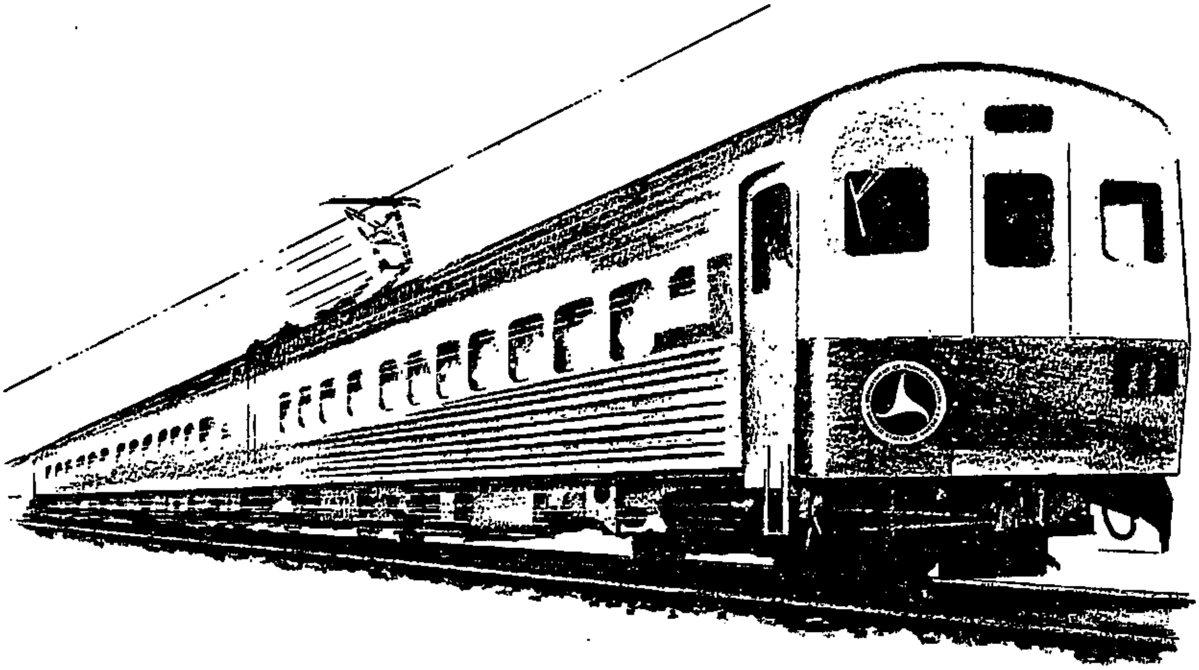
ON THE STABILITY OF THE RAILROAD TRACK IN THE VERTICAL PLANE

by

Arnold D. Kerr



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SUMMARY

The paper reviews and discusses various aspects of railroad track buckling in the vertical plane. Buckling tests of straight tracks are reviewed first. A review of the published analyses on vertical track buckling reveals that they may be grouped into two main categories. In one category, the authors assume that the track is an elastic beam which is continuously supported by a Winkler base, before as well as during buckling. In the other group, the authors assume that the track is a beam of uniform weight, which rests on a "rigid" base and that the buckling load is reached when part of the track lifts itself off the base. To clarify the validity of some of the assumptions made, two simple models which represent the assumptions made are studied first. This is followed by a review of the literature. It is shown that the assumption of continuous elastic support during buckling is not admissible. It is also shown that for buckling with lift-off, the use of linearized analyses may lead to erroneous results.

INTRODUCTION

A conventional railroad track consists of two parallel steel rail strings attached to closely spaced cross-ties, which are imbedded in a gravel base, called the ballast, as shown Fig. 1. The rail ends are joined by means of slotted web plates and bolts, thus forming an expandable joint.

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Because a rail of length L, when subjected to a temperature variation ΔT, changes its length by

$$\Delta L = \alpha \Delta T L$$

the gap-width in the joint increases during the winter and decreases during the summer. For a certain temperature increase, the gap closes. A further increase in temperature induces compression forces in the rail string, which may buckle the track.

The expandable rail joint weakens the track structurally. It increases the maintenance costs of the track and the trains, it increases the power consumption of a running train, and the frequent periodic noise generated by a train rolling over such joints is a source of discomfort to many passengers.

Therefore, it is only natural that since the early days of railroad track construction, there was a desire to eliminate some of the joints by increasing the length of the rails; with the elimination of all joints, i.e., with the use of a welded track, as the final goal.

For high speed trains, the continuously welded track is a necessity. For this reason, during the past decade, thousands of miles of welded track were installed throughout the world. However, the complete elimination of joints increases the possibility of buckling during the hot summer days and, indeed, a number of derailments caused by buckled tracks were recently reported in the literature by C.F. Rose [1]. The introduction of the welded track and the reported accidents, due to buckled tracks, make it necessary to re-examine the state of the art of the track instabilities. The purpose of the present paper is to review the literature on track buckling in the vertical plane.

TEST RESULTS

Tests to determine the response of the track, were described by various investigators already in the past century. These results and more recent ones are published in various national and international journals of railroad engineering*). An extensive collection of track test data was presented in five papers by O. Ammann and C.v. Gruenewaldt [2] [3] in 1928, 1929, 1932 and 1934, and in monographs by A. Wasiutyński [4] published in 1937, by M.T. Chlenov [5] published in 1940, and by N.B. Zverev [6] published in 1962.

In 1927, A. Wöhrl [7] reported that cases of buckling of the conventional railroad track came to his attention, although he could not find any references in the literature.

An early experimental investigation of track instability, due to axial forces, is described in Part IV of the study by Ammann and v. Gruenewaldt published in 1932. The axial forces were induced in the rails by means of two hydraulic jacks with a total capacity of up to 290 tons. Prior to loading, the joint gaps in the test tracks were filled with metal strips in order to create "continuity" in the rail and to prevent large axial displacements. In the close vicinity of the jacks, the track was preloaded vertically, in order to prevent track buckling at the jacks.

Buckling tests were conducted on tracks of the Baden-type (Badischer Oberbau) and the K-type (Reichsoberbau K) with wood and metal ties, respectively. The rail lengths used in the track were 12, 15, 30, and 60 meters.

In all tests of the straight K-type track on metal ties, buckling took place by lift-off from the ballast. The length of the buckling curve was about 30 meters.

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*) A major source of early results is the journal "Organ für die Fortschritte des Eisenbahnwesens"

Typical buckling modes are shown in Fig. 2. The occurance of cusps in the vertical buckling curves of the K-type track was attributed by the authors to the fact that the K-type joint plates were much weaker than those in the Baden-type joint. The authors also report that during the tests of the Baden-type tracks, when the buckling load was reached, the track first snapped up, stayed in this position for a short while, and then fell sideways to the ground. Since the K-type track stayed up after buckling, they attributed this sidesway to the smaller lateral rigidity of the Baden-type track.

In all of these tests, on the straight track, no noticeable side displacements were observed prior to buckling.

Results of other buckling tests were reported by J. Nemcsek [8] in 1933, F. Raab [9] in 1934, M.T. Chlenov [5] in 1940, Italian Railways [10] in 1942, M. Sonneville and M. Sergé [11] in 1948, F. Raab [12] in 1956, M. Numata [13] in 1957, the Permanent Way Society of Japan [14] in 1958, M. Numata [15] in 1960, Unyi Béla [16] in 1960, F. Birmann and F. Raab [17] in 1960, D.L. Bartlett, J. Tuora, and G.R. Smith [18] in 1961, N.B. Zverev [6] in 1962, and E.M. Bromberg [19] in 1966. In many tests, the axial forces were induced by heating the rails with electric currents. According to the published results in these tests the track buckled sideways. A possible explanation of this phenomenon was given recently by A.D. Kerr [20].

In the railroad literature, the statement is often encountered that according to analytical results, buckling in the vertical plane can not take place. Such statements seem to contradict various test results.

This situation suggests the need for a critical review of the available analyses on track buckling in the vertical plane. This is necessary, particularly in view of some obvious conceptual, as well as formulation, errors contained in the relevant literature.

GENERAL DISCUSSION OF VERTICAL BUCKLING

A uniform temperature increase in a welded straight track induces in the rails, due to constrained thermal expansions, an axial compression force N_t, as shown in Fig. 3(a). For large values of N_t, the track may buckle out vertically. In the lift-off region of length ℓ, part of the thermal expansions are released. This results in a reduction of the axial force to $\tilde{N}_t$. In the adjoining regions, each of length a, due to ballast resistance to axial displacements of the track, the constrained thermal expansions vary; so does the axial force $\tilde{N}_t < N < N_t$, as shown in Fig. 3(b)*. According to above observations, vertical buckling is a local phenomenon. That is, except for the length $(\ell + 2a)$, the track is not affected by it.

The published papers which analyze vertical track buckling, may be grouped into two main categories. In one group, the authors assume that the track is a weightless beam which is continuously supported on a Winkler base, before as well as during buckling. In the other group, the authors assume that the track is a beam of uniform weight which rests on a "rigid" base, and the buckling load is reached when part of the track lifts itself off the base.

The first point which requires clarification is, which of these two assumptions is closer to reality. In particular, it has to be established, in view of the observed lift-off during vertical buckling, if the analyses which are based on the assumption that the track remains continuously supported during buckling, will yield reasonable buckling loads.

To study these and related questions, in the following we consider first two models, which exhibit the respective buckling mechanism, but are amenable to simple analyses.

1) Model of track on elastic base

We consider first the model shown in Fig. 4. It consists of four "rigid"

*) This variation is not necessarily linear.

bars constrained at the interconnecting joints by spiral springs. These springs represent the "flexural rigidity" of the track in the vertical direction. It is assumed, that only joint 3, which is constrained by a straight spring, can move vertically. The other joints can only slide horizontally, The straight spring represents the "elastic foundation" of the track. In order to include the effect of geometric imperfections, it is assumed that in the unstressed state bars $\overline{234}$ exhibit an imperfection θ_o, as shown in Fig. 4(a).

The shown model is of one degree of freedom. Hence, its equilibrium states are described by one algebraic equation, with θ as unknown.

To determine the equilibrium states of the chosen system, we consider the free body diagram of bar $\overline{34}$, shown in Fig. 5, and set up the moment equilibrium about point 4. The resulting equation is

$$3s(\theta - \theta_o) + \frac{S}{2} L \cos\theta - P L \sin\theta = 0 \tag{1}$$

where s is the spring constant of the spiral springs. Noting that the force in the straight spring is

$$S = k L (\sin\theta - \sin\theta_o) \tag{2}$$

where k is the spring constant, equ. (1) may be written (for $\sin\theta \neq 0$) as

$$P^* = \frac{(\theta - \theta_o) + k^* (\sin\theta - \sin\theta_o)\cos\theta}{\sin\theta} \tag{3}$$

where

$$P^* = \frac{PL}{3s} \quad ; \quad k^* = \frac{kL^2}{6s} \tag{4}$$

Equilibrium equation (3) was evaluated for $0 < \theta < \frac{\pi}{2}$, $k^* = 0.1$ and 0.7 and $\theta_o = 0$ and 1°, and the determined equilibrium branches are shown in Fig. 6.

