

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.	
4. Title and Subtitle Locating portable stations to support the operation of bike sharing systems		5. Report Date 12-26-17	6. Performing Organization Code
7. Author(s) Jose Walteros Rahul Swamy		8. Performing Organization Report No.	
9. Performing Organization Name and Address University at Buffalo 1700 Capen Hall Buffalo, NY 14260-1660		10. Work Unit No.	11. Contract or Grant No. 49198-23-28
12. Sponsoring Agency Name and Address UTRC/ The City College of New York 137 th Street & Convent Avenue Marshak Hall 910 New York, NY 10031		13. Type of Report and Period Covered Final, Sept 1 2016-December 26, 2017	
15. Supplementary Notes		14. Sponsoring Agency Code	
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17. Key Words bike sharing, redistribution logistics, location coverage, vehicle routing, network flow		18. Distribution Statement	
19. Security Classif (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No of Pages 15	22. Price

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Locating portable stations to support the operation of bike sharing systems

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Abstract

Redistributing bikes has been a major challenge for the daily operation of bike sharing system around the world. Existing literature explore solution strategies that rely on pick-up-and-delivery routing as well as user incentivization approaches. The key contribution of this work is to introduce the use of portable bike stations to augment the capacity of fixed stations in the context of redistribution. A comprehensive framework to optimally locate, route and redistribute bike using portable stations is proposed using a sequence of Mixed Integer Programs. This strategic and operational decision making process is modeled in two stages. A decomposition based solution strategy is used to solve and fix the strategic decisions.

Keywords: bike sharing, redistribution logistics, location coverage, vehicle routing, network flow

1 Introduction

Bike sharing systems (BSS) have been a cost-effective and an environment friendly form of transportation. While the concept of bike sharing originated in the 1960s, they have received increased attention in the last few years with major technological advances in bike tracking capabilities and has resulted in a rapid expansion of services by both the public and the private sector all around the world [2].

On one hand, the possibility of renting bicycles for short periods of time has given the users a new way of performing their daily commuting trips, a new vibrant way of completing touristic loops, and a fast way of traversing a city in a healthy and exciting way. On the other hand, from the perspective of the city, these kind of systems have become a natural platform for reducing traffic and pollution, for promoting healthy habits within its citizens, and overall, for improving the living conditions of the city. In short, with their proven success, bike-sharing systems will continue to shape the landscape of many major cities in the years to come.

As for June 2014, bike sharing systems have been operating in 50 countries, which includes 712 cities and operating close to 806,200 bikes using 37,500 stations [8]. As of May 2011, while the largest bike

*This project was partially supported by the Region II University Transportation Research Center.

sharing system in the world is operated in Hangzhou in China by Hangzhou Public Bicycle with around 90,000 bikes [9], the largest outside of China is operated by Velib in Paris [7]. Some of the other major private operators in the world include Santander Cycles (London), Divvy (Chicago), Citi Bike (New York City), BIXI (Montreal), among many others.

The typical bike-sharing system consists of a set of fixed stations that are scattered around the city. From each of these stations, the users can withdraw a bike from the system for a given fare, use it for a given amount of time, and finally return it back to the system in any other station. The location of such stations must then be carefully chosen considering the expected demand for bicycles and available return spaces all over the city. While this basic structure is common to all BSS, different bike sharing systems implement different mechanisms for designing membership policies, bike usage and return policy, time period of operations, physical locations of stations and satisfying bike availability by means of redistribution.

Locating stations is one of the most essential decisions that influences the success of a bike sharing system. Literature in identifying optimal station locations has had various modeling considerations. One way to approach optimal station locations is through appropriate facility location models. Lin and Yang [5] proposed a hub location inventory model as an integer program to choose which set of candidate station locations to open as active stations and which pair of stations require bike lanes to be constructed between them, while ensuring a certain amount of service level. However, this model does not consider the redistribution of bikes, and assumes that bikes are available at all times. Lin et al. [6] presented a model provides an integrated view of the various costs and service quality concerns. This model is computationally intractable and they provided a greedy heuristic to find reasonable solutions. Yan et al. [11] extended this model for stochastic demands using a time-space network.

In conjunction with physical access to bike stations, the availability of bikes to users at the time of their need is a critical factor to ensure good service levels [4]. For globally optimal solutions, it is important to consider the problem of ensuring availability of bikes together with the problem of locating stations.

