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Duality of Circulation Decay Statistics and Survival Probability

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Abstract

Survival probability and circulation decay history have both been used for setting wake turbulence separation standards. Conceptually a strong correlation should exist between these two characterizations of the vortex behavior, however, the literature lacks a rigorous development in establishing this correlation. The objective of this current paper is to establish the clear linkage between survival probability and circulation decay, and illustrate how they can be mapped to one another depending on need. The impact of the study is twofold. Firstly, it provides a mean to infer circulation decay probabilistic bounds from survival probability curves, which is particularly useful in providing insight on circulation statistics at old wake ages where direct circulation measurements are often challenging. This is referred to as the reverse process. Second, in analyses that require survival probability such as encounter probability analysis, this survival probability can be inferred from known statistical decay curves. This will be referred to as the forward process.

Nomenclature

UW	Upwind vortex
DW	Downwind vortex
A bold font symbol	indicates a random variable
'*'	indicates a non-dimensional quantity.
b_o	Initial vortex pair separation
T_o	Time for a the vortex pair to sink by a distance b_o
pdf	Probability Density Function
$\boldsymbol{\Gamma_o}$	Initial circulation (Random variable)
Γ_o	Mean of the initial circulation distribution
$\sigma_{\Gamma o}$	Standard deiviation of the initial circulation distribution
SP	Survival Probability
Δt	Time separation between a leader and a follower aircraft
$\boldsymbol{\Gamma_{th}}$	Maximum allowed Circulation value in the case case of an encounter
$\boldsymbol{t_a}$	Vortex age (Random Variable). Time at which circulation reaches Γ_{th}
μ_{ta}	Mean of the age distribution
σ_{ta}	Standard deviation of the age distribution
A	Slope of the circulation linear decay curve

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I. Background and Introduction

Aircraft wake vortices, the counter-rotating flow structures trailing behind an aircraft in flight - sometimes referred to as horizontal tornados, are an unavoidable consequence of lift generation. To ensure operational safety, separation distances between aircraft are set to permit vortices to dissipate or transport away from the flight path of the following aircraft. However, if the wake vortex separations between the lead generating aircraft and a follower aircraft are set too conservatively, runway and airport capacities are adversely impacted. The objective of the ongoing international wake turbulence program is to maintain safe operation while minimizing capacity constraining effects of wake turbulence separation standards.

Historically, both circulation decay distribution and survival probability (*SP*) characteristics of wake vortex measurements have been used for devising wake vortex separation standards on a relative safety basis. In terms of communicating with the aviation community and other stakeholders, it is more useful to think of circulation as the magnitude of the "torque" like quantity that can be imposed on a following aircraft encountering the wake of a preceding aircraft's wake, and $SP(t)$ as the probability that a vortex remains discernable above either a sensor sensitivity or safety related strength threshold at a time t. The more recent examples on how both concepts have been used in relative safety cases are the A380 and B757 studies. In the case of the A380, circulation decay of the A380 data were compared with that of a reference aircraft at various separation distances. In the case of the recent B757 study, *SP* curves of the two B752 variants as well as that of the B753 were compared to set a common separation standard for the three types of the B757 (Ref 1). The effort to more rigorously establish the linkage between *SP* and circulation decay was based on the need of doing it in the forward and reverse process.

a. Inferring *SP* from a statistical circulation decay model (Forward Process): There are times when *SP* by itself is the quantity of interest and not the circulation decay curve. This is true in the case of an encounter probability analysis where the *SP* is a needed quantity. The forward process uses a known statistical circulation decay model to synthesize the corresponding *SP* curve.

b. Inferring Circulation from *SP* (Reverse Process): Wake vortex circulation data are amongst some of the most demanding measurements, whereas *SP*, which is based on detecting a vortex signature only, is an easier and more reliable measurement. This is particularly true for old vortices with circulation values approaching the measurement uncertainty limit. This implies that *SP* is a more reliable quantity than direct circulation measurements at old wake age. However, if it can be shown that circulation information can be recovered from *SP*, it represents an approach to provide at least a consistency check on the circulation data, as well as offering the potential to help interpret historical *SP* based analyses.

