DOT/FAA/TC-16/1

Federal Aviation Administration
William J. Hughes Technical Center
Aviation Research Division
Atlantic City International Airport
New Jersey 08405

Efficient and Accurate Computational Methods and Tools for Damage-Tolerance-Based Aircraft Reliability Assessment

March 2017

Final Report

U.S. Department of Transportation
Federal Aviation Administration

NOTICE

Technical Report Documentation Page

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
DOT/FAA/TC-16/1		

4. Title and Subtitle

EFFICIENT AND ACCURATE COMPUTATIONAL METHODS AND TOOLS FOR DAMAGE-TOLERANCE-BASED AIRCRAFT RELIABILITY ASSESSMENT

5. Report Date

March 2017

6. Performing Organization Code

7. Author(s)

Wu, Y.-T. and Wu, H.E.

8. Performing Organization Report No.

9. Performing Organization Name and Address

CompRel, Inc.,
1412 Whittington Dr.
Raleigh, NC

10. Work Unit No. (TRAIS)

11. Contract or Grant No.

FAA DTFACT-14-C-00001

12. Sponsoring Agency Name and Address

U.S. Department of Transportation
Federal Aviation Administration
William J. Hughes Technical Center
Aviation Research Division
Atlantic City International Airport, NJ 08405

13. Type of Report and Period Covered

Final Report
November 2013 – May 2015

14. Sponsoring Agency Code

ACE-113

15. Supplementary Notes

The Federal Aviation Administration Aviation William J. Hughes Technical Center Research Division COR was Dr. Sohrob Mottaghi.

16. Abstract

This project is part of an overall effort to apply a Damage Tolerance (DT) methodology to manage aircraft structural reliability. The DT methodology accepts the existence of initial small flaws and incorporates inspection and subsequent risk mitigation strategies to sustain structural reliability and safety. Earlier research addressed the important, yet often overlooked, technical issue—to rigorously model statistically correlated strengths in a time-dependent reliability analysis. Ignoring these correlations can lead to significant errors. When facing correlated strengths with stochastic loads, the standard Monte Carlo (MC) simulation approach was the only reliable, yet often highly inefficient, method available. To overcome the issue, a Strength-Conditioned Importance Sampling (SCIS) method for aircraft structural reliability was developed. The improved efficiency can be attributed to combining the conditional expectation method with a tailored importance sampling methodology. The earlier study suggested that only a few thousand SCIS-based samples could produce results with accuracies comparable to using MC with more than 1.e+06 samples.

This project focused on solving challenging issues associated with computing small (<<0.001) probabilities of failure and developing efficient, accurate, and robust methods. The theory of SCIS has been studied and tested thoroughly and a computer application called FlyRisk has been developed. Effective importance-sampling density functions have been proposed, implemented, and demonstrated successfully using a fracture mechanics model.

Using the SCIS samples, a kernel density estimator was combined with a fast probability integration algorithm to compute the single-flight probability of failure. This unique approach allowed a more compact and completely standalone FlyRisk code to be developed.

This project included a supporting task. This entailed developing a Bayesian parameter updating tool aimed at mitigating the lack-of-data issue by leveraging inspection results. The likelihood functions were modeled on the DT methodology and probability of detection function. The missed-inspection outcomes can be included, which in turn can significantly improve the effectiveness of the Bayesian approach.

17. Key Words	18. Distribution Statement
Damage tolerance, Importance sampling, Strength conditioned importance sampling, Markov-chain Monte Carlo, Inspection planning, Damage tolerance, Fatigue, Fracture mechanics, Risk-based maintenance optimization, Structural reliability, Reliability methods	This document is available to the U.S. public through the National Technical Information Service (NTIS), Springfield, Virginia 22161. This document is also available from the Federal Aviation Administration William J. Hughes Technical Center at actlibrary.tc.faa.gov.

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	37	

Form DOT F 1700.7 (8-72) Reproduction of completed page authorized

ACKNOWLEDGEMENTS

The reported work has extended and improved the approach originally developed in an earlier project, “Damage Tolerance Based Maintenance Planning of Aircraft Structures Subjected to Stochastic Process Random Effects,” conducted from September 2009 to April 2011. The earlier ideas and results were reported in a paper entitled “Strength-Conditioned Expectation Method for Aircraft Structural Reliability Analysis,” presented at the 52nd AIAA SDM conference in April 2011. During the first year of that project, the principal author worked for Applied Research Associates (ARA), Inc. The support by ARA is acknowledged. The Federal Aviation Administration technical monitors were Dr. M. Shiao (first year) and Dr. F.L. Abali. Their programmatic and technical assistance is gratefully acknowledged.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF ACRONYMS AND SYMBOLS

CDF	Cumulative Distribution Function
DT	Damage Tolerance
EIFS	Equivalent Initial Flaw Size
IS	Importance Sampling
KDE	Kernel Density Estimator
MC	Monte Carlo
MCMC	Markov-chain Monte Carlo
PDF	Probability Density Function
PDTA	Probabilistic Damage Tolerance Analysis
POD	Probability of Detection
POF	Probability of Failure
RBMO	Risk or Reliability-Based Maintenance Optimization
RV	Random Variable
SCEM	Strength-Conditioned Expectation Method
SCIS	Strength-Conditioned Importance Sampling
$f(x)$	PDF of random variable vector $\boldsymbol{X}$
$p(t)$	(Cumulative and unconditional) probability of failure
$p_c(t)$	(Cumulative) conditional probability of failure
$p_c(t \mid x)$	Strength-conditioned probability of failure
$p_{cs}(t)$	Conditional single-flight probability of failure
$p_{cs}(t \mid x)$	Strength-conditioned single-flight probability of failure
$p_s(t)$	Single-flight probability of failure
$R(t)$	(Cumulative and unconditional) reliability
$R_c(t)$	(Cumulative) conditional reliability
$R_c(t \mid x)$	Strength-conditioned reliability
$R_{cs}(t)$	Conditional single-flight reliability
$R_{cs}(t \mid x)$	Strength-conditioned single-flight reliability
$R_s(t)$	Single-flight reliability
ρ	Statistical correlation coefficient
X	Strength, a function of strength random variables, x
x	Strength random variables
Y	Stress variable

EXECUTIVE SUMMARY

This project is primarily based on Federal Aviation Administration (FAA) project 09-G-015, titled "Damage Tolerance Based Maintenance Planning of Aircraft Structures Subjected to Stochastic Process Random Effects," which was conducted between September 2009 and April 2011. That project focused on managing aircraft structural reliability by using the Damage Tolerance (DT) methodology combined with nondestructive inspections, and addressed an important, yet often overlooked, technical issue—to rigorously model statistically correlated strengths in a time-dependent reliability analysis. It showed that the conventional approximate approach might lead to significant errors. It also presented, for the first time, a Strength-Conditioned Importance Sampling (SCIS) method tailored to the analysis of time-dependent aircraft structural reliability. When facing correlated strengths with stochastic loads, the standard Monte Carlo method is commonly considered the only reliable solution method—but it is also known to be inefficient, especially for analyzing very small probabilities of failure. The increased efficiency of SCIS can be attributed to the conditional expectation method framework and importance sampling. The latter requires a fast sample generator such as a Markov-chain Monte Carlo (MCMC) algorithm. The earlier study suggested that only a few thousand SCIS-based MCMC samples could produce results with accuracies comparable to using MC with more than 1.e+06 samples.

In addition to addressing correlated strengths, another advantage of the SCIS framework is to cover a wide range of time-dependent strength scenarios, such as damage degradation, aging, and repair/replacement, after inspections. However, for sharp, step-function-like, strength-shifting events, such as replacements, the performance of SCIS was not well understood. Therefore, in this project, the theory of the SCIS approach has been studied more thoroughly and a computer tool, FlyRisk, has been developed. More specifically, a more effective importance sampling density function has been investigated and implemented in FlyRisk. Although the SCIS framework is capable of handling different degrees of repair qualities, for practical purposes, the FlyRisk assumed a maintenance policy that detected damaged parts are to be replaced with original parts or those repaired to "as new" condition. Other maintenance policies can be added to FlyRisk if needed.

During the development of SCIS and FlyRisk, several challenging issues associated with calculating small probabilities of failure (less than 1.e-3) were addressed. The solutions are summarized below:

- Three importance sampling density functions, denoted as h_1, h_2, and h^* in this report, were investigated and compared. Although, in theory, h^* seemed to offer the optimal performance, in practice, it was the most complicated in terms of implementation and validation. Consequently, h_2 was selected for FlyRisk implementation because of its balanced performance in speed, accuracy, and robustness. To achieve the potentially highest performance, more research will be needed to study and implement h^*.
- Using the SCIS samples, a kernel density estimator was combined with a fast probability integration scheme to compute the single-flight probability of failure. This unique approach relieves the MATLAB FlyRisk code from requiring separate reliability analysis software to estimate one of the key probability integrals in the SCIS formulation. As a result, a more compact and completely standalone FlyRisk code has been developed.

- A flexible analytical reliability function was applied to fit the reliability history for each strength realization. To compute the cumulative reliability for a large number of flight numbers over the service life, the fitted function can typically improve the reliability calculation speed by more than two orders of magnitude

This project includes a relevant supporting task, which is to develop a Bayesian parameter updating tool. The objective of the Bayesian project sought to mitigate the lack-of-data issue by leveraging inspection results. Applying Bayes' theorem to update distribution parameters is a well-used approach, but the drawback is that it may need a substantial amount of inspection data. Accordingly, likelihood functions were created based on the DT methodology and probability of detection function. Furthermore, the missed (negative results) inspection records could be included as a part of the likelihood function. The result of the task is a standalone Bayesian updating software tool.

The deliverables of this project include the final report, FlyRisk software with user's guide, and the Bayesian updating software with user's guide.

1. INTRODUCTION

This project is a continuation of earlier efforts supported by the FAA which focused on managing aircraft structural reliability by using the Damage Tolerance (DT) methodology combined with nondestructive inspections. In those efforts, the concept of Strength-Conditioned Importance Sampling (SCIS) was introduced and a Bayesian approach was developed. The reader is strongly encouraged to review [1–3] prior to this report.

During the last several decades, several major aircraft structural failures could be traced to rare material manufacturing defects. To help prevent such high-consequence failures, significant progress has been made in developing and applying DT design methodology, with an emphasis on fatigue and fracture failure modes associated with initial flaws [4–6]. The DT methodology accepts the possible existence of flaws in structures and incorporates nondestructive inspections and subsequent risk-mitigation strategies to sustain structural reliability and safety.