The main distinction between our work and existing approaches is that we allow the possibility of having portable stations to complement the system. Portable bike stations, are truck-based stations that can be moved between locations in order to: (1) serve city areas where it is not either convenient or possible to have fixed stations; (2) test locations that could potentially become fixed stations; (3) to complement existing fixed stations during events of high peaks of demand (e.g., concerts, sport events, morning and evening demand peaks).

In this project, we propose an integrated mathematical framework for operating a bike-sharing system using portable stations. This is done by solving a sequence of MILPs to determine: (1) the optimal location of fixed and portable stations and (2) the number of bikes to be added to or removed from each station every time period to satisfy the demand-supply needs.

At the first level, the optimization model determines the positions of the bike stations over the entire time horizon. This includes the positions of fixed stations from a set of candidate locations, and the positions of the portable stations for each time period. The objective of this model is to minimize the number of bikes to be added or removed from each station over all the time periods. The model transforms the origin-destination pair flow for each time period into station-station flow for the chosen station

locations. Also, the inventory level at each chosen station during each time period is reduced/increased by the net flow of bikes from/to such station, and the slack/surplus carried over to the next time period is the extra bikes to be added/removed. The objective is to minimize the slack/surplus over all time periods and chosen station locations, in order to ensure that the burden on the redistribution logistics is minimized.

2 Fixed and Portable Station Location Problem (FTSLP)

This Section introduces the fixed and portable Station Location Problem (FTSLP) and presents a mathematical model for optimally locating the fixed and portable stations from among candidate station locations. The goal of the FTSLP is to choose the locations of stations in such a way that with information on all the riders' true origin-destination pairs and the time period in which they would pick-up and drop a bike, the we minimize the number of bikes that are externally added to and removed from the chosen stations. We call the number of bikes added and removed externally to be the *redistribution load*.

If the bike-sharing system is unbalanced, the operator of the system would want to redistribute bikes among the chosen stations in such a way such that the riders are satisfied with sufficient availability of bikes (and parking spaces) and at the same time the redistribution load is kept minimal. From the operator's point of view, our proposed mathematical model aims to capture both of these priorities.

A potential rider would start their travel by searching for a bike station within an accessible distance around them (their true origin) during a certain time period. Once they locate such a bike station if there is at least one bike available at that station, they proceed to rent a bike, travel for a certain amount of time, and drop it at a bike station that is close to reach their true destination. This model assumes that we have historical knowledge on the true origin-destination flow within a geographical region of consideration. These origins and destinations are called *demand points*. The granularity of a demand point is decided by the granularity of the data available and can impact the quality of service. A demand point could be a household, a building, a street, or a block. This model assumes that there is at least one bike station that is located within an accessible (walkable) distance from each of the demand points whenever there is a need for a bike (or a parking spot). With this information, we divide the time horizon into discrete *time periods*, and aggregate the number of potential riders that travel from one demand point to another within each time period and use it as input to the model. Note that the granularity of a time period offers a trade-off between computational complexity of the model and the quality of service, which will be discussed later.

In this model, it is assumed that at the start of each time period, bikes are externally added or removed the bike stations so as to satisfy the demand for bikes or parking spaces in the next time period. This model assumes the objective that the total number of bikes to be redistributed (externally added or removed) among the bike stations for all time periods is to be minimized. In addition, there is also an upper limit on the number of bikes that a fixed or portable station can accommodate, given by the capacity of a chosen station. Note that we assume that it is possible to position a fixed station and a portable station at the same candidate location.

2.1 Model

Let N be the set of all demand points and T be the number of time periods in the planning horizon. We assume that every rider starts their journey from a location $i \in N$ and travels to a location $j \in N$. For each time period $t \in \{1, 2, \dots, T\}$, let d_{ij}^t be the aggregated number of potential riders that travel from demand point i to another demand point j . Let R be the set of candidate station locations, and K_{fix} and $K_{portable}$ be the maximum number of fixed and portable stations to be chosen from R . Suppose $S_{fix} \subseteq R$ and $S_{portable}^t \subseteq R$ are the subsets of chosen fixed and portable stations at time period t respectively. Then, $|S_{fix}| \leq K_{fix}$ and $|S_{portable}^t| \leq K_{portable}$ for every time period t . Also, let the capacity of every chosen fixed station be M_{fix} and that of a portable station be $M_{portable}$.

