Distributed Optimization of Traffic Signals in Arterial Networks

Xinyu Fei[1], Xingmin Wang[2], Xian Yu[1], Yiheng Feng[3],
Henry Liu[2], Siqian Shen[1], Yafeng Yin[2]

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[1] Department of Industrial and Operations Engineering, University of Michigan at Ann Arbor
[2] Department of Civil and Environmental Engineering, University of Michigan at Ann Arbor
[3] Lyles School of Civil Engineering, Purdue University

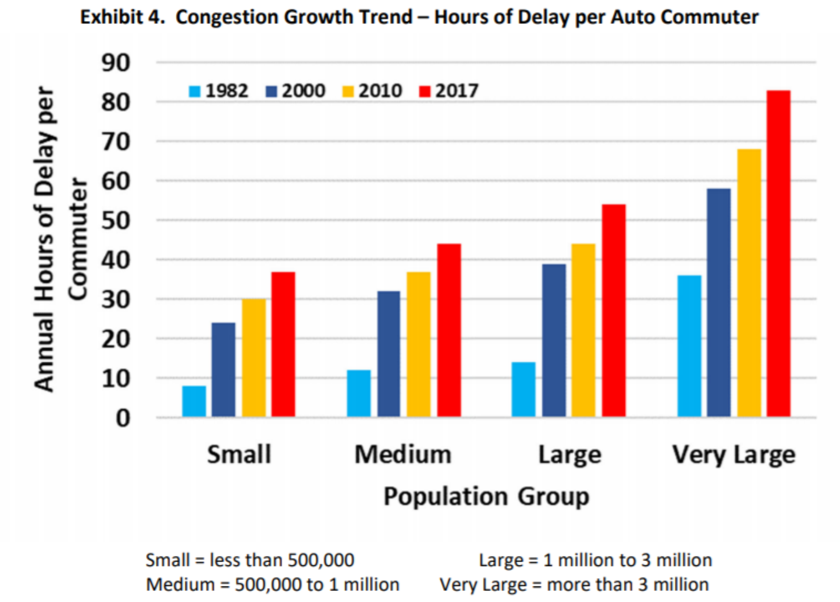
Outline

- Background
  - Traffic Congestion
- Deterministic Model
  - Overview
  - Variables, Parameters, Objective Function and Flow-balance constraints
  - Signal Constraints
- Algorithm for Deterministic Model
  - Distributed Form of Deterministic Model
  - Alternating Direction Method of Multipliers Algorithm
- Stochastic Model
- Algorithm for Stochastic Model
  - Distributed Benders Cut with cycle estimation
- Numerical Results

Background - Traffic Congestion

- Negative Impact
  - Traffic delay
  - Fuel consumption
  - Air pollution

- Approaches
  - Congestion pricing
  - Road expansion
  - Traffic signal control



2019 URBAN MOBILITY REPORT,
published by The Texas A&M Transportation Institute.

Traffic congestion in America

https://www.usnews.com/news/slideshows/worst-traffic-cities-in-america-ranked

Deterministic Model - Overview

- Spatial decomposition
  - Partition road into cells.
  - Types of cells: origin ($\mathcal{O}$), ordinary ($\mathcal{E}$), diverge ($\mathcal{V}$), intersection ($\mathcal{J}$), merge ($\mathcal{M}$), destination ($\mathcal{D}$).
- Temporal decomposition
  - Divide time horizon in time steps as $\{1, \cdots, T\}$.
- Cell Transmission Model (CTM)*

Flow capacity

Jam density

$$y_{ct} = \min\{n_{ct}, q_{ct}, q_{d(c)t}, W_{d(c)t}(N_{d(c)t} - n_{d(c)t})\}$$

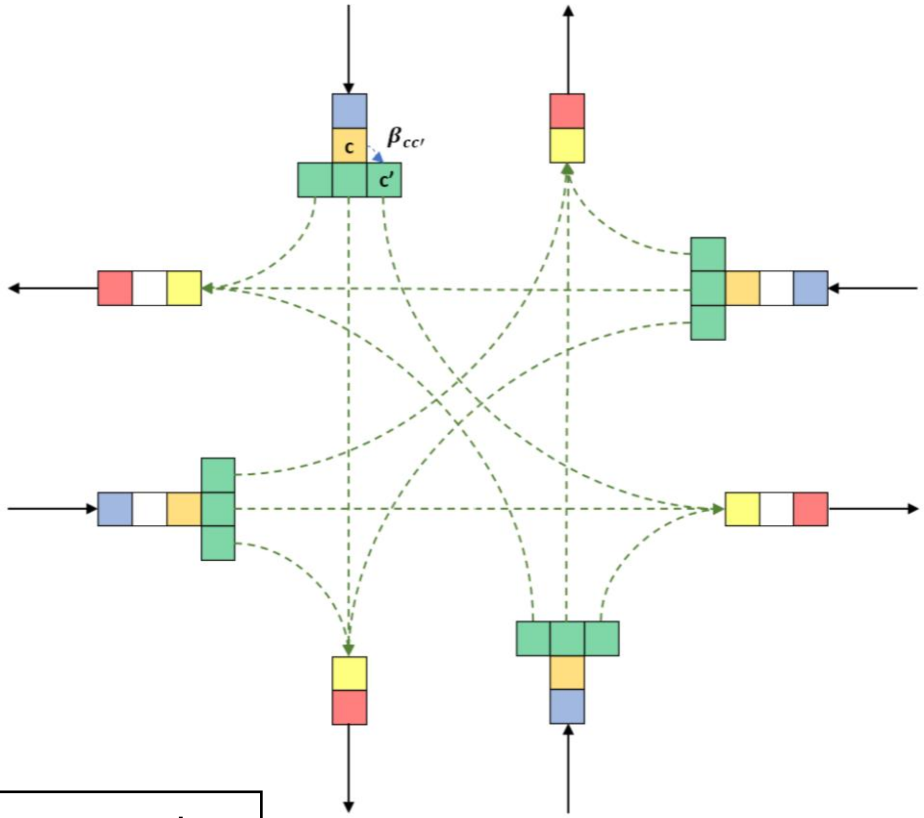
$$n_{ct+1} = n_{ct} + \bar{y}_{ct} - y_{ct}$$

Ratio between shock-wave propagation speed and the flow-free speed

Number of vehicles

Inflow

Outflow



* Daganzo, C. (1992). The cell transmission model. part II: Network traffic.