II. Analysis and Results

The present paper starts with the forward process of going from a circulation decay distribution to *SP*, followed by the inverse process of illustrating how a *SP* could be mapped to a circulation decay distribution..Although both the forward and inverse processes can be formulated analytically, the forward process of mapping from a statistical circulation decay distribution to a corresponding *SP* is treated analytically, while the inverse process is illustrated by way of an example computationally via Monte Carlo simulation. The advantage of the analytical treatment is to clearly outline the link between the circulation decay statistics and *SP*, while a Monte Carlo implementation is easier to expand to more general distribution and arbitrarily more complex decay models.

A. Inferring *SP* from a statistical circulation decay model (Forward Process)

As mentioned in chapter one of this paper, for the wake encounter frequency analysis, it is the *SP* and not the circulation decay statistics that is of interest. This is because the probability of a follower aircraft encountering the wake of the leader is a function of this wake's survival probability at a time equal to the separation time Δt. In such case, if only the statistical distribution of circulation data is available, there is a need to establish the corresponding *SP*. The purpose of this section is to show how $SP(\Delta t)$ can be analytically synthesized in the case given a statistical linear circulation decay model with a deterministic slope A and a Gaussian distribution of the initial circulation $\boldsymbol{\Gamma_o}$.

Once again, survival probability $SP(\Delta t)$ is the probability of the age $\boldsymbol{t_a}$ exceeding a time Δt (usually a separation time). The wake age $\boldsymbol{t_a}$ is here defined as the time at which circulation reaches a particular threshold $\Gamma_{th.}$.

Mathematically stated,

$$SP(\Delta t) = P(\boldsymbol{t_a} > \Delta t) = \int_{\Delta t}^{\infty} pdf(\boldsymbol{t_a}).d\boldsymbol{t_a} \tag{1}$$

With $\boldsymbol{t_a}$ the time at which $\boldsymbol{\Gamma}(\boldsymbol{t_a}) = \Gamma_{th}$

This integral is graphically represented by the orange shaded area in Figure 1 . For a linear circulation decay, the wake age $\boldsymbol{t_a}$ is linearly related to this initial circulation as follows.

$$\Gamma_{th} = A.\boldsymbol{t_a} + \boldsymbol{\Gamma_o} \;\; => \;\; \boldsymbol{t_a} = \frac{(\Gamma_{th} - \boldsymbol{\Gamma_o})}{A} \tag{2}$$

Assuming the initial circulation to be normally distributed $\boldsymbol{\Gamma_o} \sim N(\Gamma_o, \sigma_{\Gamma o})$, and given the above linear relationship between $\boldsymbol{t_a}$ and $\boldsymbol{\Gamma_o}$, it can be easily shown that the wake age is also normally distributed $\boldsymbol{t_a} \sim N(\mu_{ta}, \sigma_{ta})$ with the following expressions for the mean and standard deviation.

