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The Modified Brandeis Dice for the Traffic States Dependency: The I-Dice Problem

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Abstract

Brandeis dice example is a toy example discussed by Jaynes to explain the maximum entropy principle. A die is tossed many times and an average number of spots is revealed. If other than 3.5, die is obviously not an honest one. Probabilities are then assigned for the next toss. This example is contested on the finite sampling grounds. The authors use Maximum Relative Entropy (MRE) in Jaynes' Brandeis dice example to model traffic regimes (states) in a traffic flow. A die-like hypercube (I-dice) is generated, and instead of the spots, unit kinetic energies are given on each and every face of the cube. The number of faces for this cube is the same as the number of states for the traffic flow of a given highway segment. The faces of the I-dice represent the congested, intermediate and free-flow regimes. Probability distributions are generated via maximum relative entropy principle, a modified version of the Jaynes' MaxEnt principle. The prominent feature of this hypercube, which is called I-dice by the authors, is that it generates probabilities of the traffic states à la Boltzmann-Gibbs statistical mechanics. The probabilities for the states at each lane are computed via MRE. It is found that the probabilities do not match the observed frequencies. MRE imposes a more uniform distribution of probabilities for the speeds than the observed ones. As a result, for each state, the new speed classification is suggested by I-dice. The authors propose that I-dice may very well be used as a test-bed to check randomness in traffic flow.

1. Introduction

The increasing mobility in cities has led to the emergence of an immense number of research in traffic and transportation arena. A great variety of studies has been conducted in the literature. Some of the research topics and examples could be listed as delay studies at roundabouts [1, 2], performance evaluation of bus lines [3, 4], crash studies [5, 6], saturation flow rate [7, 8] and timing optimization [9, 10] at signalized intersections, vehicular platoon formation studies [11, 12], travel time studies [13, 14], identification of driver preferences [15, 16], city logistics [17, 18], and so on. Within such a vast number of topics, this study concentrates on modeling the vehicular traffic flow regimes concerning speeds over maximum relative entropy, covering Maxent principles' shortcomings. Thus, let us briefly consider first the maximum entropy principle as follows.

Jaynes' Maximum Entropy (MaxEnt) principle is a statistical inference method specifying probabilities under partial or incomplete information. A Lagrangian is formulated over Boltzmann-Gibbs (BG) entropy and the constraints. Jaynes had introduced the famous Brandeis dice example in his Brandeis lectures [19] to explain his MaxEnt principle. Later, BG entropy is replaced by relative entropy, and the method is called the MRE, the Maximum Relative Entropy [20-24]. MRE incorporates best of the both worlds: Bayesian inference and MaxEnt. But the discussions on MaxEnt and Bayesian are not new [25]. Williams [20] claimed that Bayesian conditioning is a case of MaxEnt inference. MaxEnt in its plain form could only process the testable information, the moments. The MRE, on the other hand, could process both data and moments.

Maximum entropy methods have many applications in variety of disciplines such as ecology [26], econometrics [27], coastal engineering [28], transportation [29, 30] in the literature. Furthermore, some of the transportation literature using Bayesian inference can be given as follows. For example, Washington et al. [31] propose Bayesian Imputation Multinomial Logit model for imputing non-chosen attribute values in the travel mode. The model embodies Bayes' theorem, the multinomial logit choice

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model and sampling based estimation. The model calibration provides a good match the observed choice behavior.

Parry & Hazelton [32] implement a likelihood based model using a Bayesian statistical approach for a day-to-day dynamic traffic. The posterior samples are obtained by approximation through using Markov Chain Monte Carlo algorithms due to impossibility of direct calculation.

Within the stochastic travel behavior study, Wei et al. [33] utilize Bayes' theorem for obtaining conditional distribution of traffic flow where the traffic network is given in Stochastic User Equilibrium. To obtain the conditional distribution i.e. posterior distribution three steps are employed. At first, the probability of route choices is obtained by the random utility theory e.g. logit based models, then by Bayes' theorem, the posterior distribution of route choices are derived, and lastly the authors find the posterior distribution of route flows. An inter-day stochastic traffic assignment problem was also probabilistically modeled in another study [34].