For every chosen station $r \in S_{fix} \cup S_{portable}^t$ and for time period $t \in 1, 2, \dots, T$, let \bar{w}_r^t and \underline{w}_r^t be the number of bikes that are externally added to and removed from r , respectively, at the beginning of t . Let I_r^t be the inventory of bikes at the start of t , right after the external addition of bikes. For every pair of stations $r, s \in S_{fix} \cup S_{portable}^t$, let b_{rs}^t be the number of bike rides from r to s that commenced during time period t . Further, for every $i \in N$, we define $R_i \subseteq R$ to be the subset of candidate stations that are *accessible* to demand point i . When a rider travels from i to j , they pick up the bike from a unique station $r \in R_i$ and drop it off at a unique station $s \in R_j$, and the path they take can be represented by $i - r - s - j$. We assume that this unique station is the closest station to the rider's demand point. Let a_{ir}^t be an assignment indicator variable which is equal to 1 when station r is the closest station (that is also a chosen station) to demand point i , and 0 otherwise. Let u_{irjs}^t be an indicator variable which takes the value 1, when a rider from i to j takes the path $i - r - s - j$ commencing during time t . We then get the relationship between the number of inter-station trips and the number of inter-demand-point trips, $b_{rs}^t = \sum_{i \in N_r} \sum_{j \in N_s \setminus \{i\}} d_{ij}^t$. Further, the inventory at each station r at the start of each time period t ($\neq 1$) depends on the inventory changes during the time period $t - 1$, and the external changes at the start of t . This can be expressed as, $I_r^t = I_r^{t-1} + \sum_{s \in R \setminus \{r\}} b_{sr}^{t-1} - \sum_{s \in R \setminus \{r\}} b_{rs}^{t-1} + \bar{w}_r^t - \underline{w}_r^t$. Hence, we have a relationship between the given demand d_{ij}^t and the number of bikes externally added and removed. Using these relationships, we define the FTSLP to be the optimization problem of choosing $S_{fix} \in R$ and $S_{portable} \in R$ such that the quantity $\sum_{r \in S_{fix} \cup S_{portable}^t} \sum_{t=2}^T \bar{w}_r^t + \underline{w}_r^t$, the total number of bikes externally added and removed, is minimum.

2.2 MIP Formulation

In this Subsection, the FTSLP is formulated as a Mixed Integer Program.

Indices

t	time period
i, j	demand points
r, s	stations
$(k)_i$	k -th closest station to demand point i

Parameters

N	set of demand points
R	set of candidate stations

T	number of time periods in the time horizon
R_i	set of candidate stations that are accessible from $i \in N$
N_r	set of demand points that can access $r \in R$
d_{ij}^t	number of trips from $i \in N$ to $j \in N$ at t
M_{fix}	capacity of a fixed station
$M_{portable}$	capacity of a portable station
K_{fix}	maximum number of fixed stations to be setup
$K_{portable}$	maximum number of portable stations possible

Decision variables

x_r	$\begin{cases} 1, & \text{if } r \text{ is chosen to be a fixed station} \\ 0, & \text{otherwise} \end{cases}$
z_r^t	$\begin{cases} 1, & \text{if } r \text{ is chosen as a portable station during } t \\ 0, & \text{otherwise} \end{cases}$
a_{ir}^t	$\begin{cases} 1, & \text{if demand from } i \text{ is covered by } r \text{ during } t \\ 0, & \text{otherwise} \end{cases}$
u_{irjs}^t	$\begin{cases} 1, & \text{if } a_{ir}^t = 1 \text{ and } a_{js}^t = 1 \\ 0, & \text{otherwise} \end{cases}$
b_{rs}^t	number of trips from r to s during t
\bar{w}_r^t	number of bikes added to candidate station r at the start of t
\underline{w}_r^t	number of bikes removed from candidate station r at the start of t
I_r^t	inventory of bikes at r at the start of t after redistribution

Formulation

$$(P) \quad \min \sum_{r \in R} \sum_{t=2}^T \bar{w}_r^t + \underline{w}_r^t \quad (2.1)$$

s.t.