$$\sigma_{ta} = \frac{\sigma_{\Gamma_o}}{A} \tag{3}$$

$$\mu_{ta} = \frac{(\Gamma_{th} - \Gamma_o)}{A} \tag{4}$$

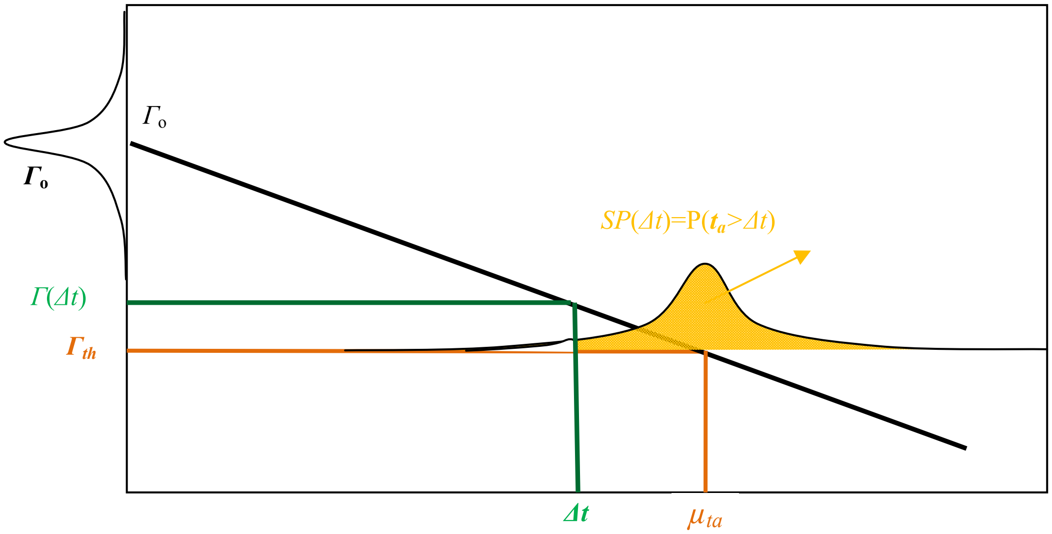

Figure 1: Graphical illustration of survival probability

Since the wake age *pdf* is Gaussian the survival probability can be calculated by making use of the Complementary Error Function *Erfc* as shown below.

$$SP(\Delta t) = P(\boldsymbol{t_a} > \Delta t) = 1 - \int_{-\infty}^{\Delta t} pdf(\boldsymbol{t_a}).d\boldsymbol{t_a} \; = 1 - \frac{1}{2}.Erfc\left(\frac{\Delta t - \mu_{ta}}{\sigma_{ta}}\right) \tag{5}$$

From the above equation, it is evident that $SP(\Delta t)$ can be determined for any separation time Δt once the age mean and standard deviation parameters (μ_{ta},σ_{ta}) are known. These parameters can be calculated using equations (3) and (4) once Γ_o, $\sigma_{\Gamma o}$, and A are found. Γ_o is known based on the aircraft type. However, the circulation distribution and decay curve are given in non-dimensional form. The only thing left is expressing the dimensional circulation $\sigma_{\Gamma o}$, and A in terms of their non-dimensional quantities.

The non dimensional circulation decay slope and initial circulation statistics are user input parameters, and are denoted by (Γ^*_o=1, $\sigma^*_{\Gamma o}$, A^*). As a side note, if Γ^*_o, $\sigma^*_{\Gamma o}$ are not explicitly provided and only the percentiles for the decay line are given (as is the case of Figure 2), then under Gaussian assumptions these parameters can be recovered as follows from the property of the Gaussian distribution.
Figure 1

The two random variables $\boldsymbol{\Gamma_o}$ and $\boldsymbol{\Gamma^*_o}$ are related linearly according to the following equation.

$$\boldsymbol{\Gamma_o} = \Gamma_o \;\; .\boldsymbol{\Gamma_o^*} \tag{6}$$

where Γ_o is the known mean initial circulation of a leader aircraft. Since $\boldsymbol{\Gamma^*_o}$ is normally distributed $\boldsymbol{\Gamma^*_o} \sim N(\Gamma^*_o=1, \sigma^*_{\Gamma o})$, $\boldsymbol{\Gamma_o}$ is also normally distributed $\boldsymbol{\Gamma_o} \sim N(\Gamma_o,\sigma_{\Gamma o})$, with the following value for the standard deviation.

$$\sigma_{\Gamma o} = \Gamma_o.\sigma^*_{\Gamma o} \tag{7}$$

The non-dimensional decay curve is given by the following equation.

$$\Gamma^*_{th} = A^*.\boldsymbol{t^*_a} + \boldsymbol{\Gamma^*_o} \tag{8}$$

In order to dimensionalize this decay curve for a particular leading aircraft, both sides of the equation should be multiplied by the nominal initial circulation Γ_o and using characteristic time T_o as follows.