Other sources of structural failures that have received less attention are environmental effects (e.g., from moisture and temperature variations) and impact damage that can significantly reduce structural strength over time. If the damage or degradation are not detected and fixed in a timely fashion, they can potentially cause unexpected structural failures.

The goal of the research was to develop a Probabilistic Damage Tolerance Analysis (PDTA) methodology and analysis tools specifically for time-dependent reliability models which involve various strength, random variables, and stochastic loading processes. Figure 1 shows the concept of a time-dependent reliability model for which the reliability is governed by the degree of overlapping in the applied load and the remaining strength, which deteriorates as the result of the growth of defects or other damages. In general, the rate of strength reduction and the rate of failure might be accelerated because of aging and environmental factors.

Walk-around inspections using specialized nondestructive evaluation devices are routinely performed to detect apparent damage; further, inspections are scheduled because smaller or hidden damage cannot be detected. After a defect or damage has been detected and fixed (repaired or replaced), the residual strength of the structure will have changed, sometimes drastically. Therefore, in building a reliability model, the quality of the fixed parts should be incorporated in the modeling of residual strength. With a proper reliability model, the inspection schedules can be optimized by managing the risk, subject to reliability, operational, and other constraints. For example, the next inspections could be based on detecting a certain damage size with a high probability of detection but before the Probability of Failure (POF) becomes unacceptable. Several Risk or Reliability-Based Maintenance Optimization (RBMO) approaches have been proposed in recent years [7–9], but these approaches did not specifically consider or address the effect of strength correlation for time-dependent reliability analysis.

With PDTA, the sources of uncertainty include systematic and random errors of the failure models, applied loads, material properties, geometries, environmental factors, defect and damage occurrence rates, and detection capability. As shown in figure 1, aircraft structural reliability models can involve strength-related Random Variables (RVs) and stress random processes. While varying among different aircraft in a fleet, the strength-related RVs, such as the fracture toughness and initial flaw size, at a certain location are essentially time independent for individual aircraft. Therefore, the single-flight reliabilities between the flights are correlated because of strength-related random variables. These correlations often make accurately computing the interval reliability difficult and time-consuming. With current PDTA practices, either the correlations are ignored or the approximate methods, such as the reliability bounds, have to be used.

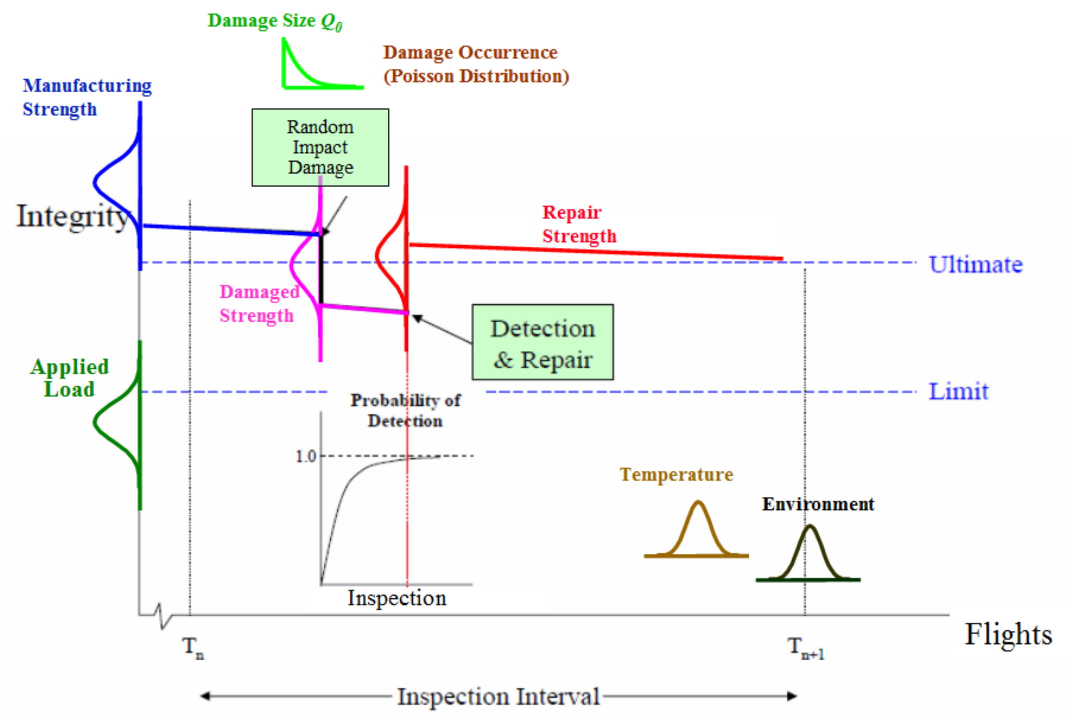

Figure 1. Time-dependent reliability model [1]

To accurately and efficiently compute time-dependent reliability and conduct RBMO, this project has developed an accurate and efficient method called Strength-Conditioned Importance Sampling (SCIS). SCIS integrates the Strength-Conditioned Expectation Method (SCEM) with a tailored Importance Sampling (IS) methodology.

Through the conditioning of the realizations of strength variables, the flight-to-flight reliabilities become independent, which allows the interval reliabilities to be computed accurately. In addition, the efficiency issue is overcome by using the Markov-chain Monte Carlo (MCMC) algorithm to generate IS samples in the failure domain. In the standard Monte Carlo (MC) approach, the required number of samples depend strongly on 1/POF, but the required number of MCMC samples to achieve the same accuracy is independent of POF. Therefore, the MCMC-based IS approach is well suited for structural reliability analysis. For very small failure probability (e.g., POF < 1.e-05), the SCIS approach can drastically reduce the number of random samples needed for MC to achieve a high level of accuracy. Furthermore, for risk-optimization purposes, the samples can be reused to compute reliabilities for various maintenance plans. The benefit of applying MCMC to RBMO problems are outlined in [10, 11].

This report documents the updated SCIS method and presents a demonstration example using a fracture mechanics model. The SCIS approach has been implemented in FlyRisk for the demonstration. The SCIS reliabilities for both the “with” and “without” inspection cases are computed and validated using SCEM MC with large sample sizes.

2. OVERVIEW OF TIME-DEPENDENT RELIABILITY MODEL

The basic assumption for the proposed methodology is that the initial structural strength, *X*, is a function of time-independent random variables, *x*, and the residual strength may change over time, *t*, because of loading (random and impact) and environmental (temperature, moisture, etc.) effects, until the structure has survived the service life, been repaired/replaced, or failed. Because the duration of each flight is relatively short compared to the structural life, *X* is assumed to be constant during each single flight. However, the applied load is a stochastic process that can vary significantly during each flight. For this study, the maximum load during any flight is characterized using a random variable, and the maximum loads are assumed to be independent between the flights.

Based on the above assumptions, a structural failure will occur if, during a flight, *i*, the strength is exceeded by the maximum of the stress, $S_i(t)$, denoted as $Y(t_i) = \arg\max_t S_i(t)$. Therefore, the single-flight probability, *P*, of survival, or reliability, for the *i*-th flight is:

$$R_s(t_i) = P[X(t_i) > Y(t_i)] \tag{1}$$

where the subscript “s” denotes “single flight.” The cumulative reliability for the duration of $t = t_1$ to t_N, denoted as $R(.)$ without a subscript, is the reliability for *N* flights and can be formulated as:

$$R(t_N) = P\left\{[X(t_1) > Y_1(t_1)] \cap [X(t_2) > Y_2(t_2)] ... \cap [X(t_N) > Y(t_N)]\right\} \tag{2}$$

By denoting $E_i = X(t_i) > Y(t_i)$, equation 2 is abbreviated as:

$$R(t_N) = P\left\{E_1 \cap E_2 ... \cap E_N\right\} \tag{3}$$

Because of the statistical correlations between $X(t_i)$'s and E_i's that result in complex joint probability density functions, an analytical solution for the cumulative reliability is generally unavailable. There are several solutions to compute $R(t_N)$, including the bounding approach [12], which provides approximate solutions, and the MC approach, which is usually time-consuming. In the following sections, the limitations of the bounding approach will be discussed and an alternative random sampling approach that is highly efficient relative to the MC approach presented.

For the remainder of this report, “cumulative reliability” will be called “reliability” and the cumulative probability of failure will be termed POF or failure probability.

2.1 RELIABILITY BOUNDS

The statistical correlation coefficient between any two events $E_i = X(t_i) - Y(t_i)$ and $E_j = X(t_j) - Y(t_j)$ is:

$$\rho_{E_iE_j} = \frac{E\left[E_iE_j\right] - E\left[E_i\right]E\left[E_j\right]}{\sigma_{E_i}\sigma_{E_i}} \tag{4}$$

Assuming $Y(t_i)$ are statistically independent and identically distributed, equation 4 leads to

$$\begin{aligned}
\rho_{E_iE_j} &= \frac{E\left[E_iE_j\right] - E\left[E_i\right]E\left[E_j\right]}{\sigma_{E_i}\sigma_{E_i}} \\
&= \frac{E\left[X_iX_j - X_iY_j - X_jY_i + Y_iY_j\right] - \left[E[X_i] - E[Y_i]\right]\left[E[X_i] - E[Y_i]\right]}{\sigma_{E_i}\sigma_{E_j}} \\
&= \frac{\left\{E\left[X_iX_j\right] - E\left[X_i\right]\cdot E\left[X_j\right]\right\} + E\left[Y_iY_j\right] - E\left[Y_i\right]\cdot E\left[Y_j\right]}{\sigma_{E_i}\sigma_{E_j}} \\
&= \frac{\rho_{X_iX_j}\sigma_{X_i}\sigma_{X_j} + 0}{\sigma_{E_i}\sigma_{E_j}}
\end{aligned} \tag{5}$$

in which σ is the standard deviation. Because a realization of $\boldsymbol{X}$ that results in a relatively higher/lower $X(t_i)$ will likely result in a higher/lower $X(t_j)$, the physical correlation coefficient $\rho_{X_iX_j}$ is expected to be positive and therefore $\rho_{E_iE_j} > 0$. The positive correlation leads to the following unimodal reliability bounds [12]:

$$\prod_{i=1}^{N} R_s(t_i) \le R(t_N) \le \min_i R_s(t_i) \tag{6}$$

Assuming no maintenance is performed, reliability should monotonically decrease because of strength deterioration. Therefore, the last single-flight reliability is the reliability upper bound, (i.e., $\min_i R_s(t_i) = R(t_N)$). The lower bound of the reliability, or the upper bound of the POF, is used for a conservative aircraft risk assessment.