$$\sum_{r \in R} x_r \leq K_{fix} \quad (2.2)$$

$$\sum_{r \in R} z_r^t \leq K_{portable} \quad t = 1, 2, \dots, T, \quad (2.3)$$

$$a_{ir}^t \leq x_r + z_r^t \quad i \in N, r \in R_i, t = 1, 2, \dots, T, \quad (2.4)$$

$$\sum_{r \in R_i} a_{ir}^t = 1 \quad i \in N, t = 1, 2, \dots, T, \quad (2.5)$$

$$a_{i,(1)i}^t \geq x_{(1)i} \quad i \in N, t = 1, 2, \dots, T, (1) \in R_i \quad (2.6)$$

$$a_{i,(k)i}^t \geq x_{(k)i} - \sum_{l=1}^{k-1} (x_{(l)i} + z_{(l)i}^t) \quad i \in N, t = 1, 2, \dots, T, k = 2, 3, \dots, |R_i| \quad (2.7)$$

$$a_{i,(1)i}^t \geq z_{(1)i}^t \quad i \in N, t = 1, 2, \dots, T, (1) \in R_i \quad (2.8)$$

$$a_{i,(k)_i}^t \geq z_{(k)_i}^t - \sum_{l=1}^{k-1} (x_{(l)_i} + z_{(l)_i}^t) \quad i \in N, t = 1, 2, \dots, T, k = 2, 3, \dots, |R_i|, \quad (2.9)$$

$$I_r^t \leq x_r M_{fix} + z_r^t M_{portable} \quad r \in R, t = 1, 2, \dots, T, \quad (2.10)$$

$$I_r^1 = \bar{w}_r^1 \quad r \in R, \quad (2.11)$$

$$I_r^t = I_r^{t-1} + \sum_{s \in R \setminus \{r\}} (b_{sr}^{t-1} - b_{rs}^{t-1}) + \bar{w}_r^t - \underline{w}_r^t \quad r \in R, t = 2, 3, \dots, T, \quad (2.12)$$

$$b_{rs}^t = \sum_{i \in N_r} \sum_{j \in N_s \setminus \{i\}} d_{ij}^t u_{irjs}^t \quad r \in R, s \in R \setminus \{r\}, t = 1, 2, \dots, T, \quad (2.13)$$

$$u_{irjs}^t \leq a_{ir}^t \quad i, j \in N, r \in R_i, s \in R_j \setminus \{r\}, t = 1, 2, \dots, T, \quad (2.14)$$

$$u_{irjs}^t \leq a_{js}^t \quad i, j \in N, r \in R_i, s \in R_j \setminus \{r\}, t = 1, 2, \dots, T, \quad (2.15)$$

$$u_{irjs}^t \geq a_{ir}^t + a_{js}^t - 1 \quad i, j \in N, r \in R_i, s \in R_j \setminus \{r\}, t = 1, 2, \dots, T, \quad (2.16)$$

$$u_{irjs}^t, b_{rs}^t, y_r, \bar{w}_r^t, \underline{w}_r^t, I_r^t, I_r^0 \geq 0 \quad i, j \in N, r \in R_i, s \in R_j \setminus \{r\}, t = 1, 2, \dots, T, \quad (2.17)$$

$$a_{ir}^t, x_r, z_r^t \in \{0, 1\} \quad i \in N, r \in R_i, t = 1, 2, \dots, T. \quad (2.18)$$

The objective function in (2.1) is to minimize the total number of bikes to be added and removed from all the chosen bike stations during the entire time horizon. Constraints (2.2) and (2.3) impose restriction on the permissible number of fixed stations and portable stations for each time period. Constraints (2.4) make sure that a demand point is assigned to a candidate station only if it chosen either as a fixed or a portable station. Constraints (2.5) ensure that every demand point is assigned to exactly one station. Constraints (2.6 - 2.9) ensure that every demand point is assigned to that chosen station that is closest to it. Constraints (2.10) make sure that the inventory of bikes at a chosen bike station does not exceed the capacity allotted to it. Constraints (2.11) and (2.12) describe the change in inventory of bikes at the bike stations as a function of the number of bikes parked and picked up by the users at the bike station, as well as the number of bikes added or removed by the redistribution trucks for every time period. It is to be noted that for every r and t , $\bar{w}_r^t - \underline{w}_r^t$ acts as the slack/surplus term, and only one of the two variables can take a non-zero value at a time. Constraints (2.13) transform the inter-demand-point flow into inter-bike-station mobility matrix for each time period. This is done by assigning demand points to accessible bike-stations. Constraints (2.14), (2.15) and (2.16) define the linearizing variable u_{irjs}^t which otherwise is a quadratic product of a_{ir}^t and a_{js}^t . Constraints (2.17) and (2.18) define the non-negative continuous and binary nature of the variables.

2.3 Benders Decomposition

Considering the complexity of the proposed formulation in section 3.2, we propose a decomposition based solution methodology to solve the FTSLP efficiently. This section describes the application of Benders Decomposition and improvements to the algorithm.