It may be seen that when the structure is initially straight ($\theta_o = 0$), it remains straight until P reaches the value P_{cr}, at which a deformed equilibrium branch bifurcates. When $\theta_o \neq 0$, the structure does not exhibit a bifurcation point. Note also the effect of the parameter k^* upon the post-buckling response.

In order to establish which of the determined equilibrium states are stable, we utilize the Lagrange energy criterion. According to this criterion, an equilibrium configuration of a conservative mechanical system is stable, if the corresponding total potential energy has a proper minimum with respect to all kinematically admissible displacements.

The considered system is conservative. Its total potential energy Π is

$$\frac{\Pi}{6s} = \frac{1}{2}\,(\theta - \theta_o)^2 + \frac{k^*}{2}(\sin\theta - \sin\theta_o)^2 - P^*\,(\cos\theta_o - \cos\theta) \qquad (5)$$

Above equation was evaluated for $\theta_o = 0$ and $k^* = 0.7$ and the results are presented as energy level curves in Fig. 7(a). The corresponding equilibrium branch based on equ. (1), is presented in Fig. 7(b).

First, it should be noted, that according to the principle of stationary total potential energy

$$\frac{\partial\Pi}{\partial\theta} = 0$$

yields the equilibrium equation (1). Hence, points on the energy level curves with a horizontal tangent, correspond to equilibrium configurations. This correspondence may be easily verified by correlating the graphs in Fig. 7.

According to the Lagrange stability criterion, minima on the energy level curves correspond to stable, and maxima as well as horizontal inflection points to unstable equilibrium configurations. Hence, according to the Π-curves of Fig. 7, the undeformed equilibrium states for $P < P_{cr}$ are stable and those for $P \geq P_{cr}$ are unstable. It also follows that the equilibrium states on the branch AL are unstable and those on branch LB are stable. (Point L is defined as the lowest point on the deformed branch.)

The shown equilibrium branch ALB exhibits a load $P_L < P_{cr}$, which should be considered as the "safe" buckling load for design purposes. This is so, because for $P < P_L$ there exists only one equilibrium position which is stable, whereas for a given $P_L < P < P_{cr}$ there exist three equilibrium positions and the undeformed system, when sufficiently perturbed, may snap into the position on the stable deformed branch with large deformations. For additional details on this subject the reader is referred to the papers by Th. v. Kármán and A.D. Kerr [21] and A.D. Kerr [22].

The response of an actual railroad track is governed not by an algebraic equation nonlinear in θ, such as (1), but by nonlinear differential equations. Because it is very difficult to solve these equations, most investigators used linearized equations. To study the effect of linearizations, equilibrium equation (1) is linearized in θ. It becomes

$$(\theta - \theta_o) + k^* (\theta - \theta_o) - P^* \theta = 0 \tag{6}$$

or rewritten, for $\theta \neq 0$,

$$P^* = \frac{(1 + k^*)(\theta - \theta_o)}{\theta} \tag{7}$$

For the initially straight system, i.e., when $\theta_o = 0$, equ. (6) reduces to

$$(1 + k^* - P^*)\,\theta = 0 \tag{8}$$

Equation (8) corresponds to the usual linear eigenvalue problem of stability theory. It is satisfied for the trivial solution $\theta = 0$. For $\theta \neq 0$, equ. (8) is satisfied when

$$P^*_{cr} = 1 + k^* \tag{9}$$

This is the Euler buckling load for the system under consideration. However, because the post-buckling curve exhibits a P_L load, it may be concluded that for the analyzed problem, P_{cr} is not a safe buckling load [22].

To show the effect of linearization on the equation with $\theta_o \neq 0$, equ. (7) was evaluated for $k^* = 0.1$ and 0.7 and $\theta_o = 1$, and the resulting branches are shown in Fig. 6. It may be seen that for small deformations the actual and linearized equilibrium branches coalesce. However, as θ increases the branches diverge and exhibit different responses. It should be noted that as $P \to P_{cr}$, on the linearized branch $\theta \to \infty$, whereas the actual branch exhibits a finite value of θ. Thus, the linearized branch for $\theta_o = 0$ or $\theta_o \neq 0$, and large θ does not represent, even approximately, the actual problem.

2) Model of track with lift-off

This model is similar to the one discussed before, except that now the weight of the system is taken into consideration and joint 3 rests on a "rigid" base and is free to lift-off, as shown in Fig. 8.

This structure was recently analyzed by A.D. Kerr [20]. Proceeding as above, it may be shown that the equilibrium equation for the non-trivial states is

$$P^* = \frac{(\theta - \theta_o) + q^* \cos\theta}{\sin\theta} \tag{10}$$

where

$$q^* = \frac{qL^2}{6s} \tag{11}$$

The corresponding equilibrium branches for $\theta_o = 0$ and different values of q^* are shown in Fig. 9. Note that for the initially straight structure, $\theta = 0$ is an equilibrium branch for any $q \geq 0$. Also of interest, is the different character of the deformed equilibrium branches for $q > 0$ as compared to the one for $q = 0$. In particular, that the branch for $q = 0$ and $\theta \geq 0$ intersects the undeformed branch at $P^* = 1.0$, whereas all branches, which take the weight into consideration, do not intersect the undeformed branch but approach it asymptotically at infinity.

To determine the stability of the determined equilibrium states, note that the total potential energy Π is

$$\frac{\Pi}{6s} = \tfrac{1}{2}\theta^2 - P^*(1 - \cos\theta) + q^*\sin\theta \tag{12}$$

The corresponding energy level curves for $q^* = 0.06$ are shown in Fig. 10(a). The corresponding equilibrium branches are shown (as solid lines) in Fig. 10(b).

According to these graphs, the undeformed equilibrium states (with $\theta = 0$) are stable for any P and $q > 0$, the equilibrium positions on branch AL are unstable, and the equilibrium positions on branch LB are stable.

It should be noted, however, that with increasing axial force P, the energy barriers $\Delta\Pi$ are decreasing. Hence, with increasing P, the disturbances which are sufficient to overcome $\Delta\Pi$ and snap the system into the deformed equilibrium state, are decreasing. Therefore, although the straight equilibrium state is theoretically stable for any $P > 0$, from a practical point of view the system becomes less stable with increasing P.

An actual track always deviates slightly from a perfectly straight line. To study the affect of very small deviations in the vertical plane, equ. (10) was evaluated for $\theta_o = 0°, \tfrac{1}{2}°, 1°, 2°, 5°$. The obtained equilibrium branches are shown in Fig. 11. It may be seen that for $\theta_o = 1°$ and P increasing from zero, the structure does not deform until P^* reaches the value P_u^*. At P_u^*, it snaps out into a strongly deformed equilibrium state on branch LB, as indicated in Fig. 11.

The load P_u is denoted in the stability literature as the "upper buckling load". Note, however, that under the influence of outside disturbances, such as vibrations or impulse loads, the system may snap-out at smaller loads $P_L < P < P_u$. Hence, also in this case, the "safe" buckling load is P_L.

To study the effect of linearization, equilibrium equation (10) was linearized. It becomes

$$P^* = \frac{(\theta - \theta_o) + q^*}{\theta} \tag{13}$$

The corresponding graphs are shown in Fig. 9 and Fig. 11 as dashed lines.

It is of importance to note that with increasing θ all linearized branches approach asymptotically the value $P^* = 1$, which is the Euler buckling load for $q = o$, although for the actual problem for which $q > o$ such a load has no meaning.

The expression of Π, which corresponds to the linearized equation, is quadratic in θ. Namely

$$\frac{\Pi}{3s} = \frac{1}{2}\theta^2 - P^*\frac{\theta^2}{2} + q^*\theta \tag{14}$$

According to the energy level curves, which are based on equ. (14), the equilibrium branches of the linearized analysis, for $\theta > 0$, are unstable. Thus, according to the linearized analysis, there does not exist a stable deformed state as shown in Fig. 8; a result which contradicts reality. This finding suggests that a linearized analysis may not be suitable for the analysis of the track with lift off.

The above discussion should be taken into consideration when evaluating the validity of the claims by H. Meier [23], F. Raab [24], and M.T. Huber [25] that, according to their analyses, the straight track can not buckle in the vertical plane.

For additional results and comments, as well as for an analysis of a model subjected to thermal loads, the reader is referred to Ref. [20]. The analyses and graphs presented in Ref. [20] reveal that the P_L value for the model subjected to a mechanical force is much lower than the corresponding P_L value due to thermal stresses. This may be an indication why in the tests by Ammann and v. Gruenewaldt [2] and Nemcsek [8], who used jacks to induce compression forces, the track buckled predominately in the vertical plane, where as in the tests which induce compression forces by heating the rails, buckling occurs predominately in the horizontal plane.

3) Comparison of results

Comparing the graphs shown in Fig. 6 with those of Fig. 9 and Fig. 11, it becomes obvious that the post-buckling response curves for these two models are very different. Since the "safe" buckling load is determined from the post-buckling response curves, it appears that at least one of the models is not suitable for the representation of vertical buckling. This situation, as well as the other findings, will be referred to in the following when reviewing the literature on vertical buckling.

THE BUCKLING ANALYSIS OF THE TRACK AS A COMPRESSED BEAM ON A CONTINUOUS WINKLER BASE

A compressed beam on a Winkler base is shown in Fig. 12. This model was utilized by F. Corini [26], M.T. Huber [27], and recently by E. Engel [28] for the analysis of vertical buckling.

To describe the beam response, they used the differential equation

$$EI \frac{d^4w}{dx^4} + P \frac{d^2w}{dx^2} + k\,w = 0 \tag{15}$$

where w is the vertical deflection of the beam, EI is its flexural rigidity in the vertical plane, P is a constant compression force, and k is the foundation modulus.

Equ. (15) is a fourth order linear ordinary differential equation with constant coefficients. A number of solutions for different boundary conditions are presented in the books by M. Hetényi [29] and A.R. Rzhanicyn [30]. For the infinite beam the critical buckling load is

$$P_{cr} = 2\sqrt{k\,EI} \tag{16}$$

and the corresponding buckling mode is an infinite sine wave, as shown in Fig. 13. This buckling mode differs greatly from the one observed in actual tests (Fig. 3), and therefore, it is reasonable to anticipate also the corresponding P_{cr} will differ substantially from the actual buckling load.

The determination of P_{cr} for the compressed infinite beam on a Winkler base, subjected also to its own weight, was discussed by A.D. Kerr [31] in 1969.

There it was shown that when the foundation response is assumed to be linear, as in equ. (15), a uniform weight of the beam has no effect upon the critical load, P_{cr}.

These observations, in conjunction with the findings of the previous section suggest that the model, in which the foundation is attached continuously to the beam before and after buckling, is not suitable for track buckling in the vertical plane.

THE BUCKLING ANALYSIS OF A TRACK AS A COMPRESSED BEAM ON A RIGID BASE

A compressed beam which rests on a "rigid" base and is subjected to its own weight q (per unit length) is shown in Fig. 14. This model was used for the analysis of vertical track buckling by H. Kayser [32], C.v. Gruenewaldt [33], A. Bloch [34], H.v. Sanden [35], K.N. Mishchenko [36], H. Lederle [37], F. Corini [26], A. Martinet [38], H. Meier [39], M.T. Huber [25], R. Lévi [40], K.N. Mishchenko [41], A.A. Krivobodrov [42], H. Rubin [43], F. Schweda [44], E. Stagni [45], R. Sauvage [46], and others.