Expressed in terms of probabilities of failure using the relation $p_s(t_i) = 1 - R_s(t_i)$, equation 6 can be converted to:

$$\max_i p_s(t_i) \le p(t_N) \le 1 - \prod_{i=1}^{N}(1 - p_s(t_i)) \tag{7}$$

For small $p_s(t_i)$, typical for aircraft, the right side of equation 7 is [12]:

$$1-\prod_{i=1}^{N}(1-p_s(t_i)) \approx \sum_{i=1}^{N} p_s(t_i) \tag{8}$$

Therefore, the upper bound of the POF, which is conservative, is approximately the sum of the single-flight POFs, $\sum p_s(t_i)$.

For example, consider a special case in which there is no strength deduction and the single-flight probability remains constant, $p_s(t_i)=\lambda$. The corresponding bounds are:

$$p(t_i) \le p(t_N) \le N \cdot p(t_i) \tag{9}$$

According to exponential failure law, for a constant failure rate (also called the up-crossing rate), λ, the time to failure has an exponential distribution and the POF is:

$$p_{_{Exact}}(t_N) = 1-e^{-\lambda t_N} \tag{10}$$

For example, consider N = 20,000 flights with a small single-flight failure rate of $\lambda = 1.0e-8$. The exact POF is $p(t_N) = 1.9998e-4$ with the bounds:

$$1.e-8 \le p(t_N) \le 2.\text{e-4} \tag{11}$$

which shows that the upper bound, more important for aircraft risk assessment, is excellent, but the lower bound has a large error.

Even if the single-flight POF is two orders of magnitude higher than the previous example, the exact interval POF is $p(t_N) = 1.98e-2$ and the bounds are:

$$1.e-6 \le p(t_N) \le 2.\text{e-2} \tag{12}$$

which shows that the upper bound is only 1% greater than the exact. For aircraft applications, the previous cases suggest that the upper bound solution $\sum p(t_i)$ can be a good approximation if there is no significant strength deterioration during the flight interval of interest. However, for the case with a decreasing strength, as in fatigue crack growth, the single-flight POF should be increasing:

$$p_s(t_N) < p_s(t_2) \ldots < p_s(t_N) \tag{13}$$

Therefore, equation 8 becomes:

$$p_s(t_N) < p(t_N) < Np_s(t_N) \tag{14}$$

When the service life approaches the wear-out stage, the $p_s(t_i)$ may increase sharply. As a result, the value of $\sum p_s(t_i)$ may be dominated by the later flights, and the bounds can be expressed as:

$$p_s(t_N) < \sum_{i=1}^{N} p_s(t_i) \ll N p_s(t_N) \tag{15}$$

Put another way, the upper bound $Np_s(t_N)$ may be too conservative.

For aircraft risk assessment, the upper bound can provide a quick estimate, but the bound may be too conservative for decreasing strengths. The strength-conditioned method described in figure 2 is computationally efficient and based on an exact formulation without simplifying assumptions.

3. STRENGTH-CONDITIONED EXPECTATION METHOD (SCEM)

The foundation of SCIS is the Conditional Expectation Method, which is a simulation method designed to estimate failure probability by averaging samples of conditioned failure probabilities. Normally, a subset of the included random variables with higher contributions to the failure probability are selected as conditioned (or controlled) variables for generating random samples [13, 14]. The SCEM always selects all strength random variables, *X*, as the conditioned variables set up independencies between the conditional E_i's and provide the opportunity for accurate reliability calculations. By conditioning on *X*, the conditional reliability, denoted as $R_c(t_i \mid x)$, can be approximated by generating random samples and taking the sampling average. Figure 2 shows the concept of SCEM for aircraft reliability analysis. It shows that each strength realization changed over time, but the stress remains a stationary random process. In this example, the conditioned reliabilities stay near 100% until after approximately 16,000 flight hours, when the random stress begins to exceed the strengths with significant probabilities. Afterwards, the conditional reliabilities rapidly drop to less than 50%.

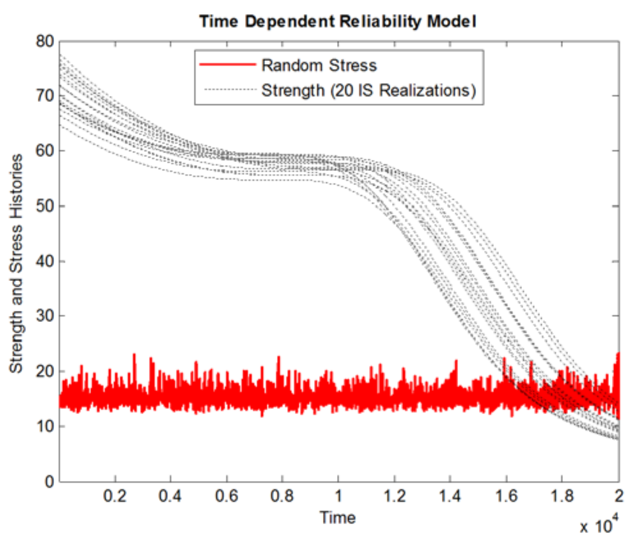

Figure 2. Concept of strength-conditioned method

For each j-th realization of $\boldsymbol{X}$, the $Y(t_i)s$ are independent. Therefore, the conditional failure events are also independent, which leads to:

$$\begin{aligned} R_c(t_N \mid x_j) &= P[(x_j(t_1) > Y(t_1)) \cap \ldots \cap (x_j(t_N) > Y(t_N))] \\ &= \prod_{i=1}^{N} P[(x_j(t_i) > Y(t_i)] \\ &= \prod_{i=1}^{N} R_{cs}(t_i \mid x_j) \end{aligned} \tag{16}$$

where $R_{cs}(t_i \mid x)$ is the conditional single-flight reliability, which is easier to compute than the unconditional single-flight reliability $R_s(t_i)$.

By integrating over the entire support of X, the unconditional reliability, or simply the reliability, is:

$$R(t_N) = \int R_c(t_N \mid \boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \tag{17}$$

which can be expressed in terms of the expectation function and estimated using the sampling average:

$$R(t_N) = E[R_c(t_N \mid \boldsymbol{x})]_{f_X} \approx \frac{1}{K} \sum_{j=1}^{K} \left[\prod_{i=1}^{N} R_c(t_i \mid \boldsymbol{x}_j) \right] \tag{18}$$

where K is the number of realizations based on the Probability Density Function (PDF) of $\boldsymbol{X}$. Finally, the POF is computed as $p(t_N) = 1 - R(t_N)$.

Similar to equation 18, the single-flight POF can be estimated using a sampling average of:

$$p_s(t_N) = E[p_{cs}(t_N \mid \boldsymbol{x})]_{f(\boldsymbol{x})} \approx \frac{1}{K} \sum_{j=1}^{K} p_{cs}(t_N \mid \boldsymbol{x}_j) \tag{19}$$

In this project, SCEM is used to provide “exact” solutions to validate the SCIS solution method.

4. SCIS

When the SCEM method is used, the strength realizations are generated by a random sampling algorithm based on the complete joint PDF of the strength variables, $f_{\boldsymbol{X}}(\boldsymbol{x})$. The realizations can be more effectively generated by the SCIS method using a selected importance sampling PDF, $h(\boldsymbol{x})$, which covers a very small, but important, part of $f_{\boldsymbol{X}}(\boldsymbol{x})$. When $h(\boldsymbol{x})$ is properly selected, the efficiency of SCIS, with respect to SCEM, can easily exceed two or more orders of magnitude for small POF (< 1.e-03).

SCIS is efficient because it generates and uses only the samples of $\boldsymbol{X}$ in a drastically reduced sampling region, Ω, that covers the domain of the events for which $P_{S_j}(t_n)$ has a relatively significant value. Using SCIS, equation 19 is approximated by:

$$\begin{aligned} p_S(t_N) &= p_\Omega \cdot p_{S|\Omega}(t_N) + p_{\overline{\Omega}} \cdot p_{S|\overline{\Omega}}(t_N) \\ &\approx p_\Omega \frac{1}{K_\Omega} \sum_{j=1}^{K_\Omega} \prod_{i=1}^{N} P_{S_j}(t_i) + p_{\overline{\Omega}} \cdot 0 \\ &\approx p_\Omega \cdot p_{S|\Omega}(t_N) \end{aligned} \tag{20}$$

in which P_Ω is the probability in Ω, $\Omega \cap \overline{\Omega} = \varnothing$, and $P_{\overline{\Omega}} = 1 - P_\Omega$, and $P_{S|\Omega}(t_N)$ is the conditional single-flight POF in Ω.

When an importance sampling density function, $h(\boldsymbol{x})$, is properly defined, the SCIS samples can be generated by an MCMC method [15–18]. The widely used Metropolis-Hastings algorithm [15, 16] was selected for this study.

In section 4.1, two sampling density functions tested successfully for SCIS are discussed. For both cases, the samples of $\boldsymbol{X}$ were generated using only the last single-flight failure probability without inspections. These samples can be used to compute the entire history of the reliability.

When applying SCIS, the number of equivalent MC samples is the number of SCIS samples multiplied by $1/p_s(t_N)$. Put another way, the efficiency (gain) factor of SCIS to MC is $1/p_s(t_N)$.

4.1 SCIS SAMPLING DENSITY h_1

The unconditional single-flight failure probability can be formulated as the integral of the strength-conditioned conditional single-flight failure probability weighted by the joint PDF of the strength random variables $\boldsymbol{X}$:

$$p_s(t) = \int_\Omega p_{cs}(t \mid \boldsymbol{x}) \cdot f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \tag{21}$$

in which $p_{cs}(t \mid \boldsymbol{x})$ is the conditional single-flight probability of failure and $f_{\boldsymbol{X}}(\boldsymbol{x})$ the joint PDF.