Deterministic Model – Variables, Parameters, Objective Function and Constraints

- Relax dynamic equation constraints
  - Transform them into linear constraints
  - Add term in objective function to maximize the flow of vehicles.
- Final objective function
  - Maximize throughput of network
  - Maximize the flow of vehicles

$$\min \quad -\sum_{c\in\mathcal{D}}\sum_{t=1}^{T} n_{ct} - \alpha \sum_{c\in\mathcal{C}}\sum_{t=1}^{T}(T-t)y_{ct}$$

- Variables

y: Number of vehicles leaving cell

n: Number of vehicles inside cell

z: Indicator of which cycle and phase of green time this time step is in

- Parameters

Q: Flow capacity

β: Turning ratio

W: Ratio between shock-wave propagation speed and the flow-free speed

N: Jam density

D: Demand

n^{init}: Initialized number of vehicles

N_{cy}: Number of cycles

$\mathcal{R}$: Set of intersections

Deterministic Model – Flow-balance Constraints

- Flow-balance constraints are different for different types of cells.
- Flow-balance constraints are for each cell and time step.

$$y_{ct} \leq n_{ct}, \ \forall c \in \mathcal{C}, \ t = 1, \cdots, T$$

Number of vehicles leaving cell is bounded by number of vehicles inside cell.

$$y_{ct} \leq Q_{ct}, \ \forall c \in \mathcal{E} \cup \mathcal{O} \cup \mathcal{M} \cup \mathcal{V}, \ t = 1, \cdots, T$$

$$y_{ct} \leq \sum_{j'=1}^{N_{cy}} (z_{1ijj't} + z_{2ijj't} - 1) Q_{ct}, \ \forall c \in \mathcal{I}_{ij}, \ \forall i \in \mathcal{R}, \ \forall j \in \mathcal{F}, \ t = 1, \cdots, T$$

Number of vehicles leaving cell is bounded by the flow capacity of this cell.

$$y_{ct} \leq Q_{c't}, \ \forall c \in \mathcal{C}/\mathcal{V}, \ \forall c' \in d(c), \ t = 1, \cdots, T$$

$$\beta_{cc'} y_{ct} \leq \sum_{j'=1}^{N_{cy}} (z_{1ijj't} + z_{2ijj't} - 1) Q_{c't}, \ \forall c \in \mathcal{V}, \ \forall i \in \mathcal{R}, \ \forall j \in \mathcal{F}, \ \forall c' \in d(c) \cap I_{ij}, \ t = 1, \cdots, T$$

Number of vehicles leaving cell is bounded by the flow capacity of processing cell.

$$y_{ct} \leq W_{c't}(N_{c't} - n_{c't}), \ \forall c \in \mathcal{C}/\mathcal{V}, \ \forall c' \in d(c), \ t = 1, \cdots, T$$

$$\beta_{cc'} y_{ct} \leq W_{c't}(N_{c't} - n_{c't}), \ \forall c \in \mathcal{V}, \ \forall c' \in d(c), \ t = 1, \cdots, T$$

Number of vehicles leaving cell is bounded by the number of vehicles that can enter processing cell.

$$n_{ct+1} = n_{ct} + \sum_{c' \in p(c)} y_{c't} - y_{ct}, \ \forall c \in \mathcal{C}/\mathcal{O}/\mathcal{I}, \ t = 1, \cdots, T$$

$$n_{ct+1} = n_{ct} + D_{ct} - y_{ct}, \ \forall c \in \mathcal{O}, \ t = 1, \cdots, T$$

$$n_{ct+1} = n_{ct} + \sum_{c' \in p(c)} \beta_{c'c} y_{c't} - y_{ct}, \ \forall c \in \mathcal{I}, \ t = 1, \cdots, T$$

Dynamic equations of flow balance.

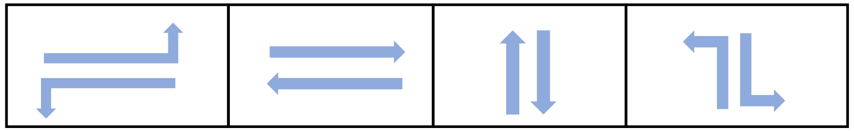
$$n_{c1} = n_c^{\text{init}}, \ \forall c \in \mathcal{C}$$

Initial constraints.

$$y_{ct} \geq 0, \ n_{ct} \geq 0, \ \forall c \in \mathcal{C}, \ t = 1, \cdots, T$$

Deterministic Model – Signal Constraints

- Optimize green time, cycle length and offset.
- Phase sequence: 1⟶2⟶3⟶4.



Phase 1 Phase 2 Phase 3 Phase 4

- Use integer variables to describe ‘if-then’ constraints.