A number of these investigators (Corini, Mishchenko, Martinet, Huber, Lederle, and Sauvage), based their analyses on the linear differential equation

$$EI \frac{d^4w}{dx^4} + P \frac{d^2w}{dx^2} = -q \tag{16}$$

where q is the uniform weight of the track. In the following, it is shown that this linear equation is not suitable for the determination of the buckling load for lift-off problems. As an example, we review first the often quoted analysis by Martinet.

Martinet assumed that the deformations of the beam part, which lifts off the base, are governed by a linear differential equation. His formulation consists of the differential equation (16) and the boundary conditions (see Fig. 15)

$$\left.\begin{aligned} & w(\pm \ell/2) = 0 \\ & \left.\frac{d^2w}{dx^2}\right|_{x=\pm \ell/2} = 0 \\ & \left.\frac{dw}{dx}\right|_{x=\pm \ell/2} = 0 \end{aligned}\right\} \tag{17}$$

Since a fourth order ordinary differential equation requires only four boundary conditions, the third condition in (17) is utilized for the determination of the unknown wave length ℓ.

The general solution of differential equation (16) is

$$w(x) = A_1 \cos(\lambda x) + A_2 \sin(\lambda x) + A_3 \frac{2x}{\ell} + A_4 - \frac{q}{2P} x^2 \tag{18}$$

where

$$\lambda = \sqrt{\frac{P}{EI}} \tag{19}$$

Because the observed deflection shapes are symmetric, it follows that for the assumed origin of the coordinate system, as shown in Fig. 15, w(x) is even in x. Thus

$$A_2 = 0 \quad ; \quad A_3 = 0 \tag{20}$$

and

$$w(x) = A_1 \cos(\lambda x) + A_4 - \frac{q}{2P} x^2 \tag{21}$$

Substitution of above expression into the first two boundary conditions in (17), and solving the resulting equations for A_1 and A_4 yields

$$A_1 = -\frac{q\,EI}{P^2 \cos\left(\frac{\lambda \ell}{2}\right)} \quad ; \quad A_4 = \frac{q\,\ell^2}{8P} + \frac{q\,EI}{P^2} \tag{22}$$

Hence

$$w(x) = q\left\{\frac{EI}{P^2}\left(1-\frac{\cos\lambda x}{\cos\frac{\lambda\ell}{2}}\right)+\frac{\ell^2}{8P}\left[1-4\left(\frac{x}{\ell}\right)^2\right]\right\} \tag{23}$$

where ℓ is an as yet undetermined quantity.

For $x = o$, above expression reduces to

$$w(o) = q\cdot\left\{\frac{EI}{P^2}\left(1-\frac{1}{\cos\frac{\lambda\ell}{2}}\right)+\frac{\ell^2}{8P}\right\} \tag{24}$$

or rewritten

$$w(o)\,\frac{EI}{q\,\ell^4} = \frac{1}{(\lambda\ell)^4}\left[\left(1-\frac{1}{\cos\frac{\lambda\ell}{2}}\right)+\frac{(\lambda\ell)^2}{8}\right] \tag{25}$$

Substitution of (23) into the third equation in (17) yields the condition for the determination of ℓ. It is

$$\mathrm{tg}\left(\frac{\lambda\ell}{2}\right) = \left(\frac{\lambda\ell}{2}\right) \tag{26}$$

Thus, the third condition in (17) is satisfied when

$$\lambda\ell = 8.985,\ \ldots,\ \ldots \tag{27}$$

Using the smallest root in (27), Martinet, noting that $P = \lambda^2 EI$, obtained the relation

$$P = 80.73\,\frac{EI}{\ell^2} \tag{28}$$

and, in conjunction with equ. (25), the relation

$$w(o) = \frac{q\ell^4}{415\,EI} = 15.7\,\frac{qEI}{P^2} \quad {}^{*)} \tag{29}$$

Taking into consideration that the force in the lift-off region drops due to buckling, from

$$N_t = \alpha E A T^o \tag{30}$$

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*) Note that in the railroad literature [47], equ. (29) is sometimes presented as $P = \eta_1\sqrt{q\,EI/w(o)}$ and equ. (28) as $\ell = \eta_2\sqrt[4]{EIw(o)/q}$ where η_1 and η_2 are coefficients.

to $\bar{N}_t = P$, as shown in Fig. 3, Martinet derived the relation

$$N_t - P = \ell\sqrt{r\left(\frac{q^2\ell^2 A}{62.566\ EI^2} - \frac{r}{4}\right)} \tag{31}$$

where A is the cross-sectional area and r is a constant friction force per unit length of track due to axial displacements in the adjoining regions. A different equation for $(N_t - P)$, is presented in Ref. [47].

For the assumed values $q = 200$ kg/m, $E = 22\times10^9$ kg/m², $I = 3\times10^{-5}\,m^4$, $A = 0.011\,m^2$, $\alpha = 10.5\times10^{-6}$ per °C, and r=400 kg/m, Martinet, utilizing equ's (29) to (31), prepared the following table:

TABLE I:

ℓ m assumed	w(o) m equ. (29)	$(N_t - P)$ kg equ. (31)	P kg equ. (28)	N_t kg $=(N_t-P)+P$	T^o equ. (30)
15	0.037	3 900	236 800	240 700	94.7
20	0.117	12 880	133 200	146 080	57.5
25	0.285	29 020	85 250	114 270	44.9
27.4	0.41	40 210	70 970	111 180	43.7
30	0.59	55 420	59 200	114 620	45.1
40	1.87	129 670	33 300	162 970	64.1
50	4.56	333 030	21 310	354 340	139

The corresponding graph is shown in Fig. 16. According to this graph, no buckling can take place for a temperature increase less than T = 43.7°C. Note that Martinet's stability criterion is of the same nature as the condition $P < P_L$, discussed above.

In order to analyze the procedure used and the results obtained by Martinet, it should be noted that equ. (23) is the deflection expression of a simply supported beam shown in Fig. 17. The corresponding graphs are shown in Fig. 18. The equilibrium branch for $w(x) > o$ is shown as solid line and the branch for $w(x) < o$, which does not exist for the problem under consideration, is shown as dashed line. The vertical axis is the equilibrium branch for the undeformed state, $w(x) \equiv o$.

According to these branches for $\lambda\ell < \pi$ there exists, for a given P, only one equilibrium position with $w(x) \equiv o$, whereas for $\lambda\ell > \pi$, for a given P also a deformed position is possible. It is of interest to note that the branch for $w(x) \not\equiv o$ approaches assymptotically the value $\lambda\ell = \pi$. Since $\lambda\ell = \pi$ is the lowest eigenvlue for the problem under consideration, it could be concluded that, for the problem shown in Fig. 17, buckling may take place for $P \geqq P_{cr}$; P_{cr} being the Euler load (for examples, see [39] p. 372 and [44] p. 255). This, however, is not correct.

To gain a better understanding of the obtained results, consider the equilibrium branches of a related problem, shown in Fig. 9. These graphs suggest that the deformed equilibrium branch in Fig. 18 approaches assymptotically the lowest eigenvalue $\lambda\ell = \pi$, because it is based on the linearized equilibrium equation (16), and that the equilibrium branch of the actual problem after reaching the corresponding load P_L, will rise monotonically. On the basis of the model study, it may also be conjectured that the branch to the left of point L will be unstable and that part of the branch to the rignt of L will be stable. Thus, the stable deformed configurations observed in tests, and shown in Fig. 16, will be on the rising branch to the right of point L.

The model study also suggests that the entire deformed equilibrium branch, shown in Fig. 18, is unstable That is, equilibrium states on this branch are not the stable deformed equilibrium configurations observed in tests.

In this connection it should be noted, that the roots of the additional condition (26) determine specific points on the equilibrium branches in Fig. 18. The point which corresponds to the lowest root is denoted by 1. The equations (28) and (29) are its coordinates. Thus, equ. (28) and (29) specify a deformed equilibrium state which is unstable.

Martinet obtained the rising branch shown in Fig. 16 by taking into consideration the drop of the axial force due to buckling deformations. He correlated the temperature increase T^o with the coordinates of the equilibrium points P, w(o), and ℓ by means of equ. (31), as shown in Table 1 and Fig. 16, and then concluded that the resulting rising w(o)-branch is stable. Although it is physically reasonable that a rising branch of this type will be stable, it is questionable if expressions (28) and (29) are valid for this branch.

In view of the deviation of the linearized branch from the actual one, as indicated in Fig. 9 and Fig. 11, it is also questionable whether the linear formulation used by Martinet, and the other investigators, is suitable for the determination of the "safe" temperature increase (43.7°C in Martinet's example).

In 1950, Mishchenko [41] reviewed the analysis by Martinet. His objection was that equ. (31), which takes into consideration the stress release due to buckling, is not sufficiently accurate. In view of the above discussion, it appears that a major objection to the Martinet analysis is that it is based on the linear differential equation (16) which may not be suitable for the determination of the point L for lift-off problem.

Whereas Martinet and various other authors based their analyses on the differential equation (16) a number of the other authors (for example, H.v. Sanden, H. Meier, and K.N. Mishchenko [41]) used the energy method. This approach is discussed in the following.

According to the principle of stationary total potential energy, the condition

$$\delta \Pi = 0 \qquad (32)$$

yields the equilibrium equation and the necessary boundary conditions. Therefore, quadratic terms in Π yield linear terms in the differential equation and the corresponding boundary conditions.

An inspection of the often quoted analysis by H. Meier [39] reveals that, for the case in which it is assumed that the axial force N_t is not affected by the deformations due to buckling, the corresponding Π used, is quadratic. Namely

$$\Pi = \int_{\ell/2}^{\ell/2} \left[\frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 - \frac{N_t}{2}\left(\frac{dw}{dx}\right)^2 + qw\right] dx \tag{33}$$

Considering $\pm\,\ell/2$ as variable end points, it may be shown, using variational calculus, that the above Π corresponds to the linear equilibrium equation (16) and the boundary conditions given in (17). Thus, Meier's "simplified" analysis [39], in which the effects of the stress release due to buckling is neglected, yields, for a straight track

$$w_{min}(o) = 16\,\frac{qEI}{N_t^2} \tag{34}$$

and

$$\ell = 2\pi\sqrt{\frac{2EI}{N_t}} \tag{35}$$

which is nothing more but an approximate solution of equ. (16) and the boundary conditions in (17). Note, however, that the axial force in Martinet's equations (28) and (29) is $\tilde{N}_t$, whereas the one in Meier's equations (34) and (35), it is N_t. Note also that $w_{min}(o)$ in (34) is identical with $w(o)$ in equ. (29), as may be seen from Fig. 18, and that equ. (35) is essentially equal to equ. (28), since $(2\sqrt{2}\pi)^2 \cong 80$.

However, unlike Martinet and Mishchenko [41], Meier uses a different stability criterion. This criterion is explained in the following on the simple model shown in Fig. 8 and its response shown in Fig. 9. It is based on the fact that if for $\theta_o = 0$ and a fixed axial load, say $P^* = 1.2$, joint 3 is forced up to the level f_1, namely to the corresponding unstable equilibrium state, then the structure may buckle out and come to rest at the corresponding point on the equilibrium branch LB.

