which at least a certain percent of measured results do not lie on the proper side of an arbitrary limit. Actually, the two plans are essentially the same, since the percent of measured results on the proper side of a certain limit depends on the closeness of the average of the measured results to the limit.

If we express the distance between the average $\overline{X}_p'$ for poor material and our acceptance limit L in sigma units, this distance will always be related to a definite percent of measured results on the proper side of the limit, regardless of the standard deviation for the measured results of tests on samples from the LOT. The acceptance procedure, essentially, is to take a small number of sampling units from a LOT, to make measurements on these units, and to estimate the true standard deviation $s'$ for the LOT from the standard deviation s computed for the measured results. Similarly, the true average $\overline{X}'$ for the LOT is estimated from the average $\overline{X}$ of the measured results.

When a plan of this type is used, the distance between the average $\overline{X}$ of the measured results and the average $\overline{X}_p'$ for poor material in sigma units is calculated by dividing the difference in measurement units by the standard deviation s for the measured results. If this distance is less than the distance between $\overline{X}_p'$ and our acceptance limit L, the LOT is rejected.

### 8.2.2 Plan for Protection Against Too Low an Average

To show how the acceptance plan outlined in Section 8.2.1 works, we shall consider the following example.

Example 8.1. We shall suppose that we have to make an acceptance decision in regard to a LOT of concrete pavement for which the specified average thickness is 9.0 inches and the allowable tolerance is 5 percent. We shall suppose also that the buyer's risk can be 10 percent on the basis of the average of the measured thicknesses of five cores, as shown in Figure 8.1. For a particular section of pavement to be accepted or rejected, we have taken five cores from random locations and have found the measured thicknesses to be 8.5, 9.0, 8.7, 9.2, and 8.6 inches. What should our decision be?

Solution. In this case, our assigned average $\overline{X}_p'$ for poor material is 9.0 - 0.05(9.0) = 8.55 inches. We first use Table 7.2 to find the value of t for $P_2$ = 100 - 10 = 90 percent when $n$ = 5 or d.f. = 4. This value is 1.533. Since we are basing our decision on the average of five measurements,

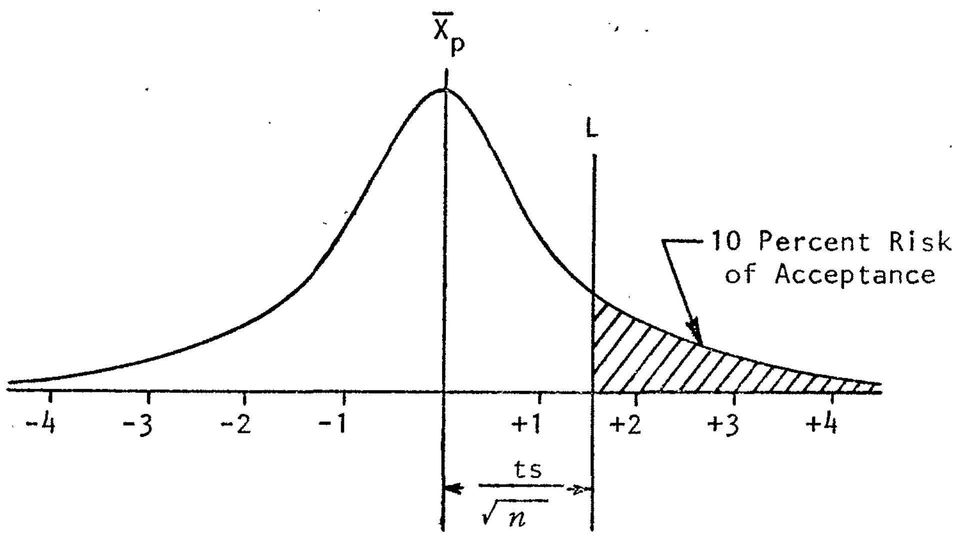
$$s_{\overline{X}} = \frac{s}{\sqrt{5}}$$

The distance between our acceptance limit and $\overline{X}_p'$ should not be less than $ts_{\overline{X}}$, which is equal to

$$\frac{1.533s}{\sqrt{5}} = \frac{1.533s}{2.236} = 0.69s$$

Figure 8.1

BUYER'S RISK OF ACCEPTANCE




For the actual measured thicknesses, the calculations are as follows:

| X | $d = \lvert\overline{X}_5 - X\rvert$ |
|---|---|
| 8.5 | 8.8 - 8.5 = 0.3 |
| 9.0 | 9.0 - 8.8 = 0.2 |
| 8.7 | 8.8 - 8.7 = 0.1 |
| 9.2 | 9.2 - 8.8 = 0.4 |
| 8.6 | 8.8 - 8.6 = 0.2 |

$\Sigma X = 44.0$

$$\overline{X}_5 = \frac{44.0}{5}$$

$$= 8.80$$

$$\overline{X}_5 - \overline{X}_p = 8.80 - 8.55 = 0.25$$

$$s = \sqrt{\frac{\Sigma d^2}{n - 1}}$$

$$= \sqrt{\frac{0.3^2 + 0.2^2 + 0.1^2 + 0.4^2 + 0.2^2}{5 - 1}}$$

$$= \sqrt{\frac{0.34}{4}}$$

$$= \sqrt{0.085}$$

$$= 0.2915$$

$$0.69s = 0.69(0.2915)$$

$$= 0.20$$

Since $\overline{X} - \overline{X}'_p = 0.25$ is greater than 0.20, we decide to accept the LOT of pavement.

Table 8.1

FACTOR FOR FINDING R FROM s

| $n$ | c |
|---|---|
| 3 | 1.91 |
| 4 | 2.23 |
| 5 | 2.47 |
| 7 | 2.83 |

For field work, it is more convenient to use the range than to use the standard deviation s. We can convert from s to R by multiplying s by a suitable factor. This plan is based on the principles given in Military Standard 414, which is used in the acceptance testing of critical military and space items. The factor that is used with this plan is designated as c. Thus,

$$R = cs \qquad (8.1)$$

or

$$s = \frac{R}{c} \qquad (8.2)$$

Values of c for various numbers of measurements $n$ are given in Table 8.1.

Example 8.2. Would the LOT in Example 8.1 be accepted on the basis of the range?

Solution. In this case,

$$0.69s = \frac{0.69R}{2.47} \approx 0.28R$$

We can therefore accept the LOT if the difference between $\overline{X}$ and $\overline{X}_p'$ is not less than 0.28R. In this case,

$$R = 9.2 - 8.5$$

$$= 0.7$$

$$0.28R = 0.28(0.7) \approx 0.20$$

$$\overline{X} - \overline{X}_p' = 8.80 - 8.55$$

$$= 0.25$$

Since $\overline{X} - \overline{X}_p'$ is greater than 0.28R, we decide to accept the LOT of pavement.

In another type of acceptance plan that is used in practical work, we compute a quantity Q by applying the equation

$$Q = \frac{\overline{X} - \overline{X}_p'}{R} \qquad (8.3)$$

If Q for a LOT is larger than a specified value Q′ which corresponds to the number of measurements and the buyer's risk that we are willing to take, we accept the LOT. Otherwise, we reject it. Values for Q′ are given in Table 8.2.

Table 8.2

ACCEPTANCE CONSTANT Q´

Risk of Accepting $\overline{X}_{p'}$

| $n$ | 1% | 5% | 10% | 20% |
|---|---|---|---|---|
| 4 | 1.02 | 0.53 | 0.37 | 0.22 |
| 5 | 0.68 | 0.38 | 0.28 | 0.17 |
| 7 | 0.42 | 0.26 | 0.19 | 0.12 |

Example 8.3. Would the LOT in Example 8.1 be accepted on the basis of the quantity Q?

Solution. By Equation 8.3,

$$Q = \frac{\overline{X} - \overline{X}_{p'}}{R}$$

$$= \frac{0.25}{0.70}$$

$$= 0.36$$

From Table 8.2, Q´ = 0.28. Since 0.36 is larger than 0.28, accept the LOT.

### 8.2.3 Protection Against Too Much Material Outside Acceptance Limit

A plan for protection against the acceptance of too much material on the wrong side of a specified limit is similar to the last one in Section 8.2.2. However, the value of Q is expressed in terms of the percent of a LOT of material that must be on the proper side of a specified limit. In Chapter 3, we computed the distance z by using the equation

$$z = \frac{X_i - \overline{X}}{s}$$

We then found the percent corresponding to the value of z in the normal table. In the acceptance plan described here, we do much the same thing. But we substitute R for s by using the factor c and get the equation

$$z = \frac{\overline{X} - \overline{X}_p'}{\frac{R}{c}}$$

which can be written

$$z = \frac{(\overline{X} - \overline{X}_p')c}{R} \qquad (8.4)$$

Since

$$Q = \frac{\overline{X} - \overline{X}_p'}{R}$$

This reduces to

$$z = Qc \qquad (8.5)$$

In Example 8.3, Q = 0.36 and c = 2.47. So z = 0.36(2.47) = 0.89. From the normal table, we find that the corresponding percent of concrete within limits is 81. For practical work, there are tables, such as Table 8.3 at the end of this chapter, in which the effect of c has been taken into account and by the use of which the percent within tolerance, designated P.W.T., is found directly from the value of Q.

In Table 8.3, sheets 1 and 2 are for values of X greater than $\overline{X}_p'$ or for positive values of Q, and sheets 3 and 4 are for values of $\overline{X}$ less than $\overline{X}_p'$ or for negative values of Q. If you examine Table 8.3, you will see that for given values of $n$ and Q, the percent within tolerance for a negative value of Q is equal to 100 minus the percent for the same positive value of Q.

On each sheet of Table 8.3, the values for percents within tolerance are shown in the left-hand column and corresponding values for Q for various numbers of measured results are given in the body of the table. In Example 8.3, $n$ = 5 and Q = 0.36. The percent within limits, which is found on the horizontal line with 0.36 in the column headed $n$ = 5, is 80. This result is very nearly the same as that we obtained by using the normal table and would be close enough for our purposes.

## 8.3 DOUBLE-LIMIT ACCEPTANCE PLAN

### 8.3.1 Use of Acceptance Limits

In some cases, it may be important that the average of the measured values for a characteristic of a material be neither too large nor too small. For example, let us suppose that we have designed a concrete mixture by the method described in ACI 613. On the basis of a single submitted sample of the fine aggregate proposed for use, we have assumed that the fineness modulus will be 2.70. If the actual average fineness modulus of the sand weighed into the plant mixture is found to be as small as 2.55 or as large as 2.85, we should be prepared to redesign the mixture. We shall suppose that we are willing to take equal risks of 5 percent that we will accept a LOT for which the average value for $\bar{X}'_p$ is either 2.55 or 2.85. This plan would be essentially the same as a single-limit acceptance plan.

Example 8.4. Let us suppose that we decide to base our decision to accept or reject a LOT on the results of gradation tests on four test portions taken from randomly selected batches at the weigh hopper. We want to state an acceptance rule.

Solution. For the given conditions, the acceptance limits can be computed in the following manner:

$$\text{L.L.} = \bar{X}'_p + \frac{ts}{\sqrt{n}}$$

$$= \bar{X}'_p + \frac{2.35s}{2}$$

$$= \bar{X}'_p + 1.18s$$

$$= \bar{X}'_p + \frac{1.18R}{2.23}$$

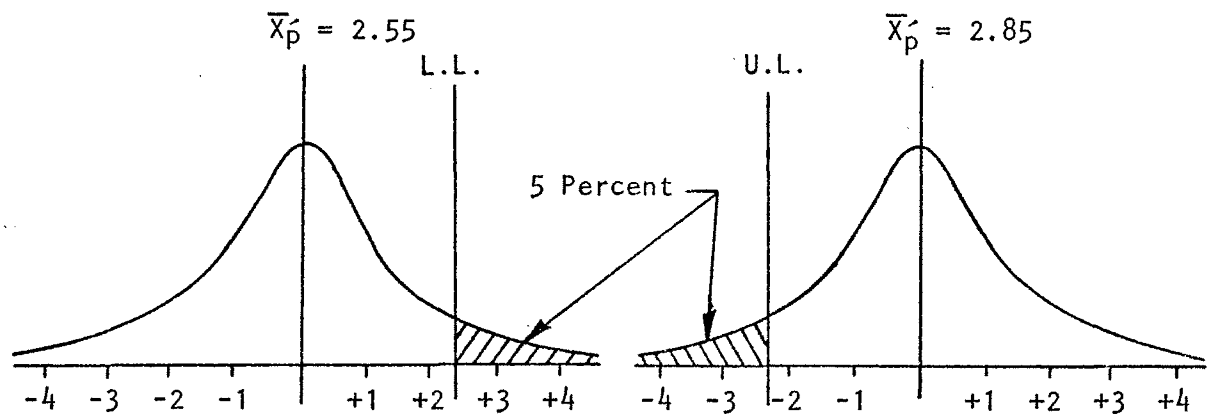
$$= \bar{X}'_p + 0.53R$$

$$\text{L.L.} = 2.55 + 0.53R$$

$$\text{U.L.} = 2.85 - 0.53R$$

The situation would look as shown in Figure 8.2.

Figure 8.2
DOUBLE-LIMIT ACCEPTANCE SITUATION




In this case, our acceptance rule will be as follows: Take four random test portions from a LOT and determine the gradation and fineness modulus. Also, determine the range R between the smallest and largest values of the fineness modulus. If the average $\overline{X}_4$ of the four measured results for a LOT is larger than 2.55 + 0.53R and smaller than 2.85 - 0.53R, accept the design for the mixture. If the average fineness modulus is outside these limits, redesign the mixture.

Example 8.5. Now let us suppose that a sand which is too coarse would cause more of a problem than a sand which is too fine, we are therefore willing to take the following risks: 10 percent for accepting a LOT for which $\overline{X}_p' = 2.55$ and 2.5 percent for accepting a LOT for which $\overline{X}_p' = 2.85$. We want to decide on an acceptance rule.

Solution. In this case the acceptance limits should be computed in the following way. For the lower limit, the value of t is that for a probability of 90 percent. For the upper limit, t is the value for a probability of 97.5 percent. Hence,

$$\text{L.L.} = 2.55 + \frac{1.64s}{2}$$

$$= 2.55 + 0.82s$$

$$= 2.55 + \frac{0.82R}{2.23}$$

$$= 2.55 + 0.37R$$

$$\text{U.L.} = 2.85 - \frac{3.18s}{2}$$

$$= 2.85 - 1.59s$$

$$= 2.85 - \frac{1.59R}{2.23}$$

$$= 2.85 - 0.71R$$

If the average of the four measured values of the fineness modulus is larger than 2.55 + 0.37R and smaller than 2.85 - 0.71R, we accept the designed mixture.

### 8.3.2 Use of Tolerance Limits

In another type of double-limit acceptance situation, a requirement of the specifications for aggregate may be that the nominal percent of the aggregate passing a 3/8-inch sieve shall be between certain limits. The procedure for selecting an acceptance rule is outlined in the next example.

Example 8.6. A specified requirement for the aggregate for a base course is that the percent passing the 3/8-inch sieve shall be between 45 and 60 percent. Since the gradation of the aggregate is not very critical, we are willing to take a 10-percent risk that we will accept a LOT for which the average percent passing the 3/8-inch sieve is less than $\overline{X}_p' = 45$ or greater than $\overline{X}_p' = 60$. Also, we decide that we will base acceptance on four gradation tests per LOT. We want to state an acceptance plan.

Solution. From Table 8.2, we find that the value of $Q'$ for a 10-percent risk and $n = 4$ is 0.37. Since we prefer to have an estimate of the percent within tolerance, we find from Table 8.3 that for a value of Q equal to 0.37, the percent within tolerance is about 78. Our acceptance rule will be as follows:

Take four random test portions from the LOT, and test each portion for gradation. If the test results indicate that more than 78 percent of the aggregate meets the requirement that the amount of aggregate passing the 3/8-inch sieve is between 45 and 60 percent, accept the LOT.

When there is a requirement of the kind in Example 8.6, the variability of the individual measurements may be so large that some test results will indicate a gradation that is too fine and other results will indicate a gradation that is too coarse. For this reason we will have to estimate the total percent within tolerance, designated T.P.W.T.

Example 8.7. Let us suppose that we make four gradation tests for a LOT of the aggregate in Example 8.6 and find that the average percent passing the 3/8-inch sieve is 52 and that the range is 15. We want to determine the total percent within tolerance.

Solution. By Equation 8.3,

$$Q_U = \frac{U.L. - \overline{X}}{R} = \frac{60 - 52}{15} = \frac{8}{15} = 0.53$$

Then from Table 8.3, the percent below the upper tolerance limit, designated $P.W.T._U$, is 90 percent.

Also,

$$Q_L = \frac{\overline{X} - L.L.}{R} = \frac{52 - 45}{15} = \frac{7}{15} = 0.47$$

and the percent above the lower tolerance limit, designated $P.W.T._L$, is 85 percent. Where both an upper limit and a lower limit must be considered, the total percent within tolerance is found by adding the percent below the upper tolerance limit to the percent above the lower tolerance limit and subtracting 100 from the sum. Thus,

$$T.P.W.T. = P.W.T._U + P.W.T._L - 100 \qquad (8.6)$$

In this example,

$$T.P.W.T. = 90 + 85 - 100$$

$$= 175 - 100$$

$$= 75 \text{ percent}$$

Since 75 percent is less than the required 78 percent, the LOT would not be acceptable at full price.

## 8.4 NON-CENTRAL t TABLES

### 8.4.1 Reason for Using Non-Central Tables

So far in this Chapter, we have been considering the design of acceptance plans with a fixed buyer's risk. We have made no assumptions in regard to the value of the true standard deviation $s'$ for the LOTs on which an acceptance decision must be based, since this value will not affect the buyer's risk. However, we must not forget that two risks are always present when we make a decision. When we apply an acceptance plan based on the t distribution,we must also consider what happens to the seller's risk. If our acceptance plan was so designed that the risk of rejecting good material, representing the process capability, was high, we would be unfair to the producer or contractor. If the acceptance plan and the Specifications should be strictly enforced, the producer or contractor would probably increase his bid price to make up for losses due to rejections or reductions in price. For this reason, we should check our acceptance plan to determine the probable size of the seller's risk.

Although we cannot fix the seller's risk when we do not know the value of $s'$ for the measured results of tests, we can estimate this value if we make some assumptions. In effect, we estimate what the seller's risk would be if

$s'$ and $\overline{X}'_g$ had some assigned values. From our study of basic acceptance plans in Chapter 6 based on the normal distribution, we know that if the distance between $\overline{X}'_p$ and the true average $\overline{X}'$ increased in the direction of increasing quality, the probability of rejection decreased. If this distance was measured in sigma units, the probability of rejection became very small when $|\overline{X} - \overline{X}'_p|$ approached 3s. The notation $|\overline{X} - \overline{X}'_p|$ means either the value of $\overline{X} - \overline{X}'_p$ or the value of $\overline{X}'_p - \overline{X}$, whichever one is positive. In fact, we could draw an operating-characteristics curve, based on a known value of $s'$, which showed the probability of rejection of LOTs having true averages at different distances from $\overline{X}'_p$ when these distances were expressed either in measurement units or in sigma units.

If we want to estimate the risks associated with an acceptance plan when the value of the true standard deviation $s'$ is unknown, we have very much the same situation. However, we can no longer use the normal table or the t table to find the risk of accepting LOTs for which the true average $\overline{X}'$ is not equal to $\overline{X}'_p$. Instead, we must use a table for the non-central t distribution which is indexed in terms of $n$ and a new quantity, designated $K_p$, which is the number of sigma units between $\overline{X}'$ and $\overline{X}'_p$. In other words,

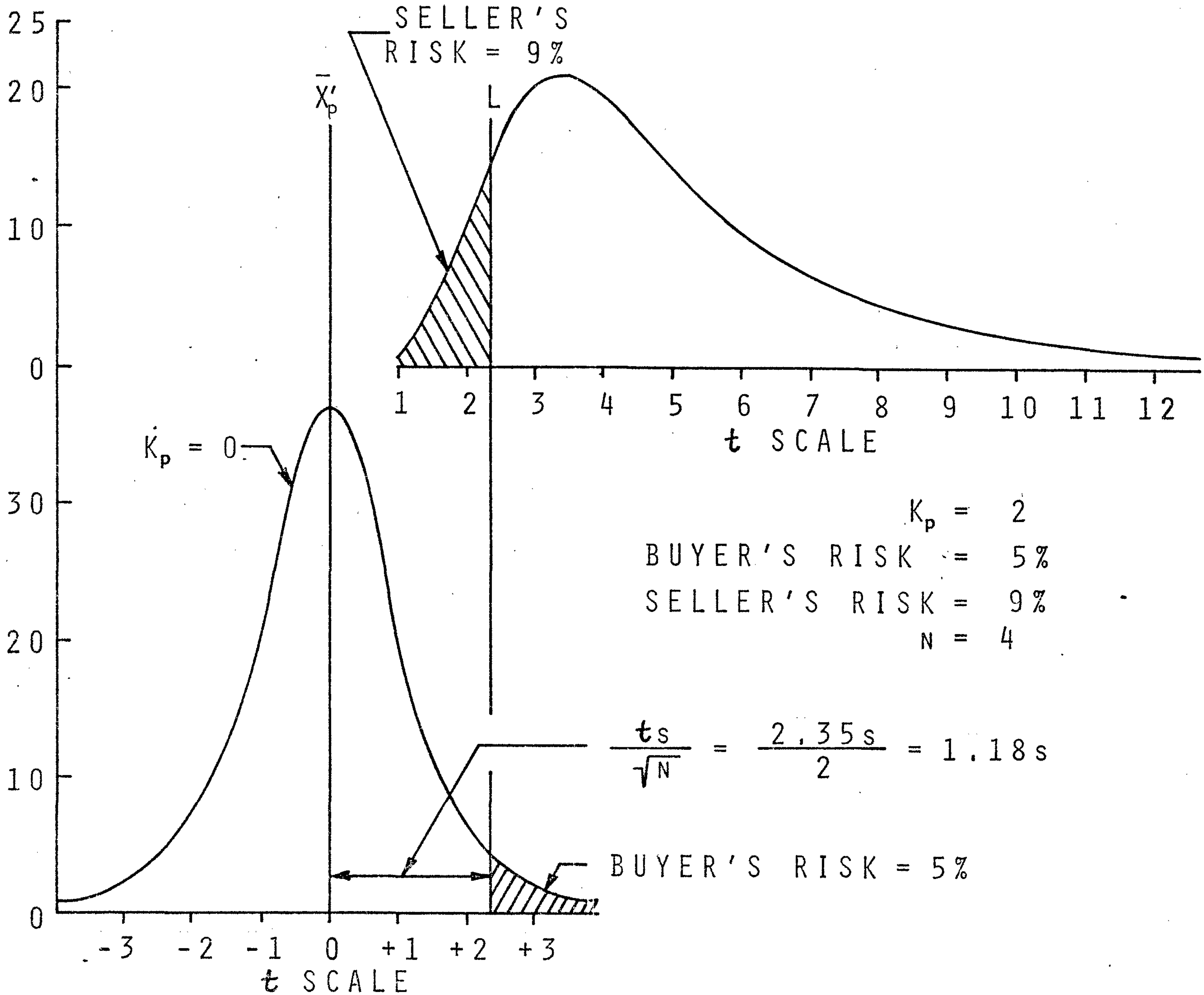
$$K_p = \frac{\overline{X}' - \overline{X}'_p}{s'} \qquad (8.7)$$

### 8.4.2 Features of Non-Central t Tables

The non-central t distribution is not symmetric. Therefore, the curve for this distribution does not have the same shape on both sides of the central value. It is skewed and has a long tail in the direction away from $\overline{X}'_p$, as shown in Figure 8.3. The shape of the curve depends on the number of measured values used for computing the sample average and on the value of $K_p$. Tables for the non-central t distribution are available. Such tables for numbers of measurements from 4 to 12 are included in this Handbook as Table 8.4 at the end of this Chapter. The number of measurements to which each sheet of the table applies is shown by a note in the upper right-hand corner of the sheet. Each value in the left-hand column headed D is a distance on the t scale measured from the vertical line at $\overline{X}'_p$. Each decimal value in the body of the table shows the portion of the total area under the proper non-central t curve that lies to the left of the vertical line at the distance

Figure 8.3

CURVE FOR NON-CENTRAL $t$ DISTRIBUTION



D. A decimal value can be converted to a percent by moving the decimal point two places to the right. In general, a value in the table corresponds to the probability of rejection of a LOT for which the average is at a distance $K_p s'$ from $\overline{X}'_p$.