To minimize the sampling region for more effective sampling, the domain of $\boldsymbol{X}$, Ω, is divided into two domains, Ω_1 and Ω_2, such that

$$p_{\Omega_1} + p_{\Omega_2} = \int_{\Omega_1} f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} + \int_{\Omega_2} f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} = 1 \tag{22}$$

Equation 21 can be rewritten as

$$
\begin{aligned}
p_s(t_N) &= p_{\Omega_1} \cdot \int_{\Omega_1} p_{cs}(t_N \mid \boldsymbol{x}) \frac{f_X(\boldsymbol{x})}{p_{\Omega_1}} d\boldsymbol{x} + p_{\Omega_2} \cdot \int_{\Omega_2} p_{cs}(t_N \mid \boldsymbol{x}) \frac{f_X(\boldsymbol{x})}{p_{\Omega_2}} d\boldsymbol{x} \\
&= p_{\Omega_1} \cdot E[p_{cs}(t_N \mid \boldsymbol{x})]_{\Omega_1} + p_{\Omega_2} \cdot E[p_{cs}(t_N \mid \boldsymbol{x})]_{\Omega_2} \\
&= p_{\Omega_1} \cdot E[p_{cs}(t_N \mid \boldsymbol{x})]_{\Omega_1} \cdot \left[1 + \frac{p_{\Omega_2}}{p_{\Omega_1}} \frac{E[p_{cs}(t_N \mid \boldsymbol{x})]_{\Omega_2}}{E[p_{cs}(t_N \mid \boldsymbol{x})]_{\Omega_1}} \right]
\end{aligned}
\tag{23}
$$

The division of Ω is defined by selecting a truncation limit of $p_{cs}(t_N \mid \boldsymbol{x}) \cdot f(\boldsymbol{x})$, P_{Limit}, such that

$$
\begin{aligned}
&X \subset \Omega_1 \text{ if } p_{cs}(t_N \mid \boldsymbol{x}) \cdot f(\boldsymbol{x}) \geq P_{Limit} \\
&X \subset \Omega_2 \text{ if } p_{cs}(t_N \mid \boldsymbol{x}) \cdot f(\boldsymbol{x}) < P_{Limit}
\end{aligned}
\tag{24}
$$

The above truncation serves to minimize the number of selected samples by neglecting the samples in the Ω_2 domain with a small, controlled error in $p_s(t_N)$. The truncation limit can be found by generating a set of pilot samples of $p_{cs}(t_N)$ using the MC or the MCMC methods. Assuming K samples have been generated, the corresponding samples in the two domains, denoted as K_1 and K_2, are approximately proportional to p_{Ω_1} and p_{Ω_2}. Therefore, equation 23 becomes:

$$
\begin{aligned}
p_s(t_N) &\approx p_{\Omega_1} E[p_{cs}(t_N \mid \boldsymbol{x})]_{\Omega_1} \left[1 + \frac{K_2}{K_1} \frac{E[p_{cs}(t_N \mid \boldsymbol{x})]_{\Omega_2}}{E[p_{cs}(t_N \mid \boldsymbol{x})]_{\Omega_1}} \right] \\
&= p_{\Omega_1} E[p_{cs}(t_N \mid \boldsymbol{x})]_{\Omega_1} \left[1 + \frac{\sum_{j=1}^{K_2} (p_{cs}(t_N \mid \boldsymbol{x}_j))_{\Omega_2}}{\sum_{j=1}^{K_1} (p_{cs}(t_N \mid \boldsymbol{x}_j))_{\Omega_1}} \right] = p_{\Omega_1} E[p_{cs}(t_N \mid \boldsymbol{x})]_{\Omega_1} (1+\varepsilon)
\end{aligned}
\tag{25}
$$

in which case ε is the relative error in $p_s(t_N)$ due to the truncation. By specifying an acceptable ε such as 0.01, a simple search procedure can be devised to find corresponding pairs of P_{Limit} and K_1 samples. Afterward, by substituting equation 19 for equation 25, and ignoring the small relative error, ε, the single-flight failure probability can be estimated as:

$$
p_s(t_N) \approx p_{\Omega_1} \cdot E\left[p_{cs}(t_N \mid \boldsymbol{x})\right]_{\Omega_1} \approx p_{\Omega_1} \frac{\sum_{j=1}^{K_1} \left[p_{cs}(t_N \mid \boldsymbol{x}_j) \right]_{\Omega_1}}{K_1}
\tag{26}
$$

which leads to:

$$p_{\Omega_1} \approx \frac{p_s(t_N)}{E\left[p_{cs}(t_N \mid \boldsymbol{x})\right]_{\Omega_1}} \approx K_1 \frac{p_s(t_N)}{\sum_{j=1}^{K_1}\left[p_{cs}(t_N \mid \boldsymbol{x}_j)\right]_{\Omega_1}} \tag{27}$$

in which the single-flight POF at t_N, $p_s(t_N) = 1 - P[X(t_N) > Y(t_N)]$ can be computed by SCEM or a fast probability integration method (presented later). The result of equation 26 then leads to the single-flight reliability $R_s(t) = 1 - p_s(t)$.

In analogues to equation 26, the POF can be formulated in terms of the strength-conditioned POF $p_{cs}(t \mid x)$ as:

$$p(t) = p_{\Omega_1} \cdot E\left[p_c(t \mid x)\right]_{\Omega_1} = p_{\Omega_1} \cdot E\left[1 - R_c(t \mid x)\right]_{\Omega_1} = p_{\Omega_1} \cdot E\left[1 - \prod_{i=1}^{t}\left[1 - p_{cs}(t \mid x)\right]\right]_{\Omega_1} \tag{28}$$

in which the product rule is valid because the strength-conditioned single-flight reliabilities are statistically independent. The unconditional reliability is $R(t) = 1 - p(t)$. The failure rate per flight, also called the hazard function, is:

$$r(t_i) = \frac{R(t_i) - R(t_{i+1})}{R(t_i)} \tag{29}$$

As discussed in section 5, the samples generated in Ω_1 can also be used for computing the single-flight POF $p_s(t_i)$ with inspections. After each inspection, $p_{cs}(t = t_1 : t_i \mid x_{j=1:K})$ should be adjusted for those simulated samples that have been inspected and repaired or replaced.

In effect, the above procedure to identify a reduced region, and generate samples in that region, is equivalent to using the following truncated sampling density:

$$h_1(\boldsymbol{x}) = \left[\frac{f_X(\boldsymbol{x})}{P_{\Omega_1}}\right]_{\Omega_1} \tag{30}$$

such that:

$$p_s(t) = p_{\Omega_1} \cdot \int_{\Omega_1} p_{cs}(t \mid \boldsymbol{x}) \cdot h(\boldsymbol{x}) d\boldsymbol{x} = p_s(t_N) \cdot \frac{E[p_{cs}(t \mid \boldsymbol{x})]_{\Omega_1}}{E\left[p_{cs}(t_N \mid \boldsymbol{x})\right]_{\Omega_1}} \tag{31}$$

Once P_{Limit} has been found, additional MCMC samples based on the truncated h_1 can be created as needed.

Sampling density h_1 was successfully used in the earlier study for SCIS. However, it requires searching for P_{Limit} first, which could be avoided by using h_2.

4.2 SCIS SAMPLING DENSITY h_2

A sampling density that seems more suitable for SCIS is one that is proportional to the conditional POF:

$$h_2(\boldsymbol{x}) = \frac{p_{cs}(t_N \mid \boldsymbol{x}) \cdot f_{\boldsymbol{X}}(\boldsymbol{x})}{p_s(t_N)} \tag{32}$$

The $h_2(\boldsymbol{x})$ is a valid PDF because $\int h_2(\boldsymbol{x})d\boldsymbol{x} = \left(\int p_{cs}(t_N \mid \boldsymbol{x}) \cdot f_{\boldsymbol{X}}(\boldsymbol{x})d\boldsymbol{x}\right) / p_s(t_N) = p_s(t_N) / p_s(t_N) = 1$. Because $p_s(t_N)$ is a constant, the sampling density can be set to be $p_{cs}(t_N) \cdot f_{\boldsymbol{X}}(\boldsymbol{x})$ for MCMC. Compared with $h_1(\boldsymbol{x})$, the key difference in using $h_2(\boldsymbol{x})$ is that the density is proportional to both $f(\boldsymbol{x})$ and $p_{cs}(t_N \mid \boldsymbol{x})$, which ensures more samples will be created in the region that matters most. In addition, there is no need to find a truncation-error threshold to define Ω_1 because insignificant risk region will be excluded automatically during the MCMC process. Therefore, $h_2(\boldsymbol{x})$ was chosen as the default sampling density for FlyRisk software implementation.

Using h_2, the single-flight failure probability at any $t \le t_N$ can be reformulated as follows:

$$p_s(t) = \int p_{cs}(t \mid x) \cdot f_{\boldsymbol{X}}(x)dx = \int \frac{p_s(t_N) \cdot p_{cs}(t \mid x)}{p_{cs}(t_N \mid x)} \cdot h_2(x)dx = p_s(t_N) \cdot E\left[\frac{p_{cs}(t \mid x)}{p_{cs}(t_N \mid x)}\right]_{h_2(x)} \tag{33}$$

from which the single-flight reliability is $R_s(t) = 1 - p_s(t)$.

Similar to equation 28, the POF can be formulated as:

$$p(t) = p_s(t_N) \cdot E\left[\frac{p_c(t \mid x)}{p_{cs}(t_N \mid x)}\right]_{h2} = p_s(t_N) \cdot E\left[\frac{1 - \prod_{i=1}^{i}\left[1 - p_{cs}(t \mid x)\right]}{p_{cs}(t_N \mid x)}\right]_{h2} \tag{34}$$

The unconditional reliability function is $R(t) = 1 - p(t)$ and the failure rate can be calculated using equation 29.

4.3 SCIS SAMPLING DENSITY h^*

Both h_1 and h_2 are used to generate SCIS samples without considering inspection effects. A third density function that takes into account the effect of repair/replacement is called h^*, where the superscript "*" indicates that the effect of inspection is included. The density h^*, as defined in equation 31, is proportional to the product of $f(x)$ and the conditional single-flight POF, with inspection effect, denoted as $p_{cs}^*(t \mid x)$:

$$h^*(\boldsymbol{x}) = \frac{p_{cs}^*(t_N \mid \boldsymbol{x}) \cdot f_{\boldsymbol{X}}(\boldsymbol{x})}{p_s^*(t_N)} \tag{35}$$

Single flight POF, with inspection effect, can be formulated as:

$$p_s^*(t_i) = \int p_{cs}^*(t_i \mid \boldsymbol{x}) \cdot f_{\boldsymbol{X}}(\boldsymbol{x}) dx = \int \frac{p_s^*(t_N) \cdot p_{cs}^*(t_i \mid \boldsymbol{x})}{p_{cs}^*(t_N \mid \boldsymbol{x})} \cdot h^*(\boldsymbol{x}) dx = p_s^*(t_N) \cdot E\left[\frac{p_{cs}^*(t_i \mid \boldsymbol{x})}{p_{cs}^*(t_N \mid \boldsymbol{x})}\right]_{h^*(\boldsymbol{x})} \tag{36}$$

To compute $p_{cs}^*(t_N \mid \boldsymbol{x})$, abbreviated as $p_{cs}^*(t_N)$, the following approximation formulas can be derived for m inspections at time $t = T_{i=1:m}$:

$$\begin{aligned} p_{cs}^*(t_N \mid \text{Inspections at } T_d = T_1 : T_m) &= p_{cs_1}(t_N \mid \text{No Detection}) + p_{cs_2}^*(t_N \mid \text{One Detection}) \\ &= \left[\prod_{i=1}^{m} PND(a(T_i))\right] \cdot p_{cs}(t_N \mid \text{No Detection}) \\ &\quad + \sum_{\text{d}=1}^{\text{m}} \left(\left[\prod_{i=1}^{d-1} PND(a(T_i))\right] \cdot POD(a(T_i)) \cdot \left[\prod_{i=d+1}^{m} PND(a(T_i))\right] \cdot p_{cs}^*(t_N \mid \text{After Repair}) \right) \end{aligned} \tag{37}$$

As shown in figure 3, where $t_e = t_N$, equation 37 sums up detection scenarios and subsequent failures, weighted by the probabilities of occurrences. Equations 35 and 37 can be used to generate MCMC samples that follow the density function of h^*.