Benders Decomposition has been widely used to solve MIPs in a variety of applications. The central idea behind the success of the algorithm is the separation of an MIP into a Master problem with "complicating" variables and a subproblem that can be easily solved. Variations of this methods has been successful in applications such as in fixed-charge network design problems [1], designing industrial distribution problems [3], vehicle routing problems, among many others.

In the proposed formulation for the FTSLP, let us examine the benefits of fixing the location decision variables (x, z) . Fixing the x and z variables will fix the station location decisions, and fixing the a variables will fix the assignment decision for every demand point. Assigning pre-set values to these variables will reduce the problem into an LP. The problem is then transformed into finding the optimal location decisions and solving the subproblem to find a fully feasible solution to the original problem. Since FTSLP is a minimization problem, the thus-generated full solution will be an upper bound to the optimal solution, and the Benders algorithm iteratively minimizes the upper bound till it reaches the optimal value. The Integer Program that fixes feasible values for (x, z, a) acts as the Master Problem (MP), and the LP that generates the full solution is the Sub-Problem (SP).

The Master problem with the location decision variables will ideally only contain constraints (2.2) - (2.3). The proposed Master Problem is given by:

$$(MP) \quad \min B \tag{2.19}$$

s.t.

$$B \geq \phi(x, z, a) \tag{2.20}$$

$$\sum_{r \in R_i} (x_r + z_r^t) \geq 1 \quad i \in N, t = 1, 2, \dots, T, \tag{2.21}$$

$$\text{Const. (2.2) - (2.3)} \tag{2.22}$$

$$x_r, z_r^t \in \{0, 1\} \quad r \in R, t = 1, 2, \dots, T. \tag{2.23}$$

The objective in (2.19) minimizes the upper bound on the FTSLP. Constraints (2.20) represent the set of Benders cuts added to the Master problem. Here, $\phi(x, z, a)$ denotes the piece-wise linear cut obtained from SP when it is solved for a given set of values for (x, z, a) . This function is essentially the dual objective function of the subproblem, as discussed further in this section. Constraints (2.21) ensure that at least one station is chosen that is accessible to every demand point. This constraint was enforced in the FTSLP as constraint (2.5) using the assignment variables.

The primal for the SP corresponding to the proposed Master problem is given by the following Primal Sub Problem (PSP) formulation.

$$(PSP) \quad \min \sum_{r \in R} \sum_{t=2}^T \bar{w}_r^t + \underline{w}_r^t \tag{2.24}$$

s.t.

$$-I_r^t \geq -x_r^* M_{fix} - z_r^{*t} M_{portable} \quad (\pi_r^t) \quad r \in R, t = 1, 2, \dots, T, \tag{2.25}$$

$$I_r^1 - \bar{w}_r^1 = 0 \quad (\beta_r^1) \quad r \in R, \tag{2.26}$$

$$I_r^t - I_r^{t-1} - \bar{w}_r^t + \underline{w}_r^t + \sum_{s \in R \setminus \{r\}} (b_{rs}^{t-1} - b_{sr}^{t-1}) = 0 \quad (\beta_r^t) \quad r \in R, t = 2, 3, \dots, T, \tag{2.27}$$

$$b_{rs}^t - \sum_{i \in N_r} \sum_{j \in N_s} d_{ij}^t u_{irjs}^t = 0 \quad (\gamma_{rs}^t) \quad r \in R, s \in R \setminus \{r\}, t = 1, \dots, T, \tag{2.28}$$

$$-u_{irjs}^t + a_{ir}^t \geq 0 \quad (\delta_{irjs}^t) \quad i, j \in N, r \in R_i, s \in R_j \setminus \{r\}, t = 1, \dots, T, \tag{2.29}$$

$$-u_{irjs}^t + a_{js}^t \geq 0 \quad (\delta_{2irjs}^t) \quad i, j \in N, \quad r \in R_i, \quad s \in R_j \setminus \{r\}, \quad t = 1, \dots, T, \quad (2.30)$$

$$u_{irjs}^t - a_{ir}^t - a_{js}^t \geq -1 \quad (\delta_{3irjs}^t) \quad i, j \in N, \quad r \in R_i, \quad s \in R_j \setminus \{r\}, \quad t = 1, \dots, T, \quad (2.31)$$

$$-a_{ir}^t \geq -x_r - z_r^t \quad (\alpha_{ir}^t) \quad i \in N, \quad r \in R_i, \quad t = 1, 2, \dots, T, \quad (2.32)$$