According to the procedure used by Meier, first the largest anticipated compression force in the track is determined from the difference of the highest anticipated temperatures in a given geographic region and the "neutral" temperature at which the rails were built in (say, $N_t = 200$ tons). This is followed by an analytical determination of the corresponding $f_1 = w_{min}(o)$ and ℓ values using equations (34) and (35). The obtained f_1 value is then compared with an f_o value which is the largest admissible track imperfection for the determined ℓ, established by observing railroad tracks in the field. If

$$f_o < f_1 \tag{36}$$

then, according to Meier, the track is safe.

Thus, for the criterion by Meier only the part AL of the deformed branch is of interest. Therefore, for small values of f_1 the linearized analysis may be sufficient, as shown in Fig. 9.

Although the knowledge of $f_1 = w_{min}(o)$ is useful for track maintenance, namely a compressed track should not be lifted up by $f \geqq f_1$, the validity of Meier's criterion (36) as a general stability criterion, is questionable. This is so because, as shown in the model study, even an initially straight track may buckle out if sufficient energy is introduced in the track to overcome the corresponding energy barrier. Such disturbances may be caused by a wave which travels ahead of a a high speed train, by small disturbances in the soil near the track, etc., and it is difficult to convert such energy inputs into f_o values.

When evaluating the procedure by Meier, also the difference in the equilibrium branches of the intially straight track and the track with initial imperfections (as shown in Fig. 11) should be noted.

Recently, E. Stagni [45] discussed Meier's results and concluded that a criterion based on the point L is safer. Realizing the shortcomings of a quadratic Π for the determination of point L, Stagni then forms a Π expression which contains also terms of fourth power. In this connection, it should be noted that this may not be sufficient for the determination of P_L, as shown recently by A.D. Kerr [22], and therefore the analysis of the track with lift-off may require a more exact formulation.

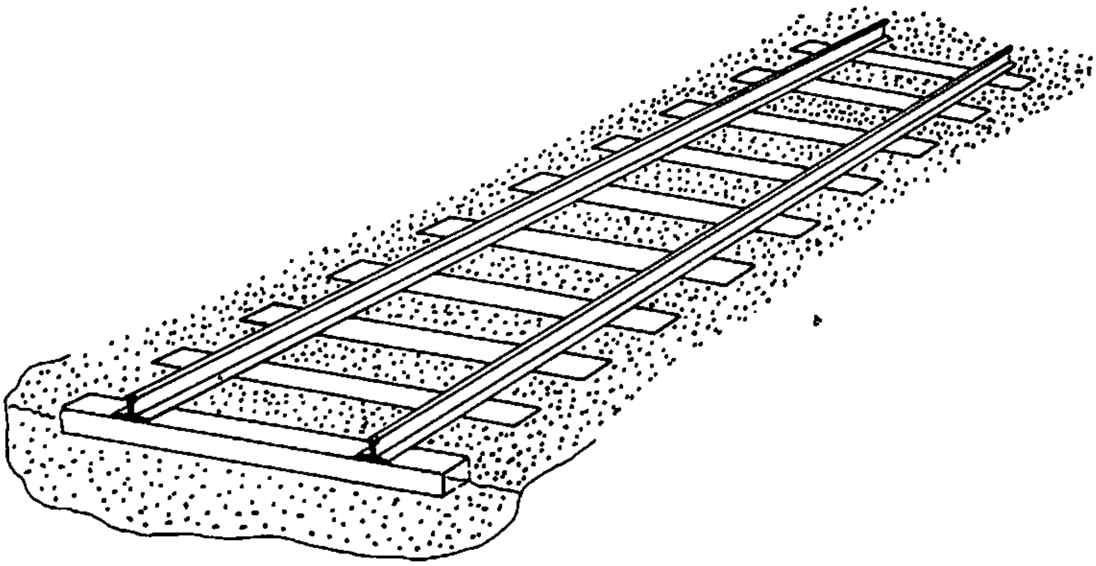


Fig. 1

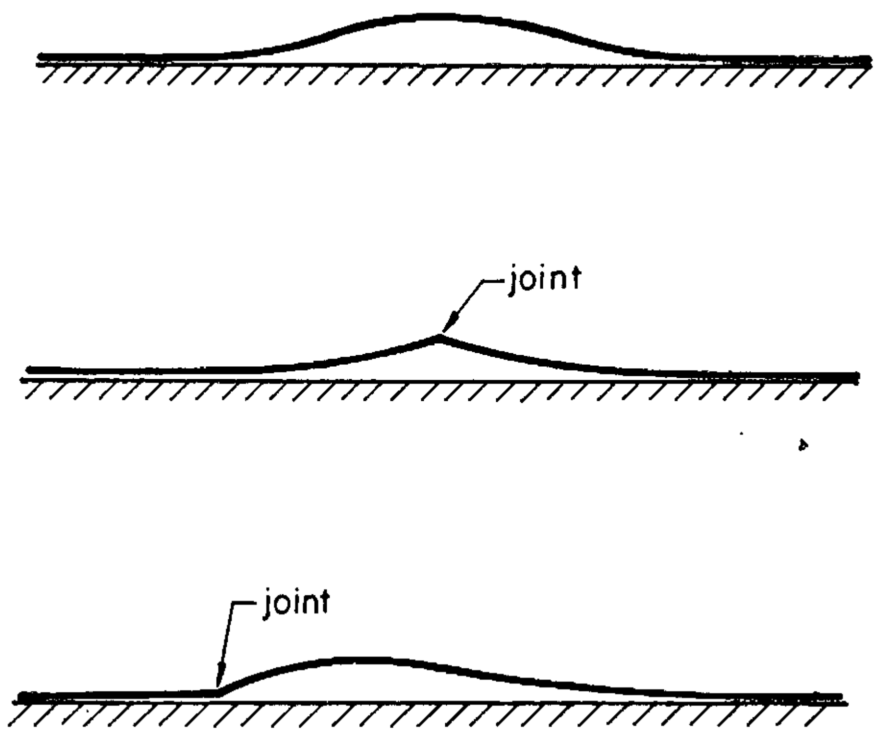
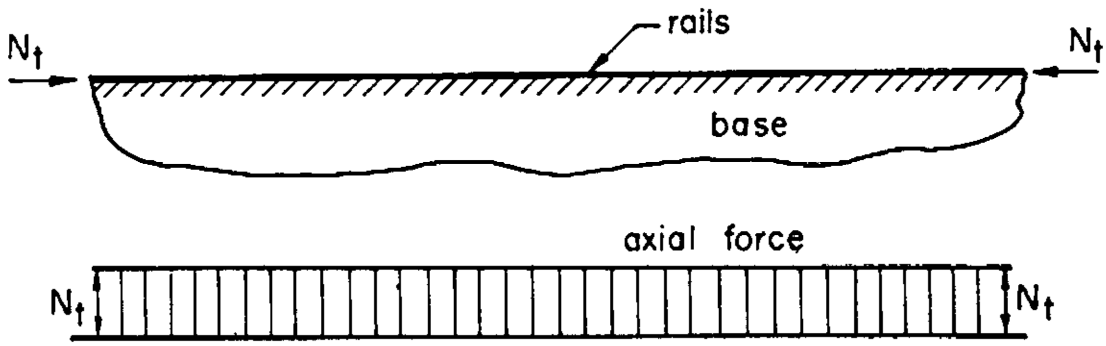
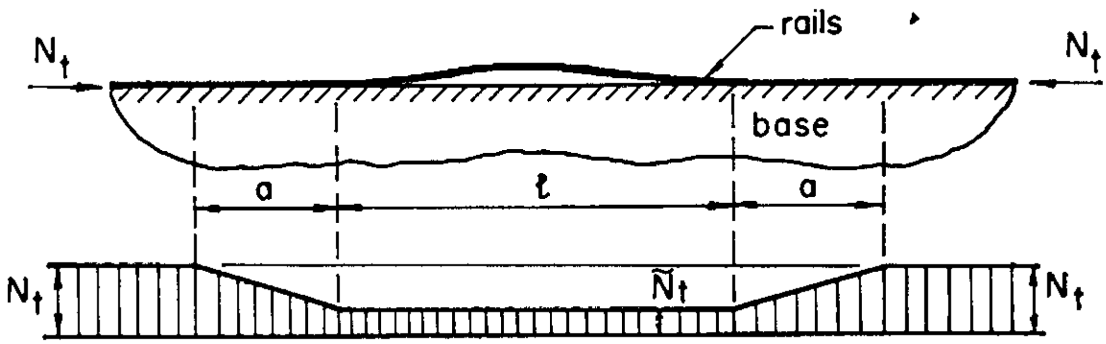