To help you understand the use of non-central t tables, we shall work out a simple example.

Example 8.8. In order to design an acceptance plan for compacted embankment, we decide that the average $\overline{X}'_p$ for a LOT having poor compaction would be only 90 percent of a reference density and the average $\overline{X}'_g$ for a LOT having good compaction would be 100 percent of this reference density. We want to take a buyer's risk of only 5 percent that we will accept a LOT for which the average density is less than $\overline{X}'_p$. We also want to estimate the seller's risk that we will reject a LOT for which the average density is greater than $\overline{X}'_g$. We are going to base our acceptance decision on four density measurements made at random locations.

Solution. To start with, we proceed exactly as in Section 8.3. Our acceptance limit will be

$$L = \overline{X}'_p + \frac{ts}{\sqrt{n}}$$

From Table 7.2 we find that the value of t for $P_2$ = 95.0 percent and $n$ = 4, or d.f. = 3 is 2.353. Then

$$L = 90 + \frac{2.353s}{2}$$

$$= 90 + 1.18s$$

Hence, if we reject all LOTs for which the average of four density measurements is less than 90 + 1.18s, we will not be taking a risk of more than 5 percent that we will accept LOTs for which the density is less than 90 percent of the reference density.

To estimate the corresponding seller's risk, we must make an assumption. We do not know what the true standard deviation $s'$ for a LOT will be. Because of the variability of materials and their compactability, $s'$ could have almost any value. But $s'$ for such a process is usually between 3 and 8. We shall assume, in this case, that $s' = 5$.

When we substitute the assumed values for $\overline{X}_g'$, $\overline{X}_p'$, and s in Equation 8.7, we get

$$K_p = \frac{100 - 90}{5} = 2$$

Since we are using the average of four measurements as a basis for acceptance, we should find the value of D for our non-central t distribution from the sheet of Table 8.4 for which $n = 4$. This is sheet No. 1, as shown by the note $n = 4$ in the upper right-hand corner.

We have already found from Table 7.2 the distance t corresponding to the 95-percent probability of rejecting a LOT for which the average density is less than $\overline{X}_p'$. This distance is 2.353 sigma units. We could have found this distance by using Table 8.4. When $\overline{X}' = \overline{X}_p'$, the value of $K_p$ is zero. The distance D corresponding to the value 0.95 in the column of Table 8.4 headed $K_p = 0.00$ lies between 2.2 and 2.4. By interpolation, the exact value would be about 2.35.

To find the probability of rejection of a LOT for which the average density is equal to $\overline{X}_g'$, or for which $K_p = 2$, we locate the value in Table 8.4 in the column headed $K_p = 2.00$ corresponding to the distance D equal to 2.35. The values for D = 2.2 and D = 2.4 are 0.0704 and 0.0986. If we interpolated for t = 2.35, we would find that the probability of rejection would be about 9 percent. This is the seller's risk of rejecting a LOT for which the actual average density is 100 percent.

The numbers in the columns on sheet No. 1 of Table 8.4 under different values of $K_p$ are really points for the operating-characteristics curve for acceptance plans based on $n = 4$ and the values of t shown in the left-hand column. We shall assume that it is accurate enough for our purposes to take t as 2.4. The corresponding probability of rejection of a LOT for which the average density is $\overline{X}_p'$ would then be 95.21 percent. The calculations for locating the points on the operating-characteristics curve are shown in Table 8.5.

After a curve has been drawn through the points located by using the values of $\overline{X}'$ and the corresponding probabilities shown in Table 8.5, the probability of rejecting or accepting a LOT for which the average density is anywhere between 90 and 100 can be determined from the curve. You must remember that the values of $\overline{X}'$ in Table 8.5 were computed by taking s as 5. For any other value of s', a similar procedure can be followed.

Table 8.5

POINTS ON OPERATING-CHARACTERISTICS CURVE

($n$ = 4 and $s'$ = 5)

| $K_p$ | $K_p s'$ | $\overline{X}' = \overline{X}'_p + K_p s'$ | Probability of Rejection | Probability of Acceptance |
|---|---|---|---|---|
| 0.00 | 0.00 | 90.00 | 95 | 5 |
| 0.25 | 1.25 | 91.25 | 90 | 10 |
| 0.50 | 2.50 | 92.50 | 81 | 19 |
| 0.75 | 3.75 | 93.75 | 69 | 31 |
| 1.00 | 5.00 | 95.00 | 55 | 45 |
| 1.25 | 6.25 | 96.25 | 41 | 59 |
| 1.50 | 7.50 | 97.50 | 28 | 72 |
| 1.75 | 8.75 | 98.75 | 17 | 83 |
| 2.00 | 10.00 | 100.00 | 10 | 90 |

## 8.5 MATHEMATICAL SIMULATION OF APPLICATION OF STATISTICAL ACCEPTANCE PLANS

For purposes of demonstration the method described here can be used to simulate the application of any sampling plan under actual field conditions for any characteristic for which the average of the measured results and the standard deviation are first determined for LOTs or sublots. Each LOT or sublot is then sampled in a random manner in accordance with the procedures of the sampling plan, and the test results so obtained are used to compute the value of $\overline{X}$ or Q that is compared with the acceptance requirement. In literature relating to statistics, the term "population" is often used to refer to a group of similar items. The procedure may be outlined as follows:

The population is generated by listing values of $\overline{X}' + zs'$ and giving each value a number in sequence. In this application,

$\overline{X}'$ = the known or assumed average of test results

z = the tabulated random normal deviates as shown in Table A-37 in Bureau of Standards Handbook 91

$s'$ = the known or assumed standard deviation for the test results

A value of $\overline{X} + zs'$ will here be called a *simulated test result*. For example, let us suppose that the average of the results of a number of tests for the 28-day compressive strength of structural concrete is 4460 psi and the standard deviation for those results is 480 psi. The first random normal deviate in Table A-37 is 0.048, and simulated test result No. 1 is then 4460 + 0.048(480) = 4483. The next deviate in the same column is -0.521, and simulated test result No. 2 is 4460 + (-0.521)(480) = 4210.

The population is generated in groups, each of which contains a number of simulated test results equal to the number of units in a LOT or sublot. Each LOT or sublot is then sampled by selecting random numbers from some suitable table of random numbers. Random numbers greater than the largest number in a group are skipped. For example, let us suppose that a population is generated by using 50 simulated test results for each of five sublots. The acceptance plan could be stated as follows:

A LOT will consist of five sublots. Each sublot will consist of not more than 50 cubic yards of concrete of the same class. A test shall be made on a single random batch from each sublot. A test shall consist of the average of the results of compressive strength tests on two cylinders made from the batch. Concrete in full compliance with strength requirements shall have a Q value in excess of 0.38 as computed by use of the equation

$$Q_L = \frac{\overline{X}_5 - 3200}{R}$$

where

$\overline{X}_5$ = average of five strength test results on samples from each of five sublots

R = absolute difference between largest and smallest test result in the group of five test results

A LOT of concrete that does not meet the specified strength requirement either shall be removed and replaced or may be accepted at a reduced price at the discretion of the engineer. If such a LOT is accepted, the basis of payment will be as shown in Table 8.6. When $Q_L$ is equal to or greater than the value shown in Column A, the percent of the LOT within tolerance will be as indicated in Column B. The percent of contract price to be paid is specified in Column C.

Table 8.6

BASIS OF PAYMENT FOR POOR CONCRETE

| A | B | C |
| --- | --- | --- |
| $Q_L = \frac{\bar{X}_5 - 3200}{R}$ | Percent of LOT Within Tolerance | Percent of Contract Price to Be Paid |
| 0.38+ | 100 - 82 | 100 |
| 0.30+ | 81 - 76 | 94 |
| 0.18+ | 75 - 66 | 85 |
| 0.01+ | 65 - 51 | 72 |
| 0.00 | 50 or less | 0 |

Example 8.9. Let us suppose that we have computed 50 simulated test results for each of five sublots of pavement concrete for which $\bar{X}' = 4460$ psi and s = 480 psi. The results for each sublot are numbered in succession from 1 to 50. The sequence for the first sublot is determined by using the 50 normal deviates in the left-hand column of Table A-37; and the deviates in the second, third, fourth, and fifth columns of that table are used for the other sublots. A single simulated test result for each sublot is then selected by using the first five random numbers less than 50 in the left-hand column of Table A-36. We want to determine the value of $Q_L$ for use in Table 8.6.

Solution. The random numbers selected from Table A-36 are 46, 44, 34, 22, and 40. Therefore, we shall use simulated test result No. 46 from sublot 1, test result No. 44 from sublot 2, and so on. The numbers of the sublots, the random numbers, and the corresponding simulated test results are shown in Table 8.7. The value of $Q_L$ computed by using the five simulated test results thus selected is found to be 0.64.

Table 8.7

VALUES FOR A TYPICAL LOT

| Sublot Number | Random Number | Simulated Test Result |
|---|---|---|
| 1 | 46 | 5100 |
| 2 | 44 | 3787 |
| 3 | 34 | 3530 |
| 4 | 22 | 4207 |
| 5 | 40 | 4379 |

$$\Sigma X = 21,003$$

$$\overline{X}_5 = 4,200$$

$$R = 1,570$$

$$Q_L = \frac{4200 - 3200}{1570} = 0.64$$

Since $Q_L = 0.64$ which is larger than our acceptance limit of $Q_L > 0.38$ we will decide to accept the LOT at full price.

In some cases it may not be considered necessary to completely simulate the acceptance procedure for illustration purposes. Under these conditions only the number of test results required for an acceptance decision need be generated by the method previously described.

## 8.6 DESIGN OF REALISTIC SPECIFICATIONS

### 8.6.1 Essential Elements of Specifications

A specification that is based on statistical methods has certain essential elements. These are:

1. A characteristic or characteristics of the material which will be measured to determine the acceptability of the material
2. A target value or desired value for each measured characteristic

3. Realistic tolerances for each target value
4. An acceptance limit or limits for each measured characteristic, or a required percent of measured results on the proper side of a single limit or between two limits
5. A size of LOT on which measurements will be made for acceptance
6. Points at which random samples will be taken for determining acceptance
7. A method of sampling
8. A method of testing
9. An acceptance rule, or a method of determining acceptability
10. Action to be taken in case requirements are not met, such as reduction in price, or rejection or removal and replacement

### 8.6.2 Reduction in Price for Deficiency in One Characteristic

Many specifications for highway materials and construction provide that in case the requirements are not satisfied, the defective material or construction shall be removed and replaced. Other specifications include the alternative provision that if the engineer gives his consent, such material or construction may be left in place without payment. These are very severe penalties. However, the penalty clause may not be strictly enforced, particularly when the State agency, or buyer, has assumed some moral responsibility for the materials to be used or for the method of manufacture or construction.

Paragraph 105.03 of the AASHO "Guide Specifications for Highway Construction" states: "All work performed and all materials furnished shall be in reasonably close conformity with the lines, grades, cross sections, dimensions and material requirements, including tolerances shown in the plans or indicated in the specifications." The term "reasonably close conformity" is not defined by AASHO. However, unbiased studies, such as those reported in the Bureau of Public Roads publication, "Quality Assurance in Highway Construction," have shown that even with reasonable tolerances, some measured results will be outside the specified limit or limits. These studies have also shown that the unacceptable values may be caused by variations due to sampling and testing rather than by actual variations in the quality level of the materials or construction. An analysis of the data indicates that a requirement that 80 percent of the measured results, within a specified limit as indicated by the methods of Section 8.2.3, would constitute "reasonably close conformity". The value 80 percent is not suggested arbitrarily. It can be shown that the percent of area under a distribution curve for the same sigma distances from the average is relatively constant up to about 80 percent regardless of the actual distribution of measured results.

Even when it is permissible for some measured results to be outside a tolerance limit, tests may indicate that the finished product is not in "reasonably close conformity" with the requirements of the plans and specifications. In such a case, the AASHO Specifications provide that reasonably satisfactory work shall be accepted and be allowed to remain in place but there shall be an appropriate adjustment in price. According to the AASHO Specifications, the amount of this reduction in price should be determined by the engineer on the basis of his judgment. This approach has many disadvantages, since different engineers on different projects might decide on different reductions in price for the same "reasonably satisfactory work". When acceptability is judged by the methods of Section 8.2.3, the percent within limits or percent within tolerance provides a numerical measure of the degree of non-compliance with the specified requirement. By matching such a percent for defective material or construction with estimated loss in value, a system of reduction in price can be developed. This system can be shown in tabular form as part of the specifications. The adjustment in price is then no longer a matter of individual judgment.

Very little information is available in regard to the effect of non-compliance with a specified requirement on the value of the completed construction. Studies in connection with concrete pavement have indicated that a reduction in thickness equal to 1 inch may reduce pavement life by about 44 percent. The same studies indicate that a reduction in concrete strength equal to 10 percent may reduce pavement life by 35 percent, while a reduction in strength equal to 20 percent may reduce pavement life by 60 percent. Other studies have indicated that a deficiency equal to 1/4 inch in a nominal 3-inch asphaltic surface course would reduce the load-carrying capacity by about 15 percent. The term "thickness" is not well defined since the sigma of measurements on both cement-concrete pavements and asphaltic-concrete surface courses has been reported to be about 0.27 inch.

Example 8.10. Let us suppose that we want to calculate an equitable adjustment of price for LOT-by-LOT acceptance of concrete pavement for which the nominal thickness is 9.0 inches and the tolerance is 0.3 inch. We shall assume that any area of the pavement 8.7 inches thick would have a normal life expectancy which is taken as 100 percent. We can assume also that an area 7.7 inches thick would have a life expectancy equal to 65 percent. We arbitrarily decide that we will pay the full contract price for a LOT of pavement for which 80 percent of the measured thicknesses can be expected to exceed 8.7 inches. However, we will pay only 65 percent of the contract price for a LOT for which our measurements indicate that there may be areas only 7.7 inches thick.

Solution. From our one-tailed table of areas for the normal curve, we see that a vertical line at a distance z equal to about 0.845 sigma unit from the average will include 80 percent of the area under the entire curve. Therefore, in order that 80 percent of the measured thicknesses in a LOT will be above 8.7 inches, the required average thickness $\overline{X}_T$ would be computed as follows:

$$\overline{X}_T = \overline{X}' + zs$$

$$= 8.7 + 0.84(0.27)$$

$$= 8.7 + 0.23$$

$$= 8.93 \text{ inches}$$

In order that practically all measured thicknesses will exceed 7.7 inches, the average thickness $\overline{X}_T$ would have to be 7.7 inches + 3s. Thus,

$$\overline{X}_T = 7.7 + 3(0.27)$$

$$= 7.7 + 0.81$$

$$= 8.51 \text{ inches}$$

It would be reasonable to decide that we will pay the full contract price for pavement for which the average thickness is 8.9 inches and will pay 65 percent of the contract price for pavement for which the average thickness is 8.5 inches. However, these average thicknesses are based on the assumption that the standard deviation for the measured thicknesses will always be 0.27 inch. This value may not be correct for any particular LOT. Also, a pavement having a highly variable thickness would be less durable than a pavement having the same average thickness but a greater degree of uniformity. The use of the percent-within-tolerance method of Section 8.3.2 is better, since this percent is a measure of both average thickness and variability. This method also allows the use of a table of graduated percents of reduction in price for nonconforming pavement.

Example 8.11. Let us suppose that we want to construct a table for the pavement considered in Example 8.10.

Solution. To construct the table, we start with the average thickness $\overline{X}_g'$ for which we are willing to pay the full contract price and with the minimum specified thickness T. For these thicknesses,

$$z = \frac{\overline{X}_g' - T}{s}$$

$$= \frac{8.9 - 8.7}{0.27}$$

$$= \frac{0.2}{0.27}$$

$$= 0.74$$

From the normal table the probability for $z = 0.74$ is 77 percent. We shall use 80 percent.

When we consider the average thickness $\overline{X}_p'$ for which we are willing to pay only 65 percent of the contract price and the minimum specified thickness T, we get

$$z = \frac{\overline{X}_p' - T}{s}$$

$$= \frac{8.5 - 8.7}{0.27}$$

$$= \frac{-0.2}{0.27}$$

$$= -0.74$$

From the normal table the probability in this case is 100 - 77 = 23 percent. We shall use 25 percent.

We could decide that we will pay the full contract price for pavement for which it is probable that 80 percent of the measured thicknesses are above 8.7 inches, and we will pay 65 percent of the contract price for pavement for which it is probable that only 25 percent of the measured thicknesses are above 8.7 inches. This is quite a range of reduction in price, and we can divide the reduction into steps. The procedure for doing this is entirely arbitrary. In general, however, it would seem that the reduction in price should be proportionately greater as the deficiency in thickness increases. In this case we can assume a reduction equal to 5 percent for the first step, a further reduction equal to 10 percent for the second step, and a further reduction equal to 20 percent for the third step. The percents of the contract price thus established are shown in the last column of Table 8.8. The percents within tolerance given in the center column of the table are selected arbitrarily. The values of Q in the first column of Table 8.8 are the values found from Table 8.3 that correspond to the percents within tolerance given in the center column of Table 8.8. In this case the notation $\overline{X}_4$ indicates that the thicknesses of four cores from each LOT will be measured and the average of these thicknesses will be used.

Table 8.8

SCHEDULE OF PAYMENT RATES

| $Q = \frac{\overline{X}_4 - T}{R}$ | Percent Within Tolerance | Percent of Contract Price to Be Paid |
|---|---|---|
| 0.40 or more | 80 or more | 100 |
| 0.39 to 0.20 | 79 - 65 | 95 |
| 0.19 to -0.07 | 64 - 45 | 85 |
| -0.08 to -0.34 | 44 - 25 | 65 |
| less than -0.34 | less than 25* | 0 |

*When the measurements indicate that less than 25 percent of the LOT is within the tolerance for thickness, the pavement shall either be removed and replaced without additional payment or be left in place without payment, as directed by the engineer.

The acceptance rule can be stated as follows: Take four cores at random locations within the LOT of pavement. Compute the value of Q by first taking the difference between the average $\overline{X}_4$ of the thicknesses of the four cores and the minimum specified thickness T, and then dividing this difference by the range R for the four thicknesses. If the computed value of Q is not less than +0.40, accept the LOT at the full contract price. If the value of Q is less than +0.40, the LOT may be accepted at a reduced price determined from the schedule of payment rates.

It should be understood that the procedure just used is only one of many possible ways of setting up a payment-rate schedule for material or construction that is acceptable but does not meet the specified requirement. However, this procedure illustrates the principles of relating the degree of nonconformance to the loss of serviceability, and of imposing sharply increasing penalties for increasing degrees of nonconformance. The objective is to encourage the supplier or contractor to exercise quality control that will insure an adequate and fairly uniform level of quality.

### 8.6.3 Combining Reductions in Price for Two or More Deficiencies

8.6.3.1 Nature of the Problem. The worth of an item of construction may depend on the measured results for two or more quality characteristics. If reductions in price are specified for failure of more than one type of measurement to fall within specified limits, the reductions in price for the various types of failure must be combined in some suitable way to arrive at the price to be paid for the item. To illustrate one way of combining reductions, the following example relating to concrete pavement is given. We shall include also methods of developing graduated scales of reduction in price for each characteristic which does not comply with the specified requirement. The important characteristics for concrete are strength, air content, and slump.

It should be understood that the purpose of this example is to illustrate a possible method of approach, and the assumed values should not be considered to be suggested standards for a similar situation.

8.6.3.2 Criteria for Strength. The worth of concrete as a material for a pavement is most influenced by the strength of the concrete. A decrease of 75 psi in the modulus of rupture may be considered to be equivalent to a decrease of about 1/2 inch in slab thickness. The AASHO design chart shows that a decrease in strength causes a very rapid decrease in serviceability, measured in terms of the total equivalent number of times a single axle load of 18,000 pounds, or 18 kips, is applied in 24 hours. Over the practical range of 28-day modulus of rupture from 400 to 700 psi, this relationship can be expressed approximately by one of the following equations:

$$T = 2.9(2)S_w \qquad (8.8)$$

or

$$T = 2.9(1.67)S_m \qquad (8.9)$$

where $T$ = total daily number of applications of single 18-kip axle loads, in hundreds of thousands

$S_w$ = working value for modulus of rupture of concrete, in pounds per square inch

$S_m$ = required 28-day modulus of rupture = $1.33S_w$

The quality of the concrete used in a pavement is usually determined on the basis of its 28-day compressive strength $f_c'$ instead of its modulus of rupture $S_m$. There is no exact correlation between these two types of strengths. However, over the limited range from 400 to 700 psi for $S_m$, a general relationship can be expressed approximately by the equation

$$f'_c = 10.33S_m - 2250 \qquad (8.10)$$

in which $f'_c$ and $S_m$ are expressed in pounds per square inch.

The serviceability of a concrete pavement can be expressed in terms of the 28-day compressive strength of the concrete by the equation

$$T = 9.3(1.63)^{f'_c} \qquad (8.11)$$

where T = total daily number of applications of single 18-kip axle loads, in hundreds of thousands

$f'_c$ = 28-day compressive strength of concrete, in thousands of pounds per square inch

The results of designed experiments indicate that a LOT of pavement concrete is acceptable if the results of five tests for compressive strength, each test result being the average of the measured strengths of two cylinders, meet the following requirements: It is probable that the 28-day compressive strength of 80 percent or more of the LOT exceeds 3000 psi, and the standard deviation is not more than 520 psi. The average compressive strength of such a LOT will be 3440 psi, and the LOT will be worth the full contract price. The relative worths of LOTs of lesser quality, as estimated by the use of Equation 8.3 and Table 8.3, are shown in Table 8.9.

Table 8.9

RELATIVE WORTH OF PAVEMENT CONCRETE BASED ON STRENGTH

| $f'_c$ (psi) | 18-Kip Axle Loads (100,000) | Percent Above 3000 psi | Percent Contract Price |
|---|---|---|---|
| 3440 or more | 50.0 or more | 80 or more | 100 |
| 3350 | 48.0 | 75 - 79 | 98 |
| 3270 | 46.0 | 70 - 74 | 96 |
| 3200 | 44.5 | 65 - 69 | 94 |
| 3130 | 43.0 | 60 - 64 | 93 |
| 3065 | 41.5 | 55 - 59 | 92 |
| 3000 | 40.5 | 50 - 54 | 90 |
| less than 3000 | | less than 50 | 0 |

8.6.3.3 Criteria for Air Content. The percent of entrained air in pavement concrete has a great effect on the durability of the pavement, especially where there is severe freezing and thawing. The best amount of entrained air is about 6 percent. It may be assumed that a reduction in the air content from 6 percent to 2 percent will reduce the factor measuring durability by about 50 percent. In any case, the proper amount of entrained air permits a reduction in water content without any reduction in slump.

The results of designed experiments show that if the air content is based on one test with a Roll-a-Meter or on the average for three tests with a Chace meter, the standard deviation for the measured results should be about 1.0 percent. When the target value for the air content is 6 percent, approximately 98 percent of the test results should be above the lower specification limit. Also, the percent of test results above the lower limit should be 93 percent when the target air content is 5.5 percent, and 84 percent when the target air content is 5.0 percent.