$$\sum_{r \in R_i} a_{ir}^t = 1 \quad (\omega_i^t) \quad i \in N, \quad t = 1, 2, \dots, T, \quad (2.33)$$

$$a_{i,(1)}^t \geq x_{(1)} \quad (\lambda_{i(1)}^t) \quad i \in N, \quad t = 1, 2, \dots, T, \quad (1) \in R_i \quad (2.34)$$

$$a_{i,(k)}^t \geq x_{(k)} - \sum_{l=1}^{k-1} (x_{(l)} + z_{(l)}^t) \quad (\lambda_{i(k)}^t) \quad i \in N, \quad t = 1, 2, \dots, T, \quad k = 2, 3, \dots, |R_i| \quad (2.35)$$

$$a_{i,(1)}^t \geq z_{(1)}^t \quad (\mu_{i1}^t) \quad i \in N, \quad t = 1, 2, \dots, T, \quad (1) \in R_i \quad (2.36)$$

$$a_{i,(k)}^t \geq z_{(k)}^t - \sum_{l=1}^{k-1} (x_{(l)} + z_{(l)}^t) \quad (\mu_{ik}^t) \quad i \in N, \quad t = 1, 2, \dots, T, \quad k = 2, 3, \dots, |R_i|, \quad (2.37)$$

$$a_{ir}^t, u_{irjs}^t, b_{rs}^t, \bar{w}_r^t, \underline{w}_r^t, I_r^t \geq 0 \quad i, j \in N, \quad r \in R_i, \quad s \in R_j \setminus \{r\}, \quad t = 1, \dots, T, \quad (2.38)$$

For a given a vector, we can immediately compute the linearizing variables as $u_{irjs}^t = a_{ir}^{*t} a_{js}^{*t}$. And given u vector, we can compute b vector using the relationship in constraint 2.27. These computations can be performed in polynomial time, i.e. in $O(|R|^2|N|^2T)$. In addition, we note that the distribution constraints exhibit a block diagonal structure such that each "block" is made up of constraints indexed a candidate station location. Therefore, the subproblem is separable by $r \in R$ and can be re-written as the following Reduced Separable Sub-Problem ($RSSP^r$).

$$(RSSP^r) \quad \min \sum_{t=2}^T \bar{w}_r^t + \underline{w}_r^t \quad (2.39)$$

s.t.

$$-I_r^t \geq -x_r^* M_{fix} - z_r^{*t} M_{portable} \quad (\pi_r^t) \quad t = 1, 2, \dots, T, \quad (2.40)$$

$$I_r^1 - \bar{w}_r^1 = 0 \quad (\beta_r^1) \quad (2.41)$$

$$I_r^t - I_r^{t-1} - \bar{w}_r^t + \underline{w}_r^t = \sum_{s \in R \setminus \{r\}} b_{sr}^{t-1} - \sum_{s \in R \setminus \{r\}} b_{rs}^{t-1} \quad (\beta_r^t) \quad t = 2, 3, \dots, T, \quad (2.42)$$

$$\bar{w}_r^t, \underline{w}_r^t, I_r^t \geq 0 \quad s \in R \setminus \{r\}, \quad i, j \in N, \quad t = 1, \dots, T, \quad (2.43)$$

The dual version of the PSP will provide the necessary Benders cut needed for the algorithm. Let π^t , β^t , γ_s^t , δ_{1sij}^t , δ_{2sij}^t and δ_{3sij}^t be the dual variables of the PSP. The dual variables are outlined alongside each of those constraints. The dual problem to the PSP, the Dual Sub-Problem (DSP) can be written as follows.

Let $\theta_{ik}^t = [x_{(k)}^* - \sum_{l=1}^{k-1} (x_{(l)}^* + z_{(l)}^{*t})]$ and $\nu_{ik}^t = [z_{(k)}^{*t} - \sum_{l=1}^{k-1} (x_{(l)}^* + z_{(l)}^{*t})]$. Let $l_i = \min\{k \in \{1, 2, \dots, |R_i|\} : x_{(k)} = 1\}$, and $q_i^t = \min\{k \in \{1, 2, \dots, |R_i|\} : z_{(k)}^t = 1\}$. Then,

$$\theta_{ik}^t = \begin{cases} 0, & \text{for } k < \min\{l_i, q_i^t\} \\ 1, & \text{for } k = \min\{l_i, q_i^t\}, \text{ if } l_i = k \\ < 0, & \text{for } k > \min\{l_i, q_i^t\} \end{cases} \quad \nu_{ik}^t = \begin{cases} 0, & \text{for } k < \min\{l_i, q_i^t\} \\ 1, & \text{for } k = \min\{l_i, q_i^t\}, \text{ if } q_i = k \\ < 0, & \text{for } k > \min\{l_i, q_i^t\} \end{cases}$$