Equation 37 assumes that the repair/replacement is "ideal" (i.e., the probability of a repaired part failing before the service time should be zero). This assumption is reasonable if the flaw size after a replacement/repair is independent of the detected flaw size. For most high-reliability and high-failure-consequence structures, the assumption of ideal repair should be reasonable for at least two reasons: (1) high-quality repairs are expected for aircraft structures and (2) the POF is small, which means the chances of finding a significant defect (exceeding a repair threshold) should also be small, especially during the early stage of the service life. Consequently, the probability of the same structural element having larger than alarming flaw sizes more than once should be relatively very small. For critical parts, however, the effect of repair on POF must be included in equation 37.

$$p^*_{cs}(t_e \mid \text{Insp. at } T_d = T_1 : T_m) = p_{cs_1}(t_e \mid \text{No Detection}) + p^*_{cs_2}(t_e \mid \text{One Detection})$$

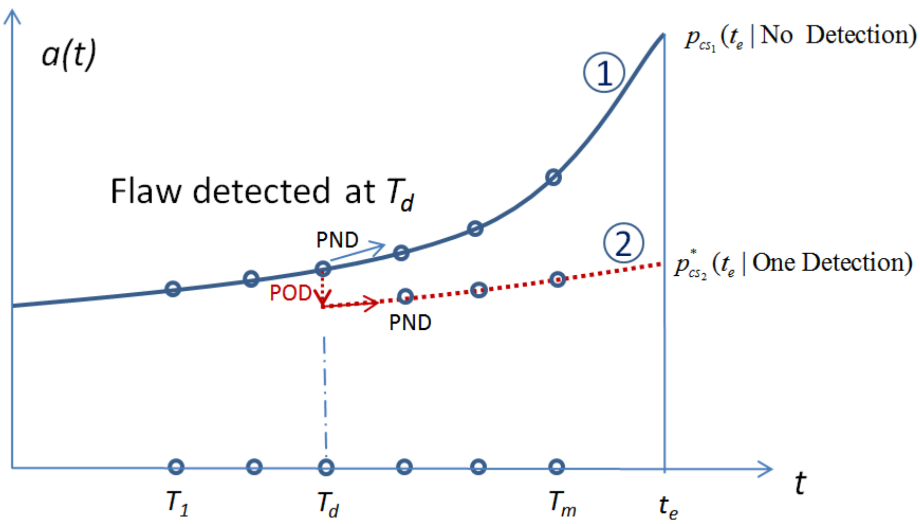

Figure 3. Two failure paths after an inspection at T_d (d = 1:m)

Although, in theory, h^* includes the inspection effect and therefore seems to offer optimal performance (i.e., requires less samples), it is also the most complicated in terms of implementation and validation. Consequently, h_2 was selected for FlyRisk implementation because of its balanced performance in speed, accuracy, and robustness. To achieve the potentially highest performance, more research will be needed to study and implement h^*.

Theoretically, a key advantage of the SCIS method is its capability to handle a broad range of strength-changing models, either gradual or sudden, including degradation, impact damage, and maintenance. However, for highly complex strength-changing events, the successful implementations of SCIS may require tailored sampling algorithms to generate high-quality SCIS samples efficiently.

5. RELIABILITY-WITH-INSPECTION ANALYSIS

5.1 TWO-STAGE SAMPLING METHODOLOGY

This section describes a two-stage sampling methodology using SCIS samples generated at a selected point in time, t_N, then using the same set of samples to simulate inspection effects and compute reliability-with-inspection for any time before t_N.

The foundation of the two-stage methodology is a maintenance simulation framework shown in figure 4 [8]. In Stage 1, the defect size samples that would lead to structural failures before the service lifetime are generated by MCMC without considering inspections. In Stage 2, inspection effects (using Probability of Detection [POD] models) are simulated using the Stage 1 samples.

The approach was built on the assumption that a structure would not be degraded because of maintenance [8]. Although special cases, such as poorly implemented maintenance practices, may cause the violation, the assumption is reasonable for safety-critical, high-reliability products. Based on this, only the fates of the potential failures need to be tracked. In the two-stage process, Stage 1 generates candidate failure samples and computes the POF assuming no inspections. Stage 2 then takes the samples to assess the risk-reduction performances of various maintenance strategies.

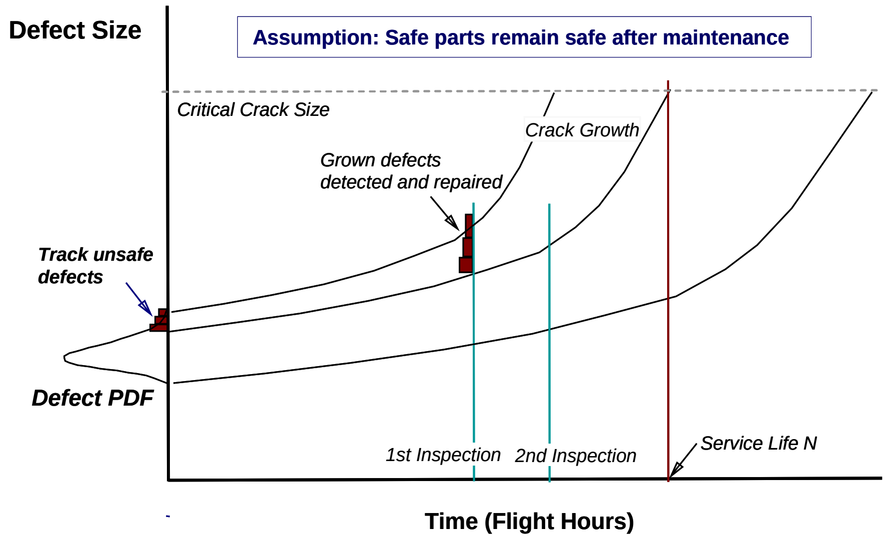

Figure 4. Illustration of two-stage methodology
Stage 1: Generate SCIS samples without inspection
Stage 2: SCIS + inspection simulation

Although the SCIS framework is capable of dealing with different degrees of repair qualities, for practical purposes, the current version of FlyRisk assumes a maintenance policy that a damaged part, if detected, is replaced by an original part or is repaired to as new condition. Other maintenance policies can be added to FlyRisk if needed.

With FlyRisk, K samples of $\boldsymbol{x}_k$ are generated from h_2 to compute $p_s(t_N)$ and $p_{cs}(t_N \mid \boldsymbol{x}_k)$, which are saved. To compute reliability-with-inspection, $p_s^{WI}(t)$, $p_{cs}(t \mid \boldsymbol{x}_k)$ with-inspection, and $p_{cs}^{WI}(t_N \mid \boldsymbol{x}_k)$ are computed using the following equation:

$$p_s^{WI}(t) = p_s(t_N) \cdot \frac{1}{K} \sum_{k=1}^{K} \frac{p_{cs}^{WI}(t \mid \boldsymbol{x}_k)}{p_{cs}(t_N \mid \boldsymbol{x}_k)} \tag{38}$$

from which the single-flight reliability is $R_s^{WI}(t) = 1 - p_s^{WI}(t)$.

Similar to equation 34, the POF with inspection can be formulated as:

$$p^{WI}(t) = p_s(t_N) \cdot E\left[\frac{p_c^{\mathrm{WI}}(t \mid x)}{p_{cs}(t_N \mid x)}\right]_{h2} = p_s(t_N) \cdot E\left[\frac{1 - \prod_{i=1}^{i}\left[1 - p_{cs}^{\mathrm{WI}}(t \mid x)\right]}{p_{cs}(t_N \mid x)}\right]_{h2} \tag{39}$$

As before, the unconditional reliability function is $R(t) = 1 - p(t)$ and the failure rate can be calculated using equation 29.

5.2 NUMERICAL ISSUES

The reliability analyses center on computing $\prod_{i=1}^{i}\left[1 - p_{cs}(t \mid x)\right]$ which requires computing $p_{cs}(t = t_1 : t_i \mid x_{j=1:K})$ for a large number of time steps (e.g., 20,000 flights) for every strength realization. To expedite the calculations, a reliability function was selected to fit the $p_{cs}(t \mid x_j)$ function using a relatively small number of points. It was found that the three-parameter Weibull reliability function was able to provide excellent fits using less than 100 points, as opposed to 20,000 points, and was able to speed up the overall reliability calculations by more than two orders of magnitude.

The single-flight POF, $p_s(t_N)$, is a key probability measure that can be computed accurately using the time-consuming SCEM method or sophisticated structural reliability analysis software. An alternative approach was developed and implemented in FlyRisk to compute $p_s(t_N)$ using the SCIS samples. The approach, summarized in section 6, combines a kernel density estimator with a fast probability integration scheme. This unique approach allows FlyRisk to be a completely standalone code with the flexibility to be modified by users.

5.3 FLYRISK EXAMPLE

Based on the fracture-mechanics demonstration problem defined in appendix A, the reliability with three inspections was analyzed. Figure 5 compares the SCIS results (without inspection) with near-exact solutions.

