Cost-Sharing Mechanisms for Ride-Sharing

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In this research project, we provided the ride-sharing industry a method to determine how to allocate the cost, such that it could potentially increase the incentive to use this service. We first identified the desirable properties that a good cost-sharing mechanism should have. Then we developed a general mechanism, which can be applied to develop specific mechanisms. Next, we proposed specific mechanisms for the static scenarios where all the information for the passengers would know in advance. We analyzed their advantages and disadvantages, so transportation planners can select according to their different needs. In addition, we incorporated the value of time and provided another mechanism for this situation. Finally, we moved to a dynamic scenario where the ride-sharing service was operated under uncertainties, and where a specific mechanism is proposed. In both scenarios, we designed and executed simulation experiments to systematically investigate the mechanisms' performances.

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About the Pacific Southwest Region University Transportation Center

The Pacific Southwest Region University Transportation Center (UTC) is the Region 9 University Transportation Center funded under the United States (U.S.) Department of Transportations University Transportation Centers Program. Established in 2016, the Pacific Southwest Region UTC (PSR) is led by the University of Southern California and includes seven partners: Long Beach State University; University of California, Davis; University of California, Irvine; University of California, Los Angeles; University of Hawaii; Northern Arizona University; Pima Community College. The Pacific Southwest Region UTC conducts an integrated, multidisciplinary program of research, education and technology transfer aimed at improving the mobility of people and goods throughout the region. Our program is organized around four themes: 1) technology to address transportation problems and improve mobility; 2) improving mobility for vulnerable populations; 3) Improving resilience and protecting the environment; and 4) managing mobility in high growth areas.

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Disclosure

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Abstract

In this research project, we seek to provide the emerging ride-sharing industry a method to determine how to allocate the cost such that it could potentially increase the incentive to use this environmentally friendly type of service. We first identify the desirable properties that a good cost-sharing mechanism shall have. Then, we develop a general framework which can be applied to develop specific mechanisms. Next, we propose specific mechanisms for the static scenario where all the information for the passengers is known in advance. We analyze their advantages and disadvantages so that transportation planners can select according to their different needs. In addition, we incorporate the value of time and provide another mechanism for this situation. Finally, we move to the dynamic scenario where the ride-sharing service is operated under uncertainties, and a specific mechanism is proposed. In both scenarios, we design and execute simulation experiments to systematically investigate the mechanisms' performances.

Cost-Sharing Mechanisms for Ride-Sharing

Executive Summary

The transportation sector represents a major part of the current and future United States (U.S.) economy, with around 9% of the U.S.' Gross Domestic Product (GDP) is directly related to transportation activity (Brumbaugh et al. 2018). Significant congestion and projected demand increase with limited infrastructure investment make necessary substantial improvements in transportation systems. Transportation planners must therefore find ways to improve transportation conditions in a cost-efficient manner. Significant advances have been made in the procurement and provision of real-time information that would be required for the effective control of a transportation system. Yet, this information is mostly used in a centralized transit system design and operation. These efforts have had limited success to date addressing congestion in most U.S. cities, which have a dispersed demand, because of a lack of high-density business and residential centers. Congestion in the U.S. continues to rise, stressing vital infrastructure, causing delayed shipments, late employees, and countless other problems. The increased adoption of dynamic cost-sharing transportation systems such as ride-sharing could help alleviate some of this traffic.

Although these new cost-sharing transportation systems are not the complete answer to congestion nationwide, their ability to augment existing public infrastructure, such as mass transit, could help to solve many congestion related problems in urban areas like Los Angeles. We seek to provide the emerging ride-sharing industry a method to determine how to allocate the cost such that it could potentially increase the incentive to use this environmentally friendly type of service. In this research report, we develop mechanisms to allocate the cost of ride-sharing to passengers and discuss the trade-offs among them. Starting from the static scenario and moving on to the dynamic scenario, we propose and analyze the different mechanisms. We have also considered the influence of value of time and develop a mechanism for this situation. We conducted experiments for both the static and dynamic settings, and our simulation results for the proposed mechanisms support our analytical analysis and each has its own advantages and disadvantages.

1. Introduction

With rapid population growth and urban development, traffic congestion has become an important issue, especially in large cities. The 2019 Annual Urban Mobility Report by Schrank et al. (2019) estimates that (a) the amount of delay endured by the average commuter at peak periods was 54 hours in 2017, up from 20 hours in 1982, and (b) the cost of congestion is more than $179 billion, nearly $1080 for every commuter in the U.S.. At the same time, according to Brown (2011), there is no public support for increased taxation to finance infrastructure capacity expansion and thus transit agencies and cities have turned to alternative approaches such as congestion pricing to raise funds. These costs to the travelers can be significant with yearly fares sometimes exceeding $1000. For example, on the 11 mile stretch of the Interstate 110 (I-110) Freeway in Los Angeles, Nelson (2013) wrote that the peak fare can be $15.40 per trip. Cities also use similar strategies to eliminate or reduce parking subsidies thus increasing the overall trip cost according to Elinson (2011). Thus, there is a need for innovative cost-efficient transportation modes that can improve transportation efficiency. Ride-sharing is one such innovative transportation mode that could at least help mitigate congestion increases, as it can tap into the significant amount of unused vehicular capacity in transportation networks.

Although the idea of carpooling/ride-sharing dates back to World War II, because of the need to conserve fuel, ride-sharing today has yet to be fully embraced by the public (Furuhata et al. 2013). However, the emergence of shared mobility services such as Uber and Lyft that utilize the latest navigation technologies has had some effect in changing the travel behavior of individuals. These phenomena as well as the significant potential of improving congestion and reducing costs partially explain the increased interest in the research community to develop models and tools to make the adoption of real-time ride-sharing systems a reality. By effectively using new communication capabilities, including mobile technology and global positioning system (GPS), there were several attempts to enable dynamic or real-time ride-sharing systems (see Chan and Shaheen 2012; Ghoseiri et al. 2010; Amey 2010 for some case studies). Dynamic ride-sharing refers to a system which supports an automatic ride-matching process between participants on short notice or even enroute (Agatz et al. 2012).

Two important research issues in the context of ride-sharing are (1) how to determine the routes and schedules of the vehicles (including how to assign passengers to vehicles) in the presence of conflicting objectives, such as maximizing the number of serviced passengers, or minimizing the operating cost and (2) how to allocate the total cost to the passengers. The first (optimization) and second (cost-sharing) problems are highly interrelated, because the routes and schedules of the vehicles determine the operating cost that needs to be shared. Conversely, the cost-sharing mechanism imposes constraints on the routes and schedules that need to be optimized, for example, because the fares of passengers should not exceed their fare quotes.

The optimization problem has received considerable attention in literature and is often solved as a vehicle routing problem with pickup and delivery (PDP) (see survey papers by Cordeau and Laporte 2007; Berbeglia et al. 2010). Agatz et al. (2011) developed an optimization-based approach to match drivers and riders in dynamic ride-sharing systems. They aimed at reducing the total system-wide vehicle miles generated by system users as well as their individual travel costs. Their results showed that even at relatively small participation rates, dynamic ride-sharing may have the potential to succeed with a sustainable ride-share population forming over time. There has been some recent research to extend the PDP to consider special features of the ride-sharing problem. For example, a review of dynamic ride-sharing systems, Agatz et al. (2012), focused on the optimization problem of finding efficient matches between passengers and drivers. This ride-matching optimization problem determines vehicle routes and the assignment of passengers to vehicles considering the conflicting objectives of maximizing the number of serviced passengers, minimizing the operating cost, and minimizing passenger inconvenience. Wang et al. (2016) considered the problem of ride-sharing routing when there are special dedicated lanes such as High Occupancy Vehicle (HOV) to reduce the travel time when there are multiple individuals sharing a trip. In this case, there is an incentive to take a detour (possibly increasing the driving distance) to pick up extra passengers to qualify for driving in an HOV lane.

The cost-sharing problem, on the other hand, has largely been neglected in the literature and is the focus of this research. One crucial component of a cost-sharing transportation system is the allocation of costs and/or savings to each participant in the system. However, this problem remains rarely studied in the literature. Without a model to allocate costs and/or savings to each participant in the ride-share, drivers have no basis to allocate their cost to the other passengers, thus having less incentive to participate.

Therefore, in this research report, we are designing mechanisms that can provide ride-sharing drivers ways to allocate the cost to the passengers such that they satisfy desirable properties for both the passengers and the drivers to participate in the ride-sharing operation. Section two reviews the previous literature. Sections four and five present a general solution framework and the specific mechanisms for allocating under the static scenario. Section six considers the value of time and Section seven extends our research to the dynamic scenario. Finally, in Section eight, we conclude our research.

2. Literature Review

In this section, we review previous literature on cost allocation and describe our contribution to the field.

For general and static cost-sharing problems in which the set of players and the cost function are both known and deterministic, Moulin mechanisms (Moulin 1999) and acyclic mechanisms (Mehta et al. 2009) are among the most studied families of cost-sharing mechanisms. These two mechanisms have built the foundation for designing truthful, and approximately budget-balanced cost-sharing mechanisms with economic efficiency. Stock exchange (Harris 2003) and ad exchange (Feldman et al. 2009) are two successful examples that facilitate real-time matching between sellers and buyers by providing opportunities for sellers (buyers) to sell (buy) trading items at the highest (lowest) prices. However, it is not straightforward to apply these systems to ride-sharing. Typically, ride-sharing participants are motivated by sharing travel costs, traveling fast using HOV lanes, and mitigating environmental concerns rather than making profits. In addition, since the final form of ride-sharing is determined only when the last passenger in a vehicle is decided, participants cannot evaluate the value of ride-sharing at the time of order submission unlike stock and ad exchanges.

Frisk et al. (2010) represents one of the earliest works that study the cost allocation problem in the transportation context. The authors compare several cost allocation methods based on solution concepts from cooperative game theory and propose a new allocation method whose aim is to equalize the participants' relative profits as much as possible. The proposed method could generate around a nine percent saving in total operating cost. Geisberger et al. (2009) suggested evenly allocating the cost among passengers while proposing an effective algorithm on solving the vehicle routing problem. Kleiner et al. (2011) contributed to the literature by providing a ride-sharing mechanism based on parallel auctions that is adaptive to individual preferences of the participants. This mechanism is proven to be incentive compatible (an important property of ride-sharing mechanisms, detailed description in Section four), and allows a trade-off between reducing vehicle miles travelled and the probability of finding matches between drivers and passengers. The work of Özener et al. (2013) studies the cost allocation problem for a vendor managed inventory (VMI) model. In a VMI model, the supplier is responsible to manage the inventory level of its customers and decides when and how much to replenish each customer. The research focuses on how to calculate the cost-to-serve for each customer. In Furuhata et al. (2015), the authors develop a cost allocation scheme for demand-responsive transport (DRT) systems. The novel cost-sharing mechanism, called Proportional Online Cost Sharing (POCS), is shown both theoretically and computationally to satisfy a list of desirable properties described in detail in Section four, including online fairness, immediate response, individual rationality, etc. This mechanism combines the traditional incremental sharing method and proportional sharing method while maintaining their advantages.

However, all the above works assume a static vehicle routing environment and separate the cost allocation mechanism from the routing of vehicles. The work of Sayarshad and Chow (2015) is one of the first studies to consider cost allocation in a dynamic vehicle routing environment. The authors illustrate a new dynamic dial-a-ride policy that features a non-myopic (considering social welfare) pricing component, assuming elastic customer demand. The pricing is shown to be socially optimal and improves social welfare in the 10 to 20 percent range. Zou (2017) solved the cost allocation problem under a dynamic vehicle routing environment. He proposed a method called Hybrid Proportional Online Cost Sharing (HPOCS) that satisfies desirable properties such as online fairness, budget balance, etc. and is based on the POCS mechanism. HPOCS assumes that each passenger has a known probability of requesting service and a given distribution of service request time. HPOCS works well when the request probability is high but fails to have economic efficiency when the request probability is low.

The reason for less literature in the dynamic setting is because of the added layer of complexity brought by uncertainty. Some desirable properties for a good cost-sharing mechanism, such as the Ex-post Incentive Compatibility property (also known as strategy proof), may not hold in a dynamic setting. This then makes assessing the performance of ride-sharing mechanisms harder, because when certain properties fail to be satisfied, we lack tools of determining whether a ride-sharing mechanism is good or not. To solve this problem, instead of providing a 100% guarantee on the properties, what we could do is to involve robust analysis on the performances of the mechanism.

In this project, we extrapolate on the current dynamic ride-sharing landscape to consider a system in which shared rides are set up through a centralized matching system that also determines the compensation from the passengers to the driver. In our proposed system, private citizens choose to use the system to share their idle capacity. The intent is for both the passengers and drivers to remotely access the system to negotiate routes and prices. An open system such as the one that is proposed can facilitate entry to the marketplace, whereby significantly reducing idle vehicle capacity.

3. Problem Description

In this section, we describe the ride-sharing problem, briefly introduce the POCS mechanism (Furuhata et al. 2015), and discuss the drawbacks of POCS when applying to the actual ride-sharing context.

Ride-sharing operations occurs when an individual would like to share his or her vehicle and split the operational cost with passengers of similar schedules and itineraries. Unlike services provided by companies like Uber and Lyft, our ride-sharing context does not involve a professional driver. We focus on the scenario where the driver actually has an origin and destination. This major difference serves as one of the motivations of this research. While Uber and Lyft drivers are providing ride-sharing services for a living (they are driving to make money) which may generate more traffic, because of a number of factors including deadhead miles (Erhardt et al. 2019), our ride-sharing context has the potential to reduce traffic because the driver is only interested in cost recovery. Therefore, the driver in our context does not spend half a day driving the streets seeking more trips. Instead, the driver picks up ride-sharing passengers on his or her way to work, home, or errands.

More formally, we have a driver who wants to share his or her own direct trip cost with a limited number of passengers in exchange for the provided service. Each passenger requests the service with his or her demand (the distance between the origin and destination) and a willingness to pay level to decide if he or she is willing to join the ride-sharing operation. At the end of the operation, passengers (with or without the driver) share the total cost of the trip together. The following better illustrates the cost allocation in our ride-sharing context.

Suppose Jack commutes between Pasadena and University of Southern California (USC) everyday for work, but he gets off work early on Wednesdays. Because Jack has some extra time from work to home on Wednesdays, he is willing to spend some time providing ride-sharing services to others on his way home. In exchange, he would like to save himself some (or all) of the travel cost to home. Jack decides that his normal travel cost to home including fuel cost and vehicle depreciation cost is about \$12. He then decides that the detour cost incurred by picking up other passengers on his way home including fuel cost and vehicle depreciation cost resulted in \$0.8 per mile. One Wednesday afternoon, on his way home, Jack picks up John, Lee, and Mary (in this sequence) and spends an extra 15-minutes driving. Their direct trip distances are six miles, eight miles, and two miles respectively. In total, Jack travelled 20 miles that Wednesday while his normal trip to home is 15 miles. What kind of cost allocation mechanism did Jack use to charge John, Lee, and Mary upon their requests? Was the charging method fair among the three of them? Was the charging method fair to Jack? These questions are the main focus of this research. In other words, we aim at determining what properties should a good ride-sharing cost allocation mechanism have and designing mechanisms to satisfy these properties.

We present the following notations that are used throughout the report:

- t: a number representing the discrete time point as requests coming in[1]
- $\mathbb{P}$: the set of passengers, $\mathbb{P} \subseteq \mathbb{N}$
- $\mathbb{P}_t$: the set of passengers who are in operation at time t, $\mathbb{P}_t \subseteq \mathbb{P}$
- π: the submit order of request of passengers in $\mathbb{P}$ which is a permutation of set $\mathbb{P}$
- k: $\in \{1, \ldots, t\}$ which represents the time point when a ride request is submitted
- $\pi(k)$: the passenger who submits his or her ride request at time k under submit order π; $\pi(0)$ represents the driver
- $\alpha_{\pi(k)}$: the alpha value of passenger $\pi(k)$, which quantifies the demand of his or her ride request; positive by assumption
- F: the direct trip cost of the driver
- $F_{\pi,t}$: the amount of the driver's direct trip cost recovered at time t under submit order π

[1] Since passengers come in sequence, we use discrete time instead of continuous time to facilitate the understanding of notations and proofs of the theorems and propositions. The discrete time points are evenly distributed.

$c_{\pi,t}$: the total cost to drive at time t under submit order π

$c^{\mathrm{p}}_{\pi,t}$: the total cost for passengers at time t under submit order π

$c^{\mathrm{d}}_{\pi,t}$: $= c_{\pi,t} - F$, the total detour cost at time t under submit order π

$c^{\mathrm{s}}_{\pi(k),t}$: the total shared cost of passenger $\pi(k)$ at time t under submit order π

$W_{\pi(k)}$: the highest cost that passenger $\pi(k)$ is willing to pay

Next, we briefly introduce the POCS mechanism (Furuhata et al. 2015), because our mechanism is built upon it. The mechanism was developed for a shuttle service to support a demand responsive transportation system. In this report we build on it to develop a mechanism for ride-sharing where the driver is no longer a professional driver. That is, the driver has a desired destination and is willing to spend some extra time to pick up additional passengers to offset their transportation costs. The idea behind POCS is that the mechanism charges every passenger by grouping them into coalitions. How the passengers are grouped into coalitions is a major part of the POCS mechanism design. In other words, upon request, every passenger starts a new coalition or joins the same coalition with the passenger immediately before him or her. This is decided by the mechanism. As a result of the POCS mechanism, everyone within the same coalition is charged proportionally to their direct travel distance from their origin to destination point: i.e. their unit prices are the same. For any two passengers that are in different coalitions, their unit prices may be different. Based on the following two major assumptions, POCS guarantees five desirable properties which we will formally define in Section four.