Results of samplings indicate that when the air content does not lie within the specified range from 4 to 7 percent, most of the failures will be on the low side. If the air content is in excess of 7 percent but is in the range from 7 to 12 percent, it can be assumed that the durability of the pavement would not be affected seriously but the strength would be reduced and the producer of the concrete would have to accept a downward adjustment in price on the strength-serviceability basis.

In Table 8.10 is shown a schedule for rates of payment on the basis of the air content of the concrete.

Table 8.10

RELATIVE WORTH OF PAVEMENT CONCRETE BASED ON AIR CONTENT

| Average Percent Air | Percent Above 4 Percent | Durability Factor | Percent of Contract Price |
|---|---|---|---|
| 5.5 | 95 | 9.5 | 100 |
| 5.0 | 85 - 94 | 9.3 | 98 |
| 4.5 | 70 - 84 | 9.1 | 96 |
| 4.0 | 50 - 69 | 8.8 | 93 |
| less than 4.0 | less than 50 | 8.1 | 85 |

8.6.3.4 Criteria for Slump. Current specification limits on slump of pavement concrete are usually 1 inch and 3 inches. However, results of a large number of tests show that the average slump used in construction was 2.3 inches and the standard deviation was 0.7 inch. On this basis a realistic specification for slump would call for a target value of 2.5 inches, and the tolerance ±2s would be given as 1.5 inches. The range would then be from 1 to 4 inches. Although slump of concrete is a function of temperature as well as of water content, a change of 1.5 inches in slump is probably equivalent, under average conditions, to a change in water content of about 0.8 gallon per cubic yard of concrete, or about 0.13 gallon per bag of cement in a six-bag mix. It can be assumed that the variation in water-cement ratio is not large enough to seriously affect the strength of the concrete. However, concrete having a slump of 4 inches will be less durable than concrete for which the slump is 2.5 inches. Concrete having a slump in excess of 4 inches can be considered unsuitable because of the increased tendency for excess mortar to collect at the surface of the pavement.

In Table 8.11 is shown a schedule for rates of payment on the basis of the slump of the concrete.

Table 8.11

RELATIVE WORTH OF PAVEMENT CONCRETE BASED ON SLUMP

| Slump (in.) | Water-Cement Ratio, by Weight | Durability Factor | Percent Between 1 in. and 4 in. | Percent Contract Price |
|---|---|---|---|---|
| 2.5 | 0.488 | 10.3 | 95 - 100 | 100 |
| 3.0 | 0.495 | 10.1 | 90 - 94 | 98 |
| 3.5 | 0.503 | 9.9 | 70 - 89 | 96 |
| 4.0 | 0.510 | 9.7 | 50 - 69 | 94 |
| | | | less than 50 | 0 |

8.6.3.5 Percents Within Limit for LOTs. In this example we shall assume that tests for compressive strength, air content, and slump have been made on 13 LOTs of pavement concrete, and the quality of each LOT is based on the results of tests on five samples from the LOT. In Table 8.12 the average of the five results for each characteristic is shown in the proper column headed $\overline{X}_5$ and the range for those five results is shown in the proper column headed

R. The percent within limit, computed by the method described in Section 8.3.2, is shown in the column headed P.W.L.

Table 8.12

SUMMARY OF PERCENT WITHIN LIMITS FOR PAVEMENT CONCRETE

| LOT No. | 28-Day Compressive Strength (psi) Specification Minimum, 3000 | | | Chace Air Content (Percent) Specification Range, 4 to 7 | | | Slump (Inches) Specification Range, 1 to 4 | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\overline{X}_5$ | R | P.W.L. | $\overline{X}_5$ | R | P.W.L. | $\overline{X}_5$ | R | P.W.L. |
| 1 | 3548 | 395 | 100 | 3.46 | 1.50 | 20 | 2.50 | 0.75 | 100 |
| 2 | 3005 | 625 | 51 | 4.23 | 1.17 | 68 | 2.60 | 0.75 | 100 |
| 3 | 3662 | 2050 | 77 | 4.43 | 1.83 | 70 | 2.85 | 1.75 | 99 |
| 4 | 3812 | 1205 | 100 | 4.07 | 1.83 | 54 | 2.10 | 1.50 | 100 |
| 5 | 3889 | 1325 | 100 | 4.80 | 3.34 | 71 | 3.15 | 2.25 | 82 |
| 6 | 3881 | 1120 | 100 | 4.88 | 1.00 | 100 | 2.85 | 0.50 | 100 |
| 7 | 4038 | 1100 | 100 | 3.84 | 2.33 | 44 | 2.45 | 4.00 | 63 |
| 8 | 4135 | 1845 | 97 | 5.77 | 2.67 | 99 | 2.75 | 2.00 | 97 |
| 9 | 4133 | 1135 | 100 | 6.60 | 3.33 | 100 | 2.15 | 1.25 | 100 |
| 10 | 3549 | 930 | 96 | 6.60 | 2.33 | 100 | 2.30 | 1.25 | 100 |
| 11 | 3924 | 475 | 100 | 6.57 | 3.50 | 100 | 2.10 | 1.75 | 98 |
| 12 | 4334 | 1380 | 100 | 6.60 | 1.17 | 100 | 2.10 | 1.50 | 100 |
| 13 | 4152 | 1160 | 100 | 6.54 | 0.17 | 100 | 2.69 | 1.50 | 100 |

8.6.3.6 Combining Price Adjustments. After the percent within limit has been determined for each characteristic, as shown in Table 8.12, the corresponding percents of contract price are found from Tables 8.9, 8.10, and 8.11. In Table 8.13 these percents are shown for each LOT and each characteristic in the second, third, and fourth columns. The three individual percents for a LOT are then multiplied together to arrive at the percent of contract price to be paid for the LOT. These results are shown in the last column of Table 8.13. Finally, the percents for the individual LOTs are added and the sum is divided by the total possible score for all the LOTs. The percent so computed is the percent of contract price to be paid for the entire amount of pavement concrete in the project. For the values shown in Table 8.13, the adjusted value is 94.5 percent.

Table 8.13

ADJUSTED PERCENT OF CONTRACT PRICE FOR PAVEMENT CONCRETE

| LOT Number | Percent of Contract Price: Compressive Strength | Slump | Air Content | Percent of Contract Price for LOT |
|---|---|---|---|---|
| 1 | 100 | 100 | 85 | 85 |
| 2 | 90 | 100 | 93 | 84 |
| 3 | 98 | 100 | 96 | 94 |
| 4 | 100 | 100 | 93 | 93 |
| 5 | 100 | 96 | 96 | 92 |
| 6 | 100 | 100 | 100 | 100 |
| 7 | 100 | 94 | 85 | 80 |
| 8 | 100 | 100 | 100 | 100 |
| 9 | 100 | 100 | 100 | 100 |
| 10 | 100 | 100 | 100 | 100 |
| 11 | 100 | 100 | 100 | 100 |
| 12 | 100 | 100 | 100 | 100 |
| 13 | 100 | 100 | 100 | 100 |
| | | | Total = | 1228 |

Sum of Percents for LOTs = 1228

Maximum Percent = 13(100) = 1300

Adjusted Percent of Contract Price $= \frac{1228}{1300} = 94.5$ percent

## 8.7 FREQUENCY OF SAMPLING AND TESTING BASED ON WORTH

Let us suppose that we have a fairly good estimate of the true average and the true standard deviation for a characteristic of the output of a process. There is a specified limit for the measured results of tests on samples, and we want to know how many samples we should take from each LOT.

Several factors must be considered in deciding on the number of samples. Although, in theory, the size of the LOT does not affect the frequency of sampling, it is necessary, from a practical standpoint, to consider the economic effects of noncompliance with a requirement of the specifications. If too large a percent of the LOT does not satisfy the requirement, either the price per unit may be reduced or it may be necessary for us to take corrective measures. In any case, the contractor will, in some way, be subjected to a penalty which will result in a dollar loss for each unit in the LOT. The total loss could be computed by multiplying the total number of units in the LOT by the loss per unit. To protect the contractor against large losses, we would want to take more measurements on a large LOT than on a small LOT. On the other hand, there is some dollar cost connected with the taking of each sample and making measurements on it. We should maintain some reasonable balance between the total cost of sampling and measuring and the total possible loss.

Another factor is the variability of the measurements in sigma units. We know that, in general, the larger the value of the standard deviation the more measurements we will have to make for the same degree of accuracy. However, the probability that the average of measurements made on samples taken from a process will be outside some limit depends on the distance between the process average $\overline{X}$ of the measured results and the limit L. If we let $\Delta$ denote this distance,

$$\Delta = \overline{X} - L \qquad \text{or} \qquad \Delta = L - \overline{X}$$

If the value of $\Delta$ is large, say greater than about two sigma units, the probability that the average of the measurements made on samples will be outside the limit L will be small.

On the basis of the preceding discussion, we can compute the required number of samples from a LOT by using the equation

$$n = \sqrt[3]{\left(\frac{s' Q \ell}{2c\Delta}\right)^2} \qquad (8.12)$$

where $n$ = number of samples from a LOT to be measured

$s'$ = assumed standard deviation of the measurements

$Q$ = number of units in the LOT

$\ell$ = reduction in worth of one unit because of a change $\Delta$ in the value of a selected characteristic

$c$ = cost of one test

$\Delta$ = change in the value of the selected characteristic

The number 3 in the angle of the radical sign means that we must find the cube root of the quantity governed by the radical sign. The cube root of a number can be found by the use of a slide rule or a suitable table. However, the value of $n$ that would be computed by applying equation 8.12 can be determined directly by means of the nomograph in Figure 8.4. To use this nomograph, it is only necessary to pass a straightedge through the proper values of the scales for $\frac{s'}{\Delta}$ and $\frac{Q\ell}{c}$ and to read the required number of samples from the scale for $n$.

<u>Example 8.12</u>. Let us suppose that a crusher produces stone having a certain grading in LOTs of 1100 tons each. The critical requirement is that 25 to 60 percent of the stone must pass the 1/2-inch sieve. If the actual percent passing the sieve is outside the specified limits, the stone must be rescreened at a cost of 20 cents per ton. It is estimated that the cost of taking samples for a test and making the test is \$2.00. We have previously made 78 gradation tests and have found that the average percent passing the 1/2-inch sieve is 48 and the standard deviation is 2.4 percent. We want to find the optimum number of samples to be taken from each LOT.

<u>Solution</u>. In equation 8.12, $s' = 2.4$ percent, $Q = 1100$ tons, $\ell = \$0.20$ per ton, $c = \$2.00$, and $\Delta$ is the difference between the average of the measured values and the nearer specified limit or $60 - 48 = 12$. If the nomograph in Figure 8.2 is not used, the calculations may be made as follows:

$$\frac{Q\ell}{2c} = \frac{1100(0.20)}{2(2.00)} = 55$$

$$\frac{s'}{\Delta} = \frac{2.4}{|L - \bar{X}|} = \frac{2.4}{12} = 0.20$$

$$n = \sqrt[3]{(0.20(55))^2} = \sqrt[3]{11^2} = \sqrt[3]{121} \approx 5$$

It is therefore desirable to make five gradation tests for each 1100-ton LOT.

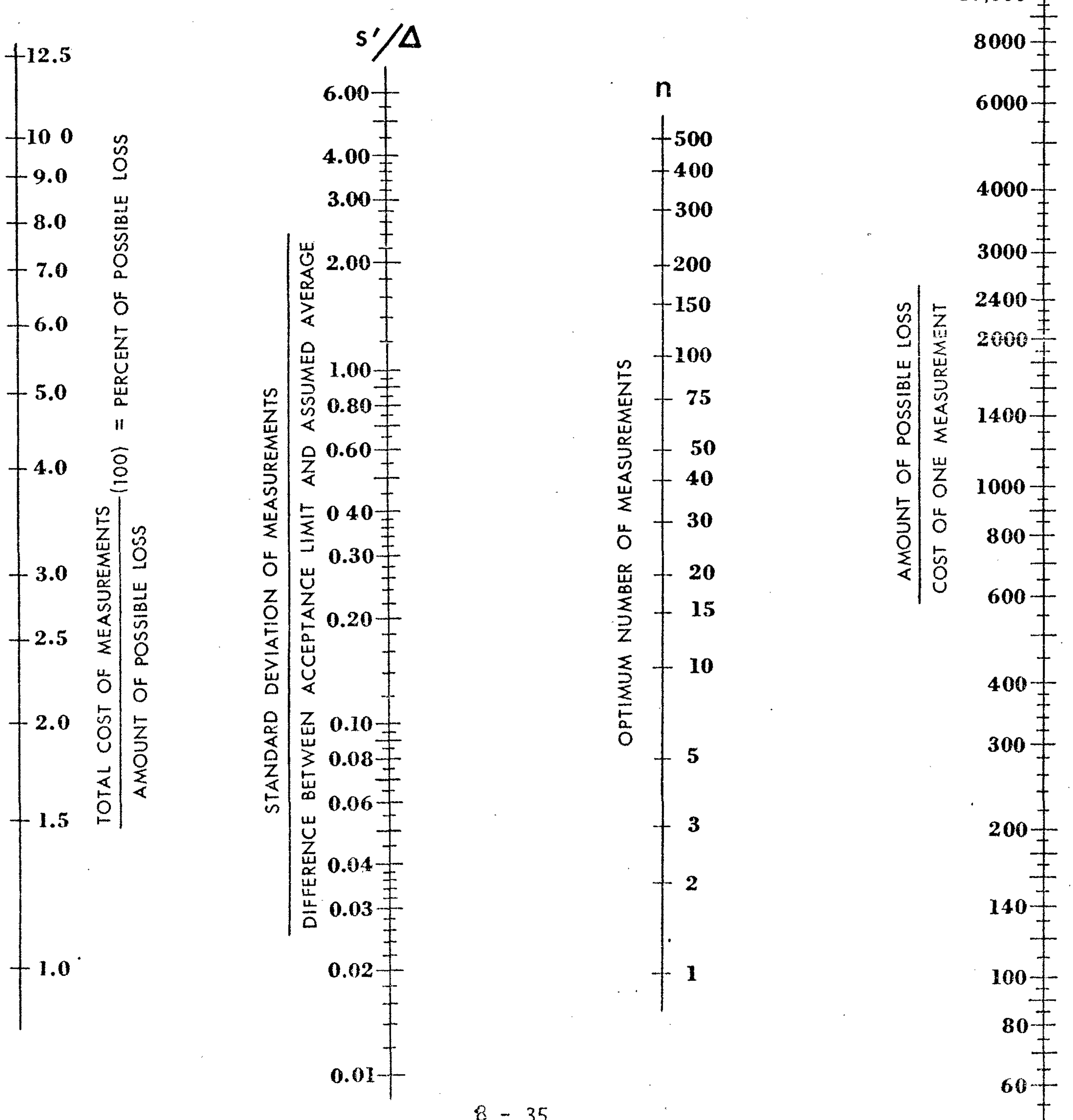
If we use the nomograph in Figure 8.4, we set a straightedge so as to pass through 0.20 on the scale for $\frac{s'}{\Delta}$ and through 110 on the scale for $\frac{Q\ell}{c}$. The straightedge will then pass very close to 5 on the scale for $n$.

To determine the reliability of basing the acceptance or rejection of a LOT on the average of the measurements on the samples, we can compute the confidence limit for a probability of 95 percent that we will make the correct

Figure 8.4

# OPTIMUM NUMBER OF MEASUREMENTS TO MINIMIZE MAXIMUM LOSS

$$n = \sqrt[3]{\left(\frac{s'QL}{2c\Delta}\right)^2}$$



decision by using the sigma that we compute from the measurements on the samples and the $P_2$ value of t in the equation

$$C.L. = \overline{X} + \frac{ts}{\sqrt{n}} \qquad (8.13)$$

In the case of the example we use the average of five test results,

$$C.L. = \overline{X}_5 + \frac{2.13s}{\sqrt{5}}$$

$$= \overline{X}_5 + 0.95s$$

Whenever the value of $\overline{X}_5 + 0.95s$ for the measured results on a particular LOT is less than the specified limit 60, we can be 95 percent confident that the actual average for the LOT is below the specified limit.

Example 8.13. Let us suppose that the specifications require that a bituminous surface course 25 feet wide is to be compacted, on the average, to at least 98 percent of laboratory Marshall specimens which have air voids of 4 percent. This permits the volume of air voids in the pavement to be not more than 4 + 2 = 6 percent. The estimated value of the standard deviation for air voids in a bituminous surface course is 1.5 percent. If the average volume of air voids in a LOT of the surface course exceeds 9 percent, it will be necessary to apply a seal coat that costs 15 cents per square yard in order to insure satisfactory durability. Also, the cost of cutting a test specimen from the pavement and making the test is \$10.00. We want to determine the best number of specimens to be cut from a mile of pavement.

Solution. In this case, s = 1.5 percent, $\ell$ = \$0.15 per square yard, and c = \$10.00. Also Q is the number of square yards in a mile of pavement, or

$$Q = \frac{5280(25)}{9} = 14{,}700 \text{ square yards}$$

and Δ is the difference between critical value of 9 percent air voids and the allowable 6 percent.

If we use the nomograph in Figure 8.4,

$$\frac{Q\ell}{c} = \frac{14{,}700(0.15)}{10.00} = 220$$

and

$$\frac{s}{\Delta} = \frac{1.5}{9.0 - (4 + 2)} = \frac{1.5}{3.0} = 0.50$$

A straightedge that passes through 0.50 on the scale for $\frac{s'}{\Delta}$ and through 220 on the scale for $\frac{Q\ell}{c}$ will cross the scale for $n$ a little below 15. So fifteen density tests should be made at random locations in each mile of pavement.

If desired, we can determine the cost of sampling and testing in terms of percent of possible loss $P_\ell$ for given values of $n$, c, Q, and $\ell$ by using the equation

$$P_\ell = \frac{nc}{Q\ell}(100) \qquad (8.14)$$

This percent $P_\ell$ can also be found easily by using the nomograph in Figure 8.4. A straightedge is passed through the values of $n$ and $\frac{Q\ell}{c}$ on the proper scales, and the required result is read on the left-hand scale.

Table 8.3

TABLE FOR ESTIMATING PERCENT OF LOT WITHIN TOLERANCE

BY RANGE METHOD

| Percent Within Tolerance | POSITIVE VALUES OF $Q_U$ OR $Q_L$ | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $n$=3 | $n$=4 | $n$=5 | $n$=6 | $n$=7 | $n$=10* | $n$=15* | $n$=25* | $n$=30* | $n$=35* | $n$=40* | $n$=50* | $n$=60* |
| 99 | 0.60 | 0.66 | 0.66 | 0.65 | 0.65 | 0.82 | 0.88 | 0.93 | 0.94 | 0.95 | 0.95 | 0.97 | 0.97 |
| 98 | 0.60 | 0.64 | 0.65 | 0.62 | 0.61 | 0.76 | 0.80 | 0.83 | 0.84 | 0.85 | 0.85 | 0.86 | 0.86 |
| 97 | 0.60 | 0.63 | 0.62 | 0.59 | 0.58 | 0.71 | 0.74 | 0.77 | 0.78 | 0.78 | 0.78 | 0.79 | 0.79 |
| 96 | 0.60 | 0.62 | 0.60 | 0.57 | 0.55 | 0.68 | 0.68 | 0.72 | 0.73 | 0.73 | 0.73 | 0.74 | 0.74 |
| 95 | 0.60 | 0.60 | 0.58 | 0.55 | 0.53 | 0.64 | 0.66 | 0.68 | 0.68 | 0.69 | 0.69 | 0.70 | 0.70 |
| 94 | 0.59 | 0.59 | 0.57 | 0.53 | 0.51 | 0.62 | 0.63 | 0.64 | 0.65 | 0.65 | 0.66 | 0.66 | 0.66 |
| 93 | 0.59 | 0.58 | 0.55 | 0.51 | 0.49 | 0.59 | 0.61 | 0.61 | 0.62 | 0.62 | 0.62 | 0.62 | 0.62 |
| 92 | 0.59 | 0.56 | 0.53 | 0.49 | 0.47 | 0.57 | 0.58 | 0.59 | 0.59 | 0.59 | 0.59 | 0.60 | 0.60 |
| 91 | 0.58 | 0.55 | 0.51 | 0.48 | 0.46 | 0.54 | 0.55 | 0.56 | 0.57 | 0.57 | 0.57 | 0.57 | 0.57 |
| 90 | 0.58 | 0.54 | 0.50 | 0.46 | 0.44 | 0.52 | 0.53 | 0.54 | 0.54 | 0.54 | 0.54 | 0.55 | 0.55 |
| 89 | 0.57 | 0.52 | 0.48 | 0.45 | 0.43 | 0.50 | 0.51 | 0.52 | 0.52 | 0.52 | 0.52 | 0.52 | 0.52 |
| 88 | 0.56 | 0.51 | 0.46 | 0.43 | 0.41 | 0.48 | 0.49 | 0.50 | 0.50 | 0.50 | 0.50 | 0.50 | 0.50 |
| 87 | 0.55 | 0.50 | 0.45 | 0.42 | 0.40 | 0.47 | 0.47 | 0.47 | 0.48 | 0.48 | 0.48 | 0.48 | 0.48 |
| 86 | 0.54 | 0.48 | 0.44 | 0.40 | 0.38 | 0.45 | 0.45 | 0.46 | 0.46 | 0.46 | 0.46 | 0.46 | 0.46 |
| 85 | 0.54 | 0.47 | 0.42 | 0.39 | 0.37 | 0.43 | 0.44 | 0.44 | 0.44 | 0.44 | 0.44 | 0.44 | 0.44 |
| 84 | 0.53 | 0.46 | 0.41 | 0.38 | 0.36 | 0.42 | 0.42 | 0.42 | 0.43 | 0.43 | 0.43 | 0.42 | 0.42 |
| 83 | 0.52 | 0.44 | 0.40 | 0.36 | 0.34 | 0.40 | 0.40 | 0.41 | 0.41 | 0.41 | 0.41 | 0.41 | 0.41 |
| 82 | 0.51 | 0.43 | 0.38 | 0.35 | 0.33 | 0.39 | 0.39 | 0.39 | 0.39 | 0.39 | 0.39 | 0.39 | 0.39 |
| 81 | 0.50 | 0.42 | 0.37 | 0.34 | 0.32 | 0.37 | 0.37 | 0.37 | 0.37 | 0.37 | 0.38 | 0.38 | 0.38 |
| 80 | 0.49 | 0.40 | 0.36 | 0.33 | 0.31 | 0.36 | 0.36 | 0.36 | 0.36 | 0.36 | 0.36 | 0.36 | 0.36 |

*When $n$ is 10 or more, the samples are arranged consecutively in subgroups of five. Then the range R of each subgroup is determined, and the average $\overline{R}$ of the ranges of all the subgroups is computed for use in finding $Q_U$ or $Q_L$.