$$\Gamma_o.\Gamma_{th}^* = A^*.\Gamma_o.\boldsymbol{t}_{\boldsymbol{a}}^* + \boldsymbol{\Gamma}_{\boldsymbol{o}}^*.\Gamma_o => \Gamma_{th} = A^*.\Gamma_o.\boldsymbol{t}_{\boldsymbol{a}}^*.\frac{T_o}{T_o} + \boldsymbol{\Gamma}_{\boldsymbol{o}}$$

$$=> \Gamma_{th} = \left(A^*.\frac{\Gamma_o}{T_o} \right).\boldsymbol{t}_{\boldsymbol{a}} + \boldsymbol{\Gamma}_{\boldsymbol{o}} \tag{9}$$

By comparing this equation to its dimensional counterpart in equation (2) the dimensional decay slope is found to be related to the non-dimensional slope according to the following equation.

$$A = A^*.\frac{\Gamma_o}{T_o} \tag{10}$$

The overall process processing from a circulation distribution to the *pdf* of wake demise, and then onto finally a *SP* is graphically illustrated below.

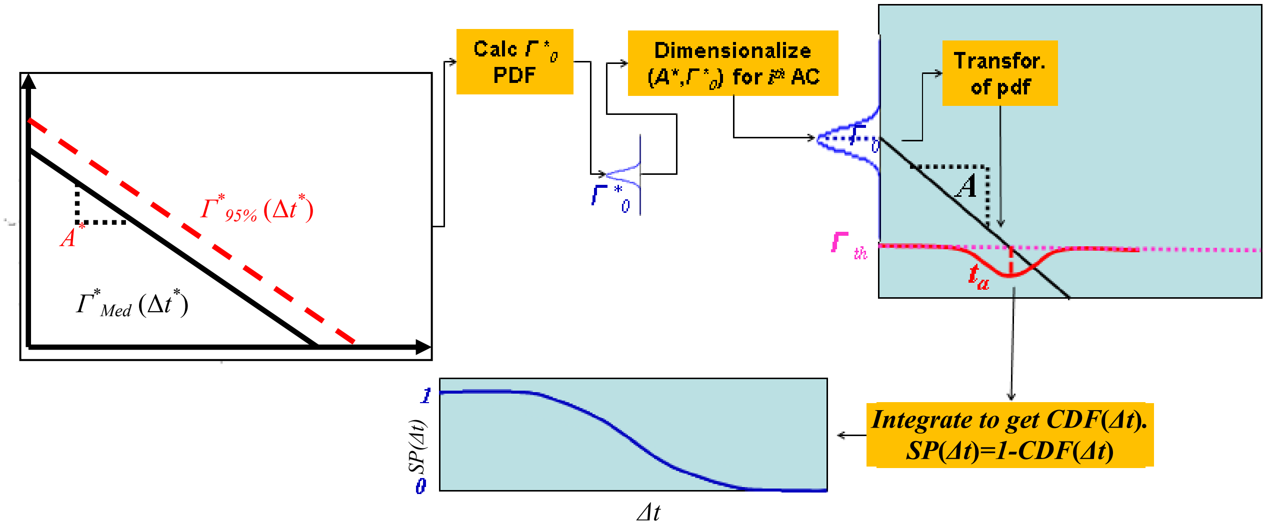

Figure 2: Block diagram outlining steps used to generate SP curve from circulation decay statistics

B. Inferring Circulation from *SP* (Reverse Process)

The formulation of the problem can be stated as follows. Given two different populations of wake vortices, there is a desire to statistically compare the associated wake circulations vs. time based on the *SP* curves representing each of the two populations. This section describes a Monte Carlo simulation approach to infer statistics on the circulation decay given the *SP* curve.