Assumption 1. *The total cost (both for the driver and for the passengers) is non-decreasing over time. That is, for all times t and t' with $t \le t'$ and all submit orders π, $c_{\pi,t} \le c_{\pi,t'}$ and $c^{\mathrm{p}}_{\pi,t} \le c^{\mathrm{p}}_{\pi,t'}$.*

Assumption 2. *The total cost (both for the driver and for the passengers) at time t is independent of the submit order. That is, for all times $t \ge 1$ and submit orders π and π' such that $\pi(1), \ldots, \pi(t)$ and $\pi'(1), \ldots, \pi'(t)$ contain the same set of passengers, $c_{\pi,t} = c_{\pi',t}$ and $c^{\mathrm{p}}_{\pi,t} = c^{\mathrm{p}}_{\pi',t}$.*

4. The Ride-Sharing Mechanism Framework

In this section, we introduce a general Ride-Sharing Mechanism Framework that satisfies certain desirable properties. It contains definitions for the desirable properties, some of which are borrowed from Furuhata et al. (2015).

We first list the extra notation we use as well as some definitions for the mechanism.

$c^{\mathrm{s1}}_{\pi(k),t}$: the total detour cost $c^{\mathrm{d}}_{\pi,t}$ shared by passenger $\pi(k)$ at time t under submit order π

$c^{\mathrm{s2}}_{\pi(k),t}$: the direct trip cost of the driver F shared by passenger $\pi(k)$ at time t under submit order π

$\beta_{\pi(k),t}$: the fraction of F that will be covered by passenger $\pi(k)$ at time t

Definition 1. For all times $t \geq 1$ and all submit orders π, the total cost for passengers $c^{\mathrm{p}}_{\pi,t}$ at time t under submit order π is the total cost of the detour that the driver takes to serve passengers $\pi(1), \ldots, \pi(t)$ plus the amount of the driver's direct trip cost covered by the passengers at time t:

$$c^{\mathrm{p}}_{\pi,t} := c^{\mathrm{d}}_{\pi,t} + F_{\pi,t} \,. \tag{1}$$

Definition 2. For all times $k \geq 1$ and all submit orders π, the marginal cost $c^{\mathrm{m}}_{\pi(k)}$ of passenger $\pi(k)$ is the increase in the total detour cost due to passenger $\pi(k)$'s ride request submission. That is, $c^{\mathrm{m}}_{\pi(k)} := c^{\mathrm{d}}_{\pi,k} - c^{\mathrm{d}}_{\pi,k-1}$.

Definition 3. For all times k_1, k_2 and t with $k_1 \leq k_2 \leq t$ and all submit orders π, the coalition cost per alpha value $c^{\mathrm{a}}_{\pi(k_1,k_2)}$ of passengers $\pi(k_1), \ldots, \pi(k_2)$ at time t under submit order π is:

$$c^{\mathrm{a}}_{\pi(k_1,k_2)} := \frac{\sum_{j=k_1}^{k_2} c^{\mathrm{m}}_{\pi(j)}}{\sum_{j=k_1}^{k_2} \alpha_{\pi(j)}} \,.$$

Definition 4. For all times k and t and all submit orders π with $k \leq t$, the total detour cost shared by passenger $\pi(k)$ at time t under submit order π is:

$$c^{\mathrm{s1}}_{\pi(k),t} := \alpha_{\pi(k)} \min_{k \leq j \leq t} \max_{1 \leq i \leq j} c^{\mathrm{a}}_{\pi(i,j)} \,.$$

Definition 5. For all times k_1, k_2 and t with $k_1 \leq k_2 \leq t$ and all submit orders π, a coalition (k_1, k_2) at time t is a group of passengers $\pi(k_1), \ldots, \pi(k_2)$ with

$$\frac{c^{s1}_{\pi(k),t}}{\alpha_{\pi(k)}} = \frac{c^{s1}_{\pi(k_1),t}}{\alpha_{\pi(k_1)}}$$

for all times k with $k_1 \leq k \leq k_2$ and

$$\frac{c^{s1}_{\pi(k),t}}{\alpha_{\pi(k)}} \neq \frac{c^{s1}_{\pi(k_1),t}}{\alpha_{\pi(k_1)}}$$

for all times k with $(k = k_1 - 1\, or\, k = k_2 + 1)$ and $1 \leq k \leq t$.

Definition 6. For all times k and t and all submit orders π with $k \leq t$, the driver's direct trip cost shared by passenger $\pi(k)$ at time t under submit order π is:

$$c^{s2}_{\pi(k),t} \coloneqq \beta_{\pi(k),t} F\,.$$

Definition 7. [2] For all times k, t with $1 \leq k \leq t$ and all submit orders π, $W_{\pi(k)}$ is the sum of $W^1_{\pi(k)}$ and $W^2_{\pi(k)}$, where $W^1_{\pi(k)}$ is the willingness to pay level corresponding to $c^{s1}_{\pi(k),t}$, and $W^2_{\pi(k)}$ is the willingness to pay level corresponding to $c^{s2}_{\pi(k),t}$.

We next introduce the five desirable properties originally described and defined in Furuhata et al. (2015). The formal definitions are presented in Appendix A. Notice that all these properties refer to the final shared costs which are the actual values passengers pay.

Online Fairness. At any time, the shared cost per demand value of any passenger is never lower than those of passengers who have requested service prior to the passenger. However, this property does not require that the initial quote per demand value provided to any passenger to

[2] This does not mean that the passenger has two willingness to pay levels. They each only have one level that is $W_{\pi(k)} = W^1_{\pi(k)} + W^2_{\pi(k)}$. And the passengers should never worry about how $W_{\pi(k)}$ is split into $W^1_{\pi(k)}$ and $W^2_{\pi(k)}$. The existence of $W^1_{\pi(k)}$ and $W^2_{\pi(k)}$ is only to prove that when the sub-mechanisms of the Ride-Sharing Mechanism Framework satisfy the Individual Rationality property, the Ride-Sharing Mechanism Framework satisfies the property as well.

be never higher than the one provided to a subsequent passenger. In other words, it can happen that a passenger who requests service late receives a lower initial quote per demand value than a prior passenger. Nevertheless, in such a situation it is guaranteed that the current shared cost per demand value of the prior passenger is never higher than the initial quote per demand value provided to a subsequent passenger.

Budget Balance. At any time, the sum of the shared costs of all passengers equals to the total travel cost of the current routing schedule, including both traveled and untraveled portions of the schedule. Here we say a mechanism is "Budget Balanced" if the costs are fully recovered. We can easily extend the definition of Budget Balance to include profits. This will not affect our mechanism design and analysis.

Immediate Response. Each passenger should be provided with an upper bound on his or her final shared cost at the time of request. Because each passenger has to make the decision of whether to accept or decline the service based on his or her willingness to pay level, this property guarantees that each passenger only has to make that decision once at the time of request, without having to worry about being charged against his or her will for a higher price than previously agreed upon.

Individual Rationality. At any time, the shared cost of any passenger who has accepted his or her initial quote never exceeds his or her willingness to pay level. Because a passenger only remains in the cooperation as long as the shared cost does not exceed his or her willingness to pay level, the Individual Rationality property guarantees that no passenger will drop out of the cooperation once he or she joins. This property also suggests that the initial quote serves as an upper bound on the final shared cost for each passenger.

Ex-Post Incentive Compatibility. The best strategy of each passenger is to request service truthfully at his or her earliest possible time, provided that all other passengers do not change their request times and whether they accept or decline their initial quotes. This property implies that no passenger can decrease his or her final shared cost by delaying his or her actual request time to be later than its truthful request time. For similar reasons as discussed under the Online

Fairness property, this property is concerned with the final shared costs rather than initial price quotes. Thus it is possible for a passenger to delay his or her actual request time and receive a lower initial quote. Even if it happens, the final shared cost of the same passenger in the delayed request case is guaranteed to be no lower than in the truthful request case.

We then discuss the issues and challenges of directly applying POCS to our ride-sharing context. Although POCS satisfies desirable properties, it is applied to a shuttle service, a different context from our ride-sharing operation. To adapt POCS to a ride-sharing operation, one could simply include the entire driver's direct trip cost into the total cost for the passengers at any time. In this simple mechanism, when $|\mathbb{P}| = 0$, the driver has no passengers and covers his or her own direct trip cost F. When $|\mathbb{P}| \geq 1$, the driver pays nothing. Then we can apply POCS to the total cost to drive $c_{\pi,t}$ to obtain the total shared cost $c^{\mathrm{s}}_{\pi(k),t}$ for each passenger.

It is easy to see that for this mechanism, all the five desirable properties hold (which is proved in Furuhata et al. 2015). However, the first passenger faces the chance of paying 100% of the driver's direct trip cost F if he or she ends up being the only passenger served, which may result in a high initial quote that could deter him or her from participating in the ride-sharing operation. To solve this issue, we need to strike a balance between compensating the driver and reducing the burden on the first passenger.

We would also like to distribute the driver's direct trip cost F among the passengers proportionally to their alpha values. This is desirable because in the ideal case, where all passengers submit their requests at the same time and their origin and destination locations are known, this is arguably the most fair and natural way to distribute F among the passengers. However, when we do not know how many passengers are going to submit requests, this property can be difficult to satisfy.

These issues and challenges serve as the motivation to develop our general mechanism framework for ride-sharing. We want to design a mechanism that satisfies the following properties:

- **Five Original Desirable Properties**: Online Fairness, Immediate Response, Individual Rationality, Budget Balance and Ex-Post Incentive Compatibility.

- **Reduced Burden for the First Passenger Property**: the initial quote for the first passenger $\pi(1)$ should not have a high beta value. In particular we want $\beta_{\pi(1),1} < 1$. If $\beta_{\pi(1),1} = 1$, passenger $\pi(1)$ may not have the incentive to join the ride-sharing operation.
- **Fairness in Sharing Driver's Cost Property**: the final share of F paid by the passengers should be proportional to their alpha values. Since we do not know in advance which passenger is going to be the last one, this means: for all times t, $\beta_{\pi(1),t} : \beta_{\pi(2),t} : \ldots : \beta_{\pi(t),t} = \alpha_{\pi(1)} : \alpha_{\pi(2)} : \ldots : \alpha_{\pi(t)}$.

POCS cannot guarantee that the five desirable properties and the Fairness in Sharing Driver's Cost Property hold at the same time, because all passengers are not guaranteed to be in the same coalition. We derive a new cost-sharing mechanism framework for ride-sharing that shares the total detour cost and the driver's trip cost F separately through 2 sub-mechanisms. This means that a passenger $\pi(k)$'s shared cost consists of two parts: 1) $c^{s1}_{\pi(k),t}$, the total detour cost shared by this passenger at time t; 2) $c^{s2}_{\pi(k),t}$, the driver's direct trip cost F shared by this passenger at time t. Note that $c^{s2}_{\pi(k),t} = \beta_{\pi(k),t}F$ and the $\{\beta_{\pi(k),t}$ values are to be determined by the specific sub-mechanisms.

Next, we introduce the Ride-Sharing Mechanism Framework built upon 2 sub-mechanisms and analyze its properties. In this framework, we constrain the beta values to satisfy two properties: the Fairness in Sharing Driver's Cost property and the Immediate Response property.

In other words, for all times t, the $\beta_{\pi(k),t}$ values are proportional to the passengers' alpha values, that is:

$$\beta_{\pi(1),t} : \beta_{\pi(2),t} : \ldots : \beta_{\pi(t),t} = \alpha_{\pi(1)} : \alpha_{\pi(2)} : \ldots : \alpha_{\pi(t)} \,. \tag{2}$$

And the $\beta_{\pi(k),t}$ values are non-increasing in t, that is, for all times k, t_1, t_2 with $k \le t_1 \le t_2$ and all submit orders π,

$$\beta_{\pi(k),t_2} \le \beta_{\pi(k),t_1} \,. \tag{3}$$

We define the Ride-Sharing Mechanism Framework as follows: For all times k, t with $k \le t$ and all submit orders π, the total shared cost for passenger $\pi(k)$ at time t under submit order π is

$$c^{s}_{\pi(k),t} = c^{s1}_{\pi(k),t} + c^{s2}_{\pi(k),t} = c^{s1}_{\pi(k),t} + \beta_{\pi(k),t}F \tag{4}$$

where $c^{s1}_{\pi(k),t}$ is calculated through POCS (see Definition 4) and $\beta_{\pi(k),t}$ satisfies Equation (2) and Equation (3). We consider specific methods of computing $\beta_{\pi(k),t}$ in Sections 5.1 - 5.3.

We show that this Ride-Sharing Mechanism Framework satisfies four out of the five original desirable properties. We can show that if $c^{s1}_{\pi(k),t}$ and $c^{s2}_{\pi(k),t}$ each satisfies one of the original desirable properties, then $c^{s}_{\pi(k),t}$ must also satisfy that original desirable property (details in Appendix C). Therefore, for the Ride-Sharing Mechanism Framework to satisfy four out of the five original desirable properties, we only need to prove that $c^{s2}_{\pi(k),t}$ under Equation (2) and Equation (3) satisfies the four properties. The proofs are provided in detail in Appendix B. Now we have proven the Ride-Sharing Mechanism Framework satisfies all five original desirable properties except for the Budget Balance property. This is because the definitions of the beta values in the Ride-Sharing Mechanism Framework do not guarantee that all beta values sum up to 1 for all times t and all submit orders π.

5. The Mechanisms

In this section, we examine three different mechanisms that fall under the Ride-Sharing Mechanism Framework and analyze their advantages and disadvantages.

5.1. Driver-out-of-Coalition Mechanism

In this mechanism, 100% of the driver's direct trip cost F is guaranteed to be transferred to the passengers and shared proportionally to the passengers' alpha values.

Definition 8. Under the Driver-out-of-Coalition Mechanism, all times k, t with $k \leq t$ and all submit orders π, the percentage $\beta_{\pi(k),t}$ of F that will be covered by passenger $\pi(k)$ at time t under submit order π is:

$$\beta_{\pi(k),t} := \frac{\alpha_{\pi(k)}}{\sum_{i=1}^{t} \alpha_{\pi(i)}} .$$

It is trivial to see that the above definition of beta values satisfies Equation (2) and Equation (3). Therefore, this Driver-out-of-Coalition Mechanism naturally satisfies four out of the five original desirable properties. We prove that this mechanism also satisfies the Budget Balance property.

THEOREM 1. *Under the Driver-out-of-Coalition Mechanism, $c^{s2}_{\pi(k),t}$ satisfies the Budget Balance property. That is, for all times $t \geq 1$ and submit orders π: $\sum_{k=1}^{t} c^{s2}_{\pi(k),t} = F$.*

Proof: Follows immediately from Definition 8, indeed we have:

$$\sum_{k=1}^{t} c^{s2}_{\pi(k),t} = F \sum_{k=1}^{t} \beta_{\pi(k),t} = F \frac{\sum_{k=1}^{t} \alpha_{\pi(k)}}{\sum_{k=1}^{t} \alpha_{\pi(k)}} = F\,. \quad \square$$

The advantages of this mechanism are:

- All five original desirable properties hold.
- The Fairness in Sharing Driver's Cost property holds.

The disadvantage of this mechanism is:

- It fails to reduce the burden of $\pi(1)$.

Proposition 1 below describes the fact that the Fairness in Sharing Driver's Cost property and the Reduced Burden for the First Passenger property are contradictory under certain circumstances.

PROPOSITION 1. *When the driver's direct trip cost F is fully recovered by the passengers, the Fairness in Sharing Driver's Cost property and the Reduced Burden for the First Passenger property cannot hold at the same time without breaking one of the five original desirable properties.*

Proof: Notice that we only consider the case where $|\mathbb{P}| \geq 2$ since $\beta_{\pi(1),t} = 1$ for all t when $|\mathbb{P}| = 1$ and the Reduced Burden of the First Passenger property is always lost. Suppose the first passenger $\pi(1)$ requests a ride at time 1, receives a quote of the entire driver's direct trip cost F and we want to reduce this burden from 1 (i.e. 100%) to x (where $x < 1$), and maintain the Fairness in Sharing Driver's Cost property. Because the alpha values of the passengers can take arbitrary positive values, it is possible that $\frac{\alpha_{\pi(1)}}{\alpha_{\pi(1)}+\alpha_{\pi(2)}} > x$, which would lead to the loss of the Immediate Response property when the second passenger submits his or her request at time 2. And when the Immediate Response property is lost, the Individual Rationality property is lost[3].

[3] Since $W^2_{\pi(k)}$ can take any value greater or equal to $c^{s2}_{\pi(k),k}$, and failing to satisfy the Immediate Response property means $c^{s2}_{\pi(k),t} > c^{s2}_{\pi(k),k}$ for $t > k$, therefore, we may have $c^{s2}_{\pi(k),t} > W^2_{\pi(k)}$ which leads to the loss of the Individual Rationality property.