Fei et al. [35] develop online travel time prediction method for freeways. One of the main contributions of the method is that Bayesian dynamic linear model is considered in the context of different traffic conditions to provide more accurate and reliable results. The method is integrated into an adaptive control system. Another study involving travel time prediction on a highway also uses Bayesian model and it is important for ITS methodologies [36]. Feng et al. [37] use Bayesian approach to update the parameters of the travel time distribution of the given data. The study estimates the travel time distribution on their case study data set. The posterior distributions are calculated via Bayesian updating process under varying conditions and iterative updating is conducted in the study. The posterior distributions of Bayes approach are also compared with estimations of expectation maximization algorithm and the results are discussed.

Wang et al. [38] proposed a newly developed Bayesian combination method (BCM) and their numerical result explains the performance of the developed method by comparing with conventional BCM. The authors compare the traffic flow prediction performances of the methods and point out that the stability and the accuracy of the proposed method are better.

Wei & Asakura [39] take Bayesian perspective into account to consider the congestion networks condition and the likelihood of the route choices. The authors put forward the idea that bi-level formulation or formerly proposed other methods in the literature cannot handle the congested networks. The study assumes that route traffic flows require stochastic user equilibrium principle and the study aims at estimating traffic flows in congested networks. Conditional probability of route flows is dealt with Bayesian approach. Another study [40] related with estimating seasonal traffic pattern is also representative for Bayesian analysis. The results of Bayesian and non-Bayesian methods for the collected data are also discussed in the study.

Further, there are some examples presenting different traffic flow phases in literature. Weber et al. [41] consider that traffic flow has two separate phases i.e. jam and free-flow and the relationship between these phases is considered within the thermodynamically traffic liquid-gas transitions. In another study, Sopasakis [42] puts forward fully stochastic traffic flow model for single-lane on a highway section. The author deals with non-equilibrium behavior, and considers Arrhenius dynamics and categorizes the traffic phases as free-flow, synchronized traffic, wide moving jams, congested traffic based on records, and the obtained model is used to predict the phases correctly. Kerner & Rehborn [43] focus on the three different kinds of traffic states i.e. free traffic flow, synchronized traffic flow and traffic jams, and explain their characteristics by experimental investigations. One of the empirical classifications of traffic states is specified by [44]. Hofleitner et al. [44] perform Dynamic Bayesian Networks for arterial traffic estimation. By five minutes time discretization, the aim is to estimate travel-time distributions from sparse measurements. They also characterize traffic conditions for each link. When characterizing the traffic, the authors consider binary traffic states i.e. undersaturated and congested states to avoid large number of model parameters.

Energy functions could be also found in some traffic flow studies in the literature. For example, Krbálek [45] considers N identical vehicles on a circle and describes the energy in the traffic system via Hamiltonian. The study involves velocity and its average in the Hamiltonian function. The author employs the distance between neighboring vehicles for the potential energy in the Hamiltonian. Likewise, the study [46] is also representative example of using energy functions in traffic flow.

As mentioned earlier, in this paper, the authors aim to model the vehicular traffic flow regimes employing the celebrated toy example over MRE, covering the MaxEnt principles' shortcomings and call the eventual die-like hypercube I-dice.

The authors of this paper had obtained traffic regimes in [47] to examine the traffic flow behavior at the selected portion of Istanbul highway. Three-regimes of traffic flow i.e. congested, intermediate and free-flow regimes are discriminated with respect to the observed speed values. Here, the kinetic energies of each regime are found. In the light of those regime energies, the main purpose is to find the probabilities of the speed values observed on the highway. Finding the probability distributions of the regime speeds via entropy maximization is the main course of action for the statistical inference from the traffic data. Then the new speed classification of the traffic regimes is proposed through I-dice model, processing both data and moments, unlike Brandeis dice which considers moments only.

In this paper, the kinetic energies are expressed through average speed values, and utilized in I-dice problem. The interactions in traffic data could be long-range, and hence the traffic flow could involve non-Markovianity, (multi)fractality etc., as discussed e.g. in [48]. In this paper, the authors inquire about the state dependency in the traffic flow, and it is recommended that I-dice model would be a convenient tool to evaluate the randomness of a given traffic problem. The difference between the given speed classification [47] and the classification proposed by I-dice asserts that the traffic states are not independent, and the randomness is not in question.