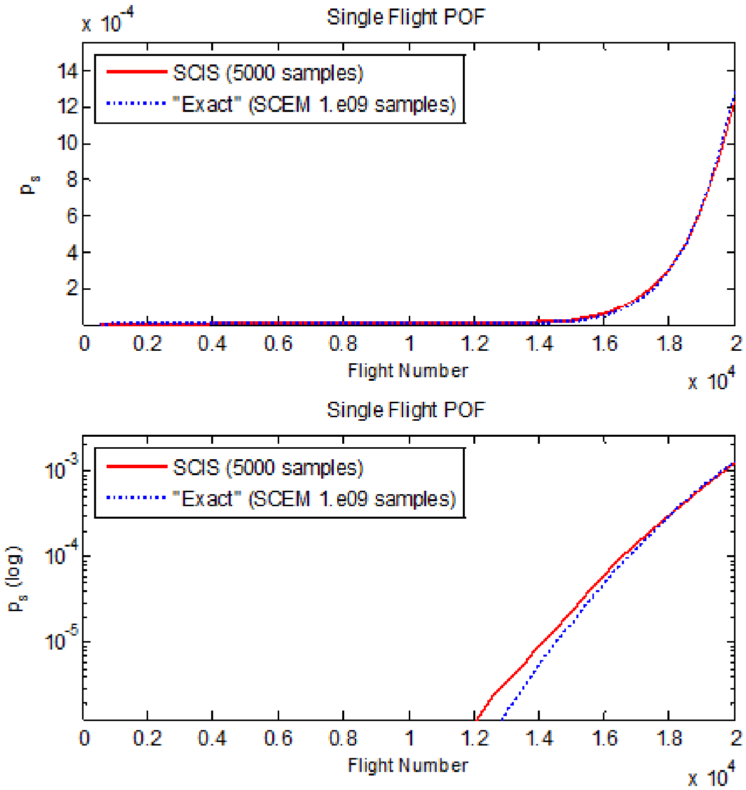

Figure 5. FlyRisk validation: comparing FlyRisk result with near-exact solutions (linear and log probability plots)

Figure 6 summarizes the FlyRisk result, which includes single-flight and cumulative POFs and hazard function (failure rate per flight). The saw-tooth shaped curves demonstrate the risk reduction effects.

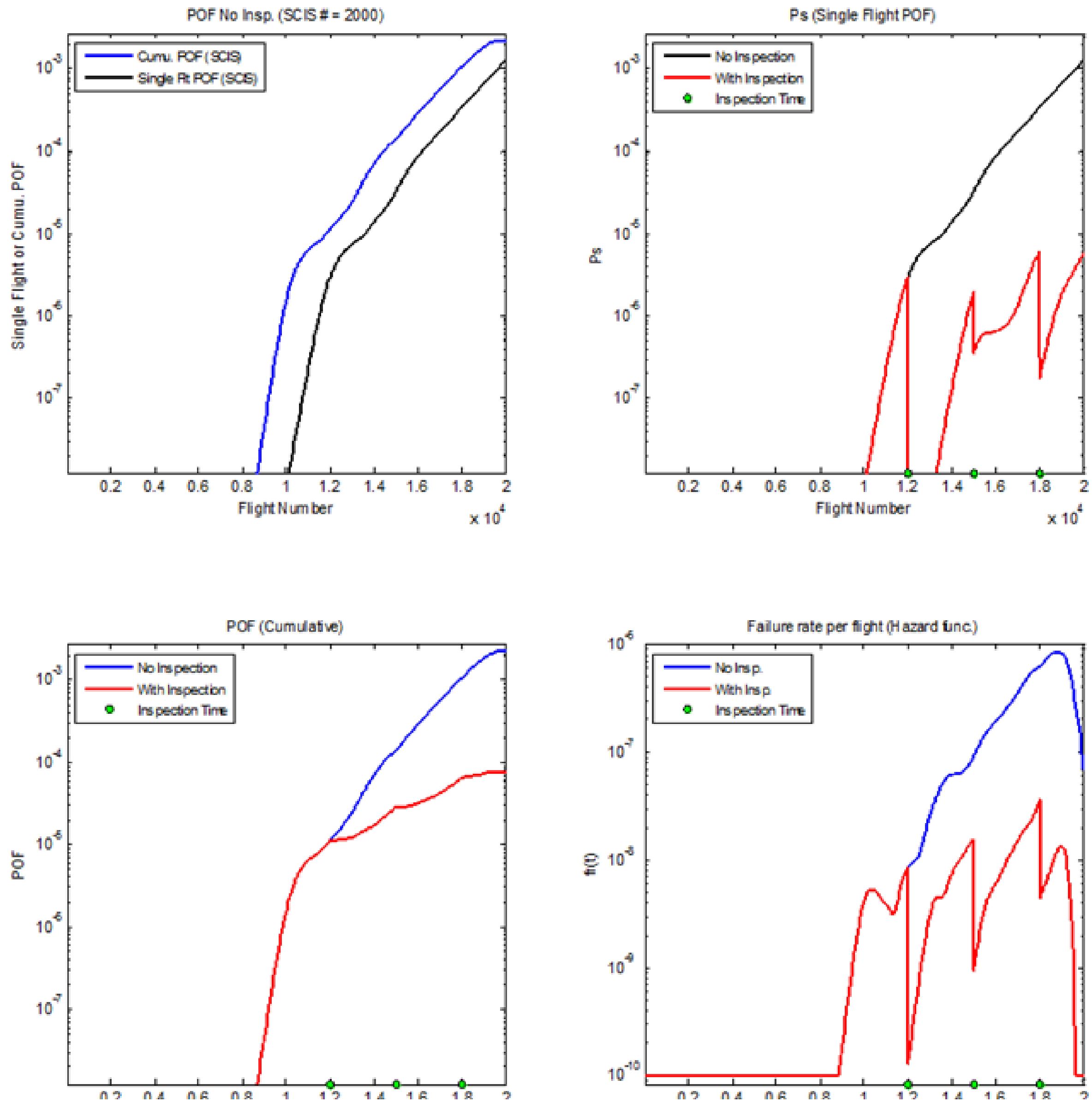

Figure 6. Reliability without and with inspections (14,000, 16,000, and 18,000 flights)

6. COMPUTE SINGLE-FLIGHT POF USING FAST PROBABILITY INTEGRATION

When using SCIS, the POF formulations involve the single-flight POF, $p_s(t_N)$. FlyRisk, a product-kernel density estimator combined with a fast probability integration scheme, was developed to compute $p_s(t_N)$. This new approach eliminates the need for a separate reliability analyzer in the MATLAB FlyRisk code, resulting in a more compact and completely standalone FlyRisk code.

The SCIS samples are generated in the standardized normal "u"-space, where u is related to X by the following parameter mapping:

$$u_i = \Phi^{-1}(F_{X_i}(x_i)) \tag{40}$$

where $F_{X_i}(x_i)$ is the original Cumulative Distribution Function (CDF) and $\Phi(u_i)$ is the standard normal CDF. The probability to be found is a probability integral:

$$p_s(t_N) = \int p_{cs}(t_N \mid \boldsymbol{u}) \cdot \phi_{\boldsymbol{u}}(\boldsymbol{u}) d\boldsymbol{u} = \int I(\boldsymbol{u}) d\boldsymbol{u} \tag{41}$$

Based on SCIS's $\boldsymbol{u}$ samples, $I(\boldsymbol{u})$ can be calculated. In addition, a Kernel Density Estimator (KDE) with a PDF of $f_K(\boldsymbol{u})$ can be found such that:

$$\int f_K(\boldsymbol{u}) d\boldsymbol{u} = 1 = \int \frac{f_K(\boldsymbol{u})}{I(\boldsymbol{u})/A} \frac{I(\boldsymbol{u})}{A} d\boldsymbol{u} \tag{42}$$

where A = $p_s(t_N)$ and therefore $I(\boldsymbol{u})/A$ is a valid PDF. From equation 42:

$$1 = \int \frac{f_K(\boldsymbol{u})}{I(\boldsymbol{u})/A} \frac{I(\boldsymbol{u})}{A} d\boldsymbol{u} = A \cdot E\left[\frac{f_K(\boldsymbol{u})}{I(\boldsymbol{u})}\right]_{I(\boldsymbol{u})/A} \tag{43}$$

which leads to:

$$p_s(t_N) = A = \frac{1}{E\left[\dfrac{f_K(\boldsymbol{u})}{I(\boldsymbol{u})}\right]_{I(\boldsymbol{u})/A}} \tag{44}$$

There are many possible models for creating $f_K(\boldsymbol{u})$. For FlyRisk, a product kernel in the following form [19] was chosen:

$$f_K(\boldsymbol{u}) = \frac{1}{k} \sum_{i=1}^{k} \left[\prod_{j=1}^{n} \frac{1}{h_j} K\left(\frac{u_j - U_{ij}}{h_j}\right) \right] \tag{45}$$

where $K(.)$ is a univariate kernel function, k is the samples of SCIS, n is the number of variables in $\boldsymbol{u}$, u_j (j = 1:n) is the point to be estimated, U_{ij} are the KDE data set, and h_j (j = 1:n) are the bandwidths of $K(.)$.

Because $E\left[f_K(\boldsymbol{u})/I(\boldsymbol{u})\right]$ in equation 44 is estimated by taking the sample average, the computed integral A is an approximate. However, if $f_K(\boldsymbol{u})$ can be optimized to fit $I(\boldsymbol{u})/A$ well, the error of A could be minimized. For FlyRisk, the biweight kernel [15] was found to be suitable and optimal h_j (j = 1:n) was found by minimizing the sum of squared errors between $f_K(\boldsymbol{u})$ and $I(\boldsymbol{u})/A$.

6.1 ILLUSTRATION EXAMPLE: ONE RANDOM VARIABLE CASE

To illustrate the KDE-based fast probability integration approach, this example used a set of 500 random samples generated from the right tail of a standard normal distribution. The analysis task was to estimate the probability of the tail region using only the samples generated from the tail and the integrand values at the locations of the samples.

The left side of figure 7 shows the locations of the samples generated from the right tail of a standard normal.

The probability to be computed is the following integral:

$$A = \int_{\Omega} \phi_{\boldsymbol{u}}(u)du = \int I(u)du \tag{46}$$

Based on the available u samples, the integrand $I(u) = \Phi(u)$ can be calculated. A KDE with a PDF of $f_K(u)$ can be created using the following form:

$$f_K(u) = \frac{1}{k}\sum_{i=1}^{k}\frac{1}{h}K\left(\frac{u - U_i}{h}\right) \tag{47}$$

The right side of figure 7 shows how the bandwidth, h, of $f_K(u)$ is optimized to fit the scaled integrand function $I(u)/A$.

From equation 44,

$$A = \frac{1}{E\left[\dfrac{f_K(u)}{I(u)}\right]_{I(u)/A}} \tag{48}$$

Using 10 runs of 500 random samples, the computed integral has an average error of less than 1%, as shown in figure 8 (cumulative moving error = 0.6%).