If we find a way to determine $\beta_{\pi(1),t}$ that the Immediate Response property holds, for example, let $\beta_{\pi(1),t} = \beta_{\pi(1),t-1} - 0.1$ and $\beta_{\pi(1),1} = 0.9$, then we may face the possibility of losing the Fairness in Sharing Driver's Cost property: $\frac{\beta_{\pi(1),2}}{\beta_{\pi(2),2}} = \frac{\beta_{\pi(1),2}}{1-\beta_{\pi(1),2}} = \frac{0.8}{0.2} \neq \frac{\alpha_{\pi(1)}}{\alpha_{\pi(2)}}$ since $\frac{\alpha_{\pi(1)}}{\alpha_{\pi(2)}}$ may take any value and we let it be 5 here. Or if we satisfy the Fairness in Sharing Driver's Cost property, We may lose the Budget Balance property: $\beta_{\pi(2),2} + \beta_{\pi(1),2} = \beta_{\pi(1),2}\frac{\alpha_{\pi(1)}+\alpha_{\pi(2)}}{\alpha_{\pi(1)}} = 0.8 \times (\frac{6}{5}) = 0.96 \leq 1$. □

One possible solution to this issue is that, if we know the maximum and minimum alpha values and assume there are more than one passenger, by letting $\beta_{\pi(1),1} = \frac{\alpha_{max}}{\alpha_{min}+\alpha_{max}}$, and setting the beta values for all $k \geq 1$ and all $t > 1$ to follow Definition 8, all the properties will hold. This is because this way of determining $\beta_{\pi(1),1}$ guarantees that $\beta_{\pi(1),t} \leq \beta_{\pi(1),1}$ for all $t > 1$. Therefore, the Immediate Response property holds when the Reduced Burden of the First Passenger property and the Fairness in Sharing Driver's Cost property hold. The only problem with this approach may be not enough reduction on the burden of $\pi(1)$ if $\frac{\alpha_{max}}{\alpha_{min}+\alpha_{max}}$ is close to 1.

5.2. Passengers Predicting Mechanism

In this mechanism, we deal with the uncertainty of the total number of passengers and the total alpha values of the passengers using prediction. We predict or estimate the total alpha value in order to normalize the share of F that is allocated to each passenger. We could either use historical data or assume the number of passengers as well as their alpha values to have some probability distributions. Another way of doing this would be to bound the total alpha value robustly so that the desirable properties hold.

First, we formally define the total alpha value.

Definition 9. For all times k, t with $1 \leq k \leq t$ and all submit orders π, the total alpha value for all passengers by time t under submit order π is:

$$A := \sum_{i=1}^{t} \alpha_{\pi(i)} \,. \tag{5}$$

The Passengers Prediction Mechanism is as follows: the system estimates A before the first request using a proposed robust method described in detail in Appendix D. Then we define for all t

$$\beta_{\pi(k),t} = \frac{\alpha_{\pi(k)}}{A} \,. \tag{6}$$

It can be seen that $\beta_{\pi(k),t}$ defined by Equation (6) are proportional to the passengers' alpha values and are non-increasing in t. Therefore, the Passengers Predicting Mechanism satisfies the requirements of the Ride-Sharing Mechanism Framework, thus satisfying the Online Fairness, Individual Rationality, Immediate Response and Ex-Post Incentive Compatibility properties.

The advantages of this mechanism are:

- Four of the five original desirable properties hold.
- The Fairness in Sharing Driver's Cost property holds.
- The Reduced Burden for the First Passenger property holds.

And the disadvantage of this mechanism is:

- It fails to satisfy the Budget Balance property. However, this mechanism will be closer to satisfying the Budget Balance property as the prediction accuracy of A increases.

5.3. Driver-in-Coalition Mechanism

In this mechanism, we do not enforce that the driver's direct trip cost F is fully recovered by the passengers. Instead, we have the driver share a portion of his or her direct trip cost F.

Definition 10. For all times k, t with $0 \le k \le t$ and all submit orders π, the percentage $\beta_{\pi(k),t}$ of F that will be shared by $\pi(k)$ ($\pi(0)$ being the driver) at time t under submit order π is:

$$\beta_{\pi(k),t} := \frac{\alpha_{\pi(k)}}{\sum_{i=0}^{t} \alpha_{\pi(i)}} \,,$$

where $\alpha_{\pi(0)}$ is the alpha value for the driver.

The analysis is the same as the analysis in Section 5.1, so the Driver-in-Coalition Mechanism satisfies the five original desirable properties. The difference between this mechanism and the Driver-out-of-Coalition Mechanism is the involvement of the driver in paying F. By involving the driver into the coalition for covering F, the initial quote for the first passenger $\pi(1)$ becomes more reasonable in that $\beta_{\pi(1),1}$ is bounded above by $\frac{\alpha_{\pi(1)}}{\alpha_{\pi(0)}+\alpha_{\pi(1)}}$. As a result, $\beta_{\pi(1),1}$ is high only when $\pi(1)$'s alpha value is too high compared to the driver's alpha value.

The advantages of the Driver-in-Coalition Mechanism are:

- All five original desirable properties hold.
- The Fairness in Sharing Driver's Cost property holds.
- It provides some reasonable reduction on the burden of the first passenger $\pi(1)$.

And the disadvantage of the Driver-in-Coalition Mechanism is:

- The driver will have to cover some portion of F no matter how large $|\mathbb{P}|$ is.

5.4. Experimental Results

In this section, we conduct simulation experiments to test the performances of the proposed mechanisms, namely 1) the Driver-out-of-Coalition (DooC) mechanism, 2) the Driver-in-Coalition (DiC) mechanism, and 3) the Passenger Prediction (PP) mechanism. We test the mechanisms on a randomized data set where each vehicle has four passengers each with a unique origin and destination. We also assume that the cost per mile is \$1, and the alpha value is the same as the distance. In addition, we have two space patterns we are interested in: a) scattered and b) clustered. "Scattered" means that the origins and destinations of the driver and the passengers are randomly scattered in the grid. "Clustered" means that the origins are generated randomly within a cluster at the bottom left corner of the grid while the destinations are within a cluster at the top right of the grid. We have in total four scenarios whose settings are summarized in Table 1. And for each scenario, we generate 100 replications and the mechanism results are averaged over these replications. Note that changing the grid size will also change the average alpha and detour. A comparison between the first two scenarios will show the impact of the gird size on the mechanism and a comparison between the last three scenarios will show the impact of clustering.

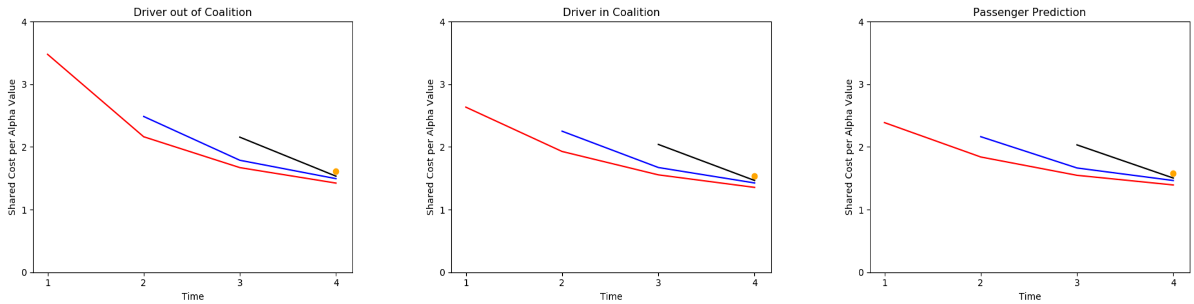
In each replication, because the number of passengers is small per vehicle, the optimal cost can be easily determined quickly through enumeration to generate the total cost. Then, we use the three proposed mechanisms to allocate a portion of the total cost to the passengers. Notice that for the PP mechanism, the mean and standard deviation of the alpha values are required, and we estimate them using 10,000 random samples of the origin and destination distributions.

Table 1 Simulation Settings for the Different Scenarios

Scenarios	**Grid Size**	**Space Pattern**
1	20 x 20	Scattered
2	40 x 40	Scattered
3	40 x 40	Clustered (5 x 5 cluster size)
4	40 x 40	Clustered (10 x 10 cluster size)

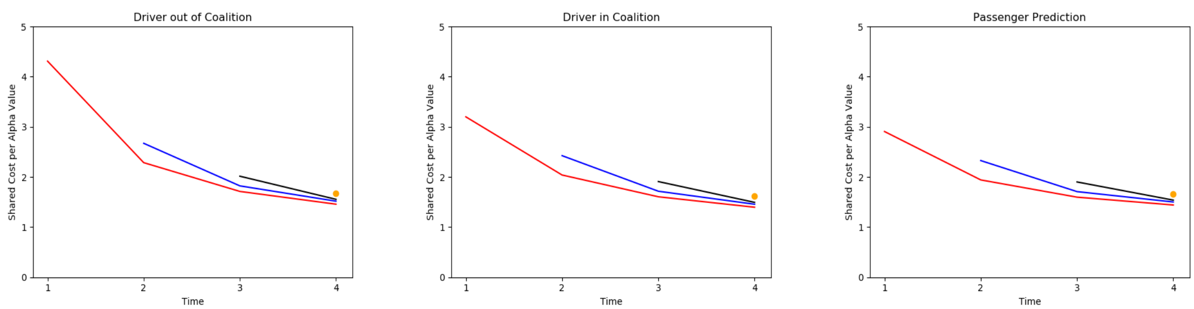
Figures 1 - 4 show the cost trajectory of all four passengers for each mechanism across the four scenarios. In each plot, the x-axis represents the time (that is, the time in which the passenger makes the request). Note that in the static case, all passenger requests are known before the vehicle departs. Thus, time 1 represents the time when the first passenger makes the request. The y-axis represents the shared cost per alpha value as the requests are being made. The red line is passenger one, the blue line is passenger two, the black line is passenger three and the yellow dot is passenger four. As we can see in these figures, the DooC mechanism puts a lot of burden on the first passenger (high initial quote). This is because of the fact that it fails to satisfy the Reduced Burden for the First Passenger property. The DiC mechanism has the lowest final price (per alpha value), because the driver in this mechanism shares the most in paying for F which reduces the total amount that passengers had to pay. When the grid size increases, the initial quotes (per alpha value) are higher because the distances are increasing so the total cost is increasing as well. When the space pattern is switched from "scattered" to "clustered", the initial quotes are significantly reduced because the alpha values for the passengers on average increase while the detour of serving a passenger is smaller because of clustering. Thus, the total cost is smaller causing the initial quotes per alpha value to decrease.

Tables 2 - 5 compare some performance measures of interest. For each table, the first two rows are the total cost of the operation and the drivers direct trip cost F, respectively. Note these two values are not impacted by the different mechanisms and depend on how the origin and destinations are generated. The mechanism differ in how they allocate the total cost among the driver and



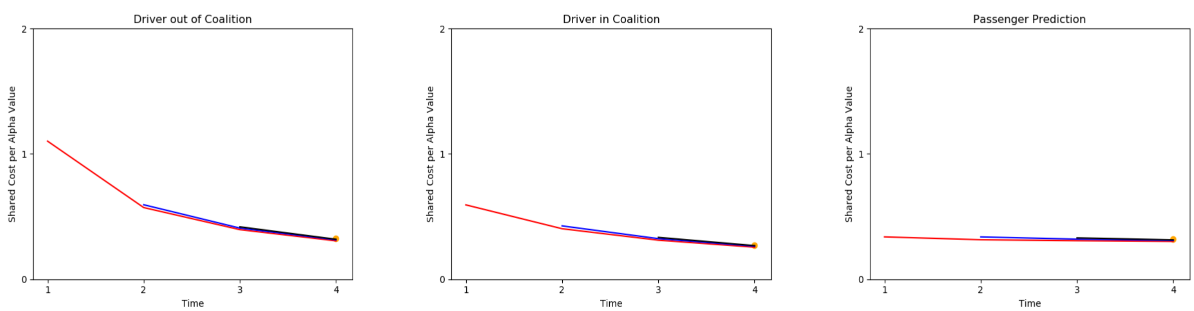
(a) DooC Mechanism **(b)** DiC Mechanism **(c)** PP Mechanism

Figure 1 Cost Trajectories for the Passengers in Scenario 1



(a) DooC Mechanism **(b)** DiC Mechanism **(c)** PP Mechanism

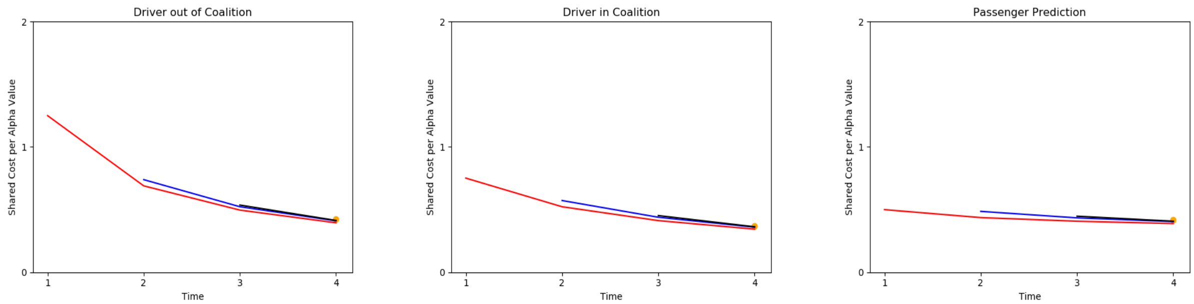
Figure 2 Cost Trajectories for the Passengers in Scenario 2



(a) DooC Mechanism **(b)** DiC Mechanism **(c)** PP Mechanism

Figure 3 Cost Trajectories for the Passengers in Scenario 3

passengers. Thus, the "Average Passenger Cost" row shows the average final shared cost per alpha value among all the passengers. Since the PP mechanism does not guarantee budget balance holds, the "% of Absolute Budget Balance Error" row shows the percentage of the absolute error of the budget balance violation. The "% of Driver's Cost Recovered" row shows how much of the driver's



(a) DooC Mechanism **(b)** DiC Mechanism **(c)** PP Mechanism

Figure 4 Cost Trajectories for the Passengers in Scenario 4

direct cost F is recovered. As we have shown, the Driver out of the Coalition mechanism results in a high initial quote for the first passenger. The last row shows the percentage reduction of this initial quote and is calculated by comparing the initial quote of the first passenger in the corresponding mechanism with that of the DooC mechanism.

Table 2 Performances of the Different Mechanisms in Scenario 1

Mechanisms	**DooC**	**DiC**	**PP**
Total Cost of the Operation	60.53	60.53	60.53
Driver's Direct Trip Cost	10.97	10.97	10.97
Average Passenger Cost	15.13	14.47	14.97
% of Absolute Budget Balance Error	0	0	3.00
% of Driver's Cost Recovered	100	79.54	93.4
% of Reduced Burden for the First Passenger	0	24.25	31.35

In all the tables, we can see that the total cost is increasing as the grid size and cluster size increases. The driver's direct trip cost increases as the grid size increases but decreases as the cluster size increases. This is because the larger the cluster size, the higher the possibility of the driver's origin and destination (OD) being closer. The average cost per passenger matches the trajectory analysis. Comparing Table 2 and Table 3, the percentage of F recovered slightly increases because as the grid gets larger, the average of the alpha values for the passengers increases which reduces

Table 3 Performances of the Different Mechanisms in Scenario 2

Mechanisms	DooC	DiC	PP
Total Cost of the Operation	124.23	124.23	124.23
Driver's Direct Trip Cost	21.09	21.09	21.09
Average Passenger Cost	31.06	29.86	31.00
% of Absolute Budget Balance Error	0	0	3.10
% of Driver's Cost Recovered	100	80.14	97.26
% of Reduced Burden for the First Passenger	0	25.77	32.56

Table 4 Performances of the Different Mechanisms in Scenario 3

Mechanisms	DooC	DiC	PP
Total Cost of the Operation	62.42	62.42	62.42
Driver's Direct Trip Cost	49.7	49.7	49.7
Average Passenger Cost	15.61	13.11	15.39
% of Absolute Budget Balance Error	0	0	1.70
% of Driver's Cost Recovered	100	79.93	98.24
% of Reduced Burden for the First Passenger	0	46.16	69.36

Table 5 Performances of the Different Mechanisms in Scenario 4

Mechanisms	DooC	DiC	PP
Total Cost of the Operation	69.61	69.61	69.61
Driver's Direct Trip Cost	42.46	42.46	42.46
Average Passenger Cost	17.40	15.26	17.17
% of Absolute Budget Balance Error	0	0	2.2
% of Driver's Cost Recovered	100	80.01	97.79
% of Reduced Burden for the First Passenger	0	39.91	60.05

the portion of the driver's alpha value within the total alpha value. The percentage of the reduced burden for the first passenger slightly increases because the detour value of the first passenger increases when the grid is larger. Comparing Table 3 and Table 4, the average cost per passenger is nearly half the price of that in the scattered pattern due to the lower detour values and larger alpha values. The percentage of the reduced burden for the first passenger increases because the passengers generally have similar alpha values (large) and detour values (small) which reduces the burden of the first passenger when using the driver in the coalition or passenger prediction mechanisms. With all other settings the same, when the cluster size increases, the average cost per passenger increases as well because the total cost is higher and the driver's direct trip cost is lower. The budget balance error of the PP mechanism is higher, because the standard deviation of alpha values is larger causing a less accurate estimation of the total alpha values A. The percentage of reduced burden decreases because the alpha value of the driver decreases so including the driver in sharing F is not as efficient. As for the PP mechanism, this percentage is also lower because the larger the cluster size, the larger the standard deviation of the alpha values, so the prediction may not be as accurate which reduces the efficiency in reducing the burden of the first passenger.

6. Ride-Sharing with Time Constraints

In this section, we discuss how to extend our mechanisms to satisfy time constraints. A ride-sharing driver has a limit on how much extra time he or she is willing to spend driving in order to provide the ride-sharing service. Also, a ride-sharing passenger would like to spend as little time as possible in the vehicle. For instance, a passenger may require that his or her trip is finished within the direct trip time plus 10 more minutes.