2. Methodology

2.1. Characteristics of the Data Obtained from the Selected Highway Segment

The authors employed the vehicle speed data obtained from a traffic observation point at the selected segment of Istanbul highway. The observation point is adjacent to the large number of residential and commercial areas. The data represent only one direction flow at three lanes for 15 consecutive days. The data were measured at two-minute intervals from 5.00 p.m. to 9.00 p.m. The measured data at each two-minute intervals correspond to the average values.

The frequency curves of the data from the observation point at the selected highway segment (Figure 1, Figure 2, Figure 3) clearly emphasize bi-modality. All three graphs of the three lanes have two distinct peaks, a transition from one peak to the other, and a gradual increase from zero to the first peak.

One may generate as many states as one wishes. But to make things a bit easier, three states are claimed to exist. The authors, thus, introduce three traffic flow regimes i.e. congested, intermediate and free-flow from the experimental study on Istanbul highway. Three lanes are investigated using data recorded at two-minute intervals. For example, up to 45 km/h, slow and fast lanes designate congested regime, whereas middle lane is considered to be in the congested regime up to 50 km/h. Furthermore, between 45 km/h and 85 km/h traffic flow is observed to be more stable and the intermediate regime occurs at the slow lane between these two limits. The speed values starting from above 85 km/h indicate the start of the free-flow regime for the slow lane. This regime begins at roughly 100 km/h on the middle and fast lanes, Table 6.

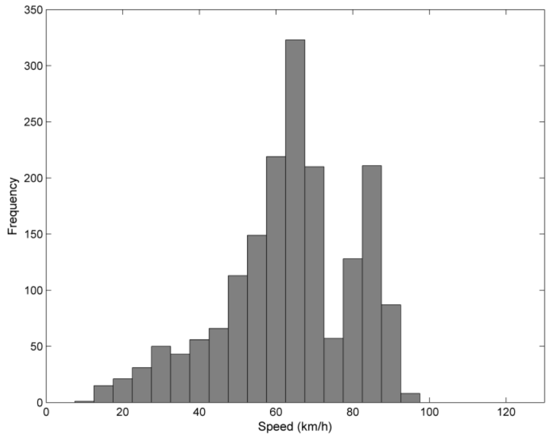

Figure 1. Speed frequency distribution at slow-lane

The authors incorporate Jaynes' maximum entropy principle in this study à la Brandeis dice problem proposed by Jaynes, with only difference being MaxEnt is replaced by MRE. The authors propose a three-faced hypercube (Figure 4), as many faces as the number of states, to implement the maximum entropy principle to the highway traffic flow. On the hypercube, each face is matched every aforementioned traffic regime.