Fig. 2

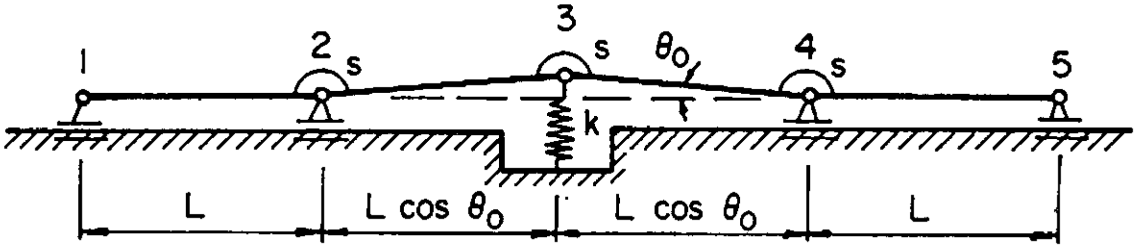


(a) BEFORE BUCKLING

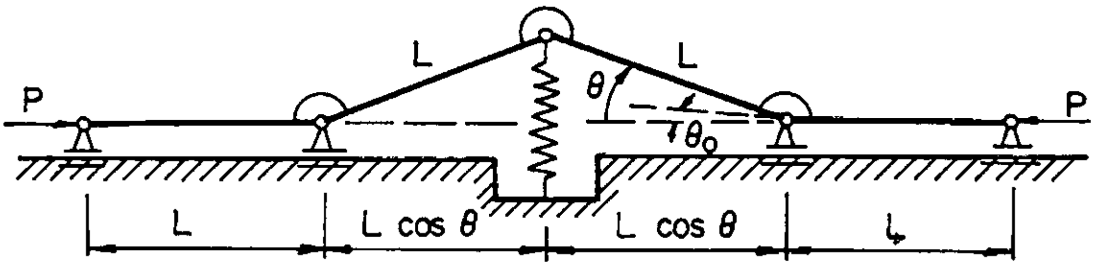


(b) AFTER BUCKLING

Fig. 3



(a) STRESSLESS EQUILIBRIUM STATE



(b) DEFORMED EQUILIBRIUM STATE

Fig. 4

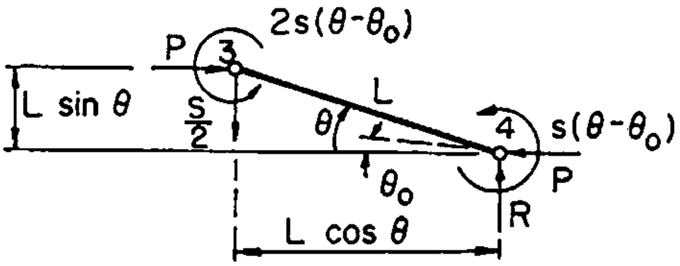


Fig. 5

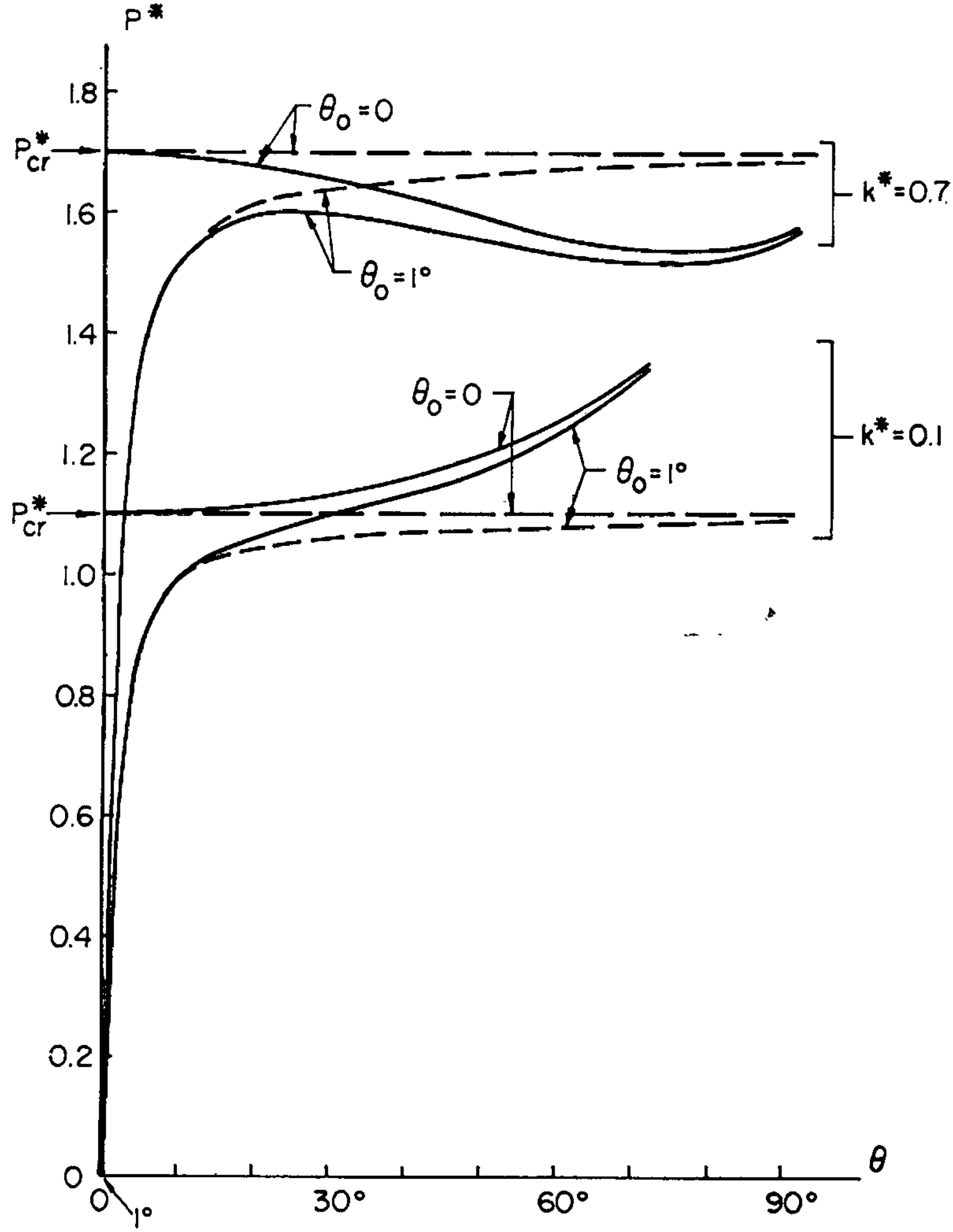
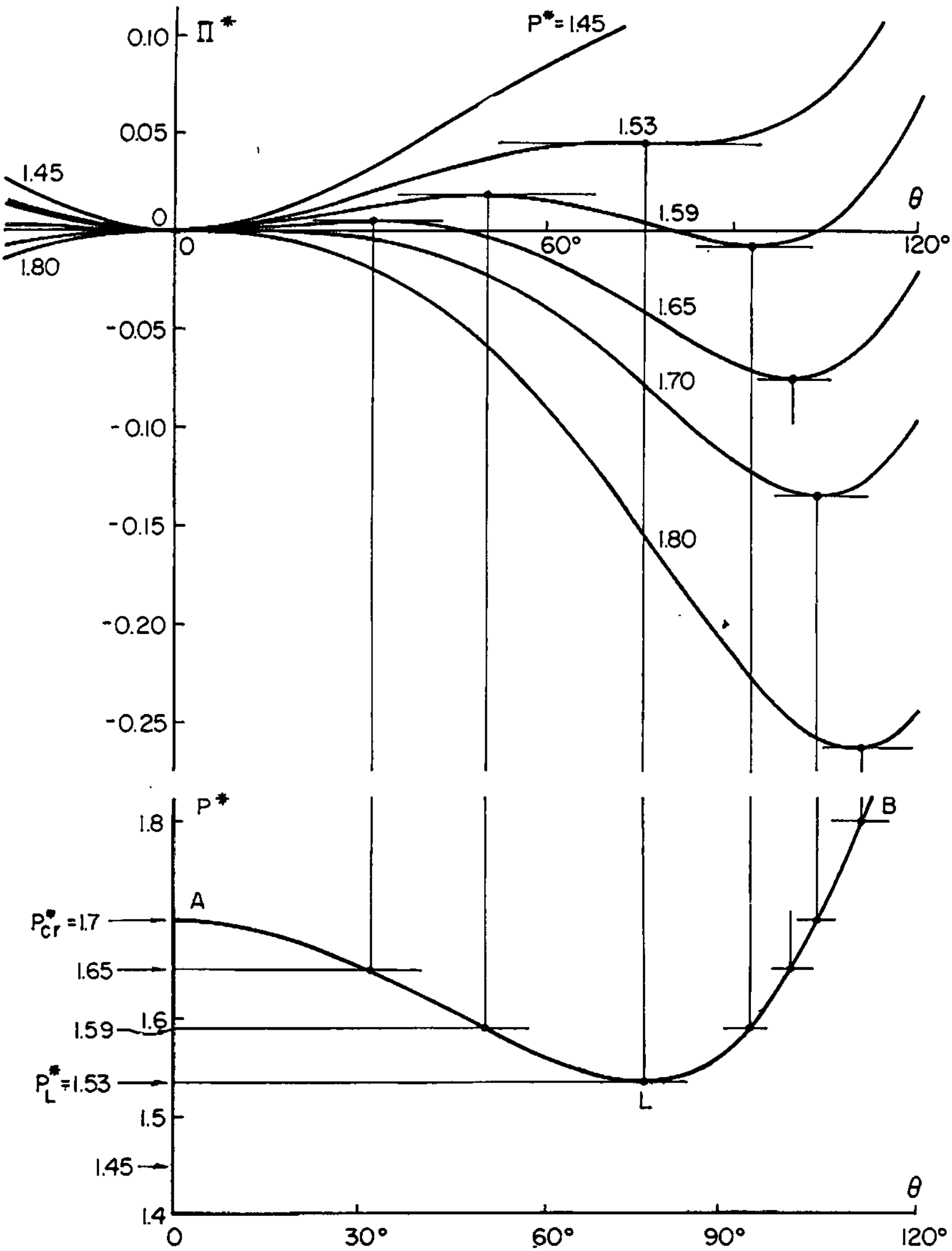
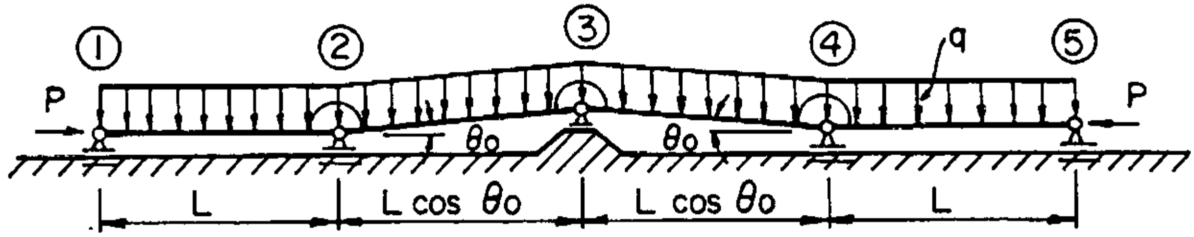


Fig. 6

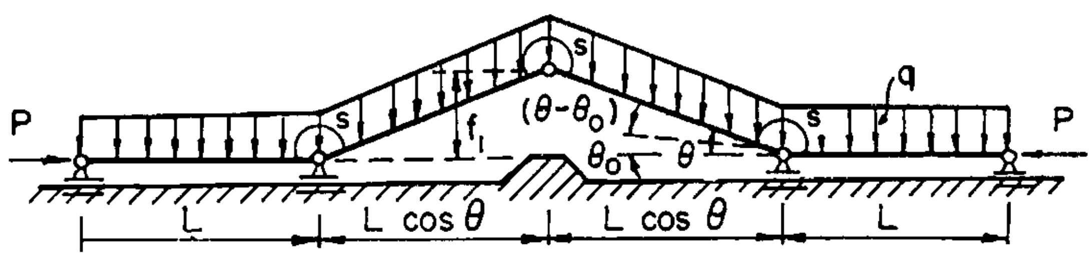


0.10
0.05
0
-0.05
-0.10
-0.15
-0.20
-0.25
Π*
P* = 1.45
1.53
1.59
1.65
1.70
1.80
1.45
1.80
θ
0
60°
120°
P*
1.8
P^*_{cr} = 1.7
1.65
1.6
1.59
P^*_L = 1.53
1.5
1.45
1.4
A
L
B
θ
0
30°
60°
90°
120°