In each iteration an initial circulation value is randomly selected based on a known distribution of initial circulations. Also, a value for the vortex age is independently selected based on a known distribution of wake age. These selections correspond to two points to get an instance of the circulation decay curve. For a linear decay model these points completely determine the decay curve. This process is repeated N_{Sim} number of times resulting in a population of decay curves adhering to the given age and initial circulation distributions. A summary of this process is outlined in Figure 3. It should be noted that the general framework of the Monte Carlo simulation is not restricted to the linear decay assumption. For a general non-linear model additional information such as Eddy Dissipation Rate (EDR), and Stratification are needed to uniquely determine the corresponding instance of the decay curve.

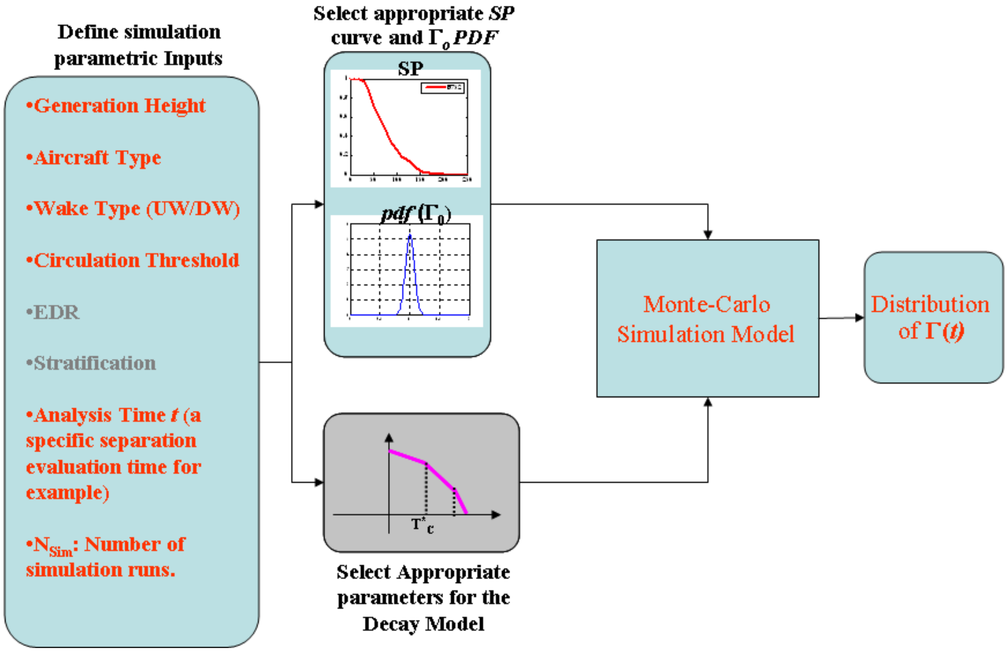

Figure 3: Block diagram illustrating Monte-Carlo simulation for a general circulation decay curve

However, this paper is mostly concerned with linear decay. This is the form of decay seen when wakes are near the ground. The process is outlined in more details for a linear decay assumption in Figure 4 below.