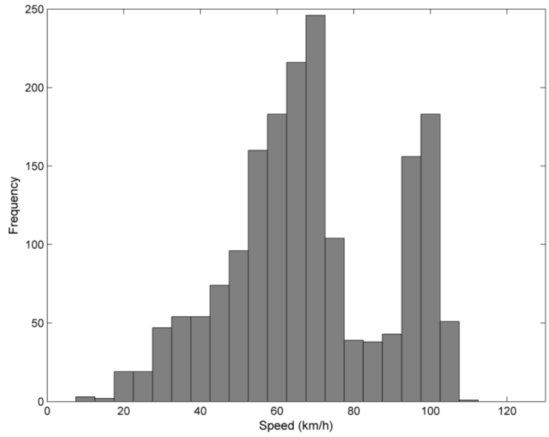

Figure 2. Speed frequency distribution at middle-lane

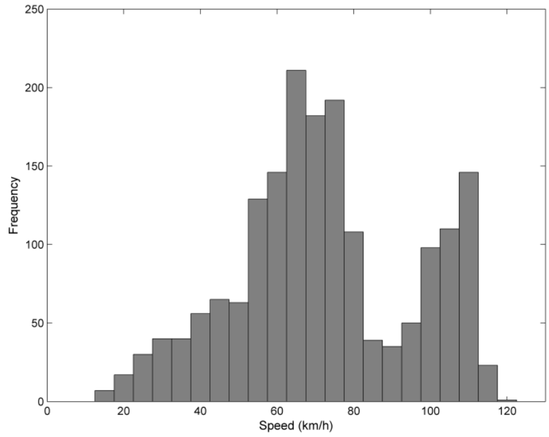

Figure 3. Speed frequency distribution at fast-lane

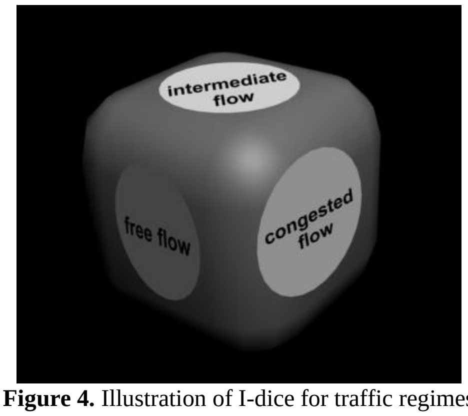

Figure 4. Illustration of I-dice for traffic regimes

2.2. Maximum Entropy Principle and Brandeis Dice Problem

Jaynes [49] explains the application of maximum entropy principle to statistical mechanics in a detailed way. Foundations of probability theory and notion of entropy is also well discussed in the study [50]. Furthermore, Uffink [51] points out that the maximum entropy principle is a particular method of statistical inference and it assigns numerical values to probabilities when certain partial or incomplete information is given. The method attains maximum (Shannon) entropy depending on probability distributions under the constraints of norm and mean energy.

The constraints are the only available information in the maximum entropy principle. Maximum-entropy probabilities of energy levels are based on the celebrated Boltzmann-Gibbs statistical mechanics [49].

The uniform probability distribution, “the maximum ignorance”, maximizes the Shannon entropy since there is no information except the constraints. This closely resembles Laplace’s “principle of insufficient reason” as information is not enough to determine whether an event is more likely or not than any other. Thus, the events are equally likely to occur without additional information and this leads to the maximum entropy principle. Application of the MaxEnt to a die is known as Brandeis dice problem. Jaynes’ Brandeis dice example is a famous application of maximum entropy principle, where a die is tossed N times and the average number of spots up is detected. One might expect an expected value of 3.5 for an unloaded die for infinite N.

According to maximum entropy principle, BG entropy is maximized using Lagrange multiplier technique subject to the norm and mean energy constraints.

$$S_{BG} = -k \sum_i p_i \ln (p_i) \tag{1}$$

where p_i is the probability of finding any system in state i, k is constant. These constraints are given as

$$\sum_i p_i = 1 \tag{2}$$

$$\sum_i p_i \, \varepsilon_i = \langle E \rangle \tag{3}$$

where ε_i is the energy of the state i, and p_i is the probability of the state i, $\langle E \rangle$ is the testable information, i.e. the expected energy.

Now, the Lagrangian is obtained, which is to be derivated with respect to p_i

$$L(p_i, \lambda, \beta) = -\sum_i p_i \ln(p_i) - \lambda(\sum_i p_i - 1) - \beta(\sum_i p_i \, \varepsilon_i - \langle E \rangle) \tag{4}$$

$$\frac{\partial}{\partial p_i}\left(-\sum_i p_i \ln(p_i) - \lambda(\sum_i p_i - 1) - \beta(\sum_i p_i \, \varepsilon_i - \langle E \rangle)\right) = 0 \tag{5}$$

where λ and β are the Lagrange multipliers.

The canonical partition function is obtained below. It is employed to find the probabilities of states in a system.

$$\sum_i e^{-\beta \varepsilon_i} = Z \tag{6}$$

where β is the inverse temperature, Z is the partition function.