Table 8.3 (Continued)

TABLE FOR ESTIMATING PERCENT OF LOT WITHIN TOLERANCE

BY RANGE METHOD

| Percent Within Tolerance | POSITIVE VALUES OF $Q_U$ OR $Q_L$ | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $n$=3 | $n$=4 | $n$=5 | $n$=6 | $n$=7 | $n$=10* | $n$=15* | $n$=25* | $n$=30* | $n$=35* | $n$=40* | $n$=50* | $n$=60* |
| 79 | 0.48 | 0.39 | 0.34 | 0.31 | 0.29 | 0.34 | 0.34 | 0.34 | 0.34 | 0.34 | 0.35 | 0.35 | 0.35 |
| 78 | 0.47 | 0.38 | 0.33 | 0.30 | 0.28 | 0.33 | 0.33 | 0.33 | 0.33 | 0.33 | 0.33 | 0.33 | 0.33 |
| 77 | 0.46 | 0.36 | 0.32 | 0.29 | 0.27 | 0.32 | 0.32 | 0.31 | 0.31 | 0.32 | 0.32 | 0.32 | 0.32 |
| 76 | 0.44 | 0.35 | 0.30 | 0.28 | 0.26 | 0.30 | 0.30 | 0.30 | 0.30 | 0.30 | 0.30 | 0.30 | 0.30 |
| 75 | 0.43 | 0.34 | 0.29 | 0.27 | 0.25 | 0.29 | 0.29 | 0.29 | 0.29 | 0.29 | 0.29 | 0.29 | 0.29 |
| 74 | 0.41 | 0.32 | 0.28 | 0.25 | 0.24 | 0.28 | 0.28 | 0.28 | 0.28 | 0.28 | 0.28 | 0.28 | 0.28 |
| 73 | 0.40 | 0.31 | 0.27 | 0.24 | 0.23 | 0.26 | 0.26 | 0.26 | 0.26 | 0.26 | 0.26 | 0.26 | 0.27 |
| 72 | 0.39 | 0.30 | 0.25 | 0.23 | 0.22 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 |
| 71 | 0.37 | 0.28 | 0.24 | 0.22 | 0.20 | 0.24 | 0.24 | 0.24 | 0.24 | 0.24 | 0.24 | 0.24 | 0.24 |
| 70 | 0.36 | 0.27 | 0.23 | 0.21 | 0.19 | 0.22 | 0.23 | 0.23 | 0.23 | 0.23 | 0.23 | 0.23 | 0.23 |
| 69 | 0.34 | 0.26 | 0.22 | 0.20 | 0.18 | 0.21 | 0.21 | 0.21 | 0.21 | 0.21 | 0.21 | 0.21 | 0.21 |
| 68 | 0.32 | 0.24 | 0.21 | 0.19 | 0.17 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 |
| 67 | 0.31 | 0.23 | 0.19 | 0.18 | 0.16 | 0.19 | 0.19 | 0.19 | 0.19 | 0.19 | 0.19 | 0.19 | 0.19 |
| 66 | 0.29 | 0.21 | 0.18 | 0.17 | 0.15 | 0.18 | 0.18 | 0.18 | 0.18 | 0.18 | 0.18 | 0.18 | 0.18 |
| 65 | 0.27 | 0.20 | 0.17 | 0.16 | 0.14 | 0.17 | 0.17 | 0.17 | 0.17 | 0.17 | 0.17 | 0.17 | 0.17 |
| 64 | 0.26 | 0.19 | 0.16 | 0.15 | 0.13 | 0.15 | 0.16 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 |
| 63 | 0.24 | 0.17 | 0.15 | 0.13 | 0.12 | 0.14 | 0.14 | 0.14 | 0.14 | 0.14 | 0.14 | 0.14 | 0.14 |
| 62 | 0.22 | 0.16 | 0.14 | 0.12 | 0.11 | 0.13 | 0.13 | 0.13 | 0.13 | 0.13 | 0.13 | 0.13 | 0.13 |
| 61 | 0.20 | 0.15 | 0.13 | 0.11 | 0.10 | 0.12 | 0.12 | 0.12 | 0.12 | 0.12 | 0.12 | 0.12 | 0.12 |
| 60 | 0.19 | 0.13 | 0.11 | 0.10 | 0.09 | 0.11 | 0.11 | 0.11 | 0.11 | 0.11 | 0.11 | 0.11 | 0.11 |
| 55 | 0.09 | 0.07 | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| 50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |

*When $n$ is 10 or more, the samples are arranged consecutively in subgroups of five. Then the range R of each subgroup is determined, and the average $\overline{R}$ of the ranges of all the subgroups is computed for use in finding $Q_U$ of $Q_L$.

Table 8.3 (Continued)

TABLE FOR ESTIMATING PERCENT OF LOT WITHIN TOLERANCE

BY RANGE METHOD

| Percent Within Tolerance | NEGATIVE VALUES OF $Q_U$ OR $Q_L$ | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | *n*=3 | *n*=4 | *n*=5 | *n*=6 | *n*=7 | *n*=10* | *n*=15* | *n*=25* | *n*=30* | *n*=35* | *n*=40* | *n*=50* | *n*=60* |
| 50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 45 | 0.09 | 0.07 | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| 40 | 0.19 | 0.13 | 0.11 | 0.10 | 0.09 | 0.11 | 0.11 | 0.11 | 0.11 | 0.11 | 0.11 | 0.11 | 0.11 |
| 39 | 0.20 | 0.15 | 0.13 | 0.11 | 0.10 | 0.12 | 0.12 | 0.12 | 0.12 | 0.12 | 0.12 | 0.12 | 0.12 |
| 38 | 0.22 | 0.16 | 0.14 | 0.12 | 0.11 | 0.13 | 0.13 | 0.13 | 0.13 | 0.13 | 0.13 | 0.13 | 0.13 |
| 37 | 0.24 | 0.17 | 0.15 | 0.13 | 0.12 | 0.14 | 0.14 | 0.14 | 0.14 | 0.14 | 0.14 | 0.14 | 0.14 |
| 36 | 0.26 | 0.19 | 0.16 | 0.15 | 0.13 | 0.15 | 0.16 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 |
| 35 | 0.27 | 0.20 | 0.17 | 0.16 | 0.14 | 0.17 | 0.17 | 0.17 | 0.17 | 0.17 | 0.17 | 0.17 | 0.17 |
| 34 | 0.29 | 0.21 | 0.18 | 0.17 | 0.15 | 0.18 | 0.18 | 0.18 | 0.18 | 0.18 | 0.18 | 0.18 | 0.18 |
| 33 | 0.31 | 0.23 | 0.19 | 0.18 | 0.16 | 0.19 | 0.19 | 0.19 | 0.19 | 0.19 | 0.19 | 0.19 | 0.19 |
| 32 | 0.32 | 0.24 | 0.21 | 0.19 | 0.17 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 |
| 31 | 0.34 | 0.26 | 0.22 | 0.20 | 0.18 | 0.21 | 0.21 | 0.21 | 0.21 | 0.21 | 0.21 | 0.21 | 0.21 |
| 30 | 0.36 | 0.27 | 0.23 | 0.21 | 0.19 | 0.22 | 0.22 | 0.22 | 0.23 | 0.23 | 0.23 | 0.23 | 0.23 |
| 29 | 0.37 | 0.28 | 0.24 | 0.22 | 0.20 | 0.24 | 0.24 | 0.24 | 0.24 | 0.24 | 0.24 | 0.24 | 0.24 |
| 28 | 0.39 | 0.30 | 0.25 | 0.23 | 0.22 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 |
| 27 | 0.40 | 0.31 | 0.27 | 0.24 | 0.23 | 0.26 | 0.26 | 0.26 | 0.26 | 0.26 | 0.26 | 0.26 | 0.26 |
| 26 | 0.41 | 0.32 | 0.28 | 0.25 | 0.24 | 0.28 | 0.28 | 0.28 | 0.28 | 0.28 | 0.28 | 0.28 | 0.28 |
| 25 | 0.43 | 0.34 | 0.29 | 0.27 | 0.25 | 0.29 | 0.29 | 0.29 | 0.29 | 0.29 | 0.29 | 0.29 | 0.29 |
| 24 | 0.44 | 0.35 | 0.30 | 0.28 | 0.26 | 0.30 | 0.30 | 0.30 | 0.30 | 0.30 | 0.30 | 0.30 | 0.30 |
| 23 | 0.46 | 0.36 | 0.32 | 0.29 | 0.27 | 0.32 | 0.32 | 0.31 | 0.31 | 0.32 | 0.32 | 0.32 | 0.32 |
| 22 | 0.47 | 0.38 | 0.33 | 0.30 | 0.28 | 0.33 | 0.33 | 0.33 | 0.33 | 0.33 | 0.33 | 0.33 | 0.33 |
| 21 | 0.48 | 0.39 | 0.34 | 0.31 | 0.29 | 0.34 | 0.34 | 0.34 | 0.34 | 0.34 | 0.35 | 0.35 | 0.35 |

*When *n* is 10 or more, the samples are arranged consecutively in subgroups of five. Then the range R of each subgroup is determined, and the average $\bar{R}$ of the ranges of all the subgroups is computed for use in finding $Q_U$ or $Q_L$.

Table 8.3 (Continued)

TABLE FOR ESTIMATING PERCENT OF LOT WITHIN TOLERANCE

BY RANGE METHOD

| Percent Within Tolerance | NEGATIVE VALUES OF $Q_U$ OR $Q_L$ | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $n$=3 | $n$=4 | $n$=5 | $n$=6 | $n$=7 | $n$=10* | $n$=15* | $n$=25* | $n$=30* | $n$=35* | $n$=40* | $n$=50* | $n$=60* |
| 20 | 0.49 | 0.40 | 0.36 | 0.33 | 0.31 | 0.36 | 0.36 | 0.36 | 0.36 | 0.36 | 0.36 | 0.36 | 0.36 |
| 19 | 0.50 | 0.42 | 0.37 | 0.34 | 0.32 | 0.37 | 0.37 | 0.37 | 0.37 | 0.37 | 0.38 | 0.38 | 0.38 |
| 18 | 0.51 | 0.43 | 0.38 | 0.35 | 0.33 | 0.39 | 0.39 | 0.39 | 0.39 | 0.39 | 0.39 | 0.39 | 0.39 |
| 17 | 0.52 | 0.44 | 0.40 | 0.36 | 0.34 | 0.40 | 0.40 | 0.41 | 0.41 | 0.41 | 0.41 | 0.41 | 0.41 |
| 16 | 0.53 | 0.46 | 0.41 | 0.38 | 0.36 | 0.42 | 0.42 | 0.42 | 0.43 | 0.43 | 0.43 | 0.42 | 0.42 |
| 15 | 0.54 | 0.47 | 0.42 | 0.39 | 0.37 | 0.43 | 0.44 | 0.44 | 0.44 | 0.44 | 0.44 | 0.44 | 0.44 |
| 14 | 0.54 | 0.48 | 0.44 | 0.40 | 0.38 | 0.45 | 0.45 | 0.46 | 0.46 | 0.46 | 0.46 | 0.46 | 0.46 |
| 13 | 0.55 | 0.50 | 0.45 | 0.42 | 0.40 | 0.47 | 0.47 | 0.47 | 0.48 | 0.48 | 0.48 | 0.48 | 0.48 |
| 12 | 0.56 | 0.51 | 0.46 | 0.43 | 0.41 | 0.48 | 0.49 | 0.50 | 0.50 | 0.50 | 0.50 | 0.50 | 0.50 |
| 11 | 0.57 | 0.52 | 0.48 | 0.45 | 0.43 | 0.50 | 0.51 | 0.52 | 0.52 | 0.52 | 0.52 | 0.52 | 0.52 |
| 10 | 0.58 | 0.54 | 0.50 | 0.46 | 0.44 | 0.52 | 0.53 | 0.54 | 0.54 | 0.54 | 0.54 | 0.55 | 0.55 |
| 9 | 0.58 | 0.55 | 0.51 | 0.48 | 0.46 | 0.54 | 0.55 | 0.56 | 0.57 | 0.57 | 0.57 | 0.57 | 0.57 |
| 8 | 0.59 | 0.56 | 0.53 | 0.49 | 0.47 | 0.57 | 0.58 | 0.59 | 0.59 | 0.59 | 0.59 | 0.60 | 0.60 |
| 7 | 0.59 | 0.58 | 0.55 | 0.51 | 0.49 | 0.59 | 0.61 | 0.61 | 0.62 | 0.62 | 0.62 | 0.62 | 0.62 |
| 6 | 0.59 | 0.59 | 0.57 | 0.53 | 0.51 | 0.62 | 0.63 | 0.64 | 0.65 | 0.65 | 0.66 | 0.66 | 0.66 |
| 5 | 0.60 | 0.60 | 0.58 | 0.55 | 0.53 | 0.64 | 0.66 | 0.68 | 0.68 | 0.69 | 0.69 | 0.70 | 0.70 |
| 4 | 0.60 | 0.62 | 0.60 | 0.57 | 0.55 | 0.68 | 0.68 | 0.72 | 0.73 | 0.73 | 0.73 | 0.74 | 0.74 |
| 3 | 0.60 | 0.63 | 0.62 | 0.59 | 0.58 | 0.71 | 0.74 | 0.77 | 0.78 | 0.78 | 0.78 | 0.79 | 0.79 |
| 2 | 0.60 | 0.64 | 0.65 | 0.62 | 0.61 | 0.76 | 0.80 | 0.83 | 0.84 | 0.85 | 0.85 | 0.86 | 0.86 |
| 1 | 0.60 | 0.66 | 0.66 | 0.65 | 0.65 | 0.82 | 0.88 | 0.93 | 0.94 | 0.95 | 0.95 | 0.97 | 0.97 |

*When $n$ is 10 or more, the samples are arranged consecutively in subgroups of five. Then the range R of each subgroup is determined, and the average $\bar{R}$ of the ranges of all the subgroups is computed for use in finding $Q_U$ or $Q_L$.

Table 8.4

$n$ = 4

PROBABILITY P FOR NON-CENTRAL t DISTRIBUTION AND GIVEN VALUES OF $K_p$ AND NUMBER OF MEASUREMENTS $n$

| D \ $K_p$ = | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 1.75 | 2.00 | 2.25 | 2.50 | 2.75 | 3.00 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2.0 | 0.0697 | 0.0287 | 0.0100 | 0.0029 | 0.0007 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.8 | 0.0848 | 0.0356 | 0.0125 | 0.0037 | 0.0009 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.6 | 0.1040 | 0.0446 | 0.0160 | 0.0048 | 0.0012 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.4 | 0.1280 | 0.0564 | 0.0207 | 0.0063 | 0.0016 | 0.0003 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.2 | 0.1581 | 0.0718 | 0.0271 | 0.0084 | 0.0021 | 0.0004 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.0 | 0.1955 | 0.0921 | 0.0360 | 0.0115 | 0.0030 | 0.0006 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.8 | 0.2411 | 0.1186 | 0.0482 | 0.0160 | 0.0043 | 0.0009 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.6 | 0.2954 | 0.1526 | 0.0651 | 0.0226 | 0.0064 | 0.0014 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.4 | 0.3580 | 0.1954 | 0.0881 | 0.0324 | 0.0096 | 0.0023 | 0.0004 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.2 | 0.4271 | 0.2476 | 0.1189 | 0.0465 | 0.0147 | 0.0037 | 0.0007 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.0 | 0.5000 | 0.3085 | 0.1586 | 0.0668 | 0.0227 | 0.0062 | 0.0013 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.2 | 0.5729 | 0.3764 | 0.2080 | 0.0948 | 0.0351 | 0.0105 | 0.0025 | 0.0005 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.4 | 0.6420 | 0.4482 | 0.2663 | 0.1318 | 0.0535 | 0.0176 | 0.0047 | 0.0010 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.6 | 0.7046 | 0.5201 | 0.3313 | 0.1781 | 0.0795 | 0.0291 | 0.0087 | 0.0021 | 0.0004 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| 0.8 | 0.7589 | 0.5887 | 0.4001 | 0.2328 | 0.1141 | 0.0465 | 0.0157 | 0.0043 | 0.0010 | 0.0002 | 0.0000 | 0.0000 | 0.0000 |
| 1.0 | 0.8045 | 0.6515 | 0.4692 | 0.2938 | 0.1573 | 0.0712 | 0.0270 | 0.0085 | 0.0022 | 0.0005 | 0.0001 | 0.0000 | 0.0000 |
| 1.2 | 0.8419 | 0.7070 | 0.5357 | 0.3583 | 0.2082 | 0.1038 | 0.0441 | 0.0159 | 0.0048 | 0.0012 | 0.0003 | 0.0000 | 0.0000 |
| 1.4 | 0.8720 | 0.7547 | 0.5972 | 0.4234 | 0.2646 | 0.1441 | 0.0680 | 0.0276 | 0.0096 | 0.0029 | 0.0007 | 0.0002 | 0.0000 |
| 1.6 | 0.8960 | 0.7949 | 0.6526 | 0.4864 | 0.3241 | 0.1910 | 0.0989 | 0.0448 | 0.0177 | 0.0061 | 0.0018 | 0.0005 | 0.0001 |
| 1.8 | 0.9152 | 0.8284 | 0.7014 | 0.5455 | 0.3843 | 0.2427 | 0.1365 | 0.0681 | 0.0301 | 0.0117 | 0.0041 | 0.0012 | 0.0003 |
| 2.0 | 0.9303 | 0.8561 | 0.7436 | 0.5997 | 0.4431 | 0.2971 | 0.1797 | 0.0976 | 0.0476 | 0.0207 | 0.0081 | 0.0028 | 0.0009 |
| 2.2 | 0.9424 | 0.8789 | 0.7798 | 0.6483 | 0.4990 | 0.3523 | 0.2268 | 0.1327 | 0.0704 | 0.0339 | 0.0147 | 0.0058 | 0.0021 |
| 2.4 | 0.9521 | 0.8977 | 0.8106 | 0.6915 | 0.5510 | 0.4066 | 0.2764 | 0.1724 | 0.0986 | 0.0516 | 0.0247 | 0.0108 | 0.0043 |
| 2.6 | 0.9598 | 0.9131 | 0.8367 | 0.7293 | 0.5985 | 0.4588 | 0.3268 | 0.2155 | 0.1314 | 0.0740 | 0.0384 | 0.0184 | 0.0082 |
| 2.8 | 0.9661 | 0.9258 | 0.8587 | 0.7623 | 0.6415 | 0.5080 | 0.3766 | 0.2606 | 0.1680 | 0.1008 | 0.0563 | 0.0292 | 0.0141 |
| 3.0 | 0.9712 | 0.9363 | 0.8774 | 0.7910 | 0.6800 | 0.5537 | 0.4250 | 0.3066 | 0.2075 | 0.1316 | 0.0782 | 0.0435 | 0.0226 |
| 3.2 | 0.9753 | 0.9451 | 0.8932 | 0.8158 | 0.7143 | 0.5957 | 0.4712 | 0.3523 | 0.2487 | 0.1655 | 0.1038 | 0.0613 | 0.0341 |
| 3.4 | 0.9788 | 0.9524 | 0.9066 | 0.8373 | 0.7447 | 0.6340 | 0.5146 | 0.3970 | 0.2907 | 0.2017 | 0.1327 | 0.0826 | 0.0487 |
| 3.6 | 0.9816 | 0.9585 | 0.9180 | 0.8559 | 0.7715 | 0.6687 | 0.5550 | 0.4401 | 0.3327 | 0.2395 | 0.1642 | 0.1071 | 0.0665 |
| 3.8 | 0.9840 | 0.9637 | 0.9277 | 0.8720 | 0.7952 | 0.6999 | 0.5924 | 0.4810 | 0.3739 | 0.2780 | 0.1976 | 0.1343 | 0.0872 |
| 4.0 | 0.9860 | 0.9681 | 0.9361 | 0.8860 | 0.8160 | 0.7280 | 0.6268 | 0.5196 | 0.4139 | 0.3166 | 0.2324 | 0.1637 | 0.1105 |
| 4.2 | 0.9877 | 0.9718 | 0.9433 | 0.8981 | 0.8345 | 0.7532 | 0.6582 | 0.5557 | 0.4524 | 0.3548 | 0.2679 | 0.1947 | 0.1362 |
| 4.4 | 0.9891 | 0.9750 | 0.9494 | 0.9087 | 0.8507 | 0.7758 | 0.6869 | 0.5892 | 0.4889 | 0.3920 | 0.3035 | 0.2269 | 0.1636 |
| 4.6 | 0.9903 | 0.9777 | 0.9548 | 0.9180 | 0.8651 | 0.7960 | 0.7129 | 0.6203 | 0.5234 | 0.4280 | 0.3389 | 0.2597 | 0.1926 |
| 4.8 | 0.9914 | 0.9801 | 0.9595 | 0.9261 | 0.8778 | 0.8140 | 0.7366 | 0.6490 | 0.5559 | 0.4625 | 0.3735 | 0.2927 | 0.2224 |
| 5.0 | 0.9923 | 0.9821 | 0.9635 | 0.9332 | 0.8890 | 0.8303 | 0.7581 | 0.6754 | 0.5863 | 0.4954 | 0.4072 | 0.3255 | 0.2529 |
| 5.2 | 0.9931 | 0.9839 | 0.9671 | 0.9395 | 0.8991 | 0.8448 | 0.7775 | 0.6996 | 0.6146 | 0.5266 | 0.4398 | 0.3578 | 0.2836 |
| 5.4 | 0.9938 | 0.9855 | 0.9702 | 0.9451 | 0.9080 | 0.8578 | 0.7952 | 0.7218 | 0.6409 | 0.5560 | 0.4710 | 0.3894 | 0.3142 |