Fig. 7



(a) BEFORE LIFT-OFF



(b) AFTER LIFT-OFF

Fig. 8

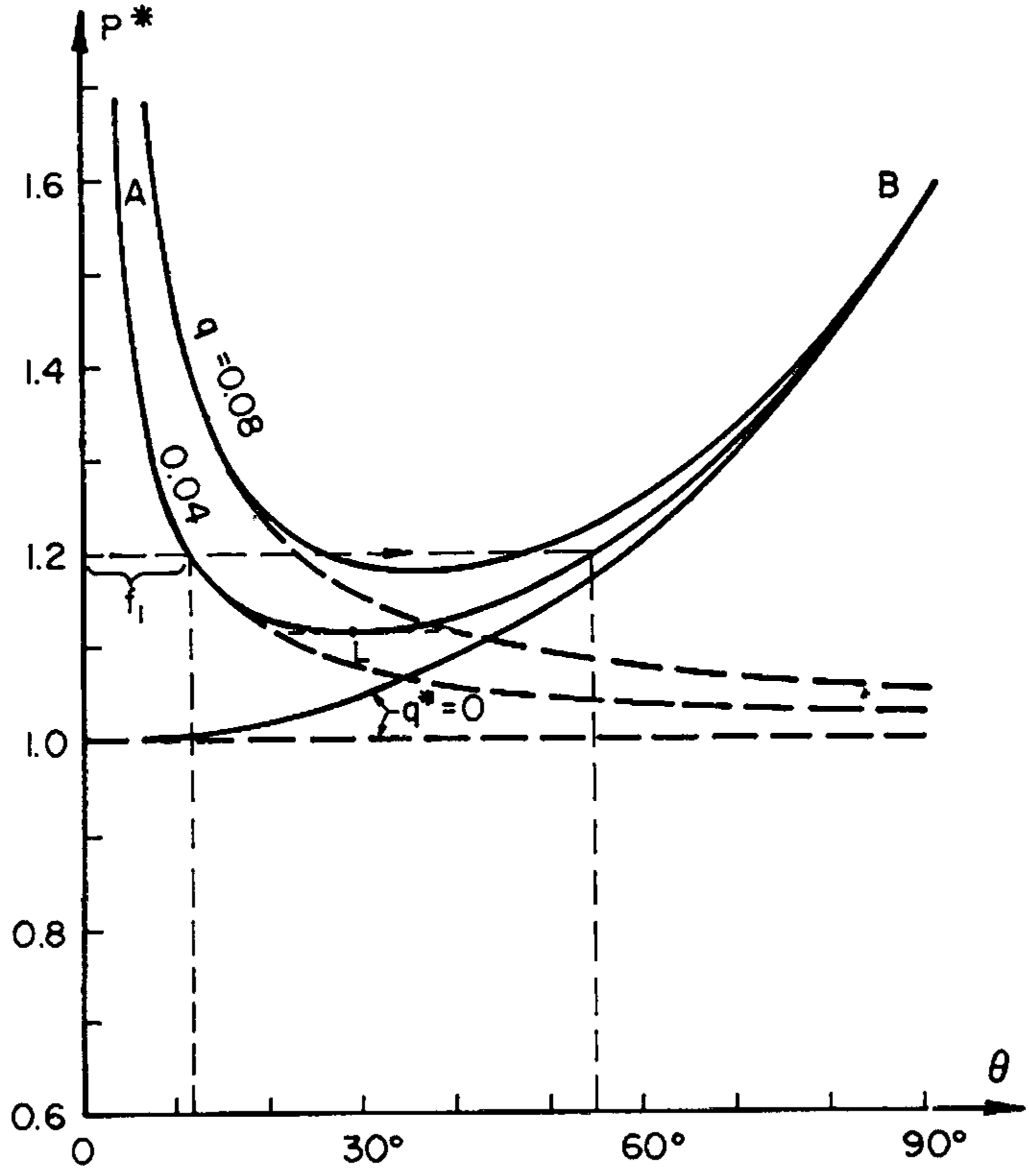
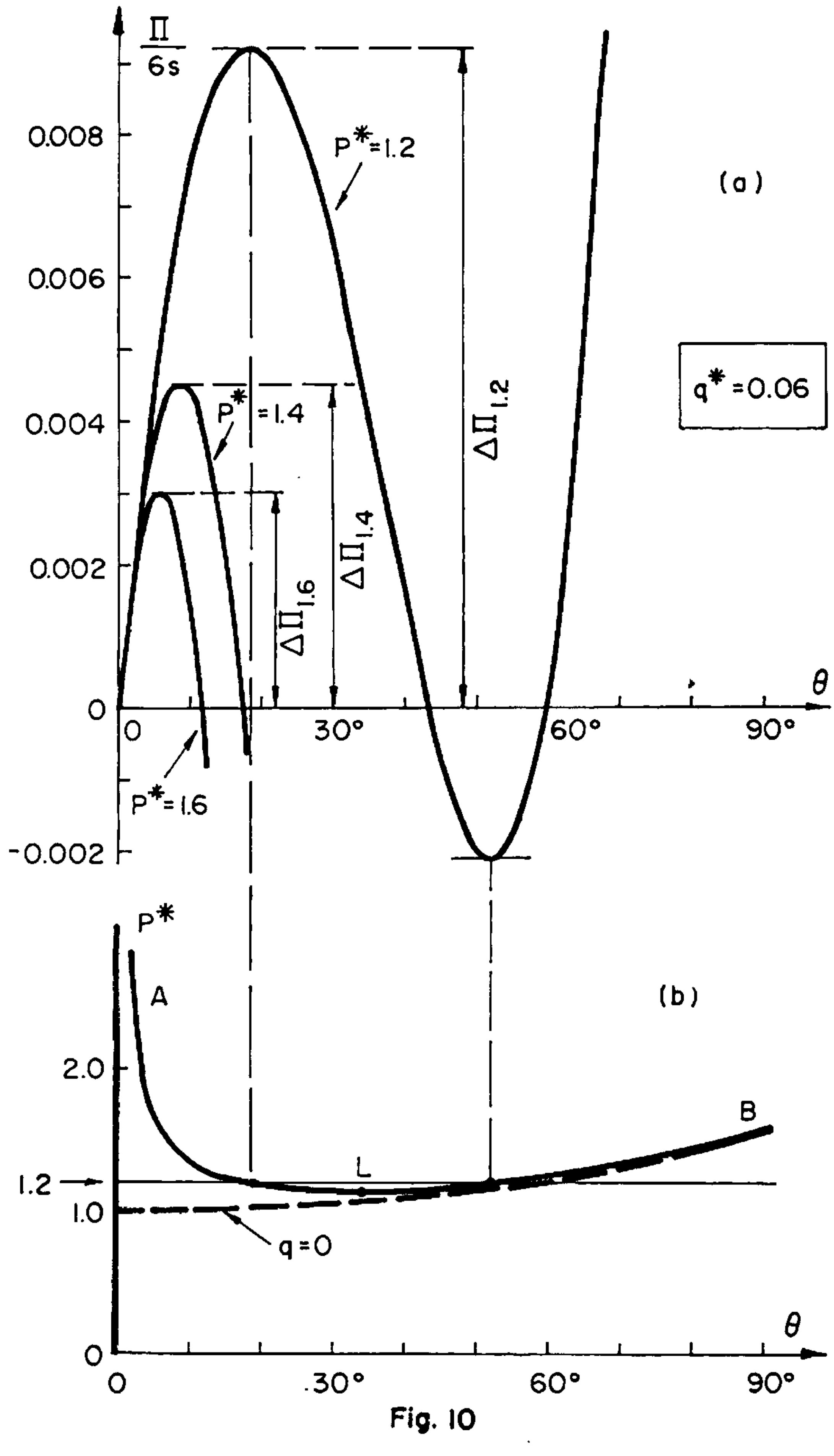


Fig. 9

(———— actual response, – – – – linearized)



Π / 6s
0.008
0.006
0.004
0.002
0
-0.002
P* = 1.2
P* = 1.4
P* = 1.6
(a)
q* = 0.06
ΔΠ1.2
ΔΠ1.4
ΔΠ1.6
θ
0
30°
60°
90°
P*
A
(b)
2.0
1.2
1.0
B
L
q = 0
θ
0
30°
60°
90°

Fig. 10

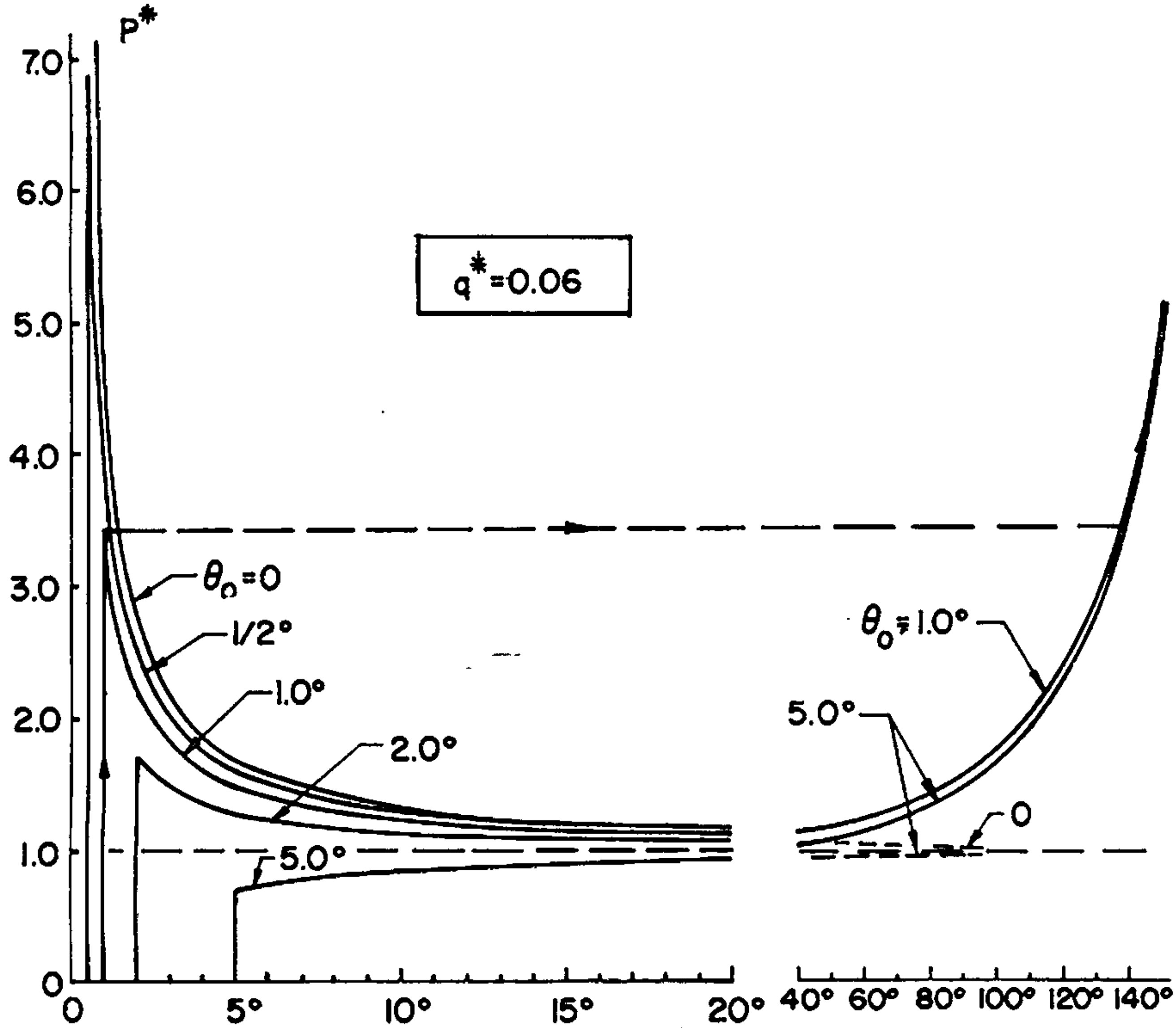


Fig. 11

(———— actual response, – – – – linearized)