The objective of DSP can be rewritten as:

$$\sum_{r \in R} x_r^* \left(\sum_{t=1}^T (-\pi_r^t M_{fix} - \sum_{i \in N_r} \alpha_{ir}^t + \sum_{i \in N_r: r=(1)_i} \lambda_{i(1)_i}^t) \right) \quad (2.44)$$

$$\begin{aligned} (DSP) \max & - \sum_{r \in R} \sum_{t=1}^T \pi_r^t (x_r^* M_{fix} + z_r^{*t} M_{portable}) - \sum_{i, j \in N} \sum_{r \in R_i} \sum_{s \in R_j \setminus \{r\}} \sum_{t=1}^T \delta_{3irjs}^t + \sum_{i \in N} \sum_{t=1}^T \left(\omega_i^t - \sum_{r \in R_i} (x_r^* + z_r^{*t}) \alpha_{ir}^t \right) \\ & + \sum_{i \in N} \sum_{t=1}^T \left(x_{(1)_i}^* \lambda_{i(1)_i}^t + \sum_{k=2}^{|R_i|} \theta_{ik}^t \lambda_{i(k)_i}^t + z_{(1)_i}^{*t} \mu_{i(1)_i}^t + \sum_{k=2}^{|R_i|} \nu_{ik}^t \mu_{i(k)_i}^t \right) \end{aligned} \quad (2.45)$$

s.t.

$$-\pi_r^t + \beta_r^t - \beta_r^{t+1} \leq 0 \quad (I_r^t) \quad r \in R, \quad t = 1, 2, \dots, T-1 \quad (2.46)$$

$$-\pi_r^T + \beta_r^T \leq 0 \quad (I_r^T) \quad r \in R \quad (2.47)$$

$$\beta_r^1 \geq 0 \quad (\bar{w}_r^1) \quad r \in R \quad (2.48)$$

$$-1 \leq \beta_r^t \leq 1 \quad (\bar{w}_r^t, \underline{w}_r^t) \quad r \in R, \quad t = 2, 3, \dots, T, \quad (2.49)$$

$$\beta_r^{t+1} - \beta_s^{t+1} + \gamma_{rs}^t \leq 0 \quad (b_{rs}^t) \quad r \in R, \quad s \in R \setminus \{r\}, \quad t = 1, 2, \dots, T-1 \quad (2.50)$$

$$\gamma_{rs}^T \leq 0 \quad (b_{rs}^T) \quad r \in R, \quad s \in R \setminus \{r\} \quad (2.51)$$

$$-d_{ij}^t \gamma_{rs}^t - \delta_{1irjs}^t - \delta_{2irjs}^t + \delta_{3irjs}^t \leq 0 \quad (u_{irjs}^t) \quad i \in N, \quad j \in N, \quad r \in R_i, \quad s \in R_j \setminus \{r\}, \quad t = 1, 2, \dots, T \quad (2.52)$$

$$\begin{aligned} \sum_{j \in N} \sum_{s \in R_j \setminus \{r\}} (\delta_{1irjs}^t + \delta_{2jsir}^t - \delta_{3irjs}^t - \delta_{3jsir}^t) \\ - \alpha_{ir}^t + \omega_i^t + \lambda_{ir}^t + \mu_{ir}^t \leq 0 \quad (a_{ir}^t) \quad i \in N, \quad r \in R_i, \quad t = 1, 2, \dots, T, \end{aligned} \quad (2.53)$$

$$\alpha_{ir}^t, \lambda_{ik}^t, \mu_{ik}^t, \pi_r^t, \delta_{1irjs}^t, \delta_{2irjs}^t, \delta_{3irjs}^t \geq 0 \quad i, j \in N, \quad r \in R_i, \quad s \in R_j \setminus \{r\}, \quad t = 1, 2, \dots, T \quad (2.54)$$

Corresponding to each constraint in the above formulation, the primal variables are written alongside. In order to obtain this objective as a function of Master Problem variables x , z and a , the traditional method is to solve the DSP in its entirety.

Lemma 2.1. DSP has a finite optimal solution.