- Variables and parameters

z: Indicator integer variable
b, e: Begin and end time of green phase
g: Green time, l: Cycle length, o: Offset
U, ϵ: Sufficient large and small parameters for ‘if-then’ constraints
G_{min}, G_{max}: Minimum and maximum green time

$$-U \cdot z_{1ijj't} + \epsilon \le t - e_{ijj'} \le U(1 - z_{1ijj't}), \ \forall i \in \mathcal{R}, \ j' = 1, \cdots, N_{cy}, \ t = 1, \cdots, T$$
$$-U \cdot z_{2ijj't} + \epsilon \le b_{ijj'} - t \le U(1 - z_{2ijj't}), \ \forall i \in \mathcal{R}, \ \forall j \in \mathcal{F}, \ t = 1, \cdots, T$$
$$\sum_{j \in \mathcal{F}} z_{1ijj't} + z_{2ijj't} \le 5, \ j' = 1, \cdots, N_{cy}, \ t = 1, \cdots, T$$

If time step t is during green time $[b_{ijj'}, e_{ijj'}]$, then $z_{1ijj't} = z_{2ijj't} = 1$.

$$o_i \le l_i, \ \forall i \in \mathcal{R}$$
$$b_{i1j'} = l_{i1j'} \cdot j' - o_i \ \forall i \in \mathcal{R}, \ j' = 1, \cdots, N_{cy}$$
$$e_{i1j'} = b_{i1j'} + g_{i1}, \ \forall i \in \mathcal{R}, \ j' = 1, \cdots, N_{cy}$$
$$b_{i2j'} = e_{i1j'}, \ \forall i \in \mathcal{R}, \ j' = 1, \cdots, N_{cy}$$
$$e_{i2j'} = b_{i2j'} + g_{i2}, \ \forall i \in \mathcal{R}, \ j' = 1, \cdots, N_{cy}$$
$$b_{i3j'} = e_{i2j'}, \ \forall i \in \mathcal{R}, \ j' = 1, \cdots, N_{cy}$$
$$e_{i3j'} = b_{i3j'} + g_{i3}, \ \forall i \in \mathcal{R}, \ j' = 1, \cdots, N_{cy}$$
$$b_{i4j'} = e_{i3j'}, \ \forall i \in \mathcal{R}, \ j' = 1, \cdots, N_{cy}$$
$$e_{i4j'} = b_{i4j'} + g_{i4}, \ \forall i \in \mathcal{R}, \ j' = 1, \cdots, N_{cy}$$
$$l_i = \sum_{j \in \mathcal{F}} g_{ij}, \ \forall i \in \mathcal{R}$$

Computation of start and end time of green phase given green time, offset and cycle length.

$$G_{\min} \le g_{ij} \le G_{\max}, \ \forall i \in \mathcal{R}, \ \forall j \in \mathcal{F}$$

Minimum and maximum green time constraints.

$$z_{1ijj't}, \ z_{2ijj't} \in \{0, 1\}, \ \forall i \in \mathcal{R}, \ \forall j \in \mathcal{F}, \ j' = 1, \cdots, N_{cy}, \ t = 1, \cdots, T$$

Deterministic Model – Distributed Formulation

- Input boundary cells: receiving inflow from cell of neighboring area.
- Output boundary cells: sending outflow to cell of neighboring area.

$$\min \quad \sum_{i=1}^{N} \left(-\sum_{c \in \mathcal{D}_i} \sum_{t=1}^{T} n_{ct} - \alpha \sum_{c \in \mathcal{C}_i} \sum_{t=1}^{T} (T-t) y_{ct} \right)$$

$$\text{s.t.} \quad \mathbf{A}_i \mathbf{x}_i = \mathbf{a}_i, \ \forall i \in \mathcal{R}$$

$$\mathbf{B}_i \mathbf{x}_i + \mathbf{v}_i = \mathbf{b}_i, \ \forall i \in \mathcal{R}$$

Write constraints for non-boundary cells into matrix form.

$$y_{ct} + s_{ct} = W N_{ct} - W u_{n_{d_c t}}, \ \forall i \in \mathcal{R}, \ \forall c \in \mathcal{B}_i^O, \ t = 1, \cdots, T$$

$$n_{ct+1} = n_{ct} + u_{y_{p_c t}} - y_{ct}, \ \forall i \in \mathcal{R}, \ \forall c \in \mathcal{B}_i^I, \ t = 1, \cdots, T$$

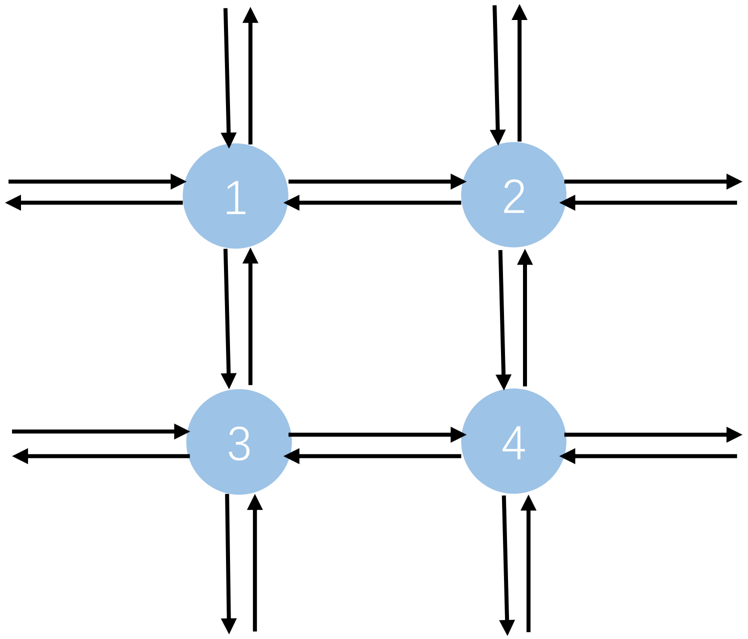
$$\mathbf{u}_i = \mathbf{x}_i, \ \forall i \in \mathcal{R}$$