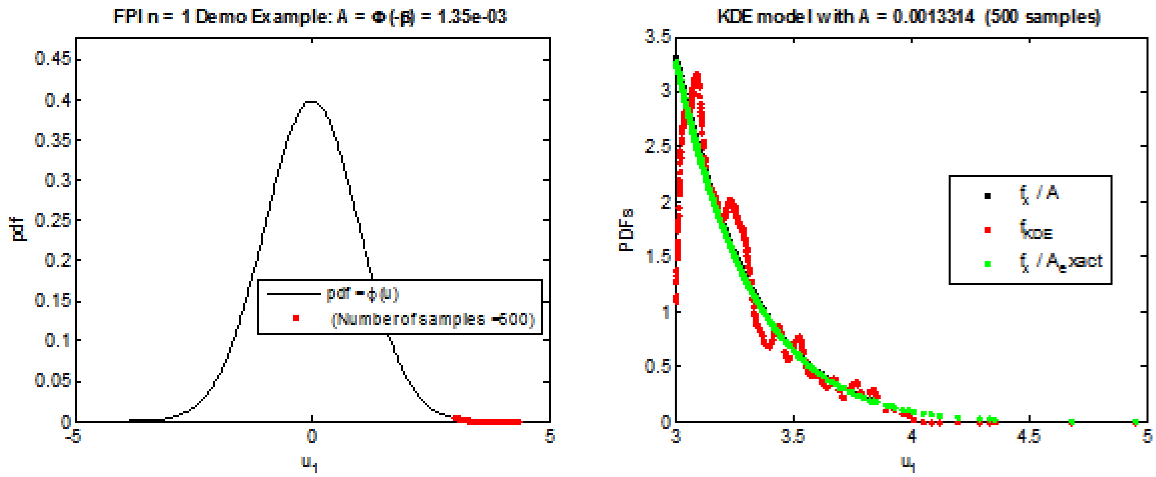

Figure 7. Original normal PDF and the best-fit KDE

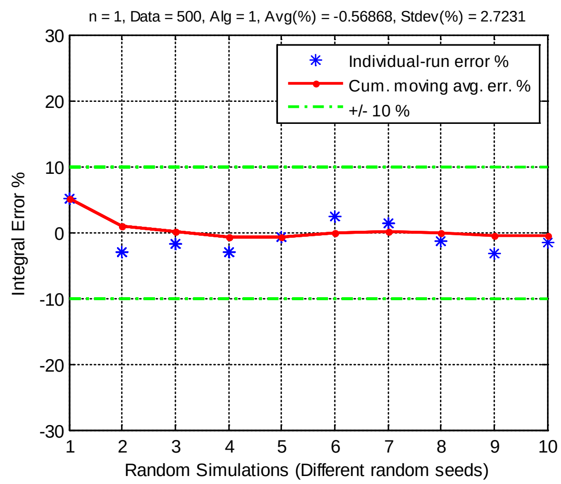

Figure 8. Ten simulations of 500 data samples for KDE test

<u>6.2 FLYRISK EXAMPLE</u>

Based on the fracture-mechanics demonstration problem defined in appendix A, $p_s(t_N)$ was computed using FlyRisk. Figure 9 shows how the optimized scaled KDE function $f_K(\boldsymbol{u})/A$ provides a reasonable fit to the integrand $I(\boldsymbol{u}) = p_{cs}(t_N \mid \boldsymbol{u}) \cdot \phi_{\boldsymbol{u}}(\boldsymbol{u})$. The error of the computed $p_s(t_N)$ was 5% by using four sets of 500 SCIS samples.

This example demonstrates that the KDE-based probability integration is a feasible approach that can provide a good $p_s(t_N)$ estimate using the SCIS samples already generated.

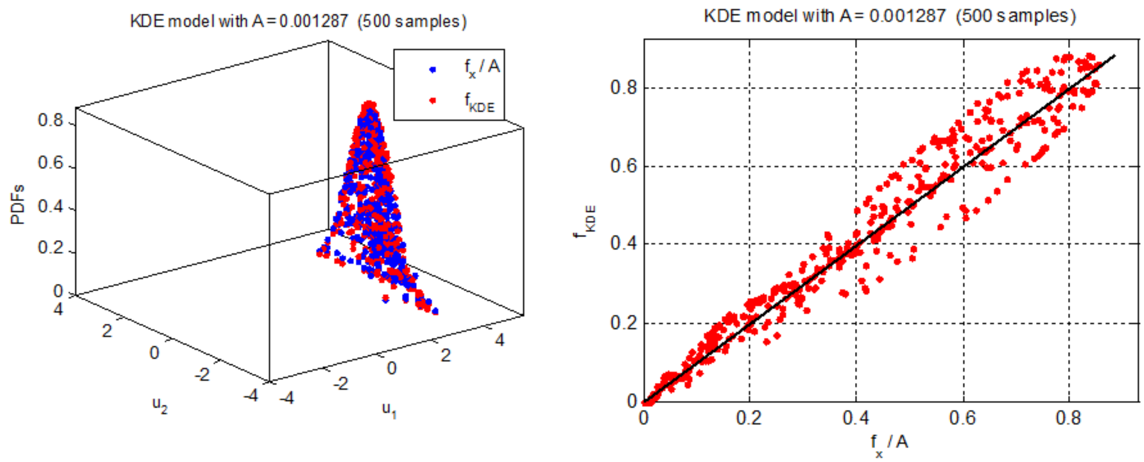

Figure 9. SCIS PDF compared with KDE
(5% error in single-flight POF based on four sets of 500 SCIS samples)

7. SUMMARY AND DISCUSSIONS

The ability to model and assess the risk of strength degradations, along with timely inspections and maintenance, can contribute significantly to maintaining the safety and reliability of aircraft structures. It is especially important to assess the effects of events that may cause significant sudden changes in the remaining strength and life of an aircraft. By monitoring structural reliability and taking necessary preventive maintenance actions at optimal times, the reliability can be sustained at more desirable levels.

Built on a probabilistic damage tolerance framework, the Strength-Conditioned Importance Sampling (SCIS) approach provides a new direction to model time-dependent reliability to assess flight risk and risk-based inspection/maintenance planning. SCIS is mathematically rigorous for treating correlated random variables between flights, capable of handling sudden strength-changing events, and computationally efficient. Because the SCIS approach is sampling-based, it is also suitable for simulating complex random events including sharp, step-function like, strength-shifting events, including repairs and replacements.

As demonstrated by FlyRisk, the SCIS approach accurately treats correlated random variables between flights and is computationally efficient as compared to the full-scale Monte Carlo or even Strength-Conditioned Expectation Method approach. The FlyRisk tool also demonstrated that the SCIS samples could be reused to compute the reliabilities for different inspection schedules. Therefore, the SCIS approach is ideally suited for risk-based inspection/maintenance planning.

Because a metal fatigue model has been used to demonstrate the SCIS approach, it should be noted that the SCIS framework can handle different materials. Theoretically, a major advantage of the SCIS method is its potential capability to treat a broad range of strength-changing models, either gradual or sudden, including degradation, impact damage, and maintenance. However, for highly complex strength-changing events, such as impacts and repair/replacements, the successful and optimal implementations of SCIS may require improved sampling algorithms to generate high-quality samples efficiently. For future research, it is important to use challenging practical problems to fully validate and demonstrate the robustness, accuracy, and efficiency of the SCIS approach.

8. REFERENCES

1. Wu, Y.-T., Shiao, M., and Seneviratne, W., "Strength-Conditioned Expectation Method for Aircraft Structural Reliability Analysis," Proceedings of the 52^{nd} AIAA/ASME/ASCE/AHS Structures, Structural Dynamics and Materials Conference, April 2011.

2. Wu, Y.-T., Shiao, M., and Millwater, H.R., "A Bayesian-Updating Computational Method for Probabilistic Damage Tolerance Analysis," Proceedings of the 51st AIAA/ASME/ASCE/AHS Structures, Structural Dynamics and Materials Conference, April 2010.

3. Wu, Y.-T., Millwater, H.R., and Abali, F., "Significance of Including Missed-Detections in Structural Reliability Updating," Proceedings of the 52nd AIAA/ASME/ASCE/AHS Structures, Structural Dynamics and Materials Conference, April 2011.

4. McClung, R.C., Leverant, G.R., and Wu, Y.-T., et al., "Development of a Probabilistic Design System for Gas Turbine Rotor Integrity," FATIGUE '99, The Seventh International Fatigue Conference, Beijing, June 1999.

5. Wu, Y.-T., Enright, M.P., and Millwater, H.R., "Probabilistic Methods for Design Assessment of Reliability with Inspection," *AIAA Journal*, Vol. 40, No. 5, pp. 937–946, 2002.

6. Millwater, H.R., Wu, Y.-T., and Fitch, S., et al., "A Probabilistically-Based Damage Tolerance Analysis Computer Program for Hard Alpha Anomalies In Titanium Rotors," Proceedings of the 45th ASME International Gas Turbine & Aero-engine Technical Congress, Munich, May 8–11, 2000.

7. Wu, Y.-T., Shiao, M., Shin, Y., and Stroud, W.J., "Reliability-Based Damage Tolerance Methodology for Rotorcraft Structures," *Transactions Journal of Materials and Manufacturing*, paper 2004-01-0681, July 2005.

8. Wu, Y.-T. and Shin, Y., "Probabilistic Damage Tolerance Methodology for Reliability Design and Inspection Optimization," Proceedings of the 45th AIAA/ASME/ASCE/AHS Structures, Structural Dynamics and Materials Conference, April 2004.

9. Wu, Y.-T., Zhao, J., Shiao, M., and Millwater, H.R., "Efficient Methods for Probabilistic Damage Tolerance Inspection Optimization," Proceedings of the 51st AIAA/ASME/ASCE/AHS Structures, Structural Dynamics and Materials Conference, April 2010.

10. Ang, A.H.-S. and Tang, W.H., *Probability Concepts in Engineering Planning and Design*, Vol. II, Decision, Risk, and Reliability, John Wiley & Sons, New York, 1984.

11. Wu, Y.-T., "MCMC-Based Simulation Method for Efficient Risk-Based Maintenance Optimization," Proceeding of the 2009 SAE World Congress, Reliability and Robust Design in Automotive Engineering: Decision Under Uncertainty/Uncertainty Modeling, SAE Technical Paper 2009-01-0566, April 2009.

12. Wu, Y.-T., Ku, A.P., and Serratella, C.M., "A Robust and Efficient Computational Method for Fatigue Reliability Update Using Inspected Data," Proceedings of the 28th International Conference on Ocean, Offshore and Arctic Engineering, OMAE2009-80034, June, 2009.