First, we list the extra notation we use in this section:

T_t^{tot}: the total operation time at time t

$T_{\pi(k)}$: the maximum time passenger $\pi(k)$ can spend in the vehicle

$L_{\pi(k),t}$: the in-vehicle time of passenger $\pi(k)$ at time t

$c^{\text{ic}}_{\pi(k),t}$: the inconvenience cost of passenger $\pi(k)$ at time t, $c^{\text{ic}}_{\pi(1),1} = 0$

$\Delta c^{\text{ic}}_{\pi(k),t}$: $= c^{\text{ic}}_{\pi(k),t} - c^{\text{ic}}_{\pi(k),t-1}$, the marginal inconvenience cost of passenger $\pi(k)$ at time t

$f_{\pi(k)}\left(L_{\pi(k),t}\right)$: a non-decreasing convex function that maps the in-vehicle time of passenger $\pi(k)$ to his or her inconvenience cost

$c^{\text{s3}}_{\pi(k),t}$: the cost of discounts shared by passenger $\pi(k)$ under a discount method at time t

$c^{\text{dis}}_{\pi(k),t}$: the discount amount provided to passenger $\pi(k)$ at time t

$c^{\text{da}}_{\pi(k_1,k_2)}$: the coalition discount cost per alpha value of passengers $\pi(k_1), \ldots, \pi(k_2)$

In this scenario with time constraints, we have two basic rules (see below) when inserting a new passenger $\pi(k), k \geq 1$. If either one of the rules is violated, passenger $\pi(k)$ will not join the ride-sharing operation.

1. T_t^{tot} does not exceed $T_{\pi(0)}$.
2. $c^{\text{s}}_{\pi(k),k}$, the initial quote price of passenger $\pi(k)$, does not exceed $W_{\pi(k)}$.

Notice that if no discount is provided in this time constraints scenario, it becomes a trivial extension from the previous mechanisms. The driver simply rejects any new passenger that causes his or her in-vehicle time to exceed $T_{\pi(0)}$. Therefore, we study the more interesting case where we allow discounts to be provided. To better illustrate the procedure, see the process flow diagram in Figure 5. In this figure, we can see that the two processes marked in yellow are the ones we need to design. We need to come up with a way to provide a discount to a passenger to compensate for his or her inconvenience cost brought by passenger $\pi(k)$'s service request. We also do the same for the new passenger $\pi(k)$.

In order to determine the discount, we need to measure how passengers value their time spent in the vehicle. We use a non-decreasing convex function $c^{\text{ic}}_{\pi(k),t} = f_{\pi(k)}\left(L_{\pi(k),t}\right)$ to quantify the inconvenience cost for passenger $\pi(k)$. Figure 6 illustrates two possibilities of the inconvenience cost function.

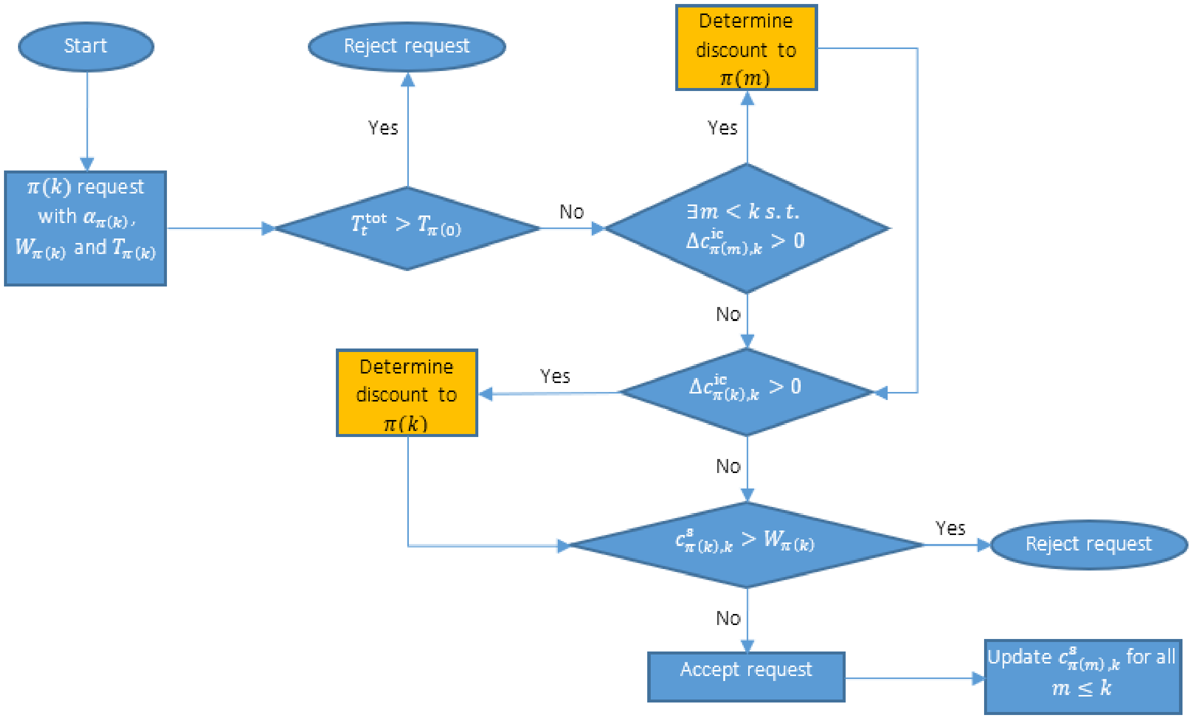
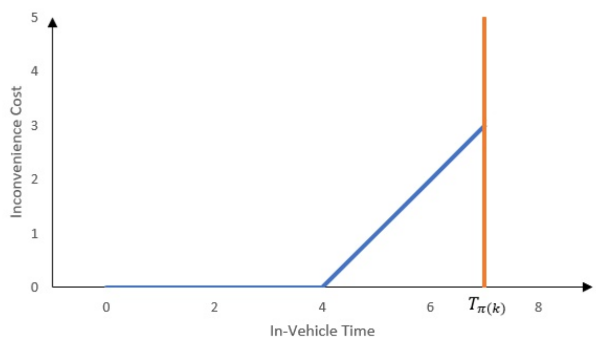
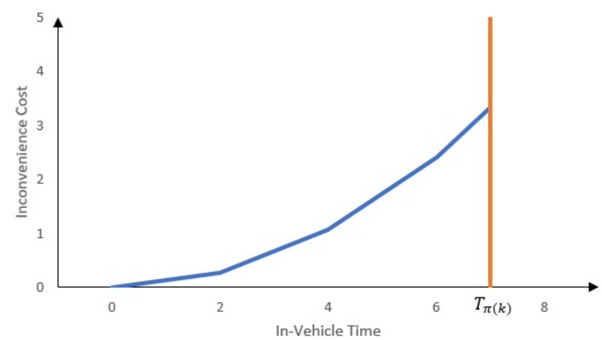


Figure 5 Process Flow Diagram



(a) Constant Cost per Unit Time



(b) Increasing Cost per Unit Time

Figure 6 Sample Convex Functions for the Inconvenience Cost

We next introduce our discount providing solutions, namely the Basic Discount Method and the Inconvenience Cost Based Discount Method. Each method outputs the discount cost $c^{\text{s3}}_{\pi(k),t}$ which serves as another cost component in the general Ride-Sharing Mechanism Framework when calculating the total shared cost $c^{\text{s}}_{\pi(k),t}$. In other words, Equation (4) becomes:

$$c^{\text{s}}_{\pi(k),t} = c^{\text{s1}}_{\pi(k),t} + c^{\text{s2}}_{\pi(k),t} + c^{\text{s3}}_{\pi(k),t} \, . \tag{7}$$

We first formally define the discount amount provided to an existing passenger as well as the new passenger whenever a new passenger requests service.

Definition 11. For all times m, k, t with $1 \leq m < k \leq t$ and all submit orders π, the discount amount provided to passenger $\pi(m)$ at time k is:

$$c^{\text{dis}}_{\pi(m),k} := \min\left(0, \left(c^{\text{s1}}_{\pi(m),k-1} + c^{\text{s2}}_{\pi(m),k-1}\right) - \left(c^{\text{s1}}_{\pi(m),k} + c^{\text{s2}}_{\pi(m),k}\right) - \Delta c^{\text{ic}}_{\pi(m),k}\right) \tag{8}$$

6.1. Basic Discount Method

In this method, the driver does not pay any portion of the discount. Instead, all the burden falls on the new passenger whose request may increase the existing passengers' inconvenience costs. Therefore, no discount is provided to this new passenger at the time of his or her request. Suppose the new passenger is $\pi(k)$ who requests service at time k. The procedure is then:

Step 1. If there exists $m < k$ such that $\Delta c^{\text{ic}}_{\pi(m),k} > 0$, go to Step 2.

Step 2. If passenger $\pi(m)$'s new shared cost before receiving the discount decreased by at least $\Delta c^{\text{ic}}_{\pi(m),k}$, go to Step 4. Otherwise, go to Step 3. In other words, if $c^{\text{dis}}_{\pi(m),k} = 0$, then $c^{\text{s3}}_{\pi(m),k} = c^{\text{s3}}_{\pi(m),k-1}$.

Step 3. If passenger $\pi(m)$'s new shared cost before receiving the discount decreased by less than $\Delta c^{\text{ic}}_{\pi(m),k}$, then this gap is provided by the new passenger $\pi(k)$. This means that if $c^{\text{dis}}_{\pi(m),k} < 0$, then $c^{\text{s3}}_{\pi(m),k} = c^{\text{s3}}_{\pi(m),k-1} + c^{\text{dis}}_{\pi(m),k}$ and $-c^{\text{dis}}_{\pi(m),k}$ is added to $c^{\text{s3}}_{\pi(k),k}$.

Step 4. Repeat Steps 1 - 3 for all existing passengers and get the value of $c^{\text{s3}}_{\pi(k),k} = \sum_{i=1}^{k-1}\left(-c^{\text{dis}}_{\pi(i),k}\right)$. If $c^{\text{s}}_{\pi(k),k} \leq W_{\pi(k)}$ and $L_{\pi(k),k} \leq T_{\pi(k)}$, then passenger $\pi(k)$ joins the ride-sharing trip. Otherwise, he or she rejects the quote by the ride-sharing operation.

We next examine if this discount method satisfies the properties listed in Section 4. For the Fairness in Sharing Driver's Cost Property and the Reduced Burden for the First Passenger Property, the discount component is independent from $c^{\text{s2}}_{\pi(k),t}$, thus there is no impact on these two properties. For the five original desirable properties, since the discount method serves as another additive component under the Ride-Sharing Mechanism Framework, based on Appendix C, we only need to examine if the $c^{s3}_{\pi(k),t}$ values generated by this discount method satisfy the five original desirable properties. As shown in the proofs in Appendix E, the Basic Discount Method satisfies three of the five original desirable properties, with the Online Fairness property and the Ex-Post Incentive Compatibility property no longer holding.

The advantages of this discount method are:

- The Fairness in Sharing Driver's Cost property and the Reduced Burden for the First Passenger property, still hold.
- The cost is easy to calculate and passengers are not responsible for the inconvenience costs that are not caused by them.

The disadvantage of this discount method is that the Online Fairness property and the Ex-Post Incentive Compatibility property are lost. This can be seen through the following counterexamples. For the Online Fairness property, suppose at time $t=2$, both passengers $\pi(1)$ and $\pi(2)$ have no inconvenience costs and $\alpha_{\pi(1)} = \alpha_{\pi(2)}$. In addition, suppose when passenger $\pi(3)$ requests service at time $t=3$, both passengers' in-vehicle time increase and their inconvenience costs go from 0 to positive values $f_{\pi(1)}\left(L_{\pi(1),3}\right)$ and $f_{\pi(2)}\left(L_{\pi(2),3}\right)$ respectively. If $f_{\pi(1)}\left(L_{\pi(1),3}\right) < f_{\pi(2)}\left(L_{\pi(2),3}\right)$ which means $\Delta c^{ic}_{\pi(1),3} < \Delta c^{ic}_{\pi(2),3}$, and this results in $c^{dis}_{\pi(1),3} > c^{dis}_{\pi(2),3}$ assuming that their total shared costs before receiving any discount decrease by the same amount. Then we have $c^{s3}_{\pi(1),3} > c^{s3}_{\pi(2),3}$, which results in $\frac{c^{s3}_{\pi(1),3}}{\alpha_{\pi(1)}} > \frac{c^{s3}_{\pi(2),3}}{\alpha_{\pi(2)}}$, thus contradicting the Online Fairness property. Here is the counter example to show that the Ex-Post Incentive Compatibility property is lost as well. Suppose passengers $\pi(3)$ and $\pi(4)$ share the same origin and destination (this also means $\alpha_{\pi(3)} = \alpha_{\pi(4)}$). Suppose the request of $\pi(3)$ increases the inconvenience costs for the existing passengers and so $c^{s3}_{\pi(3),3} > 0$. Since $\pi(4)$ has the same origin and destination as $\pi(3)$, his or her participation will not increase in-vehicle times for passengers $\pi(1), \pi(2)$ and $\pi(3)$. Thus, $c^{s3}_{\pi(4),4} = 0$. Therefore, it is beneficial for $\pi(3)$ to delay his or her request submission until right after $\pi(4)$'s. Under this new submit order $\pi^{'}$, $\pi^{'}(4) = \pi(3)$, $\pi^{'}(3) = \pi(4)$ and $c^{s3}_{\pi^{'}(4),4} = 0 < c^{s3}_{\pi(3),3}$.

6.2. Inconvenience Cost Based Discount Method

In the Inconvenience Cost Based Discount Method, the driver is also excluded in providing the discount. All the inconvenience costs are shared by all the passengers in a way that is similar to that in Definition 3; namely, they form into coalitions to share the total inconvenience cost. They then obtain their discounts based on their inconvenience costs. In other words, $c^{\mathrm{s3}}_{\pi(k),t}$ consists of two parts: 1) the amount of the total inconvenience cost passenger $\pi(k)$ accounts for; 2) the discounts provided to passenger $\pi(k)$ to compensate for his or her extra in-vehicle time.

Before we formally define $c^{\mathrm{s3}}_{\pi(k),t}$, we first define $c^{\mathrm{da}}_{\pi(k_1,k_2)}$ as:

Definition 12. For all times k_1, k_2 and t and all submit orders π with $k_1 \le k_2 \le t$, the coalition discount cost per alpha value of passengers $\pi(k_1), \ldots, \pi(k_2)$ at time t under submit order π is:

$$c^{\mathrm{da}}_{\pi(k_1,k_2)} := \frac{\sum_{i=1}^{k_2} c^{\mathrm{ic}}_{\pi(i),k_2} - \sum_{i=1}^{k_1} c^{\mathrm{ic}}_{\pi(i),k_1}}{\sum_{i=k_1}^{k_2} \alpha_{\pi(i)}}. \tag{9}$$

With the notion of coalition discount cost per alpha value, we define $c^{\mathrm{s3}}_{\pi(k),t}$ below:

Definition 13. For all times k and t and all submit orders π with $k \le t$, the cost of passenger $\pi(k)$ calculated by the Inconvenience Cost Based Discount Method at time t under submit order π is:

$$c^{\mathrm{s3}}_{\pi(k),t} := \alpha_{\pi(k)} \min_{k \le j \le t} \max_{1 \le i \le j} c^{\mathrm{da}}_{\pi(i,j)} + \left(-c^{\mathrm{ic}}_{\pi(k),t}\right) \Bigg(\tag{10}$$

Suppose the new passenger is $\pi(k)$ who requests service at time k. At the time of the request, this discount method does the following:

<u>Step 1</u>. For all $i \le k$, calculate $c^{\mathrm{ic}}_{\pi(i),k}$.

<u>Step 2</u>. Calculate the coalition discount cost per alpha value for all passengers (including the new passenger).

<u>Step 3</u>. For all $i \le k$, calculate $c^{\mathrm{s3}}_{\pi(i),k}$ based on Definition 13.

<u>Step 4</u>. Calculate $c^{\mathrm{s}}_{\pi(k),k}$ based on the results from Step 3. If $c^{\mathrm{s}}_{\pi(k),k} \le W_{\pi(k)}$, passenger $\pi(k)$ joins the ride-sharing operation. Otherwise, the passenger rejects the quote.

We next examine if this discount method satisfies the properties listed in Section 4. For the Fairness in Sharing Driver's Cost Property and the Reduced Burden for the First Passenger Property, the discount component is independent from $c^{s_2}_{\pi(k),t}$, thus these two properties still hold. For the five original desirable properties, since the discount method serves as another additive component under the Ride-Sharing Mechanism Framework, based on Appendix C, we only need to examine if the $c^{s_3}_{\pi(k),t}$ values generated by this discount method satisfy the five original desirable properties. Because $c^{s_3}_{\pi(k),t}$ itself consists of two additive parts with the first part resembling that of $c^{s_1}_{\pi(k),t}$, we only need to focus on the second part, $c^{\text{ic}}_{\pi(k),t}$, to see whether $c^{s_3}_{\pi(k),t}$ satisfies the five original desirable properties. The reason is that the total inconvenience cost for all the passengers satisfies Assumptions 1 and 2, and Equation (9) and the first part of Equation (10) are basically the same as that in POCS. Therefore, the first part of $c^{s_3}_{\pi(k),t}$ by Definition 13 satisfies all the five original desirable properties, but the second part satisfies only four of the five original desirable properties which leads to the Inconvenience Cost Based Discount Method satisfying the same properties as well, with only the Online Fairness property not holding (see proofs in Appendix E).