Maintain constraints between intersection.

$$\mathbf{v}_i \geq 0, \ \forall i \in \mathcal{R}$$

$$\mathbf{x}^{lb} \leq \mathbf{x}_i \leq \mathbf{x}^{ub}, \ \forall i \in \mathcal{R}$$

$$z_{1ijj't}, \ z_{2ijj't} \in \{0, 1\}, \ \forall i \in \mathcal{R}, \ \forall j \in \mathcal{F}, \ j' = 1, \cdots, N_{cy}, \ t = 1, \cdots, T$$



Algorithm – Alternating Direction Method of Multipliers (ADMM)

- Lagrangian function

$$
\begin{aligned}
\mathcal{L}(\mathbf{x},\mathbf{u},\mathbf{v},\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}) = \sum_{i=1}^{N}\Bigg(& -\sum_{c\in\mathcal{D}_i}\sum_{t=1}^{T} n_{ct} - \alpha\sum_{c\in\mathcal{C}_i}\sum_{t=1}^{T}(T-t)y_{ct}\Bigg) \\
& + \frac{L}{2}\sum_{i=1}^{N_I}(\|\mathbf{B}_i\mathbf{x}_i + \mathbf{v}_i - \mathbf{b}_i + \boldsymbol{\kappa}_i\|_2^2 + \|\mathbf{x}_i - \mathbf{u}_i + \boldsymbol{\nu}_i\|_2^2 \\
& + \sum_{c\in\mathcal{B}_i^O}\sum_{t=1}^{T_o}\left(y_{ct} + s_{ct} + Wu_{n_{dct}} - WN_{ct} + \lambda_{ct}\right)^2 \\
& + \sum_{c\in\mathcal{B}_i^I}\sum_{t=1}^{T_o}\left(n_{ct+1} - n_{ct} + y_{ct} - u_{y_{pct}} + \mu_{ct}\right)^2)
\end{aligned}
$$

- Algorithm steps
  - Step 1: Minimize Lagrangian function with respect to x.

    $$\mathbf{x}^{l+1} = \text{argmin}_{\mathbf{x}\in\mathcal{C}_x}\mathcal{L}(\mathbf{x},\mathbf{u}^l,\mathbf{v}^l,\boldsymbol{\kappa}^l,\boldsymbol{\lambda}^l,\boldsymbol{\mu}^l,\boldsymbol{\nu}^l)$$

  - Step 2: Minimize Lagrangian function with respect to z.

    $$\mathbf{z}^{l+1} = \text{argmin}_{\mathbf{z}\in\mathcal{C}_z}\mathcal{L}(\mathbf{x}^{l+1},\mathbf{u},\mathbf{v},\boldsymbol{\kappa}^l,\boldsymbol{\lambda}^l,\boldsymbol{\mu}^l,\boldsymbol{\nu}^l)$$

  - Step 3: Update dual variables.

$$
\begin{aligned}
\boldsymbol{\kappa}_i^{l+1} &= \mathbf{B}_i\mathbf{x}_i^{l+1} + \mathbf{u}_i^{l+1} - \mathbf{b}_i + \boldsymbol{\kappa}_i^l \\
\lambda_{ct}^{l+1} &= y_{ct}^{l+1} + s_{ct}^{l+1} + Wu_{n_{dct}}^{l+1} - WN_{ct} + \lambda_{ct}^l,\ \ c\in\mathcal{B}_i^O \\
\mu_{ct}^{l+1} &= n_{ct+1}^{l+1} - n_{ct}^{l+1} + y_{ct}^{l+1} - u_{y_{pct}}^{l+1} + \mu_{ct}^l,\ \ c\in\mathcal{B}_i^I \\
\boldsymbol{\nu}_i^{l+1} &= \mathbf{x}_i^{l+1} - \mathbf{u}_i^{l+1} + \boldsymbol{\nu}_i^l
\end{aligned}
$$

Algorithm – Alternating Direction Method of Multipliers with Heuristic

- Distributed form

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{N}\left(-\sum_{c\in\mathcal{D}_i}\sum_{t=1}^{T} n_{ct} - \alpha \sum_{c\in\mathcal{C}_i}\sum_{t=1}^{T}(T-t)y_{ct}\right) \\
\text{s.t.} \quad & \mathbf{A}_i\mathbf{x}_i = \mathbf{a}_i,\ \forall i\in\mathcal{R} \\
& \mathbf{B}_i\mathbf{x}_i + \mathbf{v}_i = \mathbf{b}_i,\ \forall i\in\mathcal{R} \\
& y_{ct} + s_{ct} = WN_{ct} - Wu_{n_{dct}},\ \forall i\in\mathcal{R},\ \forall c\in\mathcal{B}_i^O,\ t=1,\cdots,T \\
& n_{ct+1} = n_{ct} + u_{y_{pct}} - y_{ct},\ \forall i\in\mathcal{R},\ \forall c\in\mathcal{B}_i^I,\ t=1,\cdots,T \\
& \mathbf{u}_i^{(1)} = \mathbf{x}_i,\ \mathbf{u}_i^{(2)} = \mathbf{x}_i,\ \forall i\in\mathcal{R} \\
& \mathbf{u}_z^{(1)} \in [0,1]^{8N_IN_{cy}T},\ \mathbf{u}_z^{(2)} \in \{u : \|u - \tfrac{1}{2}1_{8N_IN_{cy}T}\|_2^2 = 2N_IN_{cy}T\} \\
& \mathbf{v}_i \ge 0,\ \forall i\in\mathcal{R} \\
& \mathbf{x}^{lb} \le \mathbf{x}_i \le \mathbf{x}^{ub},\ \forall i\in\mathcal{R} \\
& z_{1ijj't},\ z_{2ijj't} \in \{0,1\},\ \forall i\in\mathcal{R},\ \forall j\in\mathcal{F},\ j'=1,\cdots,N_{cy},\ t=1,\cdots,T
\end{aligned}
$$

Use two variables to estimate value of x.