Table 8.4 (Continued) $n = 5$

PROBABILITY P FOR NON-CENTRAL t DISTRIBUTION AND GIVEN VALUES OF $K_p$ AND NUMBER OF MEASUREMENTS $n$

| D \ $K_p$ = | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 1.75 | 2.00 | 2.25 | 2.50 | 2.75 | 3.00 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2.0 | 0.0581 | 0.0200 | 0.0055 | 0.0012 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.8 | 0.0731 | 0.0259 | 0.0073 | 0.0016 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.6 | 0.0924 | 0.0337 | 0.0098 | 0.0022 | 0.0004 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.4 | 0.1171 | 0.0443 | 0.0133 | 0.0031 | 0.0006 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.2 | 0.1482 | 0.0586 | 0.0183 | 0.0044 | 0.0008 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.0 | 0.1870 | 0.0776 | 0.0253 | 0.0064 | 0.0012 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.8 | 0.2343 | 0.1028 | 0.0354 | 0.0094 | 0.0019 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.6 | 0.2904 | 0.1356 | 0.0496 | 0.0140 | 0.0030 | 0.0005 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.4 | 0.3548 | 0.1771 | 0.0694 | 0.0209 | 0.0048 | 0.0008 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.2 | 0.4256 | 0.2281 | 0.0963 | 0.0314 | 0.0078 | 0.0015 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.0 | 0.5000 | 0.2881 | 0.1318 | 0.0468 | 0.0127 | 0.0026 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.2 | 0.5744 | 0.3556 | 0.1767 | 0.0687 | 0.0205 | 0.0046 | 0.0008 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.4 | 0.6452 | 0.4280 | 0.2311 | 0.0987 | 0.0327 | 0.0083 | 0.0016 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.6 | 0.7096 | 0.5017 | 0.2935 | 0.1378 | 0.0508 | 0.0145 | 0.0032 | 0.0005 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.8 | 0.7657 | 0.5734 | 0.3616 | 0.1860 | 0.0763 | 0.0246 | 0.0061 | 0.0012 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.0 | 0.8131 | 0.6400 | 0.4322 | 0.2423 | 0.1102 | 0.0400 | 0.0114 | 0.0026 | 0.0005 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| 1.2 | 0.8518 | 0.6998 | 0.5022 | 0.3047 | 0.1526 | 0.0621 | 0.0203 | 0.0053 | 0.0011 | 0.0002 | 0.0000 | 0.0000 | 0.0000 |
| 1.4 | 0.8830 | 0.7518 | 0.5688 | 0.3704 | 0.2027 | 0.0917 | 0.0339 | 0.0102 | 0.0025 | 0.0005 | 0.0001 | 0.0000 | 0.0000 |
| 1.6 | 0.9076 | 0.7959 | 0.6301 | 0.4367 | 0.2588 | 0.1290 | 0.0535 | 0.0183 | 0.0052 | 0.0012 | 0.0002 | 0.0000 | 0.0000 |
| 1.8 | 0.9269 | 0.8327 | 0.6849 | 0.5011 | 0.3187 | 0.1733 | 0.0798 | 0.0309 | 0.0100 | 0.0027 | 0.0006 | 0.0001 | 0.0000 |
| 2.0 | 0.9419 | 0.8630 | 0.7330 | 0.5618 | 0.3801 | 0.2233 | 0.1128 | 0.0487 | 0.0179 | 0.0056 | 0.0015 | 0.0003 | 0.0001 |
| 2.2 | 0.9537 | 0.8877 | 0.7744 | 0.6175 | 0.4408 | 0.2772 | 0.1521 | 0.0723 | 0.0297 | 0.0105 | 0.0032 | 0.0008 | 0.0002 |
| 2.4 | 0.9628 | 0.9077 | 0.8096 | 0.6677 | 0.4991 | 0.3331 | 0.1965 | 0.1019 | 0.0462 | 0.0183 | 0.0063 | 0.0019 | 0.0005 |
| 2.6 | 0.9700 | 0.9239 | 0.8393 | 0.7121 | 0.5539 | 0.3893 | 0.2448 | 0.1369 | 0.0678 | 0.0297 | 0.0115 | 0.0039 | 0.0012 |
| 2.8 | 0.9756 | 0.9370 | 0.8643 | 0.7511 | 0.6044 | 0.4442 | 0.2953 | 0.1765 | 0.0945 | 0.0453 | 0.0194 | 0.0074 | 0.0025 |
| 3.0 | 0.9800 | 0.9477 | 0.8851 | 0.7849 | 0.6502 | 0.4967 | 0.3467 | 0.2197 | 0.1260 | 0.0652 | 0.0305 | 0.0128 | 0.0049 |
| 3.2 | 0.9835 | 0.9563 | 0.9025 | 0.8140 | 0.6914 | 0.5461 | 0.3976 | 0.2652 | 0.1615 | 0.0896 | 0.0452 | 0.0208 | 0.0087 |
| 3.4 | 0.9864 | 0.9633 | 0.9170 | 0.8391 | 0.7280 | 0.5919 | 0.4472 | 0.3120 | 0.2003 | 0.1181 | 0.0639 | 0.0317 | 0.0144 |
| 3.6 | 0.9886 | 0.9691 | 0.9291 | 0.8606 | 0.7604 | 0.6339 | 0.4945 | 0.3589 | 0.2414 | 0.1502 | 0.0863 | 0.0458 | 0.0224 |
| 3.8 | 0.9904 | 0.9738 | 0.9393 | 0.8790 | 0.7888 | 0.6720 | 0.5391 | 0.4050 | 0.2839 | 0.1852 | 0.1123 | 0.0633 | 0.0331 |
| 4.0 | 0.9919 | 0.9777 | 0.9478 | 0.8947 | 0.8138 | 0.7064 | 0.5808 | 0.4497 | 0.3268 | 0.2224 | 0.1415 | 0.0841 | 0.0467 |
| 4.2 | 0.9932 | 0.9809 | 0.9549 | 0.9082 | 0.8356 | 0.7372 | 0.6192 | 0.4925 | 0.3696 | 0.2610 | 0.1733 | 0.1080 | 0.0632 |
| 4.4 | 0.9942 | 0.9836 | 0.9610 | 0.9197 | 0.8547 | 0.7648 | 0.6546 | 0.5330 | 0.4115 | 0.3004 | 0.2071 | 0.1347 | 0.0827 |
| 4.6 | 0.9950 | 0.9858 | 0.9661 | 0.9296 | 0.8713 | 0.7894 | 0.6868 | 0.5710 | 0.4520 | 0.3399 | 0.2424 | 0.1638 | 0.1048 |
| 4.8 | 0.9957 | 0.9877 | 0.9704 | 0.9381 | 0.8858 | 0.8113 | 0.7162 | 0.6064 | 0.4909 | 0.3789 | 0.2785 | 0.1947 | 0.1294 |
| 5.0 | 0.9963 | 0.9893 | 0.9741 | 0.9454 | 0.8985 | 0.8307 | 0.7427 | 0.6392 | 0.5278 | 0.4171 | 0.3149 | 0.2270 | 0.1561 |
| 5.2 | 0.9967 | 0.9907 | 0.9772 | 0.9518 | 0.9097 | 0.8480 | 0.7667 | 0.6695 | 0.5627 | 0.4540 | 0.3512 | 0.2602 | 0.1845 |
| 5.4 | 0.9972 | 0.9918 | 0.9800 | 0.9573 | 0.9194 | 0.8633 | 0.7884 | 0.6973 | 0.5953 | 0.4895 | 0.3870 | 0.2939 | 0.2142 |

Table 8.4 (Continued)

$n$ = 6

PROBABILITY P FOR NON-CENTRAL t DISTRIBUTION AND GIVEN VALUES OF $K_p$ AND NUMBER OF MEASUREMENTS $n$

| D \ $K_p$ = | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 1.75 | 2.00 | 2.25 | 2.50 | 2.75 | 3.00 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2.0 | 0.0510 | 0.0148 | 0.0033 | 0.0005 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.8 | 0.0659 | 0.0199 | 0.0045 | 0.0008 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.6 | 0.0852 | 0.0268 | 0.0064 | 0.0011 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.4 | 0.1102 | 0.0364 | 0.0090 | 0.0016 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.2 | 0.1419 | 0.0495 | 0.0129 | 0.0025 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.0 | 0.1816 | 0.0672 | 0.0185 | 0.0037 | 0.0005 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.8 | 0.2300 | 0.0910 | 0.0268 | 0.0057 | 0.0009 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.6 | 0.2873 | 0.1223 | 0.0386 | 0.0089 | 0.0015 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.4 | 0.3528 | 0.1622 | 0.0555 | 0.0138 | 0.0025 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.2 | 0.4247 | 0.2115 | 0.0789 | 0.0215 | 0.0042 | 0.0006 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.0 | 0.5000 | 0.2701 | 0.1103 | 0.0331 | 0.0072 | 0.0011 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.2 | 0.5753 | 0.3368 | 0.1509 | 0.0501 | 0.0121 | 0.0021 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.4 | 0.6472 | 0.4090 | 0.2011 | 0.0742 | 0.0200 | 0.0039 | 0.0005 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.6 | 0.7127 | 0.4836 | 0.2601 | 0.1066 | 0.0324 | 0.0072 | 0.0011 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.8 | 0.7700 | 0.5572 | 0.3261 | 0.1481 | 0.0507 | 0.0128 | 0.0024 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.0 | 0.8184 | 0.6266 | 0.3964 | 0.1985 | 0.0763 | 0.0221 | 0.0047 | 0.0008 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.2 | 0.8581 | 0.6897 | 0.4680 | 0.2567 | 0.1102 | 0.0363 | 0.0090 | 0.0017 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.4 | 0.8898 | 0.7451 | 0.5378 | 0.3204 | 0.1524 | 0.0567 | 0.0163 | 0.0036 | 0.0006 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| 1.6 | 0.9148 | 0.7924 | 0.6035 | 0.3870 | 0.2023 | 0.0843 | 0.0276 | 0.0071 | 0.0014 | 0.0002 | 0.0000 | 0.0000 | 0.0000 |
| 1.8 | 0.9341 | 0.8320 | 0.6633 | 0.4540 | 0.2583 | 0.1195 | 0.0443 | 0.0130 | 0.0030 | 0.0006 | 0.0001 | 0.0000 | 0.0000 |
| 2.0 | 0.9490 | 0.8646 | 0.7163 | 0.5190 | 0.3184 | 0.1618 | 0.0671 | 0.0225 | 0.0061 | 0.0013 | 0.0002 | 0.0000 | 0.0000 |
| 2.2 | 0.9605 | 0.8911 | 0.7625 | 0.5801 | 0.3804 | 0.2102 | 0.0965 | 0.0365 | 0.0113 | 0.0029 | 0.0006 | 0.0001 | 0.0000 |
| 2.4 | 0.9692 | 0.9124 | 0.8019 | 0.6362 | 0.4421 | 0.2632 | 0.1323 | 0.0557 | 0.0195 | 0.0057 | 0.0014 | 0.0003 | 0.0000 |
| 2.6 | 0.9759 | 0.9294 | 0.8352 | 0.6866 | 0.5018 | 0.3191 | 0.1739 | 0.0805 | 0.0315 | 0.0104 | 0.0029 | 0.0007 | 0.0001 |
| 2.8 | 0.9810 | 0.9430 | 0.8631 | 0.7311 | 0.5582 | 0.3759 | 0.2202 | 0.1111 | 0.0480 | 0.0178 | 0.0056 | 0.0015 | 0.0003 |
| 3.0 | 0.9850 | 0.9538 | 0.8862 | 0.7701 | 0.6103 | 0.4323 | 0.2697 | 0.1469 | 0.0694 | 0.0284 | 0.0101 | 0.0031 | 0.0008 |
| 3.2 | 0.9880 | 0.9624 | 0.9053 | 0.8037 | 0.6579 | 0.4869 | 0.3212 | 0.1871 | 0.0957 | 0.0429 | 0.0168 | 0.0058 | 0.0017 |
| 3.4 | 0.9904 | 0.9693 | 0.9211 | 0.8326 | 0.7006 | 0.5387 | 0.3732 | 0.2308 | 0.1267 | 0.0616 | 0.0264 | 0.0100 | 0.0034 |
| 3.6 | 0.9922 | 0.9749 | 0.9341 | 0.8572 | 0.7386 | 0.5870 | 0.4246 | 0.2768 | 0.1618 | 0.0845 | 0.0394 | 0.0163 | 0.0060 |
| 3.8 | 0.9937 | 0.9793 | 0.9448 | 0.8781 | 0.7721 | 0.6316 | 0.4744 | 0.3241 | 0.2003 | 0.1115 | 0.0559 | 0.0252 | 0.0102 |
| 4.0 | 0.9948 | 0.9829 | 0.9537 | 0.8959 | 0.8014 | 0.6722 | 0.5218 | 0.3716 | 0.2413 | 0.1423 | 0.0761 | 0.0369 | 0.0162 |
| 4.2 | 0.9958 | 0.9857 | 0.9610 | 0.9109 | 0.8271 | 0.7089 | 0.5666 | 0.4185 | 0.2839 | 0.1763 | 0.1000 | 0.0517 | 0.0244 |
| 4.4 | 0.9965 | 0.9881 | 0.9670 | 0.9236 | 0.8493 | 0.7418 | 0.6082 | 0.4640 | 0.3274 | 0.2128 | 0.1272 | 0.0698 | 0.0351 |
| 4.6 | 0.9971 | 0.9900 | 0.9720 | 0.9344 | 0.8686 | 0.7712 | 0.6466 | 0.5076 | 0.3709 | 0.2512 | 0.1573 | 0.0910 | 0.0486 |
| 4.8 | 0.9976 | 0.9916 | 0.9762 | 0.9435 | 0.8854 | 0.7973 | 0.6817 | 0.5489 | 0.4137 | 0.2907 | 0.1899 | 0.1152 | 0.0649 |
| 5.0 | 0.9979 | 0.9929 | 0.9796 | 0.9512 | 0.8998 | 0.8204 | 0.7137 | 0.5878 | 0.4554 | 0.3307 | 0.2244 | 0.1421 | 0.0840 |
| 5.2 | 0.9983 | 0.9939 | 0.9826 | 0.9578 | 0.9123 | 0.8408 | 0.7427 | 0.6240 | 0.4956 | 0.3706 | 0.2603 | 0.1714 | 0.1057 |
| 5.4 | 0.9985 | 0.9948 | 0.9850 | 0.9634 | 0.9232 | 0.8588 | 0.7689 | 0.6575 | 0.5339 | 0.4100 | 0.2969 | 0.2025 | 0.1299 |

Table 8.4 (Continued)

$n = 7$

PROBABILITY P FOR NON-CENTRAL t DISTRIBUTION AND GIVEN VALUES OF $K_p$ AND NUMBER OF MEASUREMENTS $n$

| D \ $K_p$ = | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 1.75 | 2.00 | 2.25 | 2.50 | 2.75 | 3.00 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2.0 | 0.0462 | 0.0115 | 0.0021 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.8 | 0.0610 | 0.0159 | 0.0030 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.6 | 0.0804 | 0.0220 | 0.0043 | 0.0006 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.4 | 0.1055 | 0.0307 | 0.0063 | 0.0009 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.2 | 0.1377 | 0.0427 | 0.0094 | 0.0014 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.0 | 0.1780 | 0.0592 | 0.0139 | 0.0023 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.8 | 0.2271 | 0.0816 | 0.0206 | 0.0036 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.6 | 0.2852 | 0.1113 | 0.0306 | 0.0058 | 0.0007 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.4 | 0.3515 | 0.1495 | 0.0449 | 0.0093 | 0.0013 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.2 | 0.4240 | 0.1971 | 0.0652 | 0.0149 | 0.0023 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.0 | 0.5000 | 0.2542 | 0.0929 | 0.0236 | 0.0041 | 0.0005 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.2 | 0.5760 | 0.3196 | 0.1294 | 0.0368 | 0.0071 | 0.0009 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.4 | 0.6485 | 0.3913 | 0.1754 | 0.0559 | 0.0123 | 0.0018 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.6 | 0.7148 | 0.4662 | 0.2306 | 0.0825 | 0.0207 | 0.0036 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.8 | 0.7729 | 0.5410 | 0.2939 | 0.1177 | 0.0336 | 0.0067 | 0.0009 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.0 | 0.8220 | 0.6125 | 0.3629 | 0.1620 | 0.0525 | 0.0121 | 0.0019 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.2 | 0.8623 | 0.6782 | 0.4348 | 0.2148 | 0.0787 | 0.0208 | 0.0039 | 0.0005 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.4 | 0.8945 | 0.7364 | 0.5065 | 0.2749 | 0.1131 | 0.0343 | 0.0076 | 0.0012 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.6 | 0.9196 | 0.7866 | 0.5752 | 0.3399 | 0.1557 | 0.0538 | 0.0138 | 0.0026 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.8 | 0.9390 | 0.8288 | 0.6390 | 0.4074 | 0.2059 | 0.0801 | 0.0236 | 0.0052 | 0.0009 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| 2.0 | 0.9538 | 0.8635 | 0.6965 | 0.4747 | 0.2621 | 0.1139 | 0.0382 | 0.0098 | 0.0019 | 0.0003 | 0.0000 | 0.0000 | 0.0000 |
| 2.2 | 0.9649 | 0.8917 | 0.7469 | 0.5396 | 0.3225 | 0.1547 | 0.0585 | 0.0173 | 0.0039 | 0.0007 | 0.0001 | 0.0000 | 0.0000 |
| 2.4 | 0.9734 | 0.9142 | 0.7904 | 0.6005 | 0.3848 | 0.2020 | 0.0852 | 0.0285 | 0.0076 | 0.0016 | 0.0003 | 0.0000 | 0.0000 |
| 2.6 | 0.9797 | 0.9321 | 0.8272 | 0.6561 | 0.4471 | 0.2542 | 0.1182 | 0.0445 | 0.0134 | 0.0033 | 0.0006 | 0.0001 | 0.0000 |
| 2.8 | 0.9844 | 0.9462 | 0.8581 | 0.7060 | 0.5076 | 0.3097 | 0.1573 | 0.0657 | 0.0224 | 0.0062 | 0.0014 | 0.0003 | 0.0000 |
| 3.0 | 0.9880 | 0.9574 | 0.8836 | 0.7499 | 0.5648 | 0.3669 | 0.2015 | 0.0924 | 0.0352 | 0.0111 | 0.0029 | 0.0006 | 0.0001 |
| 3.2 | 0.9907 | 0.9661 | 0.9046 | 0.7881 | 0.6179 | 0.4240 | 0.2497 | 0.1246 | 0.0523 | 0.0184 | 0.0054 | 0.0013 | 0.0003 |
| 3.4 | 0.9928 | 0.9729 | 0.9218 | 0.8209 | 0.6663 | 0.4798 | 0.3006 | 0.1618 | 0.0742 | 0.0289 | 0.0096 | 0.0027 | 0.0006 |
| 3.6 | 0.9943 | 0.9783 | 0.9359 | 0.8489 | 0.7098 | 0.5332 | 0.3529 | 0.2031 | 0.1009 | 0.0431 | 0.0158 | 0.0050 | 0.0013 |
| 3.8 | 0.9955 | 0.9826 | 0.9473 | 0.8727 | 0.7485 | 0.5833 | 0.4052 | 0.2476 | 0.1321 | 0.0612 | 0.0246 | 0.0086 | 0.0026 |
| 4.0 | 0.9964 | 0.9859 | 0.9566 | 0.8927 | 0.7825 | 0.6297 | 0.4565 | 0.2943 | 0.1673 | 0.0835 | 0.0365 | 0.0140 | 0.0047 |
| 4.2 | 0.9972 | 0.9886 | 0.9642 | 0.9095 | 0.8123 | 0.6721 | 0.5060 | 0.3421 | 0.2059 | 0.1099 | 0.0518 | 0.0216 | 0.0079 |
| 4.4 | 0.9977 | 0.9907 | 0.9703 | 0.9237 | 0.8381 | 0.7105 | 0.5529 | 0.3900 | 0.2471 | 0.1399 | 0.0706 | 0.0317 | 0.0127 |
| 4.6 | 0.9982 | 0.9924 | 0.9754 | 0.9355 | 0.8605 | 0.7449 | 0.5969 | 0.4371 | 0.2900 | 0.1733 | 0.0930 | 0.0448 | 0.0193 |
| 4.8 | 0.9985 | 0.9938 | 0.9795 | 0.9454 | 0.8797 | 0.7756 | 0.6377 | 0.4829 | 0.3337 | 0.2094 | 0.1188 | 0.0609 | 0.0282 |
| 5.0 | 0.9988 | 0.9948 | 0.9828 | 0.9537 | 0.8963 | 0.8029 | 0.6752 | 0.5267 | 0.3776 | 0.2475 | 0.1477 | 0.0801 | 0.0395 |
| 5.2 | 0.9990 | 0.9957 | 0.9856 | 0.9606 | 0.9105 | 0.8269 | 0.7094 | 0.5682 | 0.4210 | 0.2870 | 0.1793 | 0.1024 | 0.0534 |
| 5.4 | 0.9992 | 0.9964 | 0.9879 | 0.9665 | 0.9227 | 0.8481 | 0.7405 | 0.6071 | 0.4634 | 0.3272 | 0.2130 | 0.1275 | 0.0701 |

Table 8.4 (Continued)

$n$ = 8

PROBABILITY P FOR NON-CENTRAL t DISTRIBUTION AND GIVEN VALUES OF $K_p$ AND NUMBER OF MEASUREMENTS $n$

| D | $K_p$ = 0.00 | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 1.75 | 2.00 | 2.25 | 2.50 | 2.75 | 3.00 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2.0 | 0.0428 | 0.0092 | 0.0013 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.8 | 0.0574 | 0.0130 | 0.0020 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.6 | 0.0768 | 0.0185 | 0.0030 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.4 | 0.1021 | 0.0263 | 0.0046 | 0.0005 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.2 | 0.1346 | 0.0374 | 0.0069 | 0.0008 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.0 | 0.1753 | 0.0527 | 0.0106 | 0.0014 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.8 | 0.2250 | 0.0738 | 0.0161 | 0.0023 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.6 | 0.2837 | 0.1019 | 0.0244 | 0.0038 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.4 | 0.3505 | 0.1385 | 0.0367 | 0.0063 | 0.0007 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.2 | 0.4236 | 0.1844 | 0.0542 | 0.0104 | 0.0013 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.0 | 0.5000 | 0.2397 | 0.0786 | 0.0169 | 0.0023 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.2 | 0.5764 | 0.3038 | 0.1113 | 0.0271 | 0.0042 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.4 | 0.6495 | 0.3747 | 0.1533 | 0.0422 | 0.0076 | 0.0009 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.6 | 0.7163 | 0.4496 | 0.2047 | 0.0638 | 0.0132 | 0.0018 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.8 | 0.7750 | 0.5252 | 0.2648 | 0.0934 | 0.0221 | 0.0034 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.0 | 0.8247 | 0.5982 | 0.3317 | 0.1317 | 0.0359 | 0.0065 | 0.0008 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.2 | 0.8654 | 0.6661 | 0.4030 | 0.1790 | 0.0558 | 0.0118 | 0.0017 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.4 | 0.8979 | 0.7268 | 0.4756 | 0.2345 | 0.0830 | 0.0205 | 0.0034 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.6 | 0.9232 | 0.7795 | 0.5466 | 0.2965 | 0.1183 | 0.0336 | 0.0067 | 0.0009 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.8 | 0.9426 | 0.8241 | 0.6135 | 0.3629 | 0.1618 | 0.0526 | 0.0122 | 0.0020 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.0 | 0.9572 | 0.8608 | 0.6747 | 0.4309 | 0.2126 | 0.0782 | 0.0210 | 0.0041 | 0.0006 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| 2.2 | 0.9681 | 0.8906 | 0.7291 | 0.4982 | 0.2693 | 0.1111 | 0.0342 | 0.0078 | 0.0013 | 0.0002 | 0.0000 | 0.0000 | 0.0000 |
| 2.4 | 0.9763 | 0.9144 | 0.7764 | 0.5626 | 0.3301 | 0.1511 | 0.0527 | 0.0139 | 0.0027 | 0.0004 | 0.0000 | 0.0000 | 0.0000 |
| 2.6 | 0.9823 | 0.9332 | 0.8168 | 0.6227 | 0.3928 | 0.1975 | 0.0773 | 0.0232 | 0.0053 | 0.0009 | 0.0001 | 0.0000 | 0.0000 |
| 2.8 | 0.9867 | 0.9479 | 0.8506 | 0.6773 | 0.4555 | 0.2491 | 0.1082 | 0.0367 | 0.0097 | 0.0020 | 0.0003 | 0.0000 | 0.0000 |
| 3.0 | 0.9900 | 0.9594 | 0.8786 | 0.7260 | 0.5163 | 0.3044 | 0.1451 | 0.0551 | 0.0165 | 0.0039 | 0.0007 | 0.0001 | 0.0000 |
| 3.2 | 0.9925 | 0.9683 | 0.9017 | 0.7687 | 0.5739 | 0.3617 | 0.1875 | 0.0788 | 0.0266 | 0.0072 | 0.0016 | 0.0003 | 0.0000 |
| 3.4 | 0.9943 | 0.9752 | 0.9204 | 0.8056 | 0.6273 | 0.4193 | 0.2344 | 0.1079 | 0.0405 | 0.0124 | 0.0031 | 0.0006 | 0.0001 |
| 3.6 | 0.9956 | 0.9806 | 0.9356 | 0.8372 | 0.6761 | 0.4759 | 0.2846 | 0.1422 | 0.0588 | 0.0201 | 0.0056 | 0.0013 | 0.0002 |
| 3.8 | 0.9966 | 0.9847 | 0.9479 | 0.8640 | 0.7198 | 0.5303 | 0.3367 | 0.1812 | 0.0818 | 0.0308 | 0.0097 | 0.0025 | 0.0006 |
| 4.0 | 0.9974 | 0.9879 | 0.9578 | 0.8865 | 0.7587 | 0.5816 | 0.3894 | 0.2239 | 0.1094 | 0.0452 | 0.0157 | 0.0046 | 0.0011 |
| 4.2 | 0.9980 | 0.9904 | 0.9657 | 0.9054 | 0.7928 | 0.6292 | 0.4417 | 0.2694 | 0.1413 | 0.0634 | 0.0242 | 0.0079 | 0.0022 |
| 4.4 | 0.9984 | 0.9924 | 0.9721 | 0.9212 | 0.8225 | 0.6729 | 0.4924 | 0.3168 | 0.1772 | 0.0856 | 0.0356 | 0.0127 | 0.0039 |
| 4.6 | 0.9988 | 0.9939 | 0.9773 | 0.9343 | 0.8482 | 0.7125 | 0.5410 | 0.3650 | 0.2163 | 0.1118 | 0.0503 | 0.0196 | 0.0066 |
| 4.8 | 0.9990 | 0.9951 | 0.9814 | 0.9452 | 0.8703 | 0.7480 | 0.5868 | 0.4131 | 0.2578 | 0.1417 | 0.0683 | 0.0288 | 0.0107 |
| 5.0 | 0.9992 | 0.9960 | 0.9848 | 0.9542 | 0.8893 | 0.7797 | 0.6295 | 0.4602 | 0.3010 | 0.1749 | 0.0899 | 0.0407 | 0.0163 |
| 5.2 | 0.9994 | 0.9968 | 0.9875 | 0.9617 | 0.9055 | 0.8077 | 0.6689 | 0.5058 | 0.3451 | 0.2108 | 0.1148 | 0.0556 | 0.0239 |
| 5.4 | 0.9995 | 0.9974 | 0.9897 | 0.9679 | 0.9193 | 0.8324 | 0.7049 | 0.5493 | 0.3893 | 0.2489 | 0.1428 | 0.0734 | 0.0337 |