The advantage of this discount method is:

- The Fairness in Sharing Driver's Cost property and the Reduced Burden for the First Passenger property still hold.

The disadvantages of this discount method are:

- The Online Fairness property does not hold. The same counter-example as in Section 6.1 can be used to show that the property does not hold.
- Passengers who have high tolerance for in-vehicle time may not get any discounts while being responsible for a portion of the total inconvenience cost.

7. Ride-Sharing in a Dynamic Setting

In this section, we analyze how the dynamic setting affects the ride-sharing operation and then provide an alternative mechanism other than those in Section 5.

When we talk about the dynamic ride-sharing, we are looking at the situation where the driver knows nothing about the passengers before the operation starts. In other words, the driver receives requests from passengers as he or she is driving towards the destination. Thus a reasonable and simple routing strategy for the driver is to head towards his or her destination when no request is received and makes a detour whenever a new request is received. On the driver's way to pick up passengers, the unfinished routes can be optimized to reduce the total cost. We will use this routing strategy throughout the whole section.

When driving under the strategy described in above, we can see that the total cost is non-decreasing over time which means Assumption 1 still holds. However, when the submit sequence is different, the total cost maybe different (increases, decreases or stays the same) which means Assumption 2 is lost. As a result, all the mechanisms in Section 5 will lose the Ex-post Incentive Compatibility property because all of their $c^{s_1}_{\pi(k),t}$ are allocated according to POCS. Moreover, the main idea of POCS which is coalition cost per alpha value results in certain unfairness in allocating the total detour cost because POCS will take into coalition those passengers with big alpha values and short detour to lower down the coalition cost per alpha value for passengers with small alpha values and long detour. In this sense, we would like to introduce an alternative mechanism in this dynamic setting to address this issue.

First we introduce the idea of detour value and the coalition cost per detour value.

Definition 14. For all times $k \le t$, the detour value for passenger $\pi(k)$ under submit order π is defined as

$$d_{\pi(k)} := c_{\pi(k)} - F$$

where $c_{\pi(k)}$ is the total cost of serving passenger $\pi(k)$ only

Definition 15. For all times k_1, k_2 and t with $k_1 \le k_2 \le t$ and all submit orders π, the coalition cost per detour value $c^{\mathrm{d}}_{\pi(k_1,k_2)}$ of passengers $\pi(k_1), \ldots, \pi(k_2)$ at time t under submit order π is:

$$c^{\mathrm{d}}_{\pi(k_1,k_2)} := \frac{\sum_{j=k_1}^{k_2} c^{\mathrm{m}}_{\pi(j)}}{\sum_{j=k_1}^{k_2} d_{\pi(j)}} .$$

The Detour-Based Mechanism is then as follows: (1) For sharing the total detour cost, the mechanism uses the same structure as POCS with the only difference that the alpha value in POCS is replaced by the detour value given in Definition 14. (2) For sharing the driver's direct cost F, we use the same definition of beta values as in Definition 8. Combined together, we have the total shared cost for a passenger calculated under the Detour-Based Mechanism:

$$\begin{aligned} c_{\pi(k),t} &= c^{\mathrm{s}_1}_{\pi(k),t} + c^{\mathrm{s}_2}_{\pi(k),t} \\ &= d_{\pi(k)} \min_{k \le j \le t} \max_{1 \le i \le j} c^{\mathrm{d}}_{\pi(i,j)} + \beta_{\pi(k),t} F \end{aligned}$$

The analysis for this mechanism follows the Ride-Sharing Mechanism Framework and the properties of POCS and the Driver-Out-of-Coalition mechanism. For the total detour cost, we replaced the alpha value in POCS with the detour value and these two values are both independent of the submit order π and the time t, therefore, all the properties that POCS has in the dynamic setting are shared with the first part of this mechanism. In addition, we have shown in Section 5.1 that the $\beta_{\pi(k),t}$ by Definition 8 satisfies all the desirable properties except for the Reduced Burden for the First Passenger Property.

The advantages of the Detour-Based mechanism are:

- Four of the five original desirable properties hold.
- The Fairness in Sharing Driver's Cost property holds.

The disadvantages of this mechanism are:

- It fails to satisfy the Ex-post Incentive Compatibility property.
- It fails to reduce the burden of $\pi(1)$.

7.1. Experimental Results

In this section, we conduct simulations on the performance of the mechanisms in the dynamic setting. First, we would like to study the effect of losing the Ex-post Incentive Compatibility property in the dynamic setting. Recall, this property guarantees that a passenger does not gain by delaying their request time. Since this property is not guaranteed to hold in the dynamic setting, we study on average if a passenger can gain by delaying their request time.

Because the loss of the property comes from the POCS part of sharing the total detour cost, and all the proposed mechanisms are based on POCS, we pick one of the mechanisms, the Passengers Prediction mechanism, to present the effect of the loss of this property. The simulations settings are the same as that in the static case (Section 5.4), and we test on the data sets using a 40 x 40 grid and a space pattern of scattered and clustered. Additionally, for each replication, we delay the first passenger to become a 2^{nd}, 3^{rd} or the last in the request submit order. This results in altogether 100 * 3 = 300 samples. The results are shown in Tables 6 - 7. Each column represents the delay slots, meaning that the first passenger has delayed his or her request to become the 2^{nd}, 3^{rd} or the last in the submit order. The first three rows are the percentage of the samples that are better off, not changed and worse, while the last row shows the average change in final price per alpha value. Note that a positive price change means that the behavior of delaying one‘s request submission time on average results in a higher final price. We can see in both space patterns that although there may be cases where the passenger is better off by delaying their request time on average the passenger is worse off. These results suggest that the previous mechanisms, although they do not guarantee the Ex-post Incentive Compatibility property holds in the dynamic setting, still work well on average to discourage by removing an incentive to the passenger in delaying their request time in hopes of getting a better final price.

Table 6 Effect of Loss of Ex-post Incentive Compatibility Property (Scattered)

Delay Slots	2^{nd}	3^{4d}	4^{th}
% Better Off	37.0	39.0	24.0
% No Change	0.0	0.0	0.0
% Worse	63.0	61.0	76.0
% of Average Price Change	4.1	6.2	39.4

Table 7 Effect of Loss of Ex-post Incentive Compatibility Property (Clustered)

Delay Slots	2^{nd}	3^{4d}	4^{th}
% Better Off	38.0	34.0	29.0
% No Change	0.0	0.0	0.0
% Worse	62.0	66.0	71.0
% of Average Price Change	6.5	10.0	16.0

We next test the mechanisms on a data set from road sensor data provided by Los Angeles Metro and archived by researchers at USC who have developed the Archived Data Management System (ADMS) that collects, archives, and integrates a variety of transportation data sets from Los Angeles, Orange, San Bernardino, Riverside, and Ventura Counties. ADMS includes access to real-time traffic data sets with 9500 highway and arterial loop detectors providing data on traffic count and speed approximately every 1 minute. We selected a region within Los Angeles County that includes 33 sensors on 7 freeways: I-5, I-10, I-105, I-110, I-710, SR-60, and SR-101. Figure 7 is a map showing the locations of the sensors. Based on these traffic flows, we can generate an origin-destination (OD) matrix of demand and based on this demand matrix a OD probability matrix can be determined. In each simulation instance, for a given random number from 0 to 1, we can determine the OD associated with the random number using the OD probability matrix. Because the sensors are on the freeway which represents the cluster center of locations rather than the specific locations, we randomly generate origins and destinations within three miles of the sensors.

Next, we compare the Detour Based mechanism with the DooC mechanism because it uses the same method in sharing the driver's direct trip cost, F. Our simulation settings are as follows. The dynamic passengers arrive as a Poisson Process which means their interarrival times are exponentially distributed. We assume on average there are 10,000 passengers arriving in an hour which results in a mean of 0.006 minute for the interarrival time. We also assume that the average speed of all drivers is 36 miles per hour. We evaluate the system with 1,000 passenger requests

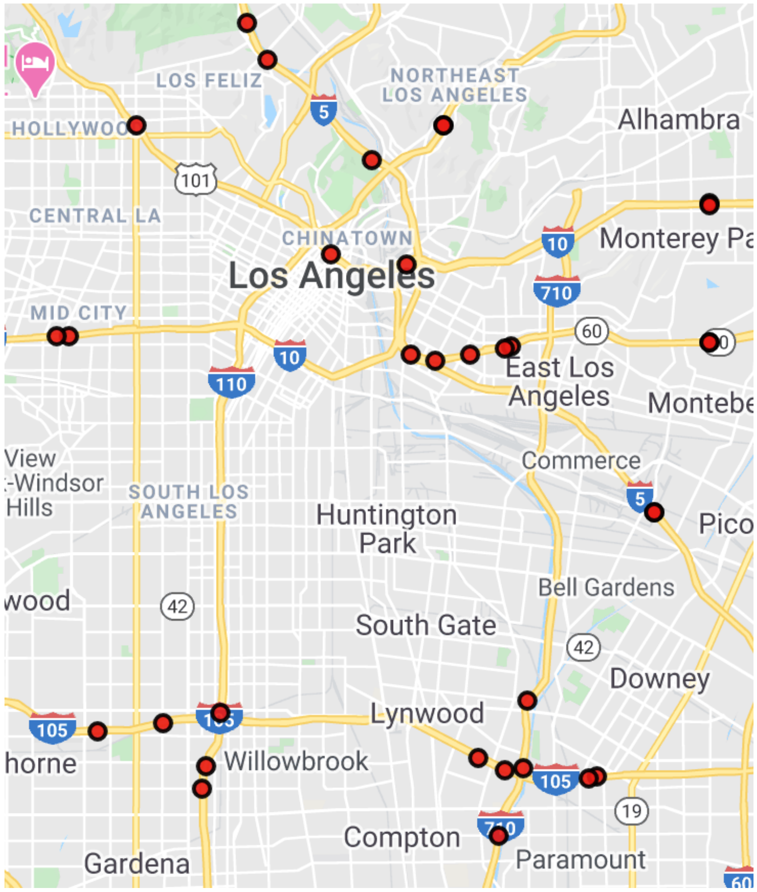


Figure 7 Map of Sensors

with 300 and 500 ride-sharing drivers. We set the time limit of the driver to be 1.5 or 2 times their direct travel time T (meaning the driver is willing to spend 50% or 100% more time in sharing the ride). In summary, our simulation settings for different scenarios are in Table 8.

Table 8 Simulation Settings for the Different Scenarios in Dynamic Setting

Scenarios	Number of Requests	Number of Drivers	Time Limit
1	1000	300	1.5T
2	1000	300	2T
3	1000	500	1.5T
4	1000	500	2T

The results averaged over 10 replications are shown in Tables 9 - 12. In each table, the first row presents the average direct trip cost per vehicle which is independent of the mechanisms used. The second row calculates the total cost of the operation per vehicle. The third row calculates the shared cost per passenger which is averaged over all the passengers served in one instance. The fourth row presents the average cost paid by the driver. The fifth row shows the percentage of passenger requests that are satisfied by the ride-sharing system.

Table 9 Performances of the DooC and Detour Based Mechanisms in Scenario 1

Mechanisms	**DooC**	**Detour**
Driver's Direct Trip Cost	7.35	7.35
Total Operation Cost per Vehicle	10.01	10.02
Shared Cost Per Passenger	4.46	4.48
Shared Cost Per Driver	1.57	1.55
% of Requests Served	56.82	56.66

Table 10 Performances of the DooC and Detour Based Mechanisms in Scenario 2

Mechanisms	**DooC**	**Detour**
Driver's Direct Trip Cost	7.31	7.31
Total Operation Cost per Vehicle	13.04	13.04
Shared Cost Per Passenger	4.47	4.47
Shared Cost Per Driver	1.29	1.29
% of Requests Served	78.79	78.83

As the tables show, as the time limit and the number of drivers increases, the percentage of requests served increases with almost all of them being served in the fourth scenario. Comparing scenarios 1 and 3, we can observe an increase in the shared cost per passenger, shared cost per driver and the percentage of requests served as the number of drivers in the system increases.

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Table 11 Performances of the DooC and Detour Based Mechanisms in Scenario 3

Mechanisms	DooC	Detour
Driver's Direct Trip Cost	7.38	7.38
Total Operation Cost per Vehicle	9.62	9.62
Shared Cost Per Passenger	4.76	4.66
Shared Cost Per Driver	2.48	2.55
% of Requests Served	75.06	75.34

Table 12 Performances of the DooC and Detour Based Mechanisms in Scenario 4

Mechanisms	DooC	Detour
Driver's Direct Trip Cost	7.35	7.35
Total Operation Cost per Vehicle	11.77	11.73
Shared Cost Per Passenger	4.81	4.75
Shared Cost Per Driver	2.67	2.69
% of Requests Served	94.53	95.10

The increase in the requests served is intuitive because with more vehicles, the system has more capacity to serve more passengers within a fixed time frame. The increase in the other two measures is counter-intuitive since with more requests served, it is reasonable to expect more passengers sharing the same ride which would reduce both the shared cost per passenger and the shared cost per driver. However, when more drivers are placed in the system, many of them end up driving alone which means their costs are not shared and this increases the average values over all the replications.

Comparing scenarios 1 and 2, we can observe that as the drivers have more time in sharing their rides, more passengers are served in the system. The total operation cost per vehicle increases while the shared cost per driver decreases. The behavior of these two measures indicates that more passengers are served by a single driver. We can see that increasing the time limit is efficient in increasing the number of passengers served compared with the method of increasing the number of drivers in the system.

8. Conclusions

In this report, we study the problem of designing cost allocation mechanisms in a ride-sharing scenario different from Uber/Lyft drivers where in our context the drivers have their own origins and destinations and they have time constraints. We first study the problem for the static case which means all the passenger requests are received before the driver starts the ride-sharing operation and then extend it to the dynamic case.

We first explore the desirable properties under the described ride-sharing scenario and then explain why mechanisms from the previous literature fail to work. We next propose a general mechanism framework that satisfies the Online Fairness, Immediate Response, Individual Rationality and Ex-Post Incentive Compatibility properties. This framework is flexible and can be adapted to serve different purposes. Based on this general mechanism framework, we then proposed three different mechanisms that each has its advantages and disadvantages. If the driver cares about the five original desirable properties, they may choose the Driver-out-of-Coalition Mechanism. If the driver cares less about the Budget Balance property but more on the Fairness in Sharing Driver's Cost and the Reduced Burden for the First Passenger properties, they may choose the Passengers Predicting Mechanism. If the driver has some flexibility in not recovering all of the direct trip cost, they may choose the Driver-in-Coalition Mechanism.

Next, we incorporate time constraints with the previously proposed mechanisms and analyze the performances of the proposed methods. Similar to the proposed mechanisms, the choice of the proposed method to deal with time constraints depends on the preferences of the driver. Finally, we analyze the performances of the mechanisms under the dynamic setting and proposed the Detour Based mechanism for allocating the total detour cost (computing $c^{s_1}_{\pi(k),t}$) in addressing a certain unfairness in POCS. Regarding the loss of the Ex-post Incentive Compatibility property, we run simulations to investigate its effect. We find that even though the property is lost, the average change in price disencourages passengers from delaying their submission request time. Regarding the performances of the Detour Based mechanism, our simulation using real world data show that it is efficient in matching passengers with drivers.

Future research can focus on the answers to the following questions. How do the different mechanisms impact the traffic system? What would be a good vehicle capacity size? How do we incorporate dynamic pricing when the system experiences high demand?

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Data Management Plan

Products of Research

This project did not involve collecting any data sets. The first set of experiments involved generating uniform random variables and the experimental section describes how to replicate the generation of these randomized data. The second set of experiments are based on road sensor data provided by Los Angeles Metro and archived in ADMS to generate an OD probability matrix.

Data Format and Content

All data sets are stored as excel files.

Data Access and Sharing

To access ADMS, please contact METRANS.

Reuse and Redistribution

Individuals must receive written authorization from METRANS to use the data stored in ADMS.

A. Appendix: Definitions of the Five Desirable Properties

The following definitions of the five desirable properties are originally described and defined in Furuhata et al. (2015) and we adopt them in our report. Notice that we adapt the definitions for both $c^{\mathrm{s1}}_{\pi(k),t}$ and $c^{\mathrm{s2}}_{\pi(k),t}$.