- Variable update
  - Update of integer variables are different

$$
\begin{aligned}
{\mathbf{u}_z^{(1)}}^{l+1} &= P_{S_b}(z^{l+1} + \nu^l) \\
P_{S_b}(a) &= \min(\mathbf{1}_{8N_IN_{cy}T}, \max(\mathbf{0}_{8N_IN_{cy}T}, a)) \\
{\mathbf{u}_z^{(2)}}^{l+1} &= P_{S_p}(z^{l+1} + \nu^l) \\
P_{S_p}(a) &= \frac{8N_IN_{cy}T^{1/2}}{2}\frac{\bar{a}}{\|\bar{a}\|_2} + \frac{1}{2}\mathbf{1}_{8N_IN_{cy}T},\ \bar{a} = a - \frac{1}{2}\mathbf{1}_{8N_IN_{cy}T}
\end{aligned}
$$

Stochastic Model

- Stochastic parameters: turning ratio (β) and demand (D).
- Assumption: known distribution of stochastic parameters.
- First-stage variables - decisions made "here and now"
  - Begin and end time of green phase (b, e), green time (g), offset (o), cycle length (l) and corresponding integer variables (z_1, z_2).
- Second-stage variables - "wait and see" recourse decisions
  - Number of vehicles leaving (y) and inside each cell (n).
- Two-stage stochastic program formulation
  - Use θ to estimate the objective value of second stage problem.
  - First-stage problem (master problem)

$$\begin{aligned} \min \quad & \sum_{k=1}^{K} p^k \theta^k \\ \text{s.t.} \quad & \text{Signal constraints and previous added cuts} \end{aligned}$$

  - Second-stage problem (subproblem)

$$\begin{aligned} \theta^k = \min \quad & -\sum_{c \in \mathcal{D}} \sum_{t=1}^{T} n_{ct}^k - \alpha \sum_{c \in \mathcal{C}} \sum_{t=1}^{T} (T-t) y_{ct}^k \\ \text{s.t.} \quad & \text{Flow constraints for each scenario} \end{aligned}$$

Algorithm – Distributed Benders Cut with Cycle Estimation

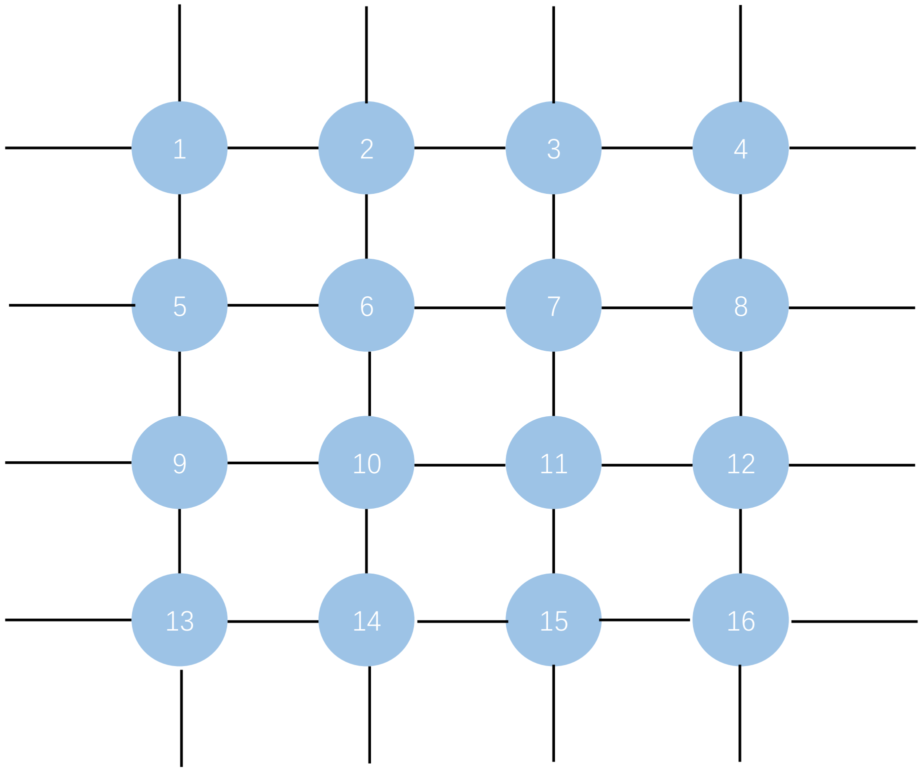
- Benders cut – for each iteration, add constraints of θ and first-stage variables based on the dual solution of subproblem to master problem. $\theta^k \geq F_D^k(z_1, z_2, \hat{\rho}^k, \hat{\sigma}^k, \hat{\pi}^k, \hat{\gamma}^k, \hat{\delta}^k, \hat{\tau}^k)$
- Distributed Benders cut
  - Signal constraints are separate for each intersection. Solve master problem for each intersection separately.

$$\min \quad \sum_{k=1}^{K} p^k \theta_i^k$$
$$\text{s.t.} \quad \text{Signal constraints and previous added cuts for each intersection}$$

  - The constraint in Benders cut can be written as summation of variables of each intersection. Add constraints to master problem of each intersection separately. $\theta_i^k \geq F_{Di}^k(z_{1i}, z_{2i}, \hat{\rho}^k, \hat{\sigma}^k, \hat{\pi}^k, \hat{\gamma}^k, \hat{\delta}^k, \hat{\tau}^k)$
- Estimation of cycle – reduce the number of integer variables
  - Use the cycle length of previous iteration to estimate the cycle each time step is in.
  - Before estimation: consider z_1, z_2 for every cycle $j' = 1, \cdots, N_{cy}$ to indicate if the time step is in cycle j'.
  - After estimation: consider z_1, z_2 for only two cycles $j' = \lfloor t/l \rfloor, \lceil t/l \rceil$ to indicate if the time step is in cycle j'.