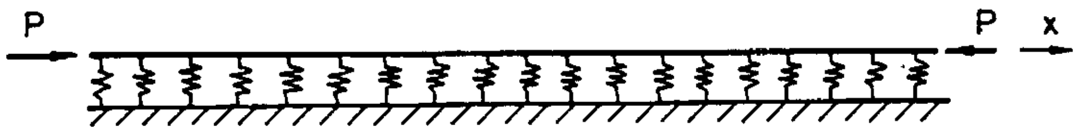


Fig. 12

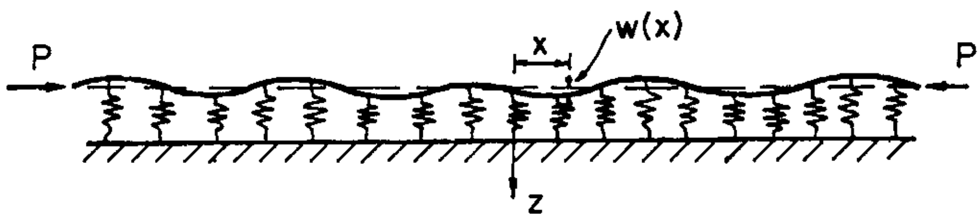


Fig. 13

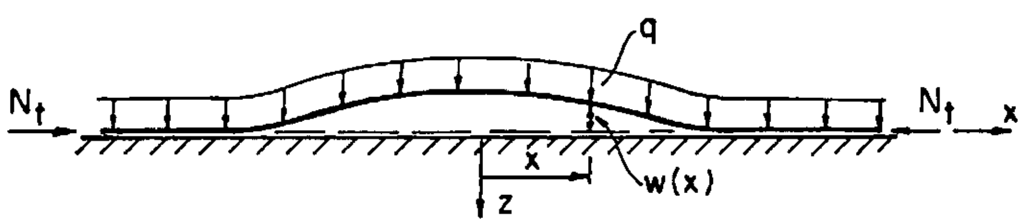


Fig. 14

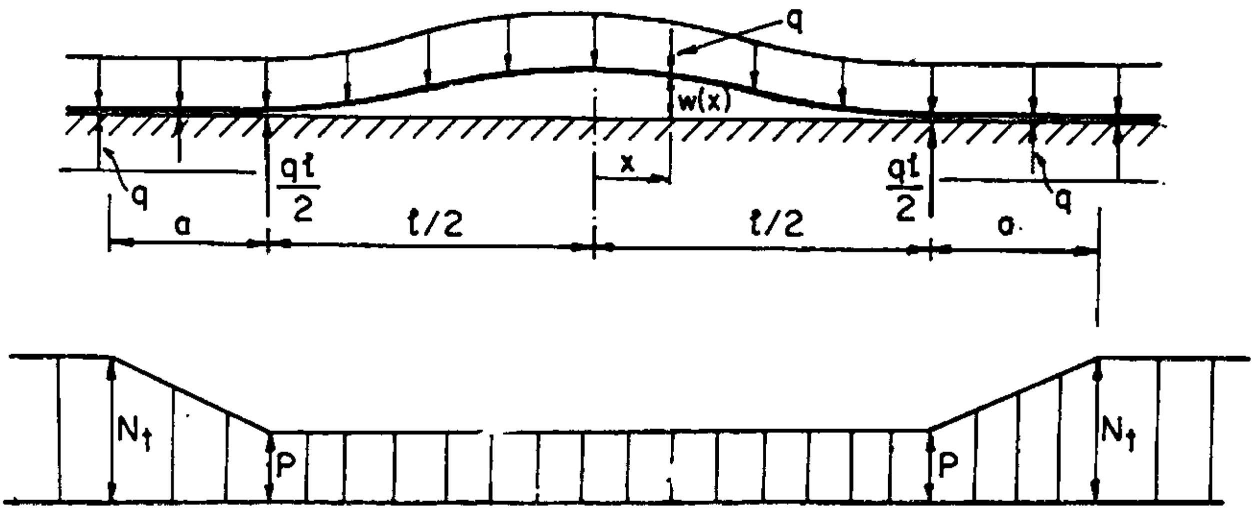

q
w(x)
x
q
$\frac{ql}{2}$
$\frac{ql}{2}$
q
a
l/2
l/2
a
N_t
P
P
N_t

Fig. 15

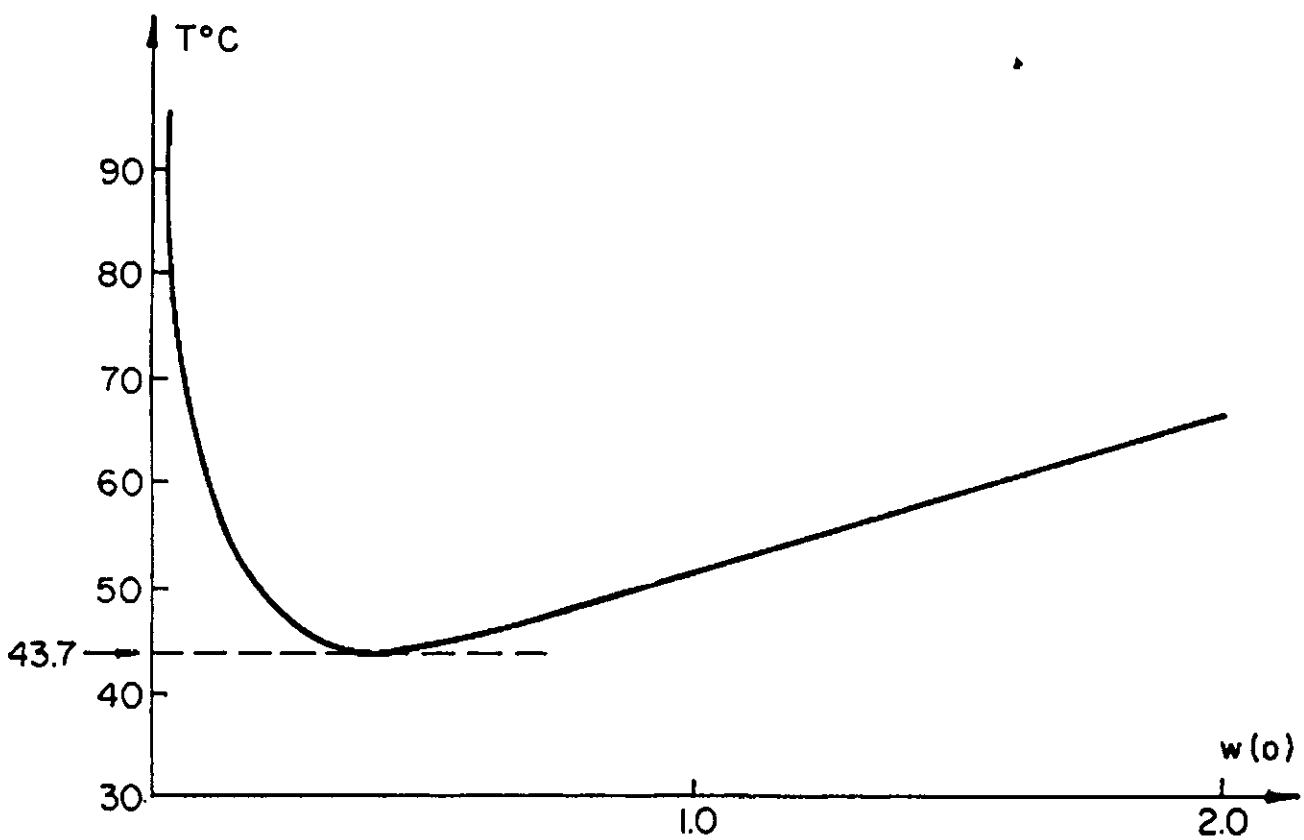

T°C
90
80
70
60
50
43.7
40
30
1.0
2.0
w(o)

Fig. 16

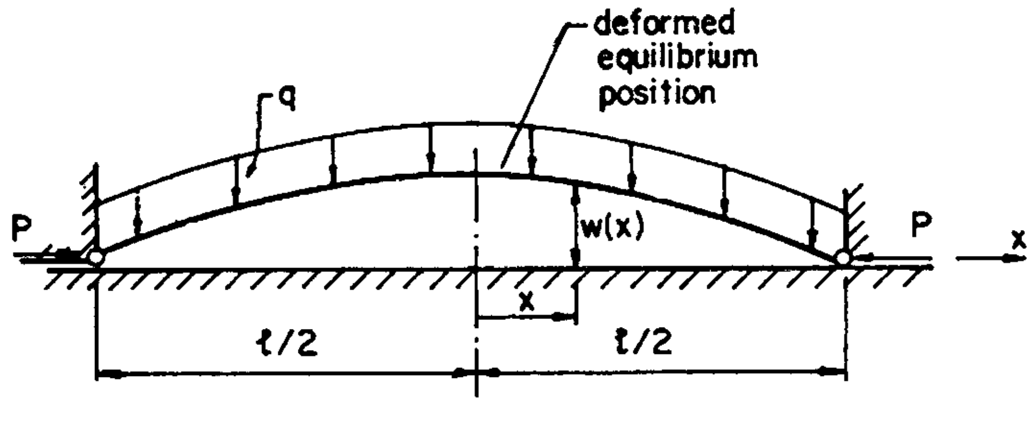

deformed
equilibrium
position
q
P
P
w(x)
x
x
ℓ/2
ℓ/2

Fig. 17

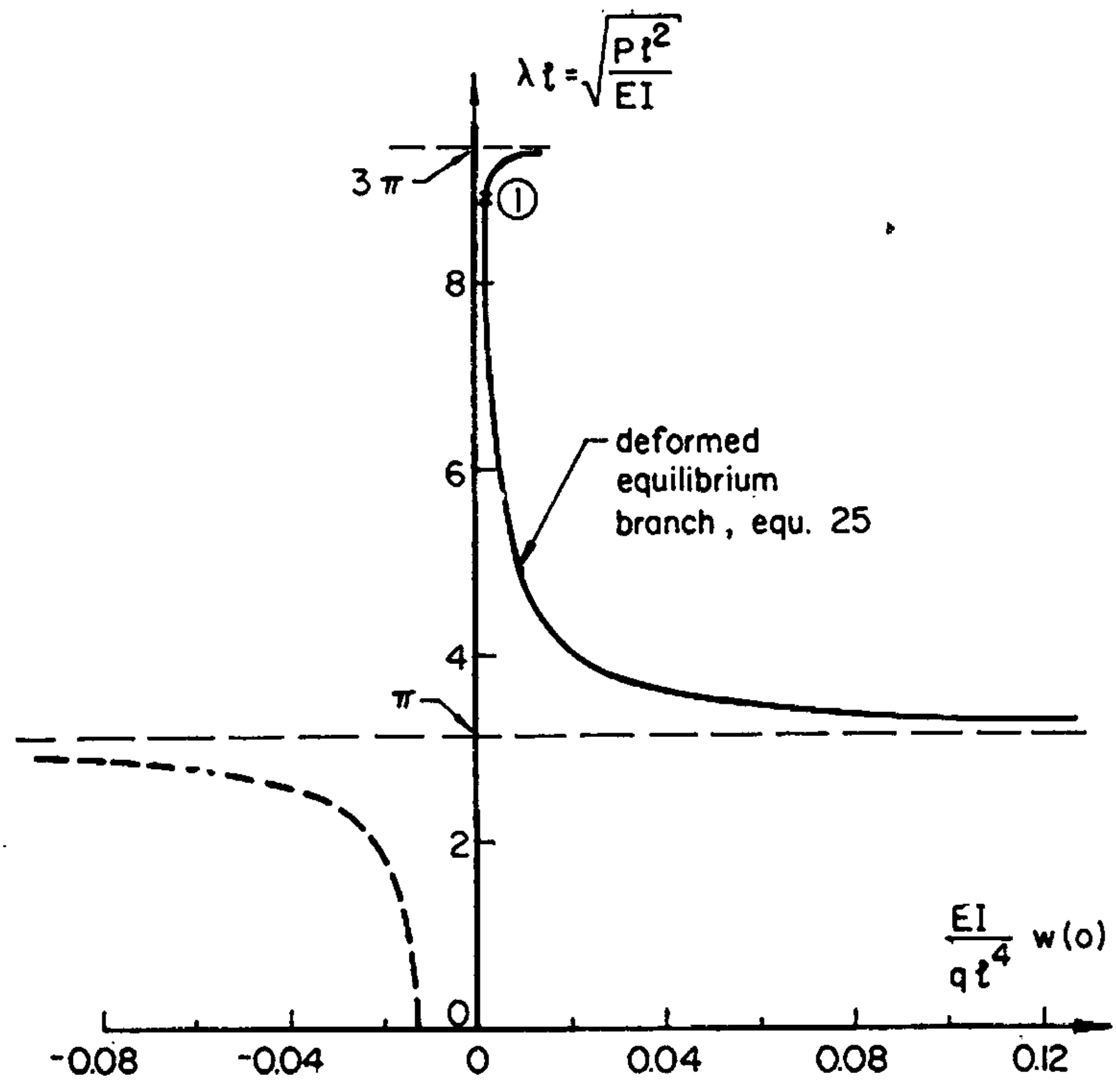

$\lambda \ell = \sqrt{\frac{P\ell^2}{EI}}$
3π
①
8
6
4
π
2
0
deformed
equilibrium
branch, equ. 25
$\frac{EI}{q\ell^4}$ w(o)
-0.08
-0.04
0
0.04
0.08
0.12