13. Ayyub, B.M., *Risk Analysis in Engineering and Economics*, CRC Press, 2003.

14. Sundararajan, C.R., *Probabilistic Structural Mechanics Handbook: Theory and Industrial Applications*, Springer, 2012.

15. Gamerman, D., *Markov Chain Monte Carlo*, Chapman & Hall, 1997.

16. Robert, C.P and Casella, G., *Monte Carlo Statistical Methods*, Springer, 2004.

17. Ang, G.L., Ang, A. H-S., and Tang, W.H., "Optimal Importance Sampling Density Estimator," *Journal of Engineering Mechanics*, Vol. 118, No. 6, June 1992, pp. 1146–1163.

18. Au, S.K. and Beck, J.L., "Estimation of Small Failure Probabilities in High Dimensions by Subset Simulation," *Probabilistic Engineering Mechanics*, Vol. 16, No. 4, pp. 263–277, 2001.

19. Givens, G.H. and Hoeting, J.A., *Computational Statistics*, John Wiley & Sons, 2005.

APPENDIX A—FLYRSIK DEMONSTRATION EXAMPLE

The analysis objective was to analyze the risk of crack growth at a fastener hole with inspections. The random variables include Equivalent Initial Flaw Size (EIFS), fracture toughness, and maximum stresses.

EIFS, a_i, has the following Weibull distribution, with η = 0.0061 in. and β = 0.996:

$$F_{a_i}(a) = 1 - \exp[-(a_i / \eta)^{\beta}] \tag{A-1}$$

The fracture toughness, K_C, is normally distributed with a mean = 35 and a standard deviation of 3.1 $ksi \cdot \sqrt{in}$. The maximum stress, *S*, has a Gumbel distribution defined in equation A-2, with A = 1.31 *ksi* and B = 14.6 *ksi*.

$$F_S(s) = \exp\{-\exp[-(s-B)/A]\} \tag{A-2}$$

The Cumulative Distribution Function (CDF) and the probability density function (PDF) of the random variables are plotted in figures A-1–A-3.

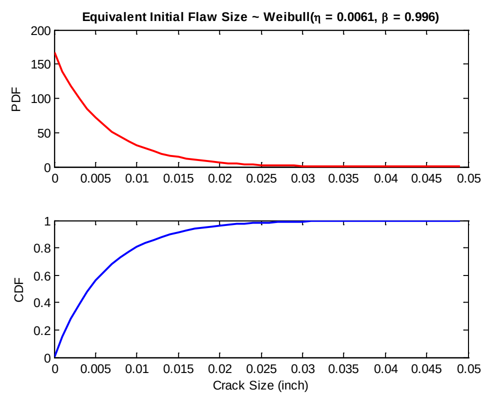

Figure A-1. PDF and CDF of EIFS

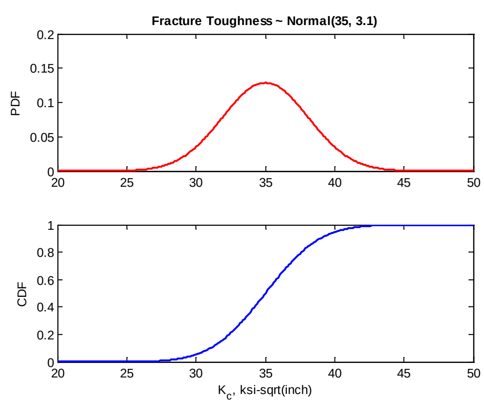

Figure A-2. PDF and CDF of fracture toughness

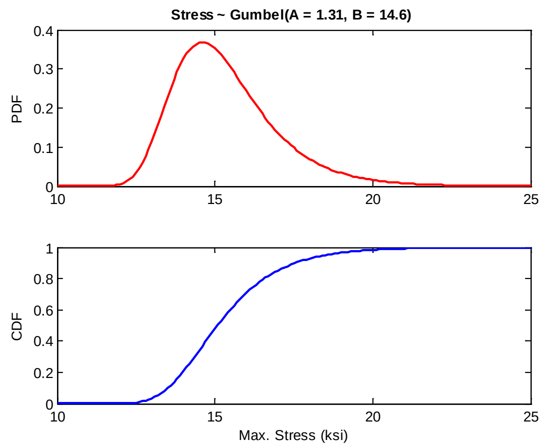

Figure A-3. PDF and CDF of maximum stress

The crack size, *a*, versus time, *t*, for damage-tolerance analysis is shown in figure A-4 as the "master" curve. For any realization of EIFS, the curve is shifted to match the crack size at time *t* = 0. For this study, the curve is fitted using equation A-3:

$$a(t) = a_o \exp(bt) = 0.0003 \cdot \exp(0.0001015t) \tag{A-3}$$

For a given EIFS, the shifted curve is:

$$a(t) = a_o \exp(bt + t_{shift}) \tag{A-4}$$

in which the time shift is:

$$t_{shift} = \frac{1}{b} \ln \frac{a_i}{a_o} \tag{A-5}$$

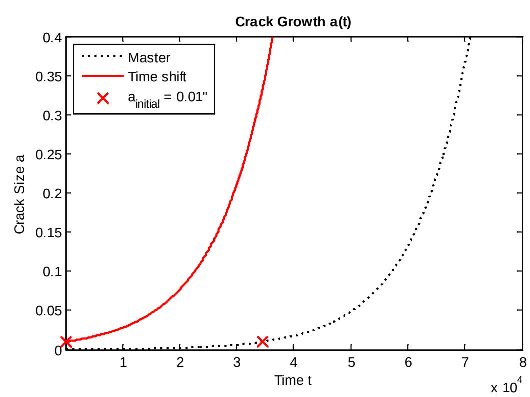

Figure A-4. Crack size versus time model

The relationship between the stress intensity factor ($R(a) = K_C / \sigma$) and the crack size is shown in figure A-5.

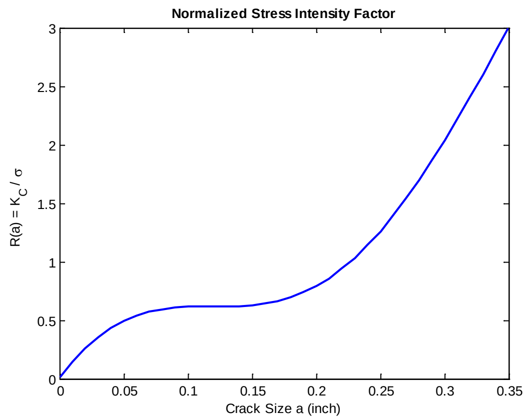

Figure A-5. Normalized stress intensity factor

Four POD curves, modeled using the following equation, were selected:

$$POD(a) = \frac{1}{1+\exp\left[-\frac{\pi}{\sqrt{3}}\frac{\ln(a-a_{\min})-\ln(a_{50}-a_{\min})}{q}\right]} \quad \text{(A-6)}$$

where a_{50} is the crack size that can be detected 50% of the time, q is a scale parameter, and $a_{\min}$ is the minimum crack size that can be detected. The parameters for the POD curves are listed in table A-1 and the curves are plotted in figure A-6.

Table A-1. Parameters for four POD curves

	POD 1	POD 2	POD 3	POD 4
a_{50}	0.05	0.05	0.10	0.10
q	0.50	0.75	0.50	0.75
a_{min}	0.00	0.00	0.00	0.00

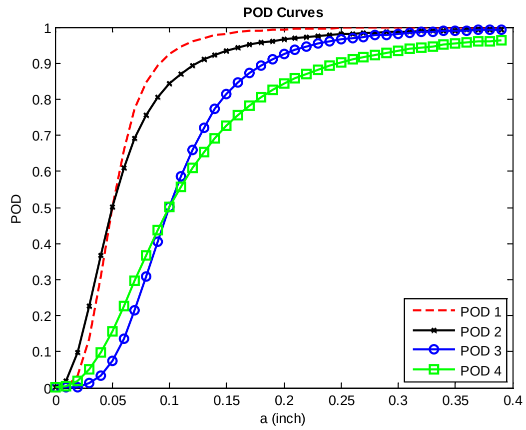

Figure A-6. Probability of detection curves

The failure limit-state function is

$$g(t) = Strength - Stress = R - S = \frac{K_C}{Y(a(t))} - S \tag{A-7}$$

and the strength-conditioned probability-of-failure is

$$P_f^c(t_i) = \Pr.\left[\frac{K_C}{Y(a(t_i))} < S\right] = 1 - F_S\left(\frac{K_C}{Y(a(t_i))}\right) \tag{A-8}$$

Figure A-7 compares the earlier Strength-Conditioned Importance Sampling (SCIS) method (for Approach 1)—which used $f(x)$ as the importance sampling density (1000 samples)—with the conventional Strength-Conditioned Expectation Method (100,000 samples) for both single-flight and cumulative probability of failure. The results validated the earlier SCIS method for the without inspection case.

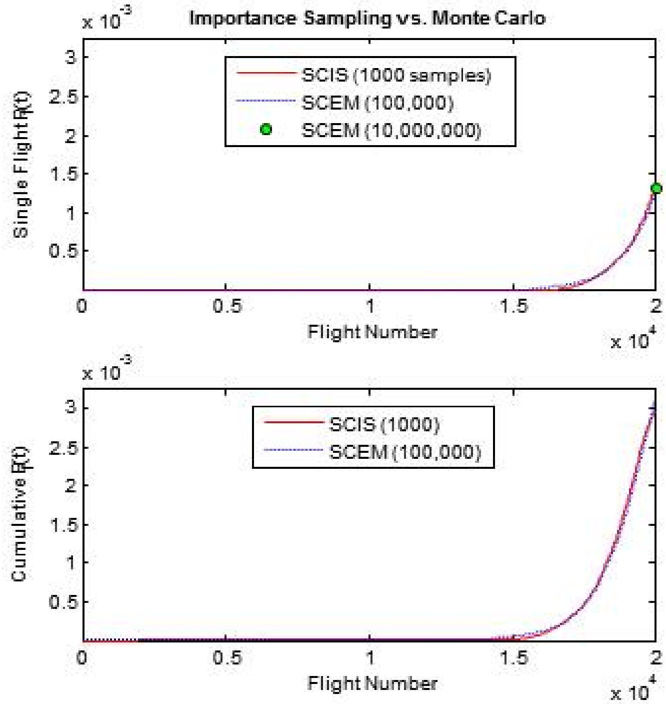

Figure A-7. Comparison of probability-of-failure using SCIS and SCEM methods (without inspection)