DEFINITION 16. Online Fairness Property. The shared costs per alpha value of a passenger is never higher than those of passengers who submit their ride requests after it, that is, for all times k_1, k_2 and t with $1 \le k_1 \le k_2 \le t$ and all submit orders π,

$$\frac{c^{\mathrm{s1}}_{\pi(k_1),t}}{\alpha_{\pi(k_1)}} \le \frac{c^{\mathrm{s1}}_{\pi(k_2),t}}{\alpha_{\pi(k_2)}}\,, \quad \frac{c^{\mathrm{s2}}_{\pi(k_1),t}}{\alpha_{\pi(k_1)}} \le \frac{c^{\mathrm{s2}}_{\pi(k_2),t}}{\alpha_{\pi(k_2)}}\,.$$

DEFINITION 17. Immediate Response Property. Passengers are provided immediately after their ride request submissions with (ideally low) upper bounds on their shared costs at any future time. That is, for all times k, t_1 and t_2 with $1 \le k \le t_1 \le t_2$ and all submit orders π,

$$c^{\mathrm{s1}}_{\pi(k),t_1} \ge c^{\mathrm{s1}}_{\pi(k),t_2}\,, \quad c^{\mathrm{s2}}_{\pi(k),t_1} \ge c^{\mathrm{s2}}_{\pi(k),t_2}\,.$$

DEFINITION 18. Individual Rationality Property. The shared costs of passengers who accepted their fare quotes never exceed their fare limits at any future time. That is, for all times k and t with $1 \le k \le t$ and all submit orders π,

$$c^{\mathrm{s1}}_{\pi(k),t} \le W^1_{\pi(k)}\,, \quad c^{\mathrm{s2}}_{\pi(k),t} \le W^2_{\pi(k)}\,.$$

DEFINITION 19. Budget Balance Property. The total cost to drive and the amount of the driver's direct trip cost recovered equal the sum of the shared costs of all passengers. That is, for all times $t \ge 1$ and all submit orders π,

$$\sum_{j=1}^{t} c^{\mathrm{s1}}_{\pi(j),t} = c^{\mathrm{d}}_{\pi,t}\,, \quad \sum_{j=1}^{t} c^{\mathrm{s2}}_{\pi(j),t} = F_{\pi,t}\,.$$

DEFINITION 20. Ex-Post Incentive Compatibility. The best strategy of every passenger is to submit his or her ride request truthfully, provided that all other passengers do not change their submit times and whether they accept or decline their fare quotes, because then it cannot decrease their shared costs by delaying their ride request submissions. That is, for all times k_1, k_2 and t with $1 \leq k_1 \leq k_2 \leq t$ and all submit orders π and $\pi^{'}$ with

$$\pi^{'}(k) = \begin{cases} \pi(k+1) & \text{if } k_1 \leq k < k_2, \\ \pi(k_1) & \text{if } k = k_2, \\ \pi(k) & \text{otherwise,} \end{cases}$$

we have $c^{s1}_{\pi(k_1),t} \leq c^{s1}_{\pi'(k_2),t}$, $c^{s2}_{\pi(k_1),t} \leq c^{s2}_{\pi'(k_2),t}$.

In other words, consider any time t, any submit order π and any passenger $\pi(k_1)$. Now assume that the passenger delays his or her ride request submission and now submits the k_2th instead of the k_1th ride request, with everything else being equal. Then, the shared costs $c^{s1}_{\pi'(k_2),t}$ and $c^{s2}_{\pi'(k_2),t}$ at time t under the new submit order π should not be lower than the shared costs $c^{s1}_{\pi(k_1),t}$ and $c^{s2}_{\pi(k_1),t}$ at time t under the previous submit order π.

B. Appendix: Proofs for $c^{s2}_{\pi(k),t}$ Satisfying the Four Properties

In this appendix section, we prove the following statement: $c^{s2}_{\pi(k),t}$ under Equation (2) and Equation (3) satisfies Online Fairness, Immediate Response, Individual Rationality and Ex-Post Incentive Compatibility properties.

THEOREM 2. *Under the Ride-Sharing Mechanism Framework, $c^{s2}_{\pi(k),t}$ satisfies the Online Fairness property. That is, for all times k_1, k_2, t with $1 \leq k_1 \leq k_2 \leq t$ and submit orders π:* $\frac{c^{s2}_{\pi(k_1),t}}{\alpha_{\pi(k_1)}} \leq \frac{c^{s2}_{\pi(k_2),t}}{\alpha_{\pi(k_2)}}$.

Proof: By Equation (2), we have:

$$\frac{\beta_{\pi(k_1),t}F}{\beta_{\pi(k_2),t}F} = \frac{\alpha_{\pi(k_1)}}{\alpha_{\pi(k_2)}} \Rightarrow \frac{\beta_{\pi(k_1),t}F}{\alpha_{\pi(k_1)}} = \frac{\beta_{\pi(k_2),t}F}{\alpha_{\pi(k_2)}} \Rightarrow \frac{c^{s2}_{\pi(k_1),t}}{\alpha_{\pi(k_1)}} = \frac{c^{s2}_{\pi(k_2),t}}{\alpha_{\pi(k_2)}}. \quad \square$$

THEOREM 3. *Under the Ride-Sharing Mechanism Framework, $c^{s2}_{\pi(k),t}$ satisfies the Immediate Response property. That is, for all times k, t_1, t_2 with $1 \leq k \leq t_1 \leq t_2$ and submit orders π:* $c^{s2}_{\pi(k),t_1} \geq c^{s2}_{\pi(k),t_2}$.

Proof: Directly by Equation (3), we have:

$$\beta_{\pi(k),t_1} F \geq \beta_{\pi(k),t_2} F \Rightarrow c^{s2}_{\pi(k),t_1} \geq c^{s2}_{\pi(k),t_2} . \quad \square$$

THEOREM 4. *Under the Ride-Sharing Mechanism Framework, $c^{s2}_{\pi(k),t}$ satisfies the Individual Rationality property. That is, for all times k, t with $1 \leq k \leq t$ and submit orders π: $c^{s2}_{\pi(k),t} \leq W^2_{\pi(k)}$.*

Proof: By Theorem 3, we know that: $c^{s2}_{\pi(k),t} \leq c^{s2}_{\pi(k),k}$. We know that $c^{s2}_{\pi(k),k} \leq W^2_{\pi(k)}$ since the passenger accepted his or her fare quote and so the property is satisfied. $\square$

THEOREM 5. *Under the Ride-Sharing Mechanism Framework, $c^{s2}_{\pi(k),t}$ satisfies the Ex-Post Incentive Compatibility property. That is, for all times k_1, k_2, t with $1 \leq k_1 \leq k_2 \leq t$ and submit orders π, π' with*

$$\pi'(k) = \begin{cases} \pi(k+1) & \text{if } k_1 \leq k < k_2, \\ \pi(k_1) & \text{if } k = k_2, \\ \pi(k) & \text{otherwise,} \end{cases}$$

we have $c^{s2}_{\pi(k_1),t} \leq c^{s2}_{\pi'(k_2),t}$.

Proof: By Equation (2), we have in both submit orders π and π', the beta values are proportional to their alpha values, that is:

$$\beta_{\pi(1),t} : \ldots : \beta_{\pi(k_1-1),t} : \beta_{\pi(k_1),t} = \alpha_{\pi(1)} : \ldots : \alpha_{\pi(k_1-1)} : \alpha_{\pi(k_1)}$$

$$\beta_{\pi'(1),t} : \ldots : \beta_{\pi'(k_2-1),t} : \beta_{\pi'(k_2),t} = \alpha_{\pi'(1)} : \ldots : \alpha_{\pi'(k_2-1)} : \alpha_{\pi'(k_2)} .$$

Since we have

$$\pi'(k) = \begin{cases} \pi(k+1) & \text{if } k_1 \leq k < k_2, \\ \pi(k_1) & \text{if } k = k_2, \\ \pi(k) & \text{otherwise,} \end{cases}$$

which results in $\alpha_{\pi'(k)} = \alpha_{\pi(k)}$ for $k < k_1$ and $\alpha_{\pi'(k_2)} = \alpha_{\pi(k_1)}$; Then we have:

$$\beta_{\pi(1),t} : \ldots : \beta_{\pi(k_1-1),t} : \beta_{\pi(k_1),t} = \beta_{\pi'(1),t} : \ldots : \beta_{\pi'(k_2-1),t} : \beta_{\pi'(k_2),t} . \tag{11}$$

Regardless of the submit order, the beta values will be determined the same way. Therefore, we have: $\beta_{\pi(k),t} = \beta_{\pi'(k),t}$ for $k < k_1$. Then, combined with Equation (11), we have:

$$\beta_{\pi(k_1),t} = \beta_{\pi'(k_2),t} \Rightarrow \beta_{\pi(k_1),t} F = \beta_{\pi'(k_2),t} F \Rightarrow c^{\mathrm{s2}}_{\pi(k_1),t} = c^{\mathrm{s2}}_{\pi'(k_2),t} . \quad \square$$

C. Appendix: Proofs for Ride-Sharing Mechanism Framework

In this appendix section, we prove the following statement: if $c^{\mathrm{s1}}_{\pi(k),t}$ and $c^{\mathrm{s2}}_{\pi(k),t}$ satisfy one of the five original desirable properties, then $c^{\mathrm{s}}_{\pi(k),t}$ will also satisfy the same property. Note that $c^{\mathrm{s1}}_{\pi(k),t}$ and $c^{\mathrm{s2}}_{\pi(k),t}$ can be the shared costs from any two sub-mechanisms.

PROPOSITION 2. *Provided that both $c^{\mathrm{s1}}_{\pi(k),t}$ and $c^{\mathrm{s2}}_{\pi(k),t}$ satisfy the Online Fairness property, $c^{\mathrm{s}}_{\pi(k),t}$ satisfies the Online Fairness property. That is, for all times k_1, k_2, t with $1 \le k_1 \le k_2 \le t$ and submit orders π:*

$$\frac{c^{\mathrm{s}}_{\pi(k_1),t}}{\alpha_{\pi(k_1)}} \le \frac{c^{\mathrm{s}}_{\pi(k_2),t}}{\alpha_{\pi(k_2)}} .$$

Proof: We know that both $c^{\mathrm{s1}}_{\pi(k),t}$ and $\beta_{\pi(k),t} F$ satisfy the Online Fairness property. That is, for all times k_1, k_2, t with $1 \le k_1 \le k_2 \le t$ and submit orders π,

$$\frac{c^{\mathrm{s1}}_{\pi(k_1),t}}{\alpha_{\pi(k_1)}} \le \frac{c^{\mathrm{s1}}_{\pi(k_2),t}}{\alpha_{\pi(k_2)}} \quad \text{and} \quad \frac{c^{\mathrm{s2}}_{\pi(k_1),t}}{\alpha_{\pi(k_1)}} \le \frac{c^{\mathrm{s2}}_{\pi(k_2),t}}{\alpha_{\pi(k_2)}} .$$

Therefore,

$$\frac{c^{\mathrm{s}}_{\pi(k_1),t}}{\alpha_{\pi(k_1)}} = \frac{c^{\mathrm{s1}}_{\pi(k_1),t}}{\alpha_{\pi(k_1)}} + \frac{c^{\mathrm{s2}}_{\pi(k_1),t}}{\alpha_{\pi(k_1)}} \le \frac{c^{\mathrm{s1}}_{\pi(k_2),t}}{\alpha_{\pi(k_2)}} + \frac{c^{\mathrm{s2}}_{\pi(k_2),t}}{\alpha_{\pi(k_2)}} = \frac{c^{\mathrm{s}}_{\pi(k_2),t}}{\alpha_{\pi(k_2)}} . \quad \square$$

PROPOSITION 3. *Provided that both $c^{\mathrm{s1}}_{\pi(k),t}$ and $\beta_{\pi(k),t} F$ satisfy the Immediate Response property, $c^{\mathrm{s}}_{\pi(k),t}$ satisfies the Immediate Response property. That is, for all times k, t_1, t_2 with $1 \le k \le t_1 \le t_2$ and submit orders π:*

$$c^{\mathrm{s}}_{\pi(k),t_1} \ge c^{\mathrm{s}}_{\pi(k),t_2} .$$

Proof: We know that both $c^{\mathrm{s1}}_{\pi(k),t}$ and $\beta_{\pi(k),t} F$ satisfy the Immediate Response property. That is, for all times k, t_1, t_2 with $1 \le k \le t_1 \le t_2$ and submit orders π,

$$c^{\mathrm{s1}}_{\pi(k),t_1} \ge c^{\mathrm{s1}}_{\pi(k),t_2} \quad \text{and} \quad c^{\mathrm{s2}}_{\pi(k),t_1} \ge c^{\mathrm{s2}}_{\pi(k),t_2} .$$

Therefore,

$$c^{\mathrm{s}}_{\pi(k),t_1} = c^{\mathrm{s1}}_{\pi(k),t_1} + c^{\mathrm{s2}}_{\pi(k),t_1} \geq c^{\mathrm{s1}}_{\pi(k),t_2} + c^{\mathrm{s2}}_{\pi(k),t_2} = c^{\mathrm{s}}_{\pi(k),t_2} . \quad \square$$

PROPOSITION 4. *Provided that both* $c^{\mathrm{s1}}_{\pi(k),t}$ *and* $c^{\mathrm{s2}}_{\pi(k),t}$ *satisfy the Individual Rationality property,* $c^{\mathrm{s}}_{\pi(k),t}$ *satisfies the Individual Rationality property. That is, for all times* k,t *with* $1 \leq k \leq t$ *and submit orders* π*:*

$$c^{\mathrm{s}}_{\pi(k),t} \leq W_{\pi(k)} .$$

Proof: We know that both $c^{\mathrm{s1}}_{\pi(k),t}$ and $c^{\mathrm{s2}}_{\pi(k),t}$ satisfy the Individual Rationality property. That is, for all times k,t with $1 \leq k \leq t$ and submit orders π:

$$c^{\mathrm{s1}}_{\pi(k),t} \leq W^1_{\pi(k)} \quad \text{and} \quad c^{\mathrm{s2}}_{\pi(k),t} \leq W^2_{\pi(k)} .$$

Since $W_{\pi(k)} = W^1_{\pi(k)} + W^2_{\pi(k)}$, we have:

$$c^{\mathrm{s}}_{\pi(k),t} = c^{\mathrm{s1}}_{\pi(k),t} + c^{\mathrm{s2}}_{\pi(k),t} \leq W^1_{\pi(k)} + W^2_{\pi(k)} = W_{\pi(k)} . \quad \square$$

PROPOSITION 5. *Provided that both* $c^{\mathrm{s1}}_{\pi(k),t}$ *and* $\beta_{\pi(k),t}F$ *satisfy the Budget Balance property,* $c^{\mathrm{s}}_{\pi(k),t}$ *satisfies the Budget Balance property. That is, for all times* $t \geq 1$ *and submit orders* π*:*

$$\sum_{j=1}^{t} c^{\mathrm{s}}_{\pi(j),t} = c^{\mathrm{p}}_{\pi,t} .$$

Proof: We know that both $c^{\mathrm{s1}}_{\pi(k),t}$ and $c^{\mathrm{s2}}_{\pi(k),t}$ satisfy the Budget Balance property. That is, for all times $t \geq 1$ and submit orders π:

$$\sum_{j=1}^{t} c^{\mathrm{s1}}_{\pi(j),t} = c^{\mathrm{d}}_{\pi,t} \quad \text{and} \quad \sum_{j=1}^{t} c^{\mathrm{s2}}_{\pi(j),t} = F_{\pi,t} .$$

We then have:

$$\sum_{j=1}^{t} c^{\mathrm{s}}_{\pi(j),t} = \sum_{j=1}^{t} (c^{\mathrm{s1}}_{\pi(j),t} + c^{\mathrm{s2}}_{\pi(j),t}) = c^{\mathrm{d}}_{\pi,t} + F_{\pi,t} = c^{\mathrm{p}}_{\pi,t} . \quad \square$$

PROPOSITION 6. *Provided that both* $c^{\mathrm{s1}}_{\pi(k),t}$ *and* $\beta_{\pi(k),t}F$ *satisfy the Ex-Post Incentive Compatibility property,* $c^{\mathrm{s}}_{\pi(k),t}$ *satisfies the Ex-Post Incentive Compatibility property. That is, for all times* k_1,k_2,t *with* $1 \leq k_1 \leq k_2 \leq t$ *and submit orders* π, π' *with*

$$\pi'(k) = \begin{cases} \pi(k+1) & \text{if } k_1 \leq k < k_2, \\ \pi(k_1) & \text{if } k = k_2, \\ \pi(k) & \text{otherwise,} \end{cases}$$

we have $c^{\mathrm{s}}_{\pi(k_1),t} \le c^{\mathrm{s}}_{\pi'(k_2),t}$.

Proof: We know that both $c^{\mathrm{s}_1}_{\pi(k),t}$ and $\beta_{\pi(k),t}F$ satisfy the Ex-Post Incentive Compatibility property. That is, fix any submit orders π, π' that satisfy the conditions above, as well as any times k_1, k_2, t with $1 \le k_1 \le k_2 \le t$; We have:

$$c^{\mathrm{s}_1}_{\pi(k_1),t} \le c^{\mathrm{s}_1}_{\pi'(k_2),t} \quad \text{and} \quad c^{\mathrm{s}_2}_{\pi(k_1),t} \le c^{\mathrm{s}_2}_{\pi'(k_2),t}\,.$$

Therefore,

$$c^{\mathrm{s}}_{\pi(k_1),t} = c^{\mathrm{s}_1}_{\pi(k_1),t} + c^{\mathrm{s}_2}_{\pi(k_1),t} \le c^{\mathrm{s}_1}_{\pi'(k_2),t} + c^{\mathrm{s}_2}_{\pi'(k_2),t} = c^{\mathrm{s}}_{\pi'(k_2),t}\,. \quad \square$$

D. Appendix: Robust Method for Total Alpha Value Estimation

In this appendix section, we formally introduce our proposed robust method. We are interested in estimating the quantiles of the distribution of the sum of α values associated with passengers that arrived on or prior to some given time $t \in \mathbb{R}_+$, after which no more requests are accepted. Since both passenger arrival times and α values are uncertain, even if the distributions associated with these quantities are perfectly known, this is a hard problem in applied probability which, to the best of our knowledge, does not have an analytical solution. If the distributions of α values and inter-arrival times are perfectly known, asymptotically accurate estimates for the quantiles can be obtained by simulation. Unfortunately, estimating these distributions accurately is not possible due to lack of data. Thus, we propose an approach inspired by modern robust queuing theory (Bandi et al. 2015, 2018) to obtain estimates for the quantiles of the distribution of the sum of α values of those passengers that arrived on or prior to time t. Our approach yields estimates that are *robust* to *ambiguity* in the distributions of the uncertain parameters.