Fig. 18

REFERENCES

[1] C.F. Rose, "Railway Accidents", Her Majesty's Stationary Office, London, 1970.

[2] O. Ammann and C.v. Gruenewaldt, "Tests of the effect of axial forces in the track", ("Versuche über die Wirkung von Längskräften im Gleis", in German), Organ für die Fortschritte des Eisenbahnwesens, Part I 1928, Part II 1929, Part III 1929, Part IV 1932.

[3] O. Ammann and C.v. Gruenewaldt, "The resistance of the track to axial and transverse displacements" ("Der Wiederstand des Gleises gegen Längs - und, Querverschiebung", in German), Organ für die Fortschritte des Eisenbahnwesens, 1934.

[4] A. Wasiutyński, "Experimental research on the elastic deformations and creep of a railroad track" ("Recherches Expérimentales sur les Déformations Elastiques et le Travail de la Superstructure des Chemins de Fer", in French), Dunod, Paris, 1937.

[5] M.T. Chelnov, "Long Rails" ("Dlynnyie Relsy", in Russian), Transzheldorizdat, Moscow, 1940.

[6] N.B. Zverev, "An experimental Investigation of the Jointless Track Response" ("Eksperimentalnoie Issledovanie Raboty Besstykogo Puti", in Russian), Trudy CNII MPS, Vyp. 244, Moscow, Transzheldorizdat, 1962.

[7] A. Wöhrl, "Experimental foundation of the Thermit rail welding in a track" ("Experimentelle Grundlagen der Thermit-Schienenschweissung auf freier Strecke", in German), Organ für die Fortschritte des Eisenbahnwesens, H.2, 1927, p. 23.

[8] J. Nemcsek, "Tests of the Royal Hungarian State Railways on the position safety of the track" ("Versuche der königlich ungarischen Staatsbahnen über die Standsicherheit des Gleises", in German), Organ für die Fortschritte des Eisenbahnwesens, H.6, 1933, p. 114.

[9] F. Raab, "The stability of the track from new points of view" ("Die Stabilität des Schienenweges unter neuen Gesichtspunkten", in German), Zeitschrift VDI, Vol. 78, 1934.

[10] Tests conducted by the Italian National Railways in 1942. Reported in Ref. 45.

[11] M. Sonneville and M. Sergé, "The Lifting of Railways-an Experimental Study" ("Le soulevement des voies ferrées-Étude expérimentale", in French), Revue Generale des Chemins de Fer, 1948.

[12] F. Raab, "Tests on the jointless track", ("Versuche am lückenlosen Gleis", in German), Eisenbahntechnische Rundschau, H.10, 1956.

[13] H. Numata, "Buckling strength of rails", Journal for Research in Railroad Technology, Japan, Nr. 7,8,9, 1957.

[14] "The test on buckling of curved track", The Permanent Way Society of Japan, Nr. 1, 1958.

[15] M. Numata, "Buckling strength of continuous welded rail", Bulletin of Int. Railway Congress Assoc., January 1960.

[16] Unyi Béla, "The Welded Track" ("Hézagnélküli Vaganyok", in Hungarian), Budapest, 1960.

[17] F. Birmann and F. Raab, "To the development of the continuously welded track - Test results of the Karlsruhe test facility; Their analysis and interpretation" ("Zur Entwicklung durchgehend verschweisster Gleise - Ergebnisse bei Versuchen auf dem Karlsruher Prufstand; Ihre Auswertung and Deutung", in German), Eisenbahntechnische Rundschau, August 1960.

[18] D.L. Bartlett, J. Tuora, and G.R. Smith, "Experiments on the stability of long welded rails", British Transport Commission, London, 1961.

[19] E.M. Bromberg, "The Stability of the Jointless Track" ("Ustoichivost Besstykogo Puti", in Russian), Transport, 1966.

[20] A.D. Kerr, "Model study of vertical track buckling", DoT Report, 1972.

[21] Th. v. Kármán and A.D. Kerr, "Instability of spherical shells subjected to external loads", Topics in Applied Mechanics, Elsevier, 1965.

[22] A.D. Kerr, "On the stability of continuously supported beams", NYU Report, 1972.

[23] H. Meier, "The stability of the jointless railroad track" ("Die Stabilität des lückenlosen Vollbahngleises", in German), Zeitschrift VDI, Vol. 78, 1934.

[24] F. Raab, "Track buckling due to thermal stresses", ("Gleisverwerfungen durch Wärmespannungen", in German), Gleistechnik und Fahrbahnbau, H.9/10, 1937.

[25] M.T. Huber, "On the danger of buckling of a straight jointless track in connection with the stability theory of an elastically supported heavy beam" ("Über die Verwerfungsgefahr eines geraden lückenlosen Gleises im Zusammenhange mit der Stabilitätstheorie des elastisch eingebetteten schweren Stabes", in German), Gleistechnik und Fahrbahnbau, H.9/10, 1941.

[26] F. Corini, "Track laying for high speeds", Bulletin of the International Railway Congress Association, Brussels, English Edition, 1936.

[27] M.T. Huber, "On the use of theory for the intepretation of test results in track buckling" ("Über die Anwendung der Theorie zur Deutung der Versuchsforschung im Gleisverwerfungsproblem", in German), Gleistechnik und Fahrbahnbau, H.15/16, 1941.

[28] E. Engel, "The position safety of the jointless railroad track" ("Die Lagesicherheit lückenloser Eisenbahngleise", in German), VDI Zeitschrift, 1960.

[29] M. Hetényi, "Beams on Elastic Foundation", The University of Michigan Press, Ann Arbor, 1946.

[30] A.R. Rzhanicyn, "Stability of Equilibrium of Elastic Systems" ("Ustoichivost Revnoviesia Uprugikh Sistem", in Russian), Izd. Gos. Tekh.-Teor. Lit., Moscow, 1955.

[31] A.D. Kerr, "Buckling of continuously supported beams", Proc. ASCE, Engineering Mechanics, February 1969.

[32] H. Kayser, "Contribution to the analysis of the jointless track" (Beitrag zur Berechnung des stosslosen Gleises", in German), Organ für die Fortschritte des Eisenbahnwesens, H.6, 1916.

[33] C.v. Gruenewaldt, "The buckling safety of the jointless track" ("Die Knicksicherheit des lückenlosen Gleises", in German), Organ für die Fortschritte des Eisenbahnwesens, H.4, 1931.

[34] A. Bloch, "The stability of the jointless track" ("Die Stabilität des lückenlosen Gleises", in German), Organ für die Fortschritte des Eisenbahnwesens, H.9, 1932.

[35] H.v. Sanden, "On the effect of thermal stresses in the railroad track" ("Über die Wirkung von Wärmespannungen im Eisenbahngleis", in German), Organ für die Fortschritte des Eisenbahnwesens, H.19, 1932.

[36] K.N. Mishchenko, "Stability analysis of the continuous railroad track under the influence of temperature effects" ("Raschet ustoichivosti neprerivnoi relsovoi kolei pod vlijaniem temperaturnykh vozdeistvii", in Russian), Trudy Moskovoskogo Instituta Inzhinerov Zheleznodorozhnogo Transporta, Nr. 21, 1932, pp. 23-41.

[37] H. Lederle, "Safety against buckling in the continuously welded track" ("Sicherheit gegen Ververfung im durchgehend geschweissten Gleise", in German), Organ für die Fortschritte des Eisenbahnwesens, H.13, 1935.

[38] A. Martinet, "Buckling of the jointless track on ballast and very long rails" ("Flambement des voies sansjoints sur ballast et rails de grande longueur", in French), Révue Générale des Chemins de Fer, 1936.

[39] H. Meier, "A smiplified method for the theoretical investigation of track buckling" ("Ein vereinfachtes Verfahren zur theoretischen Untersuchung der Gleisverwerfung", in German), Organ für die Fortschritte des Eisenbahnwesens, H.20, 1937.

[40] R. Lévi, "The deformation of the track due to heat" ("La déformation des voies par la chaleur", in French), Révue Générale des Chemins de Fer, Nr. 5, 1948.

[41] K.N. Mishchenko, "The Jointless Railroad Track" ("Besstykovyi Relsovyi Put", in Russian), Gos. Transp. Zh/D Izd. Moscow, 1950.

[42] A.A. Krivobodrov, "The stability of the railroad track when subjected to temperature effects" ("Ustoichivost zhelezodorozhnogo puti pri temperaturnom vozdeistvii na relsy", in Russian), Trudy LII ZhT'a, Vyp. 144, Moscow-Leningrad, Transzheldorizdat, 1952.

[43] H. Rubin, "The buckling safety of continuously welded new and old tracks" ("Die Verwerfungssicherheit durchgehend geschweisster neuer und alter Gleise", in German), Eisenbahntechnische Rundschau, H.9, 1955.

[44] F. Schweda, "On the critical rail temperature for the buckling of welded tracks" ("Über die kritische Schienentemperatur bei Verwerfungen durchgehend geschweisster Gleise", in German), Eisenbahntechnische Rundschau, H.6, 1961.

[45] E. Stagni, "A structural model for the study of the vertical stability of a railway" ("Un modéle de structure pour l'étude de la stabilité verticale d'une voie ferrée", in French), Bulletin Rilem, Nr. 12, 1961.

[46] R. Sauvage, "The stability of long welded rails" ("La stabilité des long rails soudés", in French), Doctoral Thesis, University of Paris, France, December 1966.

[47] V.G. Albrecht, E.M. Bromberg, K.E. Ivanov, V.N. Liashchenko, S.P. Pershin, and V.Ja. Shulga, "The jointless track and long rails" ("Besstykovoi Put i Dlinnyie Relsy", in Russian), Izd. Transport, Moscow, 1967, p. 102.