Table 8.4 (Continued)

$n = 9$

PROBABILITY P FOR NON-CENTRAL t DISTRIBUTION AND GIVEN VALUES OF $K_p$ AND NUMBER OF MEASUREMENTS $n$

| D | $K_p$ = 0.00 | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 1.75 | 2.00 | 2.25 | 2.50 | 2.75 | 3.00 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2.0 | 0.0403 | 0.0075 | 0.0009 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.8 | 0.0548 | 0.0109 | 0.0014 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.6 | 0.0741 | 0.0158 | 0.0022 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.4 | 0.0995 | 0.0229 | 0.0034 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.2 | 0.1322 | 0.0331 | 0.0052 | 0.0005 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.0 | 0.1733 | 0.0474 | 0.0082 | 0.0009 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.8 | 0.2234 | 0.0672 | 0.0127 | 0.0015 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.6 | 0.2826 | 0.0938 | 0.0197 | 0.0025 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.4 | 0.3498 | 0.1288 | 0.0301 | 0.0043 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.2 | 0.4232 | 0.1729 | 0.0453 | 0.0073 | 0.0007 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.0 | 0.5000 | 0.2266 | 0.0668 | 0.0122 | 0.0013 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.2 | 0.5768 | 0.2893 | 0.0960 | 0.0200 | 0.0025 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.4 | 0.6502 | 0.3592 | 0.1342 | 0.0319 | 0.0047 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.6 | 0.7174 | 0.4338 | 0.1818 | 0.0494 | 0.0084 | 0.0009 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.8 | 0.7766 | 0.5098 | 0.2385 | 0.0740 | 0.0146 | 0.0018 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.0 | 0.8267 | 0.5841 | 0.3030 | 0.1069 | 0.0244 | 0.0035 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.2 | 0.8678 | 0.6537 | 0.3730 | 0.1486 | 0.0392 | 0.0066 | 0.0007 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.4 | 0.9005 | 0.7167 | 0.4456 | 0.1991 | 0.0604 | 0.0120 | 0.0015 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.6 | 0.9259 | 0.7717 | 0.5180 | 0.2572 | 0.0890 | 0.0207 | 0.0032 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.8 | 0.9452 | 0.8184 | 0.5874 | 0.3212 | 0.1257 | 0.0339 | 0.0061 | 0.0007 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.0 | 0.9597 | 0.8571 | 0.6519 | 0.3886 | 0.1703 | 0.0527 | 0.0112 | 0.0016 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.2 | 0.9705 | 0.8885 | 0.7099 | 0.4569 | 0.2221 | 0.0781 | 0.0194 | 0.0033 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.4 | 0.9784 | 0.9135 | 0.7608 | 0.5239 | 0.2796 | 0.1106 | 0.0316 | 0.0064 | 0.0009 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| 2.6 | 0.9842 | 0.9333 | 0.8045 | 0.5874 | 0.3409 | 0.1502 | 0.0489 | 0.0116 | 0.0020 | 0.0002 | 0.0000 | 0.0000 | 0.0000 |
| 2.8 | 0.9884 | 0.9486 | 0.8413 | 0.6462 | 0.4039 | 0.1962 | 0.0719 | 0.0196 | 0.0039 | 0.0006 | 0.0001 | 0.0000 | 0.0000 |
| 3.0 | 0.9915 | 0.9605 | 0.8720 | 0.6993 | 0.4667 | 0.2474 | 0.1011 | 0.0313 | 0.0073 | 0.0013 | 0.0002 | 0.0000 | 0.0000 |
| 3.2 | 0.9937 | 0.9697 | 0.8971 | 0.7464 | 0.5276 | 0.3025 | 0.1364 | 0.0475 | 0.0217 | 0.0026 | 0.0004 | 0.0000 | 0.0000 |
| 3.4 | 0.9953 | 0.9767 | 0.9176 | 0.7875 | 0.5852 | 0.3598 | 0.1774 | 0.0688 | 0.0208 | 0.0049 | 0.0009 | 0.0001 | 0.0000 |
| 3.6 | 0.9965 | 0.9820 | 0.9341 | 0.8228 | 0.6386 | 0.4178 | 0.2231 | 0.0953 | 0.0323 | 0.0086 | 0.0018 | 0.0003 | 0.0000 |
| 3.8 | 0.9974 | 0.9861 | 0.9473 | 0.8528 | 0.6873 | 0.4749 | 0.2724 | 0.1271 | 0.0477 | 0.0143 | 0.0034 | 0.0007 | 0.0001 |
| 4.0 | 0.9980 | 0.9892 | 0.9579 | 0.8781 | 0.7308 | 0.5299 | 0.3242 | 0.1638 | 0.0676 | 0.0226 | 0.0061 | 0.0013 | 0.0002 |
| 4.2 | 0.9985 | 0.9916 | 0.9663 | 0.8993 | 0.7694 | 0.5820 | 0.3770 | 0.2047 | 0.0920 | 0.0340 | 0.0103 | 0.0026 | 0.0005 |
| 4.4 | 0.9989 | 0.9935 | 0.9730 | 0.9169 | 0.8032 | 0.6305 | 0.4298 | 0.2490 | 0.1209 | 0.0489 | 0.0164 | 0.0046 | 0.0010 |
| 4.6 | 0.9991 | 0.9949 | 0.9783 | 0.9314 | 0.8326 | 0.6750 | 0.4815 | 0.2956 | 0.1540 | 0.0676 | 0.0249 | 0.0077 | 0.0020 |
| 4.8 | 0.9993 | 0.9960 | 0.9826 | 0.9434 | 0.8579 | 0.7154 | 0.5313 | 0.3437 | 0.1908 | 0.0902 | 0.0362 | 0.0123 | 0.0035 |
| 5.0 | 0.9995 | 0.9968 | 0.9860 | 0.9533 | 0.8795 | 0.7517 | 0.5785 | 0.3921 | 0.2307 | 0.1168 | 0.0506 | 0.0187 | 0.0059 |
| 5.2 | 0.9996 | 0.9975 | 0.9887 | 0.9614 | 0.8980 | 0.7839 | 0.6226 | 0.4401 | 0.2728 | 0.1469 | 0.0684 | 0.0274 | 0.0095 |
| 5.4 | 0.9997 | 0.9980 | 0.9908 | 0.9681 | 0.9137 | 0.8125 | 0.6635 | 0.4870 | 0.3164 | 0.1802 | 0.0895 | 0.0387 | 0.0145 |

Table 8.4 (Continued)

$n$ = 10

PROBABILITY P FOR NON-CENTRAL t DISTRIBUTION AND GIVEN VALUES OF $K_p$ AND NUMBER OF MEASUREMENTS $n$

| D | $K_p$ = 0.00 | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 1.75 | 2.00 | 2.25 | 2.50 | 2.75 | 3.00 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2.0 | 0.0383 | 0.0062 | 0.0006 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.8 | 0.0527 | 0.0092 | 0.0010 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.6 | 0.0720 | 0.0137 | 0.0016 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.4 | 0.0975 | 0.0201 | 0.0025 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.2 | 0.1304 | 0.0295 | 0.0040 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.0 | 0.1717 | 0.0429 | 0.0064 | 0.0006 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.8 | 0.2222 | 0.0615 | 0.0102 | 0.0010 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.6 | 0.2817 | 0.0867 | 0.0160 | 0.0017 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.4 | 0.3492 | 0.1201 | 0.0249 | 0.0030 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.2 | 0.4230 | 0.1625 | 0.0381 | 0.0052 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.0 | 0.5000 | 0.2146 | 0.0569 | 0.0089 | 0.0008 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.2 | 0.5770 | 0.2758 | 0.0830 | 0.0148 | 0.0015 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.4 | 0.6508 | 0.3446 | 0.1176 | 0.0242 | 0.0029 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.6 | 0.7183 | 0.4187 | 0.1616 | 0.0383 | 0.0053 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.8 | 0.7778 | 0.4950 | 0.2149 | 0.0586 | 0.0096 | 0.0009 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.0 | 0.8283 | 0.5702 | 0.2766 | 0.0865 | 0.0165 | 0.0019 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.2 | 0.8696 | 0.6414 | 0.3448 | 0.1230 | 0.0275 | 0.0037 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.4 | 0.9025 | 0.7062 | 0.4169 | 0.1684 | 0.0436 | 0.0070 | 0.0007 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.6 | 0.9280 | 0.7633 | 0.4900 | 0.2221 | 0.0664 | 0.0126 | 0.0015 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.8 | 0.9478 | 0.8120 | 0.5613 | 0.2828 | 0.0967 | 0.0215 | 0.0030 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.0 | 0.9617 | 0.8526 | 0.6283 | 0.3485 | 0.1350 | 0.0349 | 0.0059 | 0.0006 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.2 | 0.9723 | 0.8856 | 0.6895 | 0.4167 | 0.1811 | 0.0539 | 0.0107 | 0.0014 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.4 | 0.9800 | 0.9119 | 0.7438 | 0.4851 | 0.2341 | 0.0794 | 0.0184 | 0.0029 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.6 | 0.9856 | 0.9326 | 0.7908 | 0.5513 | 0.2925 | 0.1120 | 0.0300 | 0.0055 | 0.0007 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| 2.8 | 0.9896 | 0.9486 | 0.8107 | 0.6135 | 0.3543 | 0.1515 | 0.0464 | 0.0100 | 0.0015 | 0.0002 | 0.0000 | 0.0000 | 0.0000 |
| 3.0 | 0.9925 | 0.9610 | 0.8640 | 0.6707 | 0.4176 | 0.1973 | 0.0684 | 0.0170 | 0.0030 | 0.0004 | 0.0000 | 0.0000 | 0.0000 |
| 3.2 | 0.9946 | 0.9704 | 0.8914 | 0.7219 | 0.4805 | 0.2485 | 0.0964 | 0.0274 | 0.0057 | 0.0009 | 0.0001 | 0.0000 | 0.0000 |
| 3.4 | 0.9961 | 0.9776 | 0.9136 | 0.7671 | 0.5413 | 0.3036 | 0.1304 | 0.0420 | 0.0100 | 0.0018 | 0.0002 | 0.0000 | 0.0000 |
| 3.6 | 0.9971 | 0.9830 | 0.9315 | 0.8062 | 0.5986 | 0.3610 | 0.1702 | 0.0613 | 0.0167 | 0.0034 | 0.0005 | 0.0001 | 0.0000 |
| 3.8 | 0.9979 | 0.9871 | 0.9458 | 0.8396 | 0.6516 | 0.4191 | 0.2149 | 0.0858 | 0.0263 | 0.0062 | 0.0011 | 0.0002 | 0.0000 |
| 4.0 | 0.9984 | 0.9901 | 0.9572 | 0.8678 | 0.6997 | 0.4765 | 0.2635 | 0.1155 | 0.0396 | 0.0105 | 0.0022 | 0.0003 | 0.0000 |
| 4.2 | 0.9988 | 0.9925 | 0.9661 | 0.8914 | 0.7428 | 0.5320 | 0.3149 | 0.1502 | 0.0569 | 0.0170 | 0.0040 | 0.0007 | 0.0001 |
| 4.4 | 0.9991 | 0.9942 | 0.9732 | 0.9110 | 0.7808 | 0.5846 | 0.3678 | 0.1894 | 0.0786 | 0.0261 | 0.0069 | 0.0015 | 0.0002 |
| 4.6 | 0.9994 | 0.9956 | 0.9788 | 0.9272 | 0.8140 | 0.6336 | 0.4209 | 0.2323 | 0.1048 | 0.0384 | 0.0113 | 0.0027 | 0.0005 |
| 4.8 | 0.9995 | 0.9966 | 0.9832 | 0.9405 | 0.8427 | 0.6786 | 0.4732 | 0.2781 | 0.1354 | 0.0541 | 0.0177 | 0.0047 | 0.0010 |
| 5.0 | 0.9996 | 0.9974 | 0.9867 | 0.9514 | 0.8673 | 0.7195 | 0.5238 | 0.3257 | 0.1700 | 0.0737 | 0.0264 | 0.0078 | 0.0019 |
| 5.2 | 0.9997 | 0.9979 | 0.9894 | 0.9603 | 0.8884 | 0.7561 | 0.5720 | 0.3742 | 0.2080 | 0.0971 | 0.0379 | 0.0123 | 0.0034 |
| 5.4 | 0.9998 | 0.9984 | 0.9915 | 0.9676 | 0.9062 | 0.7888 | 0.6173 | 0.4228 | 0.2487 | 0.1243 | 0.0525 | 0.0187 | 0.0056 |

Table 8.4 (Continued)

$n$ = 11

PROBABILITY P FOR NON-CENTRAL t DISTRIBUTION AND GIVEN VALUES OF $K_p$ AND NUMBER OF MEASUREMENTS $n$

| D | $K_p$ = 0.00 | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 1.75 | 2.00 | 2.25 | 2.50 | 2.75 | 3.00 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2.0 | 0.0367 | 0.0053 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.8 | 0.0510 | 0.0079 | 0.0007 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.6 | 0.0703 | 0.0119 | 0.0012 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.4 | 0.0959 | 0.0178 | 0.0019 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.2 | 0.1289 | 0.0265 | 0.0031 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.0 | 0.1704 | 0.0390 | 0.0050 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.8 | 0.2212 | 0.0565 | 0.0081 | 0.0006 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.6 | 0.2809 | 0.0804 | 0.0131 | 0.0012 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.4 | 0.3488 | 0.1123 | 0.0206 | 0.0021 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.2 | 0.4227 | 0.1531 | 0.0320 | 0.0037 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.0 | 0.5000 | 0.2035 | 0.0486 | 0.0064 | 0.0005 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.2 | 0.5773 | 0.2632 | 0.0719 | 0.0110 | 0.0009 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.4 | 0.6512 | 0.3309 | 0.1033 | 0.0183 | 0.0018 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.6 | 0.7191 | 0.4044 | 0.1437 | 0.0297 | 0.0034 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.8 | 0.7788 | 0.4807 | 0.1936 | 0.0464 | 0.0063 | 0.0005 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.0 | 0.8296 | 0.5566 | 0.2523 | 0.0700 | 0.0112 | 0.0010 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.2 | 0.8711 | 0.6290 | 0.3184 | 0.1016 | 0.0191 | 0.0020 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.4 | 0.9041 | 0.6956 | 0.3895 | 0.1419 | 0.0313 | 0.0040 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.6 | 0.9297 | 0.7546 | 0.4627 | 0.1910 | 0.0491 | 0.0075 | 0.0007 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.8 | 0.9490 | 0.8053 | 0.5352 | 0.2479 | 0.0737 | 0.0134 | 0.0015 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.0 | 0.9633 | 0.8476 | 0.6045 | 0.3110 | 0.1060 | 0.0227 | 0.0030 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.2 | 0.9738 | 0.8821 | 0.6685 | 0.3781 | 0.1462 | 0.0366 | 0.0057 | 0.0006 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.4 | 0.9813 | 0.9097 | 0.7259 | 0.4469 | 0.1940 | 0.0560 | 0.0104 | 0.0012 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.6 | 0.9868 | 0.9314 | 0.7761 | 0.5148 | 0.2483 | 0.0820 | 0.0179 | 0.0025 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.8 | 0.9906 | 0.9482 | 0.8190 | 0.5799 | 0.3077 | 0.1149 | 0.0291 | 0.0049 | 0.0005 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 3.0 | 0.9933 | 0.9610 | 0.8549 | 0.6405 | 0.3701 | 0.1546 | 0.0449 | 0.0089 | 0.0012 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| 3.2 | 0.9953 | 0.9708 | 0.8846 | 0.6956 | 0.4336 | 0.2007 | 0.0662 | 0.0152 | 0.0024 | 0.0003 | 0.0000 | 0.0000 | 0.0000 |
| 3.4 | 0.9966 | 0.9781 | 0.9087 | 0.7447 | 0.4964 | 0.2520 | 0.0934 | 0.0247 | 0.0046 | 0.0006 | 0.0001 | 0.0000 | 0.0000 |
| 3.6 | 0.9976 | 0.9836 | 0.9281 | 0.7876 | 0.5568 | 0.3072 | 0.1265 | 0.0380 | 0.0082 | 0.0013 | 0.0001 | 0.0000 | 0.0000 |
| 3.8 | 0.9983 | 0.9877 | 0.9436 | 0.8244 | 0.6136 | 0.3647 | 0.1654 | 0.0558 | 0.0138 | 0.0025 | 0.0003 | 0.0000 | 0.0000 |
| 4.0 | 0.9987 | 0.9908 | 0.9558 | 0.8557 | 0.6660 | 0.4230 | 0.2094 | 0.0786 | 0.0220 | 0.0046 | 0.0007 | 0.0001 | 0.0000 |
| 4.2 | 0.9991 | 0.9931 | 0.9654 | 0.8820 | 0.7133 | 0.4806 | 0.2575 | 0.1065 | 0.0335 | 0.0080 | 0.0014 | 0.0002 | 0.0000 |
| 4.4 | 0.9993 | 0.9948 | 0.9730 | 0.9038 | 0.7556 | 0.5363 | 0.3085 | 0.1395 | 0.0488 | 0.0131 | 0.0027 | 0.0004 | 0.0001 |
| 4.6 | 0.9995 | 0.9961 | 0.9789 | 0.9218 | 0.7927 | 0.5891 | 0.3613 | 0.1772 | 0.0683 | 0.0205 | 0.0048 | 0.0009 | 0.0001 |
| 4.8 | 0.9996 | 0.9970 | 0.9835 | 0.9366 | 0.8250 | 0.6384 | 0.4146 | 0.2189 | 0.0921 | 0.0306 | 0.0080 | 0.0016 | 0.0003 |
| 5.0 | 0.9997 | 0.9977 | 0.9871 | 0.9487 | 0.8529 | 0.6836 | 0.4673 | 0.2637 | 0.1204 | 0.0440 | 0.0128 | 0.0030 | 0.0005 |
| 5.2 | 0.9998 | 0.9983 | 0.9898 | 0.9584 | 0.8767 | 0.7247 | 0.5185 | 0.3108 | 0.1529 | 0.0610 | 0.0196 | 0.0051 | 0.0011 |
| 5.4 | 0.9998 | 0.9987 | 0.9920 | 0.9664 | 0.8969 | 0.7615 | 0.5675 | 0.3592 | 0.1890 | 0.0817 | 0.0288 | 0.0083 | 0.0019 |

Table 8.4 (Continued) $n = 12$

PROBABILITY P FOR NON-CENTRAL t DISTRIBUTION AND GIVEN VALUES OF $K_p$ AND NUMBER OF MEASUREMENTS $n$

| D | $K_p$ = 0.00 | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 1.75 | 2.00 | 2.25 | 2.50 | 2.75 | 3.00 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2.0 | 0.0354 | 0.0045 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.8 | 0.0497 | 0.0069 | 0.0005 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.6 | 0.0690 | 0.0105 | 0.0009 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.4 | 0.0945 | 0.0159 | 0.0014 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.2 | 0.1277 | 0.0239 | 0.0024 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.0 | 0.1694 | 0.0356 | 0.0040 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.8 | 0.2203 | 0.0520 | 0.0066 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.6 | 0.2803 | 0.0748 | 0.0107 | 0.0008 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.4 | 0.3484 | 0.1052 | 0.0172 | 0.0014 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.2 | 0.4226 | 0.1444 | 0.0271 | 0.0026 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.0 | 0.5000 | 0.1932 | 0.0416 | 0.0047 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.2 | 0.5774 | 0.2515 | 0.0624 | 0.0082 | 0.0005 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.4 | 0.6516 | 0.3180 | 0.0907 | 0.0139 | 0.0011 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.6 | 0.7197 | 0.3908 | 0.1279 | 0.0230 | 0.0022 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.8 | 0.7797 | 0.4669 | 0.1744 | 0.0367 | 0.0041 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.0 | 0.8306 | 0.5433 | 0.2301 | 0.0565 | 0.0075 | 0.0005 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.2 | 0.8723 | 0.6168 | 0.2938 | 0.0836 | 0.0132 | 0.0011 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.4 | 0.9055 | 0.6849 | 0.3634 | 0.1192 | 0.0224 | 0.0023 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.6 | 0.9310 | 0.7456 | 0.4363 | 0.1636 | 0.0361 | 0.0045 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.8 | 0.9503 | 0.7981 | 0.5096 | 0.2163 | 0.0558 | 0.0083 | 0.0007 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.0 | 0.9646 | 0.8422 | 0.5806 | 0.2763 | 0.0825 | 0.0146 | 0.0015 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.2 | 0.9750 | 0.8783 | 0.6471 | 0.3415 | 0.1169 | 0.0245 | 0.0030 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.4 | 0.9824 | 0.9072 | 0.7073 | 0.4098 | 0.1592 | 0.0389 | 0.0058 | 0.0005 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.6 | 0.9877 | 0.9298 | 0.7605 | 0.4786 | 0.2088 | 0.0591 | 0.0104 | 0.0011 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 2.8 | 0.9914 | 0.9474 | 0.8063 | 0.5457 | 0.2647 | 0.0857 | 0.0178 | 0.0023 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 3.0 | 0.9940 | 0.9607 | 0.8450 | 0.6093 | 0.3250 | 0.1192 | 0.0288 | 0.0045 | 0.0004 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 3.2 | 0.9958 | 0.9708 | 0.8770 | 0.6679 | 0.3880 | 0.1595 | 0.0444 | 0.0082 | 0.0010 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| 3.4 | 0.9970 | 0.9784 | 0.9030 | 0.7207 | 0.4517 | 0.2060 | 0.0652 | 0.0140 | 0.0020 | 0.0002 | 0.0000 | 0.0000 | 0.0000 |
| 3.6 | 0.9979 | 0.9840 | 0.9240 | 0.7673 | 0.5142 | 0.2576 | 0.0918 | 0.0227 | 0.0038 | 0.0004 | 0.0000 | 0.0000 | 0.0000 |
| 3.8 | 0.9985 | 0.9882 | 0.9408 | 0.8077 | 0.5740 | 0.3130 | 0.1244 | 0.0350 | 0.0069 | 0.0009 | 0.0001 | 0.0000 | 0.0000 |
| 4.0 | 0.9990 | 0.9912 | 0.9540 | 0.8422 | 0.6301 | 0.3707 | 0.1627 | 0.0517 | 0.0117 | 0.0019 | 0.0002 | 0.0000 | 0.0000 |
| 4.2 | 0.9993 | 0.9935 | 0.9643 | 0.8712 | 0.6815 | 0.4291 | 0.2061 | 0.0731 | 0.0188 | 0.0035 | 0.0005 | 0.0000 | 0.0000 |
| 4.4 | 0.9995 | 0.9952 | 0.9724 | 0.8954 | 0.7278 | 0.4868 | 0.2538 | 0.0997 | 0.0290 | 0.0062 | 0.0010 | 0.0001 | 0.0000 |
| 4.6 | 0.9996 | 0.9964 | 0.9786 | 0.9154 | 0.7690 | 0.5426 | 0.3045 | 0.1312 | 0.0426 | 0.0103 | 0.0019 | 0.0002 | 0.0000 |
| 4.8 | 0.9997 | 0.9978 | 0.9835 | 0.9318 | 0.8050 | 0.5955 | 0.3572 | 0.1676 | 0.0602 | 0.0164 | 0.0034 | 0.0005 | 0.0001 |
| 5.0 | 0.9998 | 0.9980 | 0.9872 | 0.9451 | 0.8363 | 0.6448 | 0.4107 | 0.2081 | 0.0821 | 0.0249 | 0.0058 | 0.0010 | 0.0001 |
| 5.2 | 0.9999 | 0.9985 | 0.9901 | 0.9559 | 0.8631 | 0.6900 | 0.4637 | 0.2521 | 0.1084 | 0.0364 | 0.0095 | 0.0019 | 0.0003 |
| 5.4 | 0.9999 | 0.9989 | 0.9923 | 0.9646 | 0.8859 | 0.7310 | 0.5154 | 0.2987 | 0.1389 | 0.0511 | 0.0148 | 0.0014 | 0.0006 |

# CHAPTER 9

# ATTRIBUTE SAMPLING PLANS

## 9.1 TYPES OF ATTRIBUTE SAMPLING PLANS

A LOT of similar items is often accepted or rejected by considering the presence or absence of some particular characteristic, called an *attribute*, in the individual units. When any attribute sampling plan is used, it is assumed that each LOT is a collection of N units. A random sample containing $n$ units is drawn from the LOT, and each of these chosen units is inspected. In some cases, it is sufficient merely to ascertain whether or not the governing attribute is present in the unit, and the unit is then classified as being acceptable or defective. In other cases, it is necessary to count the number of defects in each unit that is inspected.