Our proposed approach proceeds as follows. First, we design *uncertainty sets* for the passenger interarrival times and for the α values. The uncertain parameters are guaranteed to materialize in these sets with some user chosen probability. Second, we formulate an optimization problem whose optimal objective value we refer to as "robust sum of α values" and corresponds to an estimate of the desired quantile. Then, we show that the optimal objective value of this problem can be computed in closed form. Finally, we present the performance of our robust approach.

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The Model

Let $T_i \in \mathbb{R}_+,\ i = 1, 2, \ldots, n$ denote the (uncertain) arrival time of the ith passenger. Thus, the sequence $\{T_1, T_2, \ldots, T_n\}$ collects the interarrival times of all passengers. Suppose that λ is the arrival rate of passengers in the system. Similarly, let $\alpha_i \in \mathbb{R}_+$, $i = 1, 2, \ldots, n$ denote the (uncertain) α value of the ith passenger to arrive and let $\bar{\alpha}$ denote the expectation of the distribution of α values. The n here is a number large enough that can simulate the population of the passengers, and by the central limit theorem, larger n yields smaller standard deviation.

We propose to restrict the α values to lie in the uncertain set

$$\mathcal{U}_{\mathrm{a}} := \left\{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n \ : \ -\Gamma_{\mathrm{a}} \leq \frac{\sum_{\ell=1}^{i} \alpha_\ell - i\bar{\alpha}}{(i)^{\frac{1}{\tau_{\mathrm{a}}}}} \leq \Gamma_{\mathrm{a}}, \quad \forall i = 1, \ldots, n \right\}, \tag{12}$$

where Γ_{a} is a budget of uncertainty parameter and $\tau_{\mathrm{a}} \in (1, 2]$ is a parameter modeling heavy-tailed probability distributions. Accordingly, we restrict the interarrival times to lie in the set

$$\mathcal{U}_{\mathrm{t}} := \left\{ (T_1, \ldots, T_n) \in \mathbb{R}_+^n \ : \ -\Gamma_{\mathrm{t}} \leq \frac{\sum_{\ell=1}^{i} T_\ell - \frac{i}{\lambda}}{(i)^{\frac{1}{\tau_{\mathrm{t}}}}} \leq \Gamma_{\mathrm{t}}, \quad \forall i = 1, \ldots, n \right\}, \tag{13}$$

where Γ_{t} is a budget of uncertainty parameter and $\tau_{\mathrm{t}} \in (1, 2]$ is a parameter modeling heavy-tailed probability distributions.

The parameters Γ_{a} and Γ_{t} are chosen to guarantee that with some prescribed probability p, $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_n)$ and $\boldsymbol{T} := (T_1, \ldots, T_n)$ will materialize in these regions. Suppose for example if we assume that α values are normally distributed, and we are looking to estimate the $p = 95$th quantile of the sum of α values of passengers that arrived before time t. Then, we can set $\tau_{\mathrm{a}} = 2$ and $\Gamma_{\mathrm{a}} = 1.64$ (corresponding to 95 percentile of truncated normal distribution since our alpha values are non-negative).

We define the robust sum of α values as the maximum value that the sum of all α's associated with passengers that arrived prior to time t can take, subject to $\boldsymbol{\alpha}$ and $\boldsymbol{T}$ both lying in their respective uncertainty sets. It is given as the optimal value of the optimization problem

$$\begin{aligned} \max \ & \sum_{i=1}^{n} \alpha_i \mathbb{I}\left(\sum_{\ell=1}^{i} T_\ell \leq t \right) \\ \text{s.t.} \ & \boldsymbol{\alpha} \in \mathcal{U}_{\mathrm{a}}, \boldsymbol{T} \in \mathcal{U}_{\mathrm{t}}. \end{aligned} \tag{14}$$

As we will see, this problem admits an analytical solution for the specific choices of uncertainty sets in Equations (12) and (13).

For the case of general convex uncertainty sets, our problem can be formulated as the following mixed-integer convex problem

$$\begin{aligned} \max \ & \sum_{i=1}^{n} \alpha_i y_i \\ \text{s.t.} \ & \boldsymbol{y} \in \{0,1\}^n,\ \boldsymbol{\alpha} \in \mathcal{U}_{\mathrm{a}}, \boldsymbol{T} \in \mathcal{U}_{\mathrm{t}} \\ & \sum_{\ell=1}^{i} T_\ell - t \le M(1-y_i) \quad \forall i = 1,\ldots,n, \end{aligned} \tag{15}$$

where M is a "big-M" constant. Note that this problem is an MILP if the uncertainty sets are both polyhedral. In this formulation, we use the auxiliary binary variables y_i, $i=1,\ldots,n$. The big-M constraints in Problem (15) ensure that, at an optimal solution, $y_i = 1$ if and only if $\sum_{\ell=1}^{i} T_\ell \le t$ so that the ith passenger has arrived on or before time t. We prove that these two formulations are equivalent.

Proposition 7. *Problems* (14) *and* (15) *are equivalent. In particular, the two problems have the same optimal objective value and any feasible solution to Problem* (14) *(resp.* (15)*) can be used to construct a feasible solution to Problem* (15) *(resp.* (14)*) with the same cost.*

Proof. Let $(\boldsymbol{\alpha}^\star, \boldsymbol{T}^\star)$ be feasible in Problem (14) and define $\boldsymbol{y}^\star$ through

$$y_i^\star := \mathbb{I}\left(\sum_{\ell=1}^{i} T_\ell^\star \le t\right)$$

We show that the triple $(\boldsymbol{\alpha}^\star, \boldsymbol{T}^\star, \boldsymbol{y}^\star)$ is feasible in Problem (15) with the same cost. Fix $i' \in \{1,\ldots,n\}$. If $\sum_{\ell=1}^{i'} T_\ell^\star \le t$, then, $y_{i'}^\star = 1$ and it follows that

$$\sum_{\ell=1}^{i'} T_\ell^\star - t \le 0 = M(1 - y_{i'}^\star).$$

If, on the other hand, $\sum_{\ell=1}^{i'} T_\ell^\star > t$, then, $y_{i'}^\star = 0$ and, for M sufficiently large, it holds that

$$\sum_{\ell=1}^{i'} T_\ell^\star - t \le M = M(1 - y_{i'}^\star).$$

Since the choice of i' was arbitrary, we conclude that $(\boldsymbol{y}^\star, \boldsymbol{\alpha}^\star, \boldsymbol{T}^\star)$ is feasible in Problem (15). Moreover, by definition of $\boldsymbol{y}^\star$, it holds that

$$\sum_{i=1}^{n} \alpha_i^\star y_i^\star = \sum_{i=1}^{n} \alpha_i^\star \mathbb{I}\left(\sum_{\ell=1}^{i} T_\ell^\star \leq t\right)$$

We have constructed a feasible solution to Problem (15) that attains the same cost as that attained by $(\boldsymbol{\alpha}^\star, \boldsymbol{T}^\star)$ in Problem (14). Since the choice of $(\boldsymbol{\alpha}^\star, \boldsymbol{T}^\star)$ was arbitrary, Problem (15) upper bounds Problem (14).

For the converse, let $(\boldsymbol{y}^\star, \boldsymbol{\alpha}^\star, \boldsymbol{T}^\star)$ be optimal in Problem (15). Since $\mathcal{U}_{\mathrm{a}} \subseteq \mathbb{R}_+^n$, we may assume without loss of generality that

$$y_i^\star = 1 \quad \forall i \text{ such that } \sum_{\ell=1}^{i} T_\ell \leq t.$$

Indeed, if there exists some i' such that $\sum_{\ell=1}^{i'} T_\ell \leq t$ and $y_{i'}^\star = 0$, we can always increase $y_{i'}^\star$ to 1 and remain feasible and optimal, since the objective value will not decrease in the process. From the feasibility of $\boldsymbol{y}^\star$ in Problem (15), it must hold that $y_i^\star = 0$ for all i such that $\sum_{\ell=1}^{i} T_\ell > t$. We conclude that $y_i^\star = \mathbb{I}\left(\sum_{\ell=1}^{i} T_\ell^\star \leq t\right)$. Therefore, $(\boldsymbol{\alpha}^\star, \boldsymbol{T}^\star)$ is feasible in Problem (14) and attains the same cost as that attained by $(\boldsymbol{y}^\star, \boldsymbol{\alpha}^\star, \boldsymbol{T}^\star)$ in Problem (15). We conclude that the two problems are equivalent. □

Analytical Solution Proof

We show that for the special case where the uncertainty sets are defined as in Equations (12) and (13), the optimization Problem (14) can be solved analytically.

Proposition 8. *An optimal solution* $(\boldsymbol{\alpha}^\star, \boldsymbol{T}^\star)$ *to Problem* (14) *is given by:*

$$T_i^\star := \max\left(0, \frac{i}{\lambda} - \Gamma_{\mathrm{t}}(i)^{\frac{1}{\tau_{\mathrm{t}}}} - \sum_{\ell=1}^{i-1} T_\ell^\star\right) \quad \forall i = 1, \ldots, n \tag{16}$$

and

$$\alpha_i^\star = i\bar{\alpha} + \Gamma_{\mathrm{a}}(i)^{\frac{1}{\tau_{\mathrm{a}}}} - \sum_{\ell=1}^{i-1} \alpha_\ell^\star \quad \forall i = 1, \ldots, n. \tag{17}$$

Proof: The proof is in two parts. First, we show that for any fixed $\boldsymbol{\alpha}$ feasible in Problem (14), $\boldsymbol{T}^\star$ results in an objective value no smaller than that attained by any other feasible $\boldsymbol{T}$. Second, we show that for any fixed $\boldsymbol{T}$ feasible in Problem (14), $\boldsymbol{\alpha}^\star$ results in an objective value no smaller than that attained by any other feasible $\boldsymbol{\alpha}$.

Fix $\tilde{\boldsymbol{\alpha}} \in \mathcal{U}_{\mathrm{a}}$ in Problem (14). Then, Problem (14) reduces to

$$\begin{aligned} \max \ & \sum_{i=1}^{n} \tilde{\alpha}_i \mathbb{I}\left(\sum_{\ell=1}^{i} T_\ell \leq t \right) \\ \text{s.t.} \ & \boldsymbol{T} \in \mathcal{U}_{\mathrm{t}}. \end{aligned} \tag{18}$$

We show that $\boldsymbol{T}^\star$ defined in the premise of the proposition is optimal in Problem (18). First, we show that $\boldsymbol{T}^\star$ is feasible in Problem (18).

According to Equation (16), for all $i \in \{1, \ldots, n\}$, we have

$$\begin{aligned} \sum_{\ell=1}^{i} T_\ell^\star &= T_i^\star + \sum_{\ell=1}^{i-1} T_\ell^\star \\ &= \sum_{\ell=1}^{i-1} T_\ell^\star + \max\left(0, \frac{i}{\lambda} - \Gamma_{\mathrm{t}}(i)^{\frac{1}{\tau_{\mathrm{t}}}} - \sum_{\ell=1}^{i-1} T_\ell^\star \right) \\ &= \max\left(\sum_{\ell=1}^{i-1} T_\ell^\star, \frac{i}{\lambda} - \Gamma_{\mathrm{t}}(i)^{\frac{1}{\tau_{\mathrm{t}}}} \right). \end{aligned} \tag{19}$$

We now show by induction that $\sum_{\ell=1}^{i} T_\ell^\star$ satisfies the constraints in the definition of $\mathcal{U}_{\mathrm{t}}$, i.e., that $\boldsymbol{T}^\star \in \mathcal{U}_{\mathrm{t}}$. We first show the base case. Since Γ_{t} is positive, we have

$$T_1^\star = \max\left(0, \frac{1}{\lambda} - \Gamma_{\mathrm{t}}(1)^{\frac{1}{\tau_{\mathrm{t}}}}\right) \leq \frac{1}{\lambda} + \Gamma_{\mathrm{t}}(1)^{\frac{1}{\tau_{\mathrm{t}}}}$$

and

$$T_1^\star = \max\left(0, \frac{1}{\lambda} - \Gamma_{\mathrm{t}}(1)^{\frac{1}{\tau_{\mathrm{t}}}}\right) \geq \frac{1}{\lambda} - \Gamma_{\mathrm{t}}(1)^{\frac{1}{\tau_{\mathrm{t}}}}.$$

Next, we show the induction step. Fix $i \in \{1, \ldots, n\}$ and suppose that

$$\frac{i-1}{\lambda} - \Gamma_{\mathrm{t}}(i-1)^{\frac{1}{\tau_{\mathrm{t}}}} \leq \sum_{\ell=1}^{i-1} T_\ell^\star \leq \frac{i-1}{\lambda} + \Gamma_{\mathrm{t}}(i-1)^{\frac{1}{\tau_{\mathrm{t}}}}.$$

Then,

$$\sum_{\ell=1}^{i} T_\ell^\star = \max\left(\sum_{\ell=1}^{i-1} T_\ell^\star, \frac{i}{\lambda} - \Gamma_{\mathrm{t}}(i)^{\frac{1}{\tau_{\mathrm{t}}}} \right), \tag{20}$$

and there are two cases. If $\sum_{\ell=1}^{i-1} T^\star_\ell \le \frac{i}{\lambda} - \Gamma_{\mathrm{t}}(i)^{\frac{1}{\tau_{\mathrm{t}}}}$, then it immediately follows that

$$\frac{i}{\lambda} - \Gamma_{\mathrm{t}}(i)^{\frac{1}{\tau_{\mathrm{t}}}} = \sum_{\ell=1}^{i} T^\star_\ell \le \frac{i}{\lambda} + \Gamma_{\mathrm{t}}(i)^{\frac{1}{\tau_{\mathrm{t}}}}$$

On the other hand, if $\sum_{\ell=1}^{i-1} T^\star_\ell > \frac{i}{\lambda} - \Gamma_{\mathrm{t}}(i)^{\frac{1}{\tau_{\mathrm{t}}}}$, we then have

$$\frac{i}{\lambda} - \Gamma_{\mathrm{t}}(n)^{\frac{1}{\tau_{\mathrm{t}}}} < \sum_{\ell=1}^{i-1} T^\star_\ell \le \frac{i-1}{\lambda} + \Gamma_{\mathrm{t}}(i-1)^{\frac{1}{\tau_{\mathrm{t}}}} < \frac{i}{\lambda} + \Gamma_{\mathrm{t}}(i)^{\frac{1}{\tau_{\mathrm{t}}}}.$$

where the second inequality comes from Equation (20) and the last inequality holds due to the fact that $f(i) = \frac{i}{\lambda} + \Gamma_{\mathrm{t}}(i)^{\frac{1}{\tau_{\mathrm{t}}}}$ is increasing in i. Thus we have proven that $\boldsymbol{T}^\star$ by Equation (16) is a feasible solution to Problem (18). We then show that no other feasible $\boldsymbol{T}$ will result in a better objective value than the $\boldsymbol{T}^\star$ constructed above. For any $\boldsymbol{T}$ that is within the uncertainty set $\mathcal{U}_{\mathrm{t}}$, we have

$$-\Gamma_{\mathrm{t}}(i)^{\frac{1}{\tau_{\mathrm{t}}}} + \frac{i}{\lambda} \le \sum_{\ell=1}^{i} T_\ell \le \Gamma_{\mathrm{t}}(i)^{\frac{1}{\tau_{\mathrm{t}}}} + \frac{i}{\lambda} \quad \forall i = 1, \dots, n\,.$$

Suppose there exists a feasible $\boldsymbol{T}' \in \mathcal{U}_{\mathrm{t}}$ that yields a strictly better objective value. Define $i^\star$ and i' as

$$i^\star := \left\{ \max_{i=1,\dots,n} i : \mathbb{I}\left(\sum_{\ell=1}^{i} T^\star_\ell \le t \right) = 1 \right\} \tag{21}$$

and

$$i' := \left\{ \max_{i=1,\dots,n} i : \mathbb{I}\left(\sum_{\ell=1}^{i} T'_\ell \le t \right) = 1 \right\}.$$

Then the existence of $\boldsymbol{T}'$ implies that $i' \ge i^\star + 1$ which in turn results in

$$\sum_{\ell=1}^{i^\star} T^\star_\ell \le t \text{ and } \sum_{\ell=1}^{i^\star+1} T^\star_\ell > t,$$
$$\sum_{\ell=1}^{i^\star} T'_\ell \le t \text{ and } \sum_{\ell=1}^{i^\star+1} T'_\ell \le t.$$

We then have

$$\sum_{\ell=1}^{i^\star+1} T'_\ell \le t < \sum_{\ell=1}^{i^\star+1} T^\star_\ell\,.$$

Since $\boldsymbol{T}^\star$ is constructed such that $\sum_{\ell=1}^{i^\star+1} T^\star_\ell = \frac{i+1}{\lambda} - \Gamma_{\mathrm{t}}(i+1)^{\frac{1}{\tau_{\mathrm{t}}}}$, we have

$$\sum_{\ell=1}^{i^\star+1} T'_\ell < \frac{i+1}{\lambda} - \Gamma_{\mathrm{t}}(i+1)^{\frac{1}{\tau_{\mathrm{t}}}}\,,$$

which contradicts with $\boldsymbol{T}^{'}$ being feasible.

Similarly, we show that $\boldsymbol{\alpha}^{\star}$ constructed in Proposition 8 is an optimal solution for Problem (14).