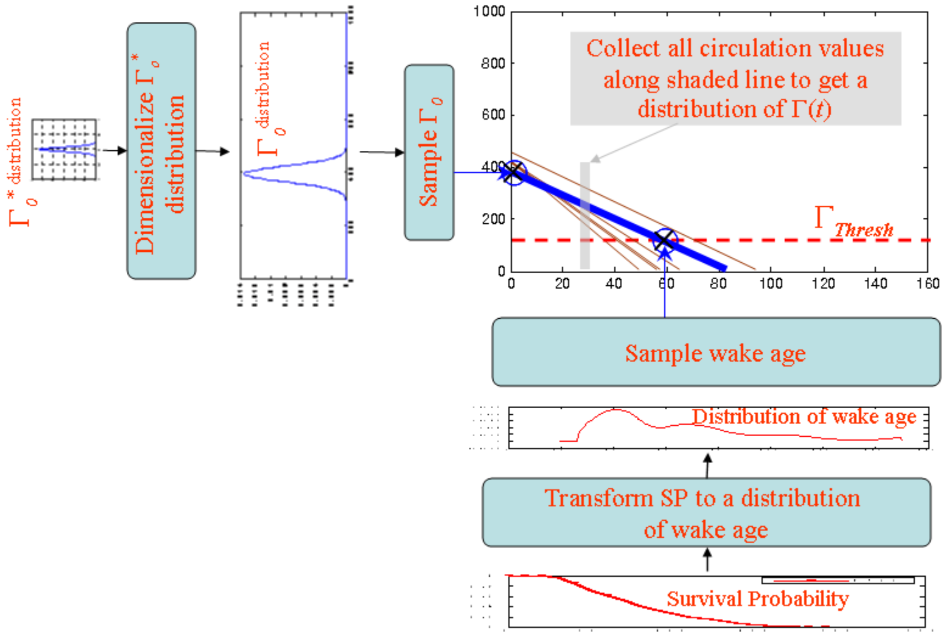

Figure 4: Details of the Monte Carlo simulation for the particular case of a linear decay.

As mentioned earlier in this section, this simulation is based on the assumption that the *pdf*s of initial circulation Γ_o and wake age t_a are both known.

The *pdf* of Γ_o: Since the magnitude of initial circulation is well above the measurement uncertainty, initial circulation measurements are reliable enough to estimate the parameters of the Γ_o distribution. This distribution is usually provided in non-dimensional form since it is built using data from range of airplane types. In the case of a non-dimensional Gaussian distribution the parameters of the dimensional distribution for the airplane of interest can be determined using equations (6) and (7) of the previous section.

The *pdf* of $\boldsymbol{t_a}$: Finally, it will be shown how the age distribution is obtained given the *SP* curve. Recall that *SP*(Δt) is the probability that a vortex remains discernable at time Δt .This definition is shown mathematically in equation (1) and is repeated here for clarity.

$$SP(\Delta t) = P(\boldsymbol{t_a} > \Delta t) = \int_{\Delta t}^{\infty} pdf(\boldsymbol{t_a}).d\boldsymbol{t_a}$$

Where *pdf* is the probability density function and t_a represents wake age. Therefore, *pdf*(t_a) is the derivative of *SP*(Δt) multiplied by (-1). This is found by interpolating the *SP* curve and numerically calculating the derivative.

The *SP* compiled from a near-ground-effect (NGE) dataset for three aircraft (respectively denoted as SP_A, SP_B and SP_C) are used to provide the sample calculations and their linkage to circulation distributions.
Note that for any wake age the difference between SP_A,SP_B is visually greater than or equal to that between SP_A and SP_C.. If the *SP* is a surrogate to circulation distribution, then the corresponding relative difference between aircraft A and B would then also be greater than that between aircraft A and C.

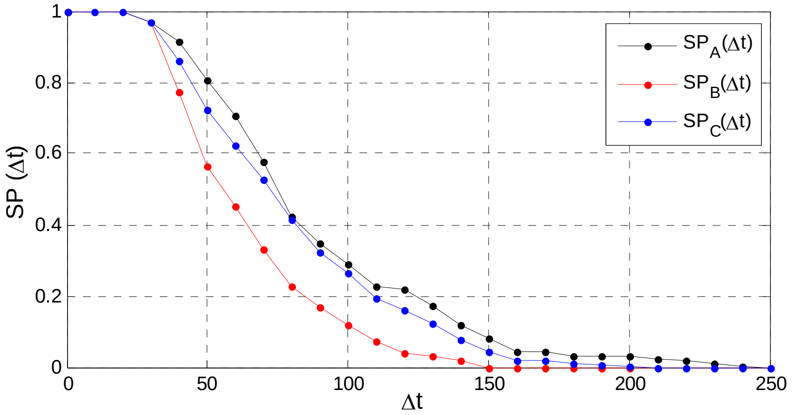

Figure 5: Overlaid Survival probability curves for aircraft A, B and C

In the Monte Carlo simulation results that follow, a linear circulation decay is assumed. The number of iterations is 10,000, and a 100 m^2/s circulation threshold value is used. In addition, the initial non-dimensional circulation distribution for each aircraft is assumed Gaussian, with a mean of unity and one-sigma of 0.075 as estimated from Ref 2. Figure 6 shows the result of the circulation decay statistics corresponding to the SP input of Figure 5. These results are presented in terms of circulation median, 10^{th}, and 90^{th} percentiles of the circulation data Aircrafts A, B and C with A treated as a reference Aircraft. All the circulations are normalized to the median initial circulation of the reference aircraft. For clarity, the simulation output is plotted only for the first 100 seconds.