There are a number of types of attribute sampling plans. The general types are single sampling plans, double sampling plans, and multiple sampling plans. Each of these general types includes plans for reduced or tightened inspection. It is beyond the scope of this Handbook to attempt to cover all types of attribute sampling plans, and only single sampling plans will be discussed here. When a single sampling plan is used, acceptance or rejection of a LOT is based on the following principle: If either the number of defective units in the sample or the total number of defects does not exceed a specified acceptance number c, the LOT is accepted. If the number c is exceeded, the LOT is rejected. A single sampling plan is completely described by the numbers N, $n$, and c.

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To dig deeper - See Military Standard MIL-STD-105D, Sampling Procedures and Tables for Inspection by Attributes; and *Quality Control and Industrial Statistics* by Duncan, A.J., Richard D. Irwin, Inc., 3rd ed., 1965, p. 145 - 333.

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## 9.2 USES OF ATTRIBUTE SAMPLING PLANS

With respect to the acceptance of a highway material or construction, an attribute sampling plan is most useful when the attribute of interest cannot be measured or does not have to be measured, but the unit can be classified as acceptable or defective by visual inspection. An attribute sampling plan may also be used when single measurements of the attribute are not distributed normally. However, the most general use of such a plan is in a go or no-go situation. Examples of suitable situations are the following: A unit of concrete pipe is either satisfactory or defective with respect to the presence of spalls or exposed reinforcing or with respect to failure to meet a strength test. Particles of aggregate either have or do not have fractured faces, and either meet or do not meet certain requirements for hardness. The number of irregularities in a concrete pavement slab either is less than or greater than a permissible limit. Also, since there is some evidence that the distribution of single measurements of slab thickness is not always normal, an attribute sampling plan may be used in connection with the lengths of cores from a concrete pavement.

A concrete pavement or a bridge deck can be rated by making a visual inspection to count the number of cracks, spalls, and popouts in a panel. Asphalt pavement can be rated by considering the number of cracks and the amount of rutting and raveling. Different weights can be assigned to different defects, and an acceptable quality level can be established. If the number of defects in a pavement exceeds the allowable number, the pavement would be scheduled for early maintenance.

## 9.3 ADVANTAGES AND DISADVANTAGES OF ATTRIBUTE SAMPLING PLAN

In a case where either an acceptance plan based on variables or a plan based on an attribute would be appropriate, the choice would depend on several considerations. An attribute sampling plan requires practically no computations and is adaptable to control charting. The usual inspection process is to subject each item in the sample to a rapid visual examination or to use a simple gage to determine whether or not a certain dimension meets specified requirements. No elaborate testing or measuring equipment is needed, and comparatively little time is required for the inspection of a large number of items. Moreover, it is often possible to note the presence or absence of two or more types of defects during a single inspection. To help in the choice of a suitable buyer's risk or seller's risk, many tables and operating-characteristics curves are available.

The great disadvantage of an attribute sampling plan is that much available information is not obtained. Since the purpose of the inspection is simply to classify an item as good or bad or as go or no-go, the inspection reports do not show the average level and the variability of a characteristic. As a result there is no clue in regard to the type of corrective action that can or should be taken.

In general, an attribute single sampling plan is much less efficient than a sampling plan based on the variations in measured test results. To obtain a certain buyer's risk or seller's risk, the number of samples needed for an attribute sampling plan may be 30 percent greater than the number needed for a plan based on the distribution of variables.

## 9.4 QUALITY LEVELS AND RISKS

When an attribute sampling plan is used, the quality level of a LOT is usually determined by the number of defective units per hundred, called the percent defective, or by the number of defects per hundred units. These terms are defined by the following equations:

$$\text{Percent Defective} = \frac{\text{Number of Defective Units}}{\text{Number of Units Inspected}}(100) \qquad (9.1)$$

$$\text{Defects Per 100 Units} = \frac{\text{Total Number of Defects}}{\text{Number of Units Inspected}}(100) \qquad (9.2)$$

In connection with an attribute sampling plan, the criterion for good material or construction is an *acceptable quality level* (A.Q.L.). This level is the maximum percent defective or the maximum number of defects per hundred units that can be considered satisfactory as a *process average*. (Values of the A.Q.L. less than 10.0 apply either to percent defective or defects per hundred units. Values of the A.Q.L. greater than 10.0 apply only to defects per hundred units). When a certain value of the A.Q.L. is specified, the acceptance plan will accept most of the LOTs submitted providing that the process average percent defective is not greater than the specified A.Q.L. The A.Q.L. alone does not indicate the degree of protection to the buyer in regard to acceptance of individual LOTs, but relates to what might be expected for a series of LOTs. The degree of protection that a particular attribute sampling plan provides for the buyer is best shown by the *operating-characteristics curve* (O.C. curve) for that plan, as will be explained in Section 9.5.

The criterion for poor material or construction is LOT tolerance percent defective (L.T.P.D.). The L.T.P.D. is the maximum percent defective that can be tolerated for material or construction which will still meet engineering requirements with respect to serviceability. This percent defective is sometimes called the rejectable quality level (R.Q.L.). Accordingly, attributes plans are designed so that there is a very small probability of acceptance of LOTs when the process average percent defective is equal to or exceeds the L.T.P.D.

The alpha ($\alpha$) or seller's risk is related to the A.Q.L., while the beta ($\beta$) or buyer's risk is related to the size of the L.T.P.D. It is customary to set the seller's risk ($\alpha$) at 5 percent and the buyer's risk ($\beta$) at 10 percent. In other words, when the process average percent defective is less than the A.Q.L., LOTs will be accepted 95 percent of the time. In the case of the sampling plans in the Military Standard MIL-STD-105D, the seller's risk is not constant but is larger than 5 percent for small LOTs and less than 5 percent for large LOTs. The relationship of the A.Q.L. to the seller's risk and the relationship of the L.T.P.D. to the buyer's risk are shown in Figure 9.1.

## 9.5 CHOICE OF ACCEPTABLE QUALITY LEVEL

Many engineers would be reluctant to knowingly accept a LOT of material containing defective units, but there is a relatively high probability that such LOTs will be accepted, even when each unit in the LOT is inspected. Therefore, although some purchasers specify zero defects, it is the more general practice to allow a small percent defective or a small number of defects per hundred units in each LOT, indicated by the specified A.Q.L. The choice of the size of the A.Q.L. is largely a matter of judgment and will depend on the buyer's or engineer's estimate of the effect of defects on safety or serviceability and the effect of a small A.Q.L. on bid prices. Probably the best guide is the process average for LOTs that have been found to be satisfactory in the past.

One method of choosing an appropriate value for the A.Q.L. is shown in Figure 9.2. This method is similar to the decision-function method described in Section 8.7 and is based on the relationship of the cost of inspection to the loss associated with accepting defective units. For example, if concrete pipe is to be placed in a shallow trench, the cost of replacing a single length, or unit, might be estimated at 4 times the cost of inspecting and testing a unit. The use of Figure 9.2 in this case would permit a large A.Q.L. and a small number of units in the sample for accepting LOTs of pipe.

Figure 9.1

OPERATING - CHARACTERISTICS CURVE FOR SINGLE SAMPLING PLAN FOR ATTRIBUTES WHEN N = 20 AND c = 3

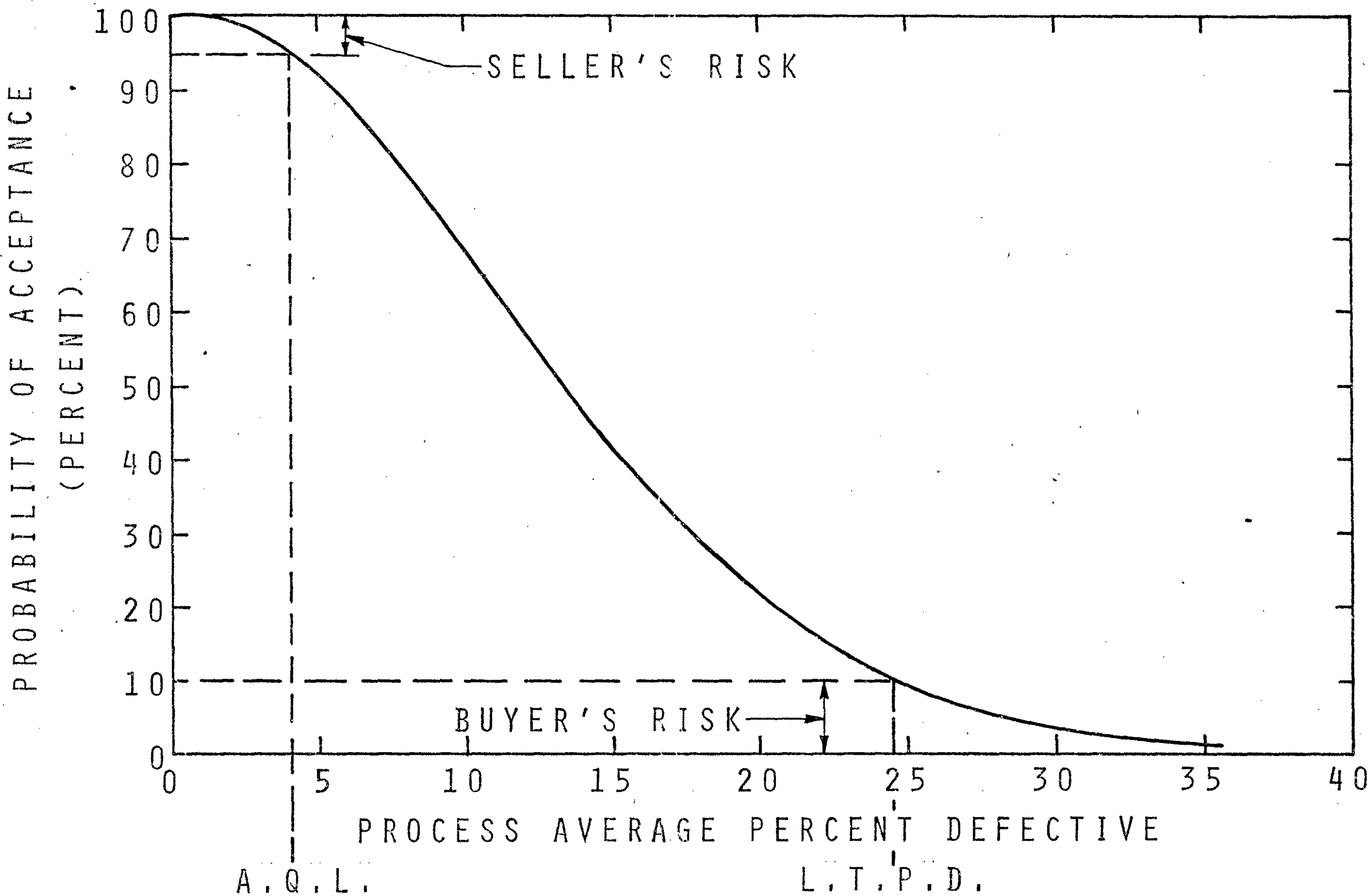
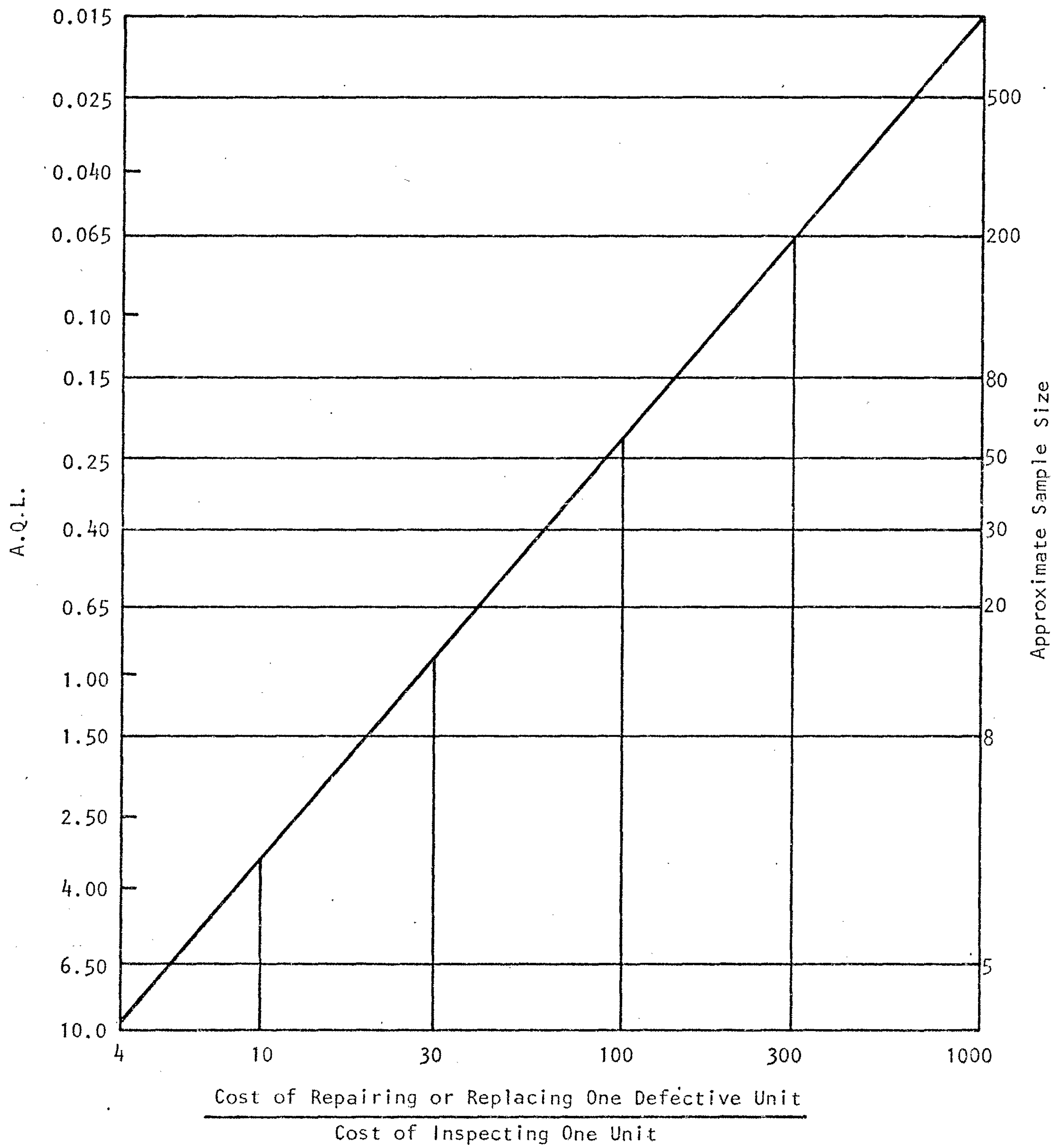


Figure 9.2

RELATIONSHIP OF A.Q.L. TO INSPECTION COST AND SAMPLE SIZE



However, if the pipe is to be placed under a high embankment and the cost of replacement might be 100 times the cost of inspection and testing, the use of Figure 9.2 would require a much smaller A.Q.L. and a large number of units in the sample for acceptance.

Probably a more practical approach in choosing an appropriate value for the A.Q.L. is to examine the operating-characteristics curves shown for the various plans in MIL-STD-105D. Typical O.C. curves are shown in Figure 9.3. By observing the values on the bottom scale for the process average percent defective corresponding to the position of the intersection of lines projected from 10 and 95 percent probability of acceptance, the engineer can decide on the plan having an A.Q.L. and a L.T.P.D. appropriate for a given situation.

Another method of choosing an acceptance plan is to refer to the tabulated values for operating-characteristics curves for single sampling plans. For example, the tabulated values for a LOT size of 151 to 280 and a sample size of 32 are shown in Table X-G-1 on page 42 of MIL-STD-105D. The extreme left-hand column headed $P_a$ shows the buyer's risk. The numbers in the top row are the A.Q.L. and the numbers in the body of the table are the L.T.P.D. Suppose that we are willing to take a β risk of 10 percent of accepting a LOT that is about 10 percent defective. Looking in the row starting with $P_a = 10$ we find L.T.P.D. = 11.6 in the column headed A.Q.L. = 1.5. Referring to Table X-G-2 on page 43 we see that the acceptance number is 1. This means that if we inspected a sample of 32 items and found one defective item we would accept, if more than one, we would reject.

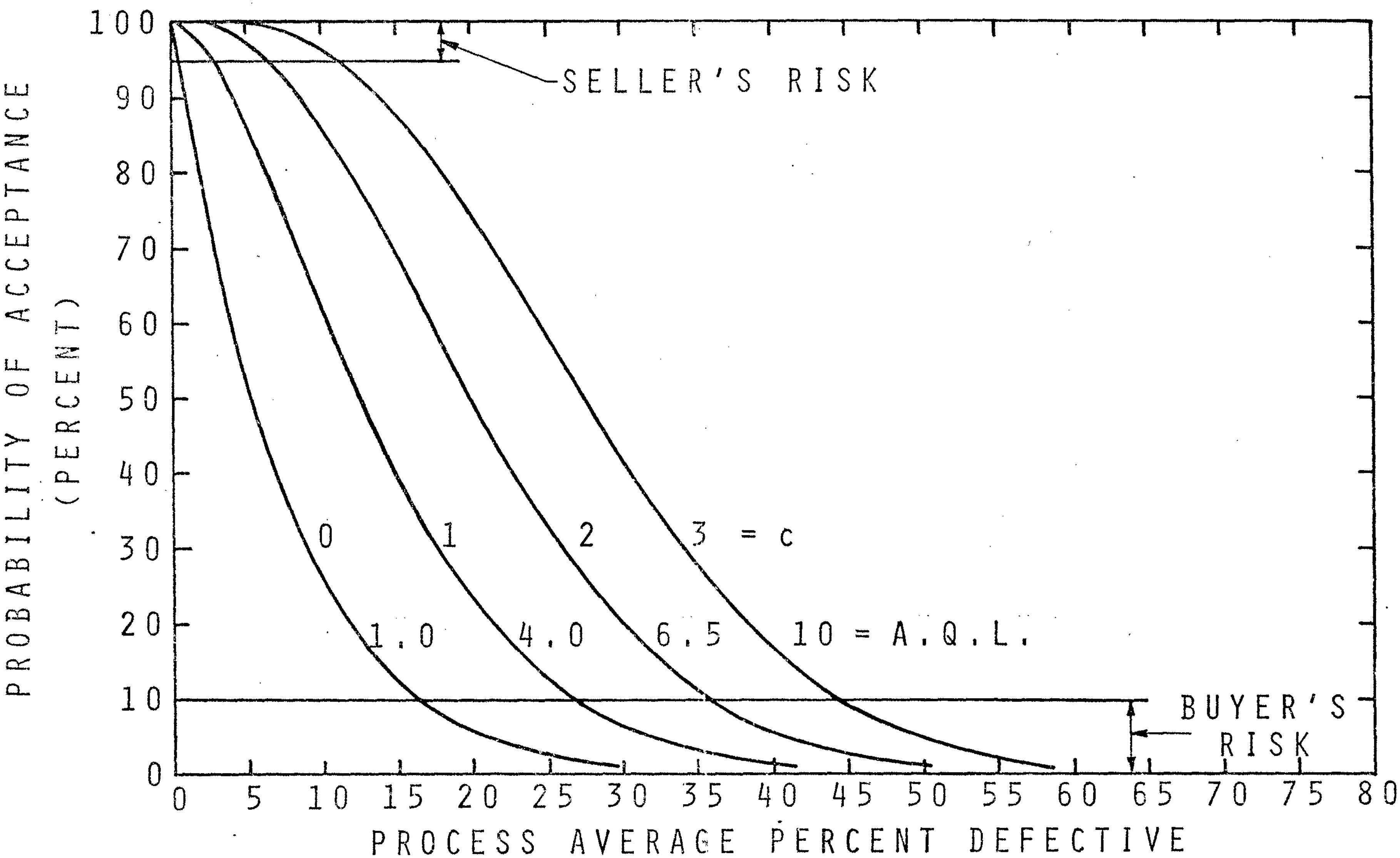
We could also look at Table VI-A on page 24. In the row for sample size 32 in this table we see that one limiting L.T.P.D. with a β risk of 10 percent is about 12 percent and this corresponds to an A.Q.L. of 1.5 percent.

## 9.6 CONSTRUCTION OF OPERATING-CHARACTERISTICS CURVES

The number of possible O.C. curves is unlimited. If a plan with a suitable value of the A.Q.L. or the L.T.P.D. cannot be located in the Military Standard MIL-STD-105D or some other source, an O.C. curve that will pass through any two selected points can be found by varying $n$ and c. One point can be located at the desired A.Q.L. and the corresponding percent probability of acceptance, and the other point can be located at the maximum permissible value of the buyer's risk and the desired L.T.P.D. It is beyond the scope of

Figure 9.3

OPERATING - CHARACTERISTICS CURVES FOR N = 3



this Handbook to include the tables needed for plotting such a curve. However, the method is illustrated in Example 9.5.

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To dig deeper - See *Engineering Statistics* by Bowker and Lieberman, Prentice-Hall, Inc., or *Quality Control and Industrial Statistics* by Duncan, A. J., Richard D. Irwin, Inc.

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## 9.7 USE OF TABLES

In a practical situation the values of the sample size $n$ and the acceptance number c for use in the attribute sampling plan are obtained from tables, such as Table 9.1, which has been adapted from Table II-A in MIL-STD-105D. To use such a table the number of items or units to be included in a LOT is calculated or estimated. In a table similar to Table 9.1, the number of units in a LOT are shown in the left-hand column, and the corresponding number of units in the sample to be inspected is given in the second column. Each of the items in the sample is located by the use of a table of random numbers, and it is inspected for defects. The number of defective units or the total number of defects is noted and compared with the allowable number in the table. The acceptable quality levels are represented by the numbers in the top row. A relatively small value for the A.Q.L. represents allowable percent defective in a LOT, and a large value for the A.Q.L. represents the total number of defects per 100 units. In the next row below the values of the A.Q.L., the abbreviation Ac stands for Accept and the abbreviation Re stands for Reject.