As a result, the probability distribution of any state of the system is expressed by the canonical partition function in the following form

$$p_i = \frac{e^{-\beta \varepsilon_i}}{Z} \tag{7}$$

Consider Brandeis dice problem in which an ordinary six-faced die is tossed a certain number of times. An average number of spots is computed from N trials. For an honest die, this average is 3.5.

$$\sum_i p_i \varepsilon_i = 3.5 \tag{8}$$

The expected value is given by the equation below

$$\langle E \rangle = \frac{e^{-\beta}+2e^{-2\beta}+3e^{-3\beta}+4e^{-4\beta}+5e^{-5\beta}+6e^{-6\beta}}{e^{-\beta}+e^{-2\beta}+e^{-3\beta}+e^{-4\beta}+e^{-5\beta}+e^{-6\beta}} \tag{9}$$

Depending on this information, β and Z values are calculated as 0 and 6, respectively. By substituting these values into the probability equation, the probability values assigned to each face are obtained as 1/6. Namely, any face could be expected with equal probability.

2.3. Maximum Relative Entropy (MRE)

The MaxEnt method by Jaynes was conceived to assign probabilities to events, as just mentioned above. A new revised technique, maximum relative entropy, MRE [24], could update the probabilities if required. This update is based on the constraints which could themselves be updated.

Giffin [24] deals with maximum relative entropy (MRE) and shows how data and moment constraints are considered and processed in a given problem. Giffin [24] states that MRE method involves both Bayes rule and MaxEnt principles. To setup the MRE formulation, Bayes rule provides data constraints, whereas MaxEnt produces moment constraints. Two approaches i.e. simultaneous and sequential updating are discussed in the study. In both approaches, the application of the constraints is posed differently. By the nature of the MRE technique, a caveat is timely: The order of the constraints changes the outcome, hence they are not commutative. The order of the constraints is a problem when the constraints are processed sequentially, but not when processed simultaneously.

So far, the nature of information has dictated what technique to follow: Data could be processed by Bayes’ rule, not the moment constraints. Vice versa is true for MaxEnt. MaxEnt could only process the moments. MRE now combines both techniques and could work with data and moments. One of the major differences from the MaxEnt is that entropic form is replaced by the relative entropy as follows.

The relative entropy is maximized on the joint posterior. The reference probability is the old joint prior. The constraints, $C1$ is the normalization constraint, whereas $C2$ is the energy constraint.

$$S = -\int P(y,\theta) log \frac{P(y,\theta)}{P_{old}(y,\theta)} dy d\theta \tag{10}$$

where $\theta = \{\theta_1, \theta_2, \ldots\ldots \theta_i\}$ are the probabilities, and $y = \{y_1, y_2, \ldots\ldots y_i\}$ are the instances for *i*th-face of the cube to turn up in a total of *n* trials.

Subject to

$$C1: \int P(y,\theta)\, dy d\theta = 1 \tag{11}$$

And,

$$C2: \int P(y,\theta)\, f(\theta) dy d\theta = \langle E \rangle \tag{12}$$

where $f(\theta)$ is the energy function of (θ).

Maximize the Lagrangian below

$$L = -\int P(y,\theta) log \frac{P(y,\theta)}{P_{old}(y,\theta)} - \alpha[\int P(y,\theta)\, dyd\theta - 1] - \beta[\int P(y,\theta)\, f(\theta) dyd\theta - \langle E \rangle] \quad (13)$$

where α and β are the Lagrange multipliers.

To find

$$P(y,\theta) = P_{old}(y,\theta) \frac{e^{-\beta f(\theta)}}{Z} \quad (14)$$

Where,

$$P_{old}(y,\theta) = P_{old}(\theta) P_{old}(y|\theta) \quad (15)$$

And the likelihood is no other than

$$P(y_1, y_2, \ldots \ldots y_i | \theta_1, \theta_2, \ldots \ldots \theta_i) = \frac{n!}{y_1!, y_2! \ldots . y_i!} \theta_1^{y_1} \theta_2^{y_2} \theta_i^{y_i} \quad (16)$$

That is the multinomial distribution, in a total of *n* trials.

The partition function is

$$Z = \int P_{old}(\theta)\, e^{-\beta f(\theta)} d\theta \quad (17)$$

$$\frac{\partial \ln Z}{\partial(-\beta)} = \langle E \rangle \quad (18)$$

Now the posterior distribution, given the moment is

$$P(\theta | C1, C2) = P_{old}(\theta) \frac{e^{-\beta f(\theta)}}{Z} \quad (19)$$

Which concludes the problem of finding the posterior distribution, given the expected value.