Fix $\tilde{\boldsymbol{T}} \in \mathcal{U}_{\mathrm{a}}$ in Problem (14). Then, Problem (14) reduces to

$$\begin{aligned} &\max \sum_{i=1}^{n} \alpha_i \mathbb{I}\left(\sum_{\ell=1}^{i} \tilde{T}_\ell \leq t\right) \\ &\text{s.t.}\ \ \boldsymbol{\alpha} \in \mathcal{U}_{\mathrm{a}}. \end{aligned} \tag{22}$$

First, we show that $\boldsymbol{\alpha}^{\star}$ defined in the premise of the proposition is feasible in Problem (22). According to Equation (17), for all $i = 1, \ldots, n$, we have

$$\alpha_i^{\star} = \bar{\alpha} + \Gamma_{\mathrm{a}}\left(i^{\frac{1}{\tau_{\mathrm{a}}}} - (i-1)^{\frac{1}{\tau_{\mathrm{a}}}}\right) \geq 0$$

and

$$\sum_{\ell=1}^{i} \alpha_i^{\star} = i\bar{\alpha} + \Gamma_{\mathrm{a}}(i)^{\frac{1}{\tau_{\mathrm{a}}}}$$

which is within the uncertainty set $\mathcal{U}_{\mathrm{a}}$ given by Equation (12). Thus $\boldsymbol{\alpha}^{\star}$ constructed by Equation (17) is a feasible solution to Problem (22). We next show that no other feasible $\boldsymbol{\alpha}$ can achieve a strictly higher objective value than the $\boldsymbol{\alpha}^{\star}$ constructed. Assume there exists a $\boldsymbol{\alpha}^{'}$ that is feasible and achieves a strictly better objective value than our constructed $\boldsymbol{\alpha}^{\star}$. Then we have

$$\sum_{i=1}^{i^{\star}} \alpha_i^{'} > \sum_{i=1}^{i^{\star}} \alpha_i^{\star} = i^{\star}\bar{\alpha} + \Gamma_{\mathrm{a}}(i^{\star})^{\frac{1}{\tau_{\mathrm{a}}}},$$

which contradicts with the assumption that $\boldsymbol{\alpha}^{'}$ is feasible. □

THEOREM 6. *The analytical optimal solution to Problem* (14) *is given by*

$$A^{\star} = \Gamma_{\mathrm{a}} \cdot \sqrt{i^{\star}} + i^{\star}\bar{\alpha}, \tag{23}$$

where $i^{\star}$ is derived by the following steps:1) Let $t = \frac{i^{\star}}{\lambda} - \Gamma_{\mathrm{t}}(i^{\star})^{\frac{1}{\tau_{\mathrm{t}}}}$; 2) Solve for $i^{\star}$ since all other values in this equation is given; 3) If the solved $i^{\star}$ is a non-negative integer, we are done. If not, we round it down to get the desired $i^{\star}$. Using $\tau_{\mathrm{t}} = 2$ as an example, we shall have:

$$i^{\star} = \lfloor \frac{\Gamma_{\mathrm{t}}^2 + \sqrt{\Gamma_{\mathrm{t}}^4 + 4\mu^2 t^2}}{2\mu^2} \rfloor \tag{24}$$

where $\mu = \frac{1}{\lambda}$.

Proof: By Proposition 8, we have the optimal solutions $\boldsymbol{T}^\star$ and $\boldsymbol{\alpha}^\star$ ready to compute $A^\star$. According to Proposition 7, $i^\star$ is uniquely defined by $\boldsymbol{T}^\star$. To solve for $i^\star$, we follow Definition (21) and we have:

$$\sum_{\ell=1}^{i^\star} T_\ell^\star \leq t \Rightarrow -\Gamma_{\mathrm{t}}(i^\star)^{\frac{1}{\tau_{\mathrm{t}}}} + \frac{i^\star}{\lambda} \leq t\,.$$

By letting $-\Gamma_{\mathrm{t}}(i^\star)^{\frac{1}{\tau_{\mathrm{t}}}} + \frac{i^\star}{\lambda} = t$, we can solve for $i^\star$. This is exactly the same as the procedures provided in the Theorem. With $i^\star$ calculated, we shall have:

$$A^\star = \sum_{\ell=1}^{i^\star} \alpha_{i^\star}^\star = i^\star \bar{\alpha} + \Gamma_{\mathrm{a}}(i^\star)^{\frac{1}{\tau_{\mathrm{a}}}}\,. \quad \square$$

Numerical Performance

In this section, we present the performance of our formulation of uncertainty sets. We compare our results with that of the simulation approach. The simulation approach is to draw samples of interarrival times and α values from assumed distribution and then generate the summation of the α values. We show that our robust approach in average has a similar performance to the simulation approach, but our approach has advantages in computational efficiency.

We test the performance of our uncertainty set formation with the following parameters:

Γ_t	τ_t	λ	Γ_α	τ_α	$\bar{\alpha}$	σ_α
3.4	2	0.5	$1.64 * \sigma_\alpha$	2	5	5

We let $\tau_t = 2$ to model exponential interarrival times. $\Gamma_t = 3.4$ is the variability information parameter corresponding to the 95% quantile of gamma distribution. We set $\tau_\alpha = 2$ because we assume alpha values are normally distributed. $\Gamma_\alpha = 1.64 * \sigma_\alpha$ is the variability information parameter corresponding to the 95% quantile of the assumed normal distribution with standard deviation σ_α. The 1.64 is the q value of truncated normal at 95% quantile.

The results are shown in Figure 8. We tested and compared the absolute error of the robust estimation approach and the simulation approach both for the 95% case and the expectation case. Notice that since we have two uncertainties in our model, we fix the uncertainty of the interarrival

				Γ	1.64		Γ	0.05	
Interarrival Time		Alpha Values		95%			50%		
TRUE	ASSUMED	TRUE	ASSUMED	Γ_α	A_Est	A_Sim	Γ_α	A_Est	A_Sim
exp(2)	exp(2)	normal(5,5)	normal(5,1)	1.64	20.54%	25.74%	0.05	22.20%	22.57%
exp(2)	exp(2)	normal(5,5)	normal(5,2)	3.28	15.58%	23.54%	0.1	21.49%	21.79%
exp(2)	exp(2)	normal(5,5)	normal(5,3)	4.92	7.93%	18.56%	0.15	17.04%	17.83%
exp(2)	exp(2)	normal(5,5)	normal(5,4)	6.56	2.60%	10.03%	0.2	9.11%	9.78%
exp(2)	exp(2)	normal(5,5)	normal(5,5)	8.2	14.74%	0.00%	0.25	0.00%	0.00%
exp(2)	exp(2)	normal(5,5)	normal(5,6)	9.84	27.75%	10.76%	0.3	11.33%	10.74%
exp(2)	exp(2)	normal(5,5)	exp(5)	8.2	1.99%	16.58%	0.25	21.51%	22.68%
exp(2)	exp(2)	normal(5,2)	exp(5)	8.2	29.15%	9.17%	0.25	0.00%	1.43%
				Average Error	15.04%	14.30%		12.84%	13.35%

Figure 8 The Results of the Different Assumptions

time first (assuming that we know the true distribution) so that we can test how the different assumptions of the distributions of the alpha values could affect our estimation error.

As one can see in Figure 8, the robust estimation approach performed better when our assumption of standard deviation is smaller than that of the true distribution. This is because in our MILP model, we tend to maximize the total alpha values A, so when we would like to have 95% of the case our estimation is larger than the true value, the estimation is supposed to be large. As for the expectation scenario, we chose $\Gamma_\alpha = 0.05 * \sigma_\alpha$ because, when we estimate the expectation, $\Gamma_\alpha = 0 * \sigma_\alpha$ which would not include any robustness. And since the absolute error goes up as Γ_α goes up, we change 0 to a small enough number which is 0.05. In both scenarios, we observe that our uncertainty set approach with a closed form solution, has similar average errors compared to that of the simulation approach.

E. Appendix: Proofs for Discount Methods

In this appendix section, we provide proofs in detail for the performances of the discount methods proposed in Sections 6.1 and 6.2.

Proofs for Basic Discount Method

THEOREM 7. *Under the Basic Discount Method, $c^{s3}_{\pi(k),t}$ satisfies the Immediate Response property. That is, for all times k, t_1, t_2 with $1 \le k \le t_1 \le t_2$ and submit orders π: $c^{s3}_{\pi(k),t_1} \ge c^{s3}_{\pi(k),t_2}$.*

Proof: Directly by Step 2 and Step 3, we have that passenger $\pi(k)$ has a non-negative $c^{\mathrm{s3}}_{\pi(k),k}$ for all $1 \le k \le t$ and that $c^{\mathrm{dis}}_{\pi(k),t} \le 0$. Since $c^{\mathrm{s3}}_{\pi(k),t} = c^{\mathrm{s3}}_{\pi(k),t-1} + c^{\mathrm{dis}}_{\pi(k),t}$ for all $k < t$, we have

$$c^{\mathrm{s3}}_{\pi(k),t_2} = c^{\mathrm{s3}}_{\pi(k),t_1} + \sum_{t=t_1+1}^{t_2} c^{\mathrm{dis}}_{\pi(k),t} \le c^{\mathrm{s3}}_{\pi(k),t_1}\,. \quad \square$$

THEOREM 8. [4] *Under the Basic Discount Method, $c^{\mathrm{s3}}_{\pi(k),t}$ satisfies the Individual Rationality property. That is, for all times k, t with $1 \le k \le t$ and submit orders π: $c^{\mathrm{s3}}_{\pi(k),t} \le W^3_{\pi(k)}$.*

Proof: By Theorem 7, we know that $c^{\mathrm{s3}}_{\pi(k),t} \le c^{\mathrm{s3}}_{\pi(k),k}$. We know that $c^{\mathrm{s3}}_{\pi(k),k} \le W^3_{\pi(k)}$ since the passenger accepted his or her fare quote and so the property is satisfied. $\square$

THEOREM 9. *Under the Basic Discount Method, $c^{\mathrm{s3}}_{\pi(k),t}$ satisfies the Budget Balance property. That is, for all times $t \ge 1$ and submit orders π: $\sum_{i=1}^{t} c^{\mathrm{s3}}_{\pi(i),t} = 0$.*

Proof: Directly by Step 3 and Step 4, we have that all the discounts provided to the existing passengers when a new passenger requests service is covered by the new passenger. This means that:

$$\begin{aligned}
\sum_{i=1}^{t} c^{\mathrm{s3}}_{\pi(i),t} &= \sum_{i=1}^{t} \left(c^{\mathrm{s3}}_{\pi(i),i} + \sum_{j=i+1}^{t} c^{\mathrm{dis}}_{\pi(i),j} \right) \\
&= \sum_{i=1}^{t} c^{\mathrm{s3}}_{\pi(i),i} + \sum_{i=1}^{t-1} \sum_{j=i+1}^{t} c^{\mathrm{dis}}_{\pi(i),j} \\
&= \sum_{i=1}^{t} c^{\mathrm{s3}}_{\pi(i),i} + \sum_{j=2}^{t} \sum_{i=1}^{j-1} c^{\mathrm{dis}}_{\pi(i),j} \\
&= \sum_{i=2}^{t} c^{\mathrm{s3}}_{\pi(i),i} + \sum_{j=2}^{t} \sum_{i=1}^{j-1} c^{\mathrm{dis}}_{\pi(i),j} \\
&= \sum_{i=2}^{t} c^{\mathrm{s3}}_{\pi(i),i} - \sum_{j=2}^{t} c^{\mathrm{s3}}_{\pi(j),j} \\
&= 0\,,
\end{aligned}$$

where the fourth equality holds because there are no existing passengers when the first passenger $\pi(1)$ requests service, thus we have $c^{\mathrm{s3}}_{\pi(1),1} = 0$. $\square$

[4] Similar to Definition 7, this does not mean that the passenger has another willingness to pay level. They each only have one level that is $W_{\pi(k)} = W^1_{\pi(k)} + W^2_{\pi(k)} + W^3_{\pi(k)}$. The existence of $W^3_{\pi(k)}$ is only to facilitate the proof denotation.

Proofs for Inconvenience Cost Based Discount Method

Theorem 10. *Under the Inconvenience Cost Based Discount Method, $c^{\mathrm{s3}}_{\pi(k),t}$ satisfies the Immediate Response property. That is, for all times k, t_1, t_2 with $1 \le k \le t_1 \le t_2$ and submit orders π: $c^{\mathrm{s3}}_{\pi(k),t_1} \ge c^{\mathrm{s3}}_{\pi(k),t_2}$.*

Proof: The first part of $c^{\mathrm{s3}}_{\pi(k),t}$ already satisfies this property. For the second part, since $c^{\mathrm{ic}}_{\pi(k),t} = f_{\pi(k)}\left(T^{\mathrm{tot}}_t\right)$, and f is a non-decreasing convex function, we have that $-c^{\mathrm{ic}}_{\pi(k),t_1} \ge -c^{\mathrm{ic}}_{\pi(k),t_2}$ because $t_1 \le t_2$. And this leads to:

$$\begin{aligned} c^{\mathrm{s3}}_{\pi(k),t_1} &= \alpha_{\pi(k)} \min_{k \le j \le t_1} \max_{1 \le i \le j} c^{\mathrm{da}}_{\pi(i,j)} + \left(-c^{\mathrm{ic}}_{\pi(k),t_1}\right) \\ &\ge \alpha_{\pi(k)} \min_{k \le j \le t_2} \max_{1 \le i \le j} c^{\mathrm{da}}_{\pi(i,j)} + \left(-c^{\mathrm{ic}}_{\pi(k),t_2}\right) \\ &= c^{\mathrm{s3}}_{\pi(k),t_2} . \quad \square \end{aligned}$$

Theorem 11. *Under the Inconvenience Cost Based Discount Method, $c^{\mathrm{s3}}_{\pi(k),t}$ satisfies the Individual Rationality property. That is, for all times k, t with $1 \le k \le t$ and submit orders π: $c^{\mathrm{s3}}_{\pi(k),t} \le W^3_{\pi(k)}$.*

Proof: By Theorem 10, we know that $c^{\mathrm{s3}}_{\pi(k),t} \le c^{\mathrm{s3}}_{\pi(k),k}$. We also know that $c^{\mathrm{s3}}_{\pi(k),k} \le W^3_{\pi(k)}$ since the passenger accepted his or her fare quote, and so the property is satisfied. $\square$

Theorem 12. *Under the Inconvenience Cost Based Discount Method, $c^{\mathrm{s3}}_{\pi(k),t}$ satisfies the Budget Balance property. That is, for all times $t \ge 1$ and submit orders π: $\sum_{i=1}^{t} c^{\mathrm{s3}}_{\pi(i),t} = 0$.*

Proof: Since $\alpha_{\pi(k)} \min_{k \le j \le t} \max_{1 \le i \le j} c^{\mathrm{da}}_{\pi(i,j)}$ satisfies the Budget Balance property, then we have

$$\sum_{k=1}^{t} \alpha_{\pi(k)} \min_{k \le j \le t} \max_{1 \le i \le j} c^{\mathrm{da}}_{\pi(i,j)} = \sum_{k=1}^{t} c^{\mathrm{ic}}_{\pi(k),t} .$$

In other words, the summation of the value for passengers in the same coalition is the marginal inconvenience cost for that coalition. Summing this value over all the passengers equals to summing the marginal inconvenience cost for all the coalitions which equals to the total inconvenience cost. Then, we have:

$$\sum_{i=1}^{t} c^{\mathrm{s3}}_{\pi(i),t} = \sum_{k=1}^{t} c^{\mathrm{ic}}_{\pi(k),t} - \sum_{k=1}^{t} c^{\mathrm{ic}}_{\pi(k),t} = 0 . \quad \square$$

THEOREM 13. *Under the Inconvenience Cost Based Discount Method, $c^{\mathrm{s3}}_{\pi(k),t}$ satisfies the Ex-Post Incentive Compatibility property. That is, for all times k_1, k_2, t with $1 \leq k_1 \leq k_2 \leq t$ and submit orders $\pi, \pi^{'}$ with*

$$\pi^{'}(k) = \begin{cases} \pi(k+1) & \text{if } k_1 \leq k < k_2, \\ \pi(k_1) & \text{if } k = k_2, \\ \pi(k) & \text{otherwise,} \end{cases}$$

we have $c^{\mathrm{s3}}_{\pi(k_1),t} \leq c^{\mathrm{s3}}_{\pi'(k_2),t}$.

Proof: Recall that the total inconvenience cost for all the passengers satisfy Assumptions 1 and 2. This means that $\sum_{k=1}^{t} c^{\mathrm{ic}}_{\pi(k),t}$ is independent of the submit order and is non-decreasing in time. And since $\pi(k_1)$ and $\pi'(k_2)$ both refer to the same passenger, then we have $c^{\mathrm{ic}}_{\pi(k_1),t} = c^{\mathrm{ic}}_{\pi'(k_2),t} \Rightarrow -c^{\mathrm{ic}}_{\pi(k_1),t} \leq -c^{\mathrm{ic}}_{\pi'(k_2),t}$. Combined with the fact that the first part of $c^{\mathrm{s3}}_{\pi(k),t}$ satisfies this property, we have:

$$\begin{aligned} c^{\mathrm{s3}}_{\pi(k_1),t} &= \alpha_{\pi(k_1)} \min_{k_1 \leq j \leq t} \max_{1 \leq i \leq j} c^{\mathrm{da}}_{\pi(i,j)} + \left(-c^{\mathrm{ic}}_{\pi(k_1),t}\right) \\ &\leq \alpha_{\pi'(k_2)} \min_{k_2 \leq j \leq t} \max_{1 \leq i \leq j} c^{\mathrm{da}}_{\pi(i,j)} + \left(-c^{\mathrm{ic}}_{\pi'(k_2),t}\right) \\ &= c^{\mathrm{s3}}_{\pi'(k_2),t}. \quad \square \end{aligned}$$