Each number in the body of the table represents either the number of defective units or the number of defects per 100 units. If the actual number for the given conditions is not greater than the number under the heading Ac, the LOT should be accepted. If the actual number is equal to or greater than the number under the heading Re, the LOT should be rejected. The arrows are used to show the smallest or largest sample size to be used in connection with the A.Q.L. regardless of LOT size. For an A.Q.L. of 1.0 the smallest sample size is 13 units while for an A.Q.L. of 15 defects per 100 units the largest sample size is 80 units. The acceptance number appears in the same row as the appropriate sample number under the column heading for the A.Q.L. For instance, if there are 20 units in the sample to be inspected and the acceptable quality level is 2.5 percent, the numbers under the headings Ac and Re would be 1 and 2. The meaning of these limits is as follows: If we have used random numbers

Table 9.1

SINGLE SAMPLING PLANS FOR ATTRIBUTES

| LOT or Batch Size | Sample Size | Acceptable Quality Levels (normal inspection) | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.010 | 0.015 | 0.025 | 0.040 | 0.065 | 0.10 | 0.15 | 0.25 | 0.40 | 0.65 | 1.0 | 1.5 | 2.5 | 4.0 | 6.5 | 10 | 15 | 25 | 40 | 65 | 100 | 150 | 250 | 400 | 650 | 1000 |
| | | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re | Ac Re |
| 2 to 8 | 2 | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | | 0 1 | ↓ | ↓ | 1 2 | 2 3 | 3 4 | 5 6 | 7 8 | 10 11 | 14 15 | 21 22 | 30 31 |
| 9 to 15 | 3 | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | 0 1 | ↑ | ↓ | 1 2 | 2 3 | 3 4 | 5 6 | 7 8 | 10 11 | 14 15 | 21 22 | 30 31 | 44 45 |
| 16 to 25 | 5 | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | 0 1 | ↑ | ↓ | 1 2 | 2 3 | 3 4 | 5 6 | 7 8 | 10 11 | 14 15 | 21 22 | 30 31 | 44 45 | ↑ |
| 26 to 50 | 8 | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | 0 1 | ↑ | ↓ | 1 2 | 2 3 | 3 4 | 5 6 | 7 8 | 10 11 | 14 15 | 21 22 | 30 31 | 44 45 | ↑ | ↑ |
| 51 to 90 | 13 | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | 0 1 | ↑ | ↓ | 1 2 | 2 3 | 3 4 | 5 6 | 7 8 | 10 11 | 14 15 | 21 22 | 30 31 | 44 45 | ↑ | ↑ | ↑ |
| 91 to 150 | 20 | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | 0 1 | ↑ | ↓ | 1 2 | 2 3 | 3 4 | 5 6 | 7 8 | 10 11 | 14 15 | 21 22 | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ |
| 151 to 280 | 32 | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | 0 1 | ↑ | ↓ | 1 2 | 2 3 | 3 4 | 5 6 | 7 8 | 10 11 | 14 15 | 21 22 | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ |
| 281 to 500 | 50 | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | 0 1 | ↑ | ↓ | 1 2 | 2 3 | 3 4 | 5 6 | 7 8 | 10 11 | 14 15 | 21 22 | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ |
| 501 to 1200 | 80 | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | 0 1 | ↑ | ↓ | 1 2 | 2 3 | 3 4 | 5 6 | 7 8 | 10 11 | 14 15 | 21 22 | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ |
| 1201 to 3200 | 125 | ↓ | ↓ | ↓ | ↓ | ↓ | 0 1 | ↑ | ↓ | 1 2 | 2 3 | 3 4 | 5 6 | 7 8 | 10 11 | 14 15 | 21 22 | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ |
| 3201 to 10000 | 200 | ↓ | ↓ | ↓ | ↓ | 0 1 | ↑ | ↓ | 1 2 | 2 3 | 3 4 | 5 6 | 7 8 | 10 11 | 14 15 | 21 22 | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ |
| 10001 to 35000 | 315 | ↓ | ↓ | ↓ | 0 1 | ↑ | ↓ | 1 2 | 2 3 | 3 4 | 5 6 | 7 8 | 10 11 | 14 15 | 21 22 | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ |
| 35001 to 150000 | 500 | ↓ | ↓ | 0 1 | ↑ | ↓ | 1 2 | 2 3 | 3 4 | 5 6 | 7 8 | 10 11 | 14 15 | 21 22 | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ |
| 150001 to 500000 | 800 | ↓ | 0 1 | ↑ | ↓ | 1 2 | 2 3 | 3 4 | 5 6 | 7 8 | 10 11 | 14 15 | 21 22 | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ |
| 500001 and over | 1250 | 0 1 | ↑ | | 1 2 | 2 3 | 3 4 | 5 6 | 7 8 | 10 11 | 14 15 | 21 22 | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | ↑ |

LEGEND:

↓ = Use first sampling plan below arrow. If sample size equals, or exceeds, LOT or batch size, do 100 percent inspection.

↑ = Use first sampling plan above arrow.

Ac = Acceptance number

Re = Rejection number

to select a sample consisting of 20 items from a LOT containing 100 items and we find one item defective, we would accept the LOT. However, if we find two defective items, we would reject the LOT.

Example 9.1. Let us suppose that we have a LOT of concrete pipe containing 400 units or pieces of pipe, and that the specified acceptable quality level for percent defective with respect to large spalls chipped from the bells is 10 percent. We want to select a suitable number of units in the sample to be inspected, and to determine the greatest permissible number of defective units in the sample from an acceptable LOT.

Solution. We see from Table 9.1 that for a LOT containing between 281 and 500 units, the number of units in the sample should be 50. When the acceptable quality level is 10 percent and 50 units are inspected, the number of units that can be defective is found on the horizontal line for which the sample size is 50 and in the column headed Ac under the value 10 for acceptable quality level. This allowable number is 10. In other words, if the actual number of defective units in the 50 inspected is 10 or less, the LOT would be accepted. On the other hand, if there were 11 or more defective units, the LOT would be rejected.

Example 9.2. We want to design an attribute sampling plan for determining whether or not a 5-pound sample of No. 67 crushed gravel (Simplified Practice Gradation) indicates compliance with the specification requirement that 75 percent of all particles retained on a No. 4 sieve must have at least one crushed face.

Solution. From previous experience we estimate that the average weight of the individual particles retained on the No. 4 sieve will be about 5 grams. Then, since there are 454 grams in 1 pound, the estimated number of particles in the 5-pound sample is

$$N = \frac{5(454)}{5} = 454$$

According to Table 9.1 the sample size for a batch size of 281 to 500 particles should be 50 particles. To obtain this sample we would first obtain a test portion weighing about 1 pound by splitting the 5-pound sample in some suitable manner. From the 1-pound test portion we would select a random sample consisting of 50 particles and would inspect each of these particles for the presence of a fractured face. In Table 9.1 the rejection number for an A.Q.L. equal to 25 percent is 22. If we find that more than 21 particles in the sample containing 50 particles do not have one fractured face, we would decide that the aggregate in the 5-pound sample does not comply with the specification requirement.

Example 9.3. As the application of an attribute sampling plan, let us suppose that we want to design an acceptance plan for newly constructed concrete pavement. On the average, 6000 feet of two-lane pavement are constructed in one day. We will test the pavement the next day after construction by the use of a rolling straightedge which is 16 feet long and will mark the high spots so that they may be easily counted. We will consider the total equivalent length of one lane constructed in one day to be a LOT, and we will divide each LOT into units 100 feet long. We decide that an acceptable pavement will have not more than three deviations of 1/4 inch or more from the 16-foot straightedge in a 100-foot unit. We will pay the full contract price for a LOT in which the number of defective units is not more than 4.0 percent and will accept at a proportionately reduced price a LOT in which the number of defective units is between 4.0 and 10.0 percent.

Solution. The length of one traffic lane constructed in one day is 2(6000) = 12,000 feet, and the number of 100-foot units in a LOT is 120. According to Table 9.1, the number of 100-foot units in each LOT to be inspected would be 20. These units will be selected by the use of a table of random numbers. The selected units will be located on the pavement, and the trueness of the surface will be tested by moving the rolling straightedge along a line parallel to the centerline and at a random distance from it. After all 20 units in a LOT have been tested, the inspector will count the number of units that show three or more 1/4-inch deviations. Looking in the body of Table 9.1 we see that for 4.0 percent defective, only two of the 20 units can have more than the allowable number of 1/4-inch deviations.

Example 9.4. To illustrate the use of the values of defects per hundred units in Table 9.1, we include this hypothetical example involving a sampling plan for acceptance of the wearing course of an asphalt pavement with respect to smoothness.

To determine the degree of smoothness, the pavement will be divided into LOTs having lengths ranging from approximately 5000 to 9000 linear feet, and each LOT will consist of units that are 100 feet long. Since the number of units in a LOT will be between 51 and 90, the size of a sample will be 13 units. The units in the sample will be located at random distances measured parallel to the centerline of the pavement, and the measurements for smoothness will be made along lines that are parallel to the centerline and at random distances from the centerline.

A defect in smoothness shall be considered to be any of the following:

More than two deviations of 1/8 inch or more from a straightedge with a span length less than 5 feet.

More than three deviations of 1/4 inch or more from a straightedge with a span length less than 16 feet.

More than one deviation of 3/8 inch from a stringline with a 50-foot span.

More than one deviation greater than 3/8 inch above or below the theoretical elevation at a point.

In a LOT of wearing course with acceptable smoothness, the total number of defects per hundred units shall not exceed 100. If the wearing course does not meet this requirement, one of the following three penalties shall be chosen: 1) The course shall be removed and replaced; 2) It shall be overlaid with a minimum of 1 inch of additional course; 3) It may be accepted at the price shown in Table 9.2.

To implement such an acceptance plan, 13 units 100 feet long would be selected by the use of a table of random numbers and inspected for defects. From Table 9.1 if the total number of defects in the sample of 13 units is less than 21 the pavement can be considered to have less than 100 defects per 100 units. If the total number of defects in the sample is more than 21 but less than 30 the pavement can be considered to have from 100 to 150 defects per hundred units, if more than 30 but less than 44, from 150 to 250 defects per hundred units, and if more than 44 defects are found in the sample the pavement can be considered to have 400 or more defects per hundred units.

It is emphasized that the critical number of units used in this example is entirely hypothetical. The actual number used in practice would depend on the results of sampling smooth and rough riding pavements.

Table 9.2

REDUCED PAYMENT SCHEDULE FOR ASPHALT WEARING COURSE

| Number of Defects Per 100 Units | Percent of Contract Price to Be Paid |
|---|---|
| 100 | 100 |
| 150 | 75 |
| 250 | 50 |
| More than 250 | Remove and Replace or overlay as directed |

## 9.8 SPECIAL SAMPLING PLAN

An attribute single sampling plan with respect to slab thickness for acceptance of a section of a lane of concrete pavement less than 1.5 miles in length has been developed by the Engineering Research and Development Bureau of New York State Department of Transportation.

The step-by-step procedure for the use of this plan, which is illustrated schematically in Figure 9.4, is outlined as follows:

"1. Determine the total length of pavement to be cored and randomly select 18 core locations. If a core location falls closer than 1.5 ft from a joint or edge, move the core location at least 1.5 ft away from the joint or edge.

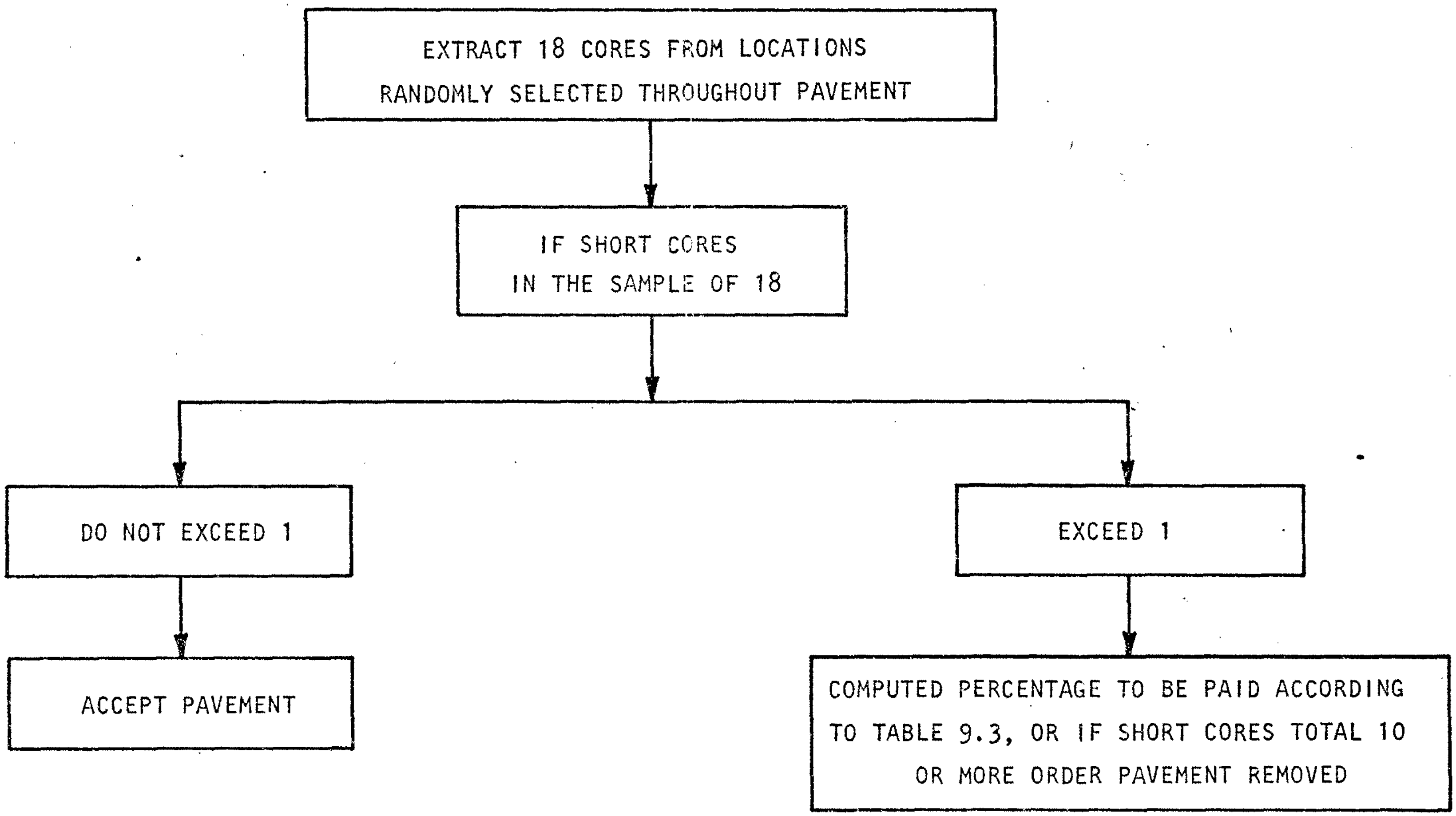
"2. If one short core or none are found, accept the pavement.

"3. If two or more short cores are found whose length deviates from the specified thickness by less than 1 in., determine payment according to Table" 9.3.

"4. If any core is shorter than specified thickness by 1 in. or more, consider this due to a mistake in paving. Screen around these cores, and if a mistake is proved remove the deficient portions of pavement. Exclude them and the screening cores from the normal sample size. Take additional random cores to replace those that triggered the screening and accept the remaining pavement as outlined in Steps 1 through 3.

Figure 9.4

SCHEMATIC OPERATION OF A SINGLE SAMPLING PLAN
FOR PAVEMENT SHORTER THAN 1.5 LANE MILES



"5. If mistakes are not found, include the original short cores that triggered the screening in the sample size and accept the whole pavement as outlined in Steps 1 through 3."

Table 9.3

REDUCED PAYMENT SCHEDULE FOR CONCRETE PAVEMENT

| Short Cores In Sample | Percent of Contract Price to Be Paid |
|---|---|
| 0 or 1 | 100 |
| 2 or 3 | 90 |
| 4 or 5 | 80 |
| 6 or 7 | 70 |
| 8 or 9 | 60 |
| 10 or more | Remove pavement |

The method of finding the parameters for this plan will be illustrated by using tables in *Engineering Statistics* by Bowker and Lieberman. The tables referred to in the following example are in that book.

Example 9.5. We are given the following selected values:

Seller's risk ($\alpha$) = 0.05

Buyer's risk ($\beta$) = 0.10

A.Q.L. = $p_1'$ = 0.02

L.T.P.D. = $p_2'$ = 0.22

Solution. The first step in designing the sampling plan is to obtain the value of the ratio $p_2'/p_1'$. Thus

$$\frac{p_2'}{p_1'} = \frac{0.22}{0.02} = 11$$

Looking in Table 13.14 in the column headed $\alpha = 0.05$ and $\beta = 0.10$ for a number that is equal to or just less than 11, we find the number 10.946. Also, in the column headed c in the same row, we find our acceptance number, which is 1.

Now we refer to Table 13.13. In the column headed P(A) = 0.959 (one minus alpha) and in the row for c = 1, we find the number 0.355. To get the sample size $n$, we divide this number by $p_1'$. Thus,

$$n = \frac{0.355}{0.02} \approx 18$$

To determine values for plotting the operating-characteristics curve from this plan, we again refer to Table 13.13. As shown in the first column in Table 9.4 in our Handbook, we list the numbers in the top row in Table 13.13, which are the probabilities of acceptance P(A) for various values of the L.T.P.D. To find the corresponding values of the L.T.P.D., we use the numbers found in the row designated c = 1 and divide each of these numbers by the sample size $n$, or 18. The results are listed in the second column in Table 9.4. The curve is plotted in Figure 9.5.

Table 9.4

COMPUTATIONS FOR OPERATING-CHARACTERISTICS CURVE

| P(A) | L.T.P.D. |
|---|---|
| 0.995 | $\frac{0.103}{18} = 0.006$ |
| 0.990 | $\frac{0.149}{18} = 0.008$ |
| 0.975 | $\frac{0.242}{18} = 0.013$ |
| 0.950 | $\frac{0.355}{18} = 0.020$ |
| 0.900 | $\frac{0.532}{18} = 0.030$ |
| 0.750 | $\frac{0.961}{18} = 0.053$ |
| 0.050 | $\frac{1.678}{18} = 0.093$ |
| 0.250 | $\frac{2.693}{18} = 0.150$ |
| 0.100 | $\frac{3.890}{18} = 0.216$ |
| 0.050 | $\frac{4.744}{18} = 0.264$ |
| 0.025 | $\frac{5.572}{18} = 0.310$ |
| 0.010 | $\frac{6.638}{18} = 0.369$ |
| 0.005 | $\frac{7.430}{18} = 0.413$ |

Figure 9.5

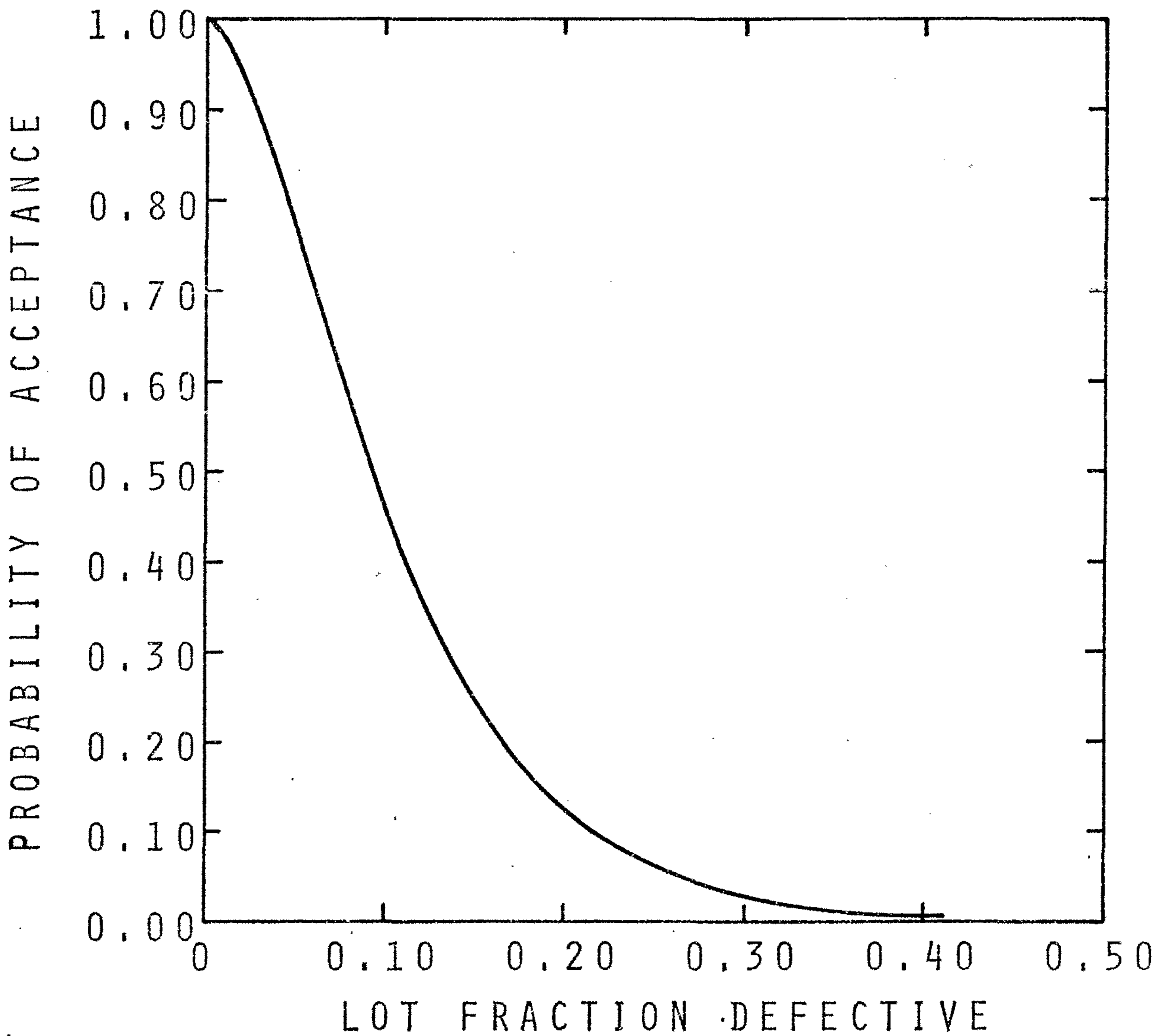
OPERATING-CHARACTERISTICS CURVE FOR SINGLE SAMPLING PLAN FOR SHORT SECTIONS OF CONCRETE PAVEMENT

(DEFICIENCY IN THICKNESS OF CONCRETE SLABS)

$\alpha = 0.05$ $\quad P_1' = 0.02$

$\beta = 0.010$ $\quad P_2' = 0.22$

$N = 18$ $\quad c = 1$



# GLOSSARY

ACCEPTABLE QUALITY LEVEL - The maximum percent defective or maximum number of defects per 100 units that can be considered satisfactory as a process average.

ALPHA (α) RISK - The risk that acceptable material or construction will be rejected, or a Type I error will be made.

ATTRIBUTE - A particular characteristic of a unit of material or construction.

BETA (β) RISK - The risk that unacceptable material or construction will be accepted, or a Type II error will be made.

BUYER'S RISK - See beta risk.

DEGREES OF FREEDOM - The number of measurements less the number of constants derived from them.

INFINITY - An unlimited distance.

ITERATION - The method of successive trials.

LOT - A definite amount of material or construction, such as a truckload of concrete or a certain area of pavement.

MAJOR REQUIREMENT - One which if not met would result in material loss of performance.

MINOR REQUIREMENT - One which if not met would not materially reduce performance.

OC CURVE - See operating-characteristics curve.

OPERATING-CHARACTERISTICS CURVE - One that shows for an accepted sampling plan the relation between the probability of acceptance and the quality of cemented LOTs.

PLUS OR MINUS - Added to or subtracted from.

PROCESS AVERAGE - The average percent defective or average number of defects per 100 units in the output of a process.

SELLER'S RISK - Sell alpha risk.

SIMULATED TEST RESULT - Result produced by mathematical process to represent those expected in actual practice.

TYPE I ERROR - The incorrect rejection of acceptable material or construction.

TYPE II ERROR - The incorrect acception of unacceptable material or construction.