Should the data then be provided,

$$P(\theta | C1, C2, C3) = P_{old}(\theta) P(y|\theta) \frac{e^{-\beta f(\theta)}}{Z'} \quad (20)$$

where $C3$: $P(y)$ is the application of observation data and Z'.

$$Z' = \int P_{old}(\theta) P(y|\theta)\, e^{-\beta f(\theta)} d\theta \quad (21)$$

Please note the difference between Z and Z'. This is why the sequencing changes the eventual distributions.

Before presenting the next section, the flowchart illustrating the steps undertaken in this study is briefly provided for the readers below (Figure 5). The results are discussed in the following section.

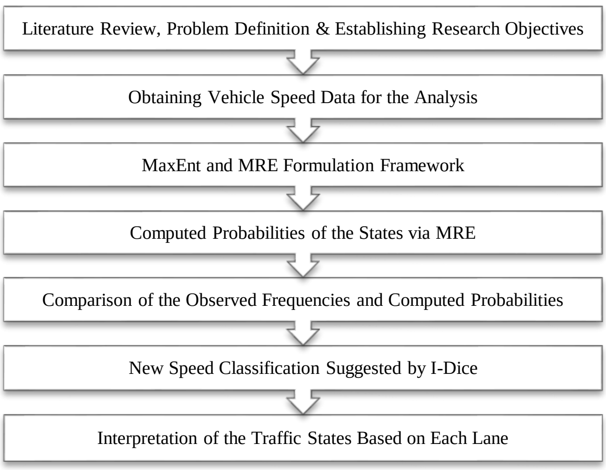

Figure 5. Flowchart illustrating the steps

3. Results and Discussion

In this section, MRE is used in Jaynes' Brandeis dice example to model traffic regimes in a traffic flow. Along a similar vein, a die-like hypercube is generated. Unit kinetic energies replace the spots on each and every face of the cube. *Iyte* is alma mater of one of the authors, and the initial of the university name inspires us to call this cube shortly I-dice. The number of the faces of this cube would be the same as the number of states for the traffic flow on a given highway segment in Istanbul. I-dice may have less or more than 6 faces, by virtue of the number of the regimes, and it has 3 in this I-dice toy example. These regimes are the congested, intermediate and free-flow traffic conditions.

In this part, the kinetic energies of the three traffic regimes are calculated. The energy representation of the traffic flow system could be provided by a simple kinetic energy expression composed of average speeds of the states. The average speeds for different regimes for each lane are computed by the frequency count, Table 1. The histograms may be inspected at the Figures 1, 2 and 3.

Table 1. Average Speeds For Each Lane (km/h)

Regime	Slow Lane	Middle Lane	Fast Lane
Congested	35.3	37.9	34.3
Intermediate	65.4	71.5	69.2
Free-Flow	87.6	102	106.9

The kinetic energy per state is given per unit mass. The speeds are the measured average speeds of the states.

$$E_i = \frac{1}{2}(V_i)^2 \quad (22)$$

where V_i is the average speeds of the states and E_i is the energy of the *i*th-state.

If the states of the energies are divided by E_1 or the energy of the congested regimes of each lane, the following scaled energy form is obtained

$$E_s = \frac{E_i}{E_1} \quad (23)$$

where E_s is the scaled energy.

And tabulated in Table 2.

Table 2. Scaled Energies For Each Lane, E_s

Regime	Slow Lane	Middle Lane	Fast Lane
Congested	1	1	1
Intermediate	3.43	3.56	4.07
Free-Flow	6.16	7.25	9.18

The traffic regimes could be visualized on I-dice as illustrated in Figure 4. Table 3 is composed by again frequency count and individual values are found out of the total vehicle passed within the two-minute intervals during the specified hours of 15 consecutive days. For example, for the slow lane and congested regime, 16 % of the vehicles belonged to this interval, i.e. less than 45 km/h.

Table 3. Observed Probabilities For Each Lane

Regime	Slow Lane	Middle Lane	Fast Lane
Congested	0.16	0.21	0.146
Intermediate	0.785	0.758	0.69
Free-Flow	0.055	0.032	0.164

Table 4 is formed using the formula

$$\sum_i p_i E_s = \langle E \rangle \tag{24}$$

Table 4. Average Energy For Each Lane, $\langle E \rangle$

	Slow Lane	Middle Lane	Fast Lane
Average Energy	3.19	3.14	4.46

where Tables 2 and 3 are made use of.