Note that, as expected from observing the SP curves, there exists a better overlapping of the circulation distributions between Aircraft A and C over that of A and B. It is important to note that although the circulation decay is assumed linear, the percentiles become increasingly nonlinear at later vortex age. This is because at later ages only slowly decaying vortices are included in the percentile calculations since faster decaying vortices would have died by that time. So for example, the nonlinear shape of the median does not indicate that the vortices decay slower after a certain age but just that the slowly decaying vortices go into the calculation of the median at that age.

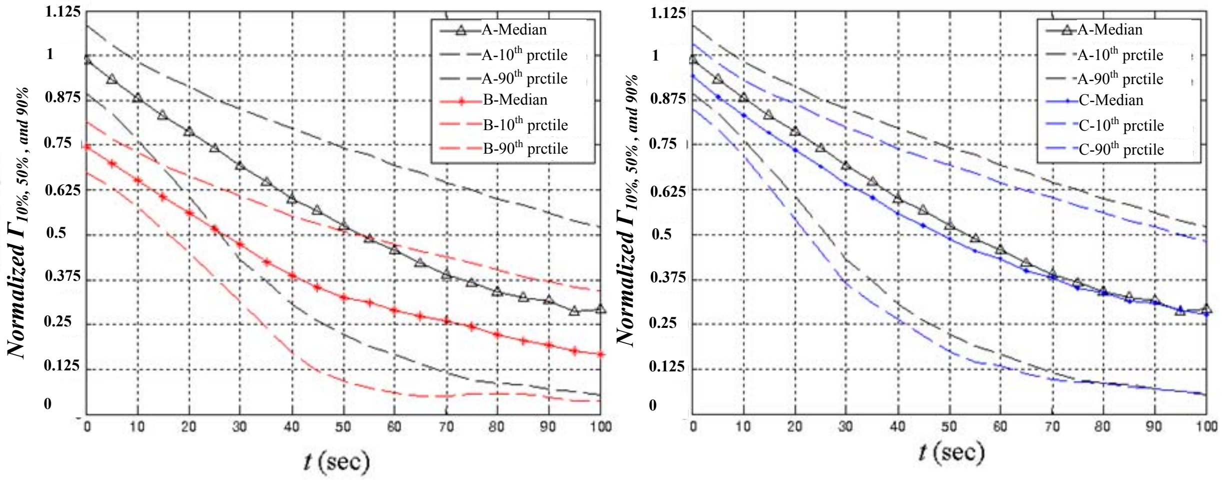

Figure 6: Results in terms of 10th, 50th and 90th percentiles of the circulation decay output of the Monte Carlo simulation for the three aircraft with A treated as the reference aircraft. Therefore all circulation values are normalized too the initial circulation median of aircraft A.

Finally, it is important to note also that if a conditional SP were constructed from data based on additional factors such as a protection corridor, a time window or additional meteorological parameters, the general process described could allow the determination circulation decay statistics tailored to a particular operational concept.

III. Conclusion

The linkage between survival probability and circulation decay statistics has been established analytically and numerically using Monte Carlo simulations.. By way of an example calculation, it is illustrated that a larger *SP* difference between data from two aircraft at a given time maps directly to a corresponding larger difference in the circulation statistics. Hence, *SP* is a surrogate to circulation distribution. However, if there is a need to find a maximum encounter probability, it can be obtained via circulation decay distribution. Finally, there exists a large amount of analysis literature based on *SP*, and the present paper provides a framework to gain further insight in those studies (Refs 3-6)

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