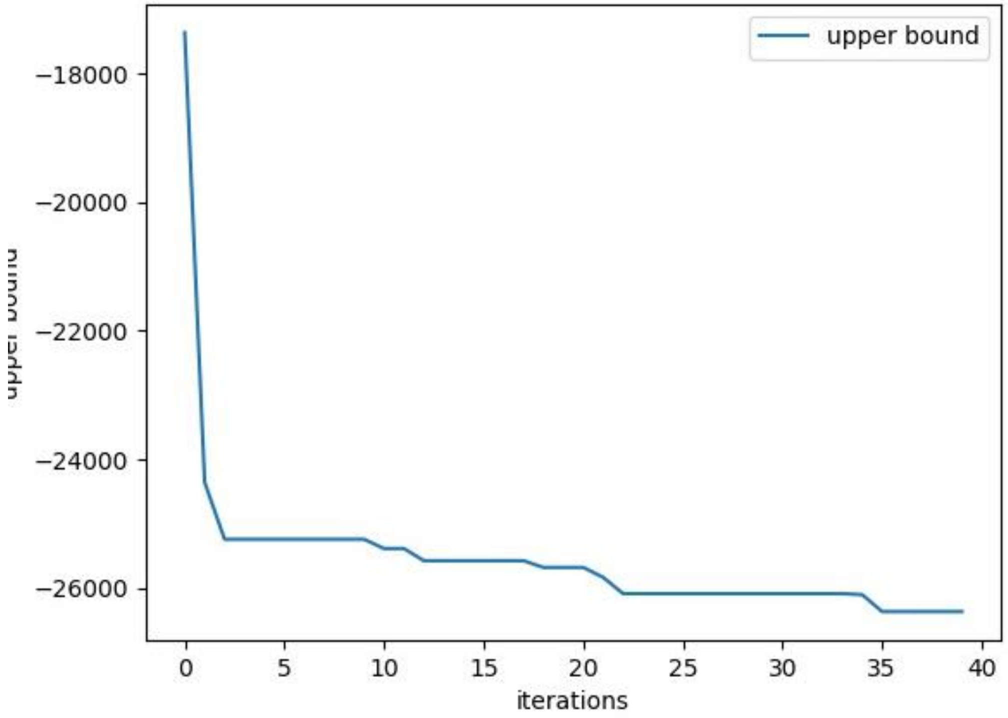
Result

- Settings
  - Grid network size: 4 × 4, time horizon: 200s, number of scenarios: 10
  - Flow capacity: Q = 2 for intersection cells, Q = 4 for other cells.
  - Jam density: N = 8 for intersection cells, N = 16 for other cells.
  - Minimum green time: 12s, maximum green time: 32s
  - Demand: Poisson distribution with mean randomly generated from [0.4,1.2] (west-east direction) and [0.2, 0.6] (south-north direction)
  - Turning ratio: Randomly selected from given possible turning ratio set
- Problem size
  - 12800 integer variables, 501728 continuous variables, 1176448 constraints.
- Performance
  - Gurobi: no feasible solution with 7200s time limit.
  - Benders cut: cannot solve master problem with 7200s time limit after 1 iteration.
  - Distributed Benders cut: Obtain solution after 40 iterations (14000s).
  - Distributed Benders cut with cycle estimation: Obtain solution after 40 iterations (9500s).



Result of Distributed Benders Cut with Cycle Estimation

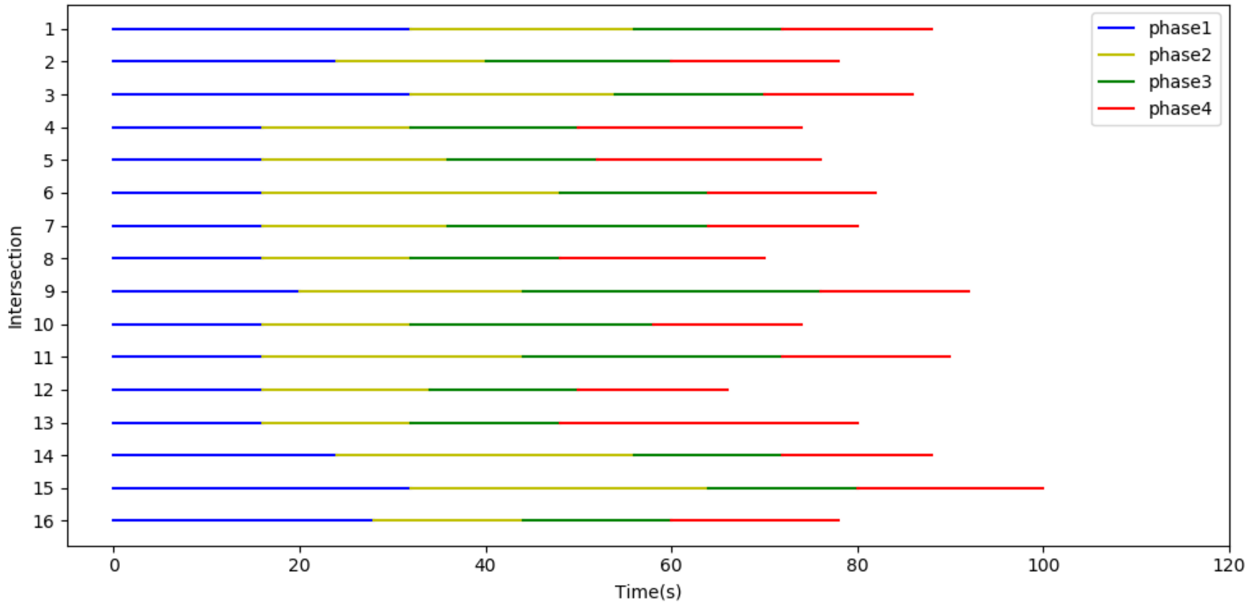
- Grid network size: 4 × 4, time horizon: 200s (50 time steps), number of scenarios: 10