As expected, the slow lane has a lower expected energy compared to the fast lane, though the middle lane is open for discussion. Table 5 is computed by MRE formulation framework given in Section 2.3. Here, beta values are calculated first, then partition functions could be found out. Probabilities are computed via these partition functions for the lanes. Table 6 lists the state characterization, which was performed earlier [47].

Table 5. Computed Probabilities For Each Lane via MRE

Regime	Slow Lane	Middle Lane	Fast Lane
Congested	0.3997	0.4618	0.6226
Intermediate	0.3315	0.3319	0.2934
Free-Flow	0.2688	0.2063	0.0839
β	0.077	0.129	0.245
Z	0.772	0.634	0.4190

Table 6. Current Speed Classification For Each State (km/h)

Regime	Slow Lane	Middle Lane	Fast Lane
Congested	< 45	< 50	< 45
Intermediate	45-85	50-100	45-100
Free-Flow	> 85	> 100	>100

Please contrast Table 3 to Table 5 in that the observed frequencies did not match the computed probabilities for the states. For the MRE technique to work, one only needs Tables 2 and 4. The reasons behind this mismatch could be argued between Tables 3 and 5. What the authors believe is that I-dice formulation of the traffic states suggests a new classification of the speed regimes based on Table 7, in stark contrast to the earlier regime classification provided on Table 6. If the researchers like to determine the classification of regimes based on their representative speed ranges, MRE technique could compute the probabilities, and by working backwards, the energies of each state may be specified. The observed data in our case did not reflect Boltzmann-Gibbsian tendencies, in that energies of the states are not inversely proportional to their probabilities. If the states are organized based on Table 7, the new distribution now follows a BG statistics inserted in MRE. MRE imposes a more uniform distribution of probabilities for the speeds than the observed ones, Tables 3 and 5.

Table 7. Speed Classification Suggested by I-dice For Each State (km/h)

Regime	Slow Lane	Middle Lane	Fast Lane
Congested	< 60	< 62	< 72
Intermediate	60-80	62-90	72-107
Free-Flow	> 80	> 90	> 107

4. Conclusions

Modelling traffic states à la Brandeis dice and calling it I-dice is proposed in this study. Jaynes's toy example, Brandeis dice example, has generated a controversy despite its simplicity. The motive behind the critics was that even though Brandeis dice example has shown the inner workings of MaxEnt principle, the use of frequency count for probability is simply not a well-posed problem. In this respect, I-dice example could also be based on the same grounds, but we aware the fact that we only tried to adapt MRE principle into our toy example. One may see a comparison of the techniques, and other constructive arguments in [51].

Here, the celebrated Brandeis dice problem using MRE is now converted into I-dice model in a traffic flow setting. This model could take advantage of data as well as moments. I-dice example is mainly proposed to model the traffic states on a hyper-dimensional cube. A three-lane and three-state example was given from a highway segment. All the frequency counts, average speeds and expected energies are computed and tabulated. Before getting started, (the observed) probabilities of the states were already known. Expected energies were generated for each lane.

The following inferences are drawn from this study:

- Processing the moments through MRE technique, it is seen that the computed probabilities are different from the observed ones. This has shown that if the classification of the states is done, as the authors have done using Langevin analysis [47], then the probabilities did not follow an inverse relationship with energies as is the case with Boltzmann-Gibbs thermostatistics.
- This is to say that with the current characterization of the regimes, BG statistics is not applicable with the selected highway portion. But the authors wished to find out what would be the new classification of the states so that they now follow BG theory embedded in MRE.
- Based on this framework, the state characterization for each lane is obtained. In other words, the speed classification suggested by I-dice for each state is consistent with MRE findings.
- One now may use I-dice method to classify the traffic states in terms of speeds once their probabilities are found out, and determine the randomness and Markovianity in the state classification.
- The histograms for each lane could also depict the non-Gaussianity of the speed values, hence would involve long-range correlations and non-Markovianity. This is quantitatively verified over a mismatch of the speed classifications in this study.

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