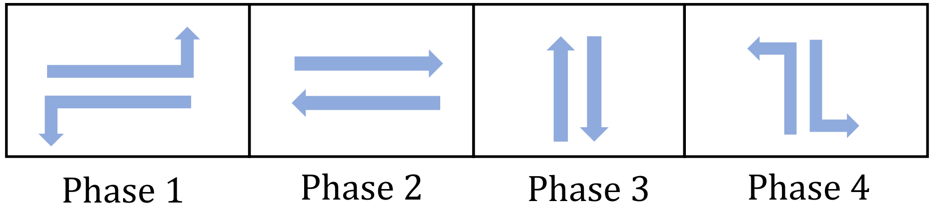
- Average time of solving master problem: 35s
- Average time of solving sub problem: 163s
- Average time of each iteration: 198s

Original Signal Control Plan Provided by Distributed Benders Cut with Cycle Estimation

- Each line is corresponding to an intersection.
- Different color segment indicates the time period of different phases.

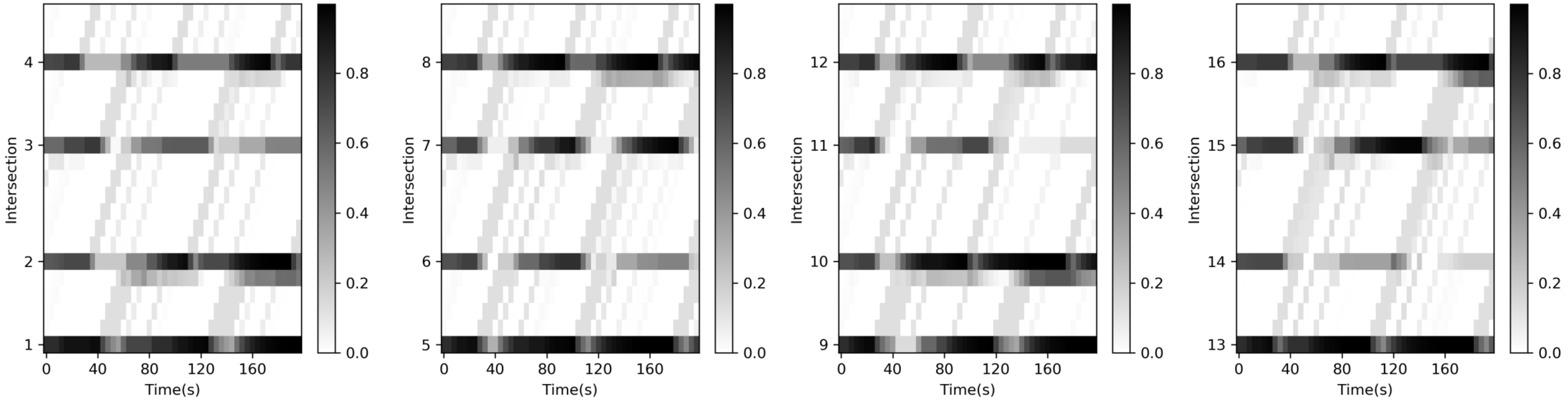


Signal control plan of intersections



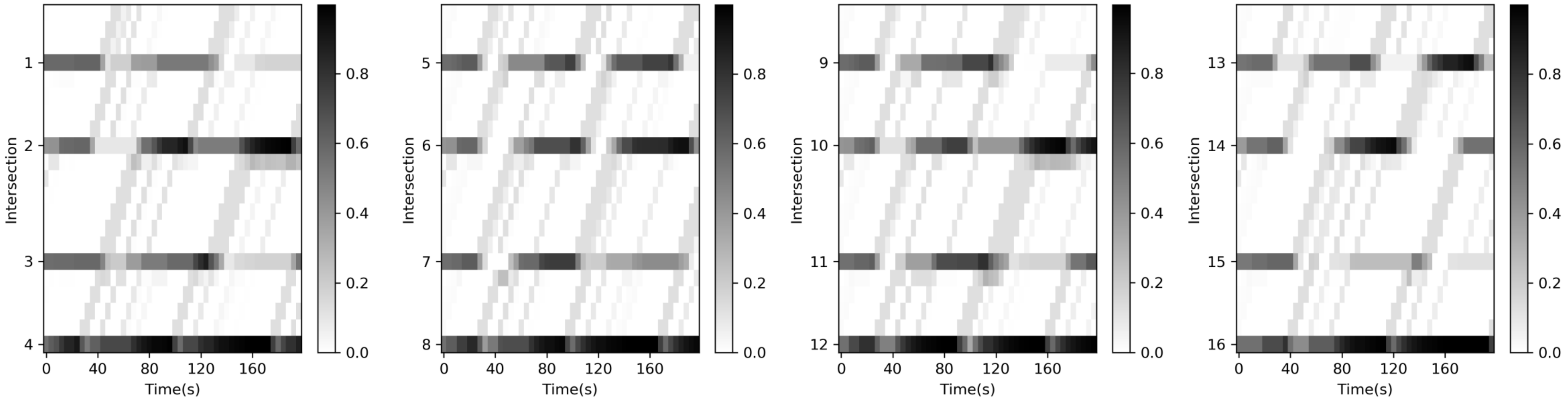
Ratio of Number of Vehicles and Jam Density in Cells Simulated Based on Original Signal Control Plan

- Each graph is corresponding to a corridor.
- The moving direction of corridor is from west to east.



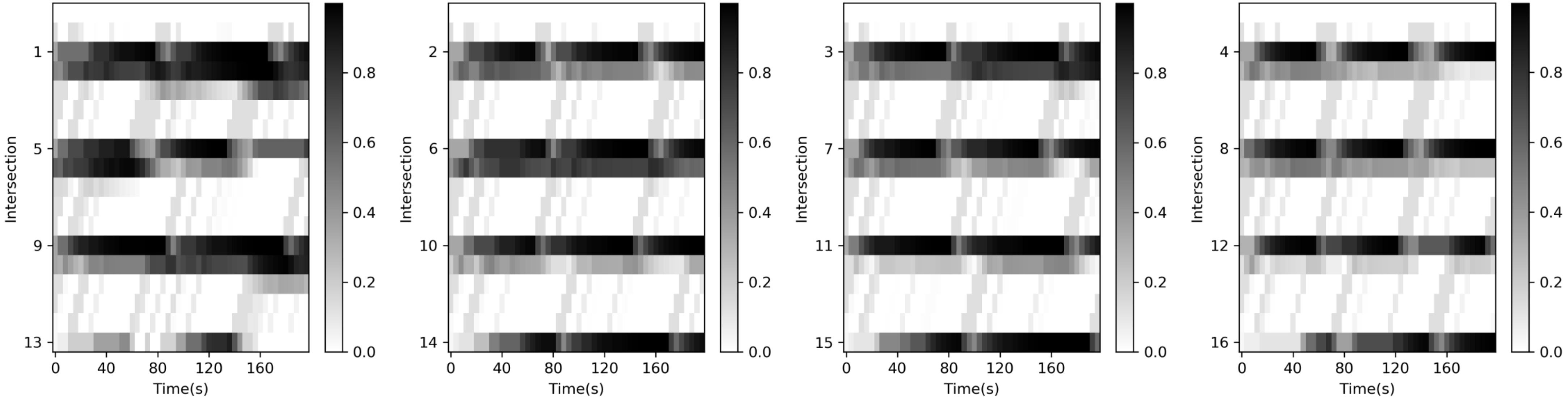
Ratio of Number of Vehicles and Jam Density in Cells Simulated Based on Original Signal Control Plan

- Each graph is corresponding to a corridor.
- The moving direction of corridor is from east to west.



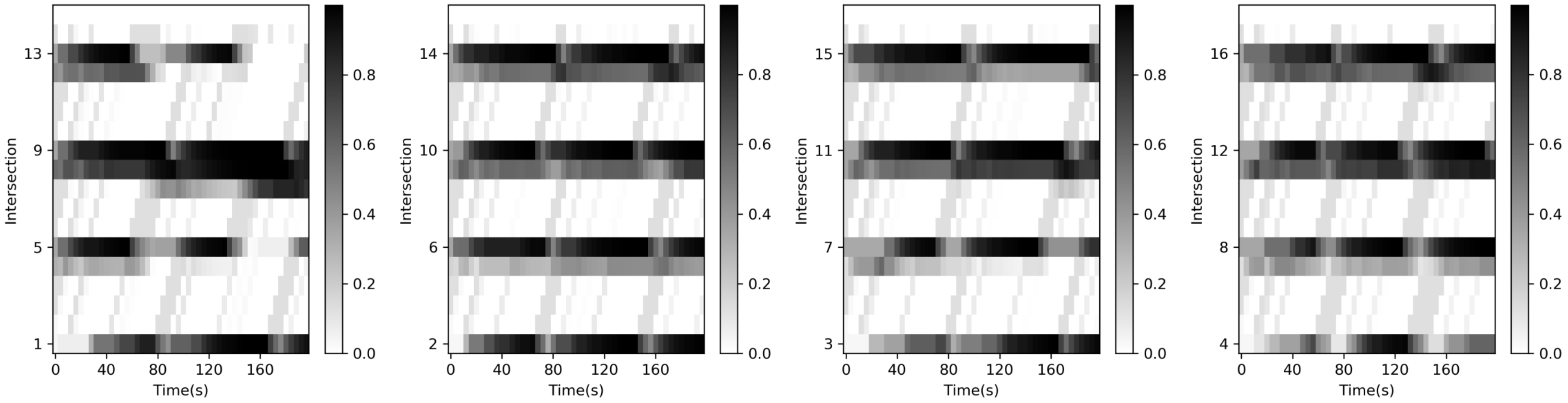
Ratio of Number of Vehicles and Jam Density in Cells Simulated Based on Original Signal Control Plan

- Each graph is corresponding to a corridor.
- The moving direction of corridor is from north to south.



Ratio of Number of Vehicles and Jam Density in Cells Simulated Based on Original Signal Control Plan

- Each graph is corresponding to a corridor.
- The moving direction of corridor is from south to north.



Future Work

- Apply model and algorithm to solve larger instance.
- Employ parallel computing to solve problem in each iteration.

Thank you!
Questions?