Proof. From Karush-Kuhn-Tucker (KKT) conditions, we know that solving a linear program to optimality, DSP in this case, will result in one of the three outcomes: the problem being infeasible, the optimal solution being unbounded or the optimal solution being finite.

It is clear upon inspection that setting all the variables to be *zero* satisfies the constraints. The solution space has at least one feasible solution, and hence is not infeasible.

To prove that the maximization problem is not unbounded, we are interested in deriving a finite upper bound on the optimal solution. For ease of notation, let $C_r^t = (x_r^* M_{fix} + z_r^{*t} M_{portable})$. In the objective

function, each of the C_r^t parameters are non-negative since the capacity allotted to a candidate station location is non-negative. From constraints 2.46 and 2.47, we gather that $(\beta_r^t - \beta_r^{t+1})$ is a lower bound on $\pi_r^t \forall t = 1, 2, \dots, T-1$ and β_r^T is a lower bound on π_r^T . Combining that with constraints 2.48 and 2.49, -2 is a lower bound on π_r^t (when $\beta_r^t = -1$ and $\beta_r^{t+1} = 1$). Hence, $(\sum_{r \in R} \sum_{t=1}^T 2 C_r^t)$ is a finite upper bound on $(-\sum_{r \in R} \sum_{t=1}^T \pi_r^t C_r^t)$, the first term in the objective function.

Let each element in the second term in the objective function be denoted as $B_{irjs}^t = (a_{ir}^{*t} + a_{js}^{*t} - 1)\delta_{3irjs}^t - a_{ir}^{*t}\delta_{1irjs}^t - a_{js}^{*t}\delta_{2irjs}^t$. For every tuple of indices (i, r, j, s, t) , let us consider 4 cases.

- **Case 1** $a_{ir}^{*t} = 1, a_{js}^{*t} = 1$

Using constraints 2.52 and 2.51, $B_{irjs}^t = \delta_{3irjs}^t - \delta_{1irjs}^t - \delta_{2irjs}^t \leq \gamma_{rs}^t \leq 0$.

- **Case 2** $a_{ir}^{*t} = 1, a_{js}^{*t} = 0$

Using the non-negativity constraint 2.54, $B_{irjs}^t = -\delta_{1irjs}^t \leq 0$.

- **Case 3** $a_{ir}^{*t} = 0, a_{js}^{*t} = 1$

Using the non-negativity constraint 2.54, $B_{irjs}^t = -\delta_{2irjs}^t \leq 0$.

- **Case 4** $a_{ir}^{*t} = 0, a_{js}^{*t} = 0$

Using the non-negativity constraint 2.54, $B_{irjs}^t = -\delta_{3irjs}^t \leq 0$.

Hence, *zero* is an upper bound on $\sum_{r \in R} \sum_{s \in R} \sum_{i, j \in N} \sum_{t=1}^T B_{irjs}^t$. Since both terms in the objective function have a finite upper bound, the optimal solution for the maximization problem is not unbounded. Since the problem is neither infeasible nor unbounded, it has a finite optimal solution. \square

The Strong Duality Theorem [10] states that the primal and dual optimal solution to a linear program are equal. Hence, Lemma 2.1 affirms that the primal sub problem (PSP) has a finite optimal solution. This further implies that we can restrict our focus to adding Benders optimality cuts, and not feasibility cuts.

Utilizing the separable nature of the sub problem, we propose an alternate approach to solving the DSP in it's entirety. We first solving each of the separated sub problems, $RSSP^r$, to optimality and obtain the dual solutions corresponding to each $RSSP^r$. The dual solutions we get are the π and β vectors. Then, we use Complementary Slackness conditions to generate the rest of the dual solutions necessary to develop the $\phi(x, z, a)$ function. We formalize this procedure in Theorem 2.2.

Lemma 2.2. The objective function of the DSP is a function of (x, z, a) and is the Benders optimality cut, $\phi(x, z, a)$, that is to be added to the MP. This function can be derived from optimal dual solutions of the $RSSP^r \forall r \in R$.

Proof. The Benders cut to be derived is given by the objective function in 2.45, which we had earlier denote as $\phi(x, z, a)$. The unknowns in the function are the π , δ_1 , δ_2 and δ_3 variables. Upon solving $RSSP^r \forall r \in R$, we get the π and β variables. The idea is to derive the rest of the unknown variables using known values for the β variables, given (x, z, a) parameters and Complementary Slackness (C.S.) conditions. It is to be noted that it is sufficient to ensure C.S. conditions only for those extra constraints in the PSP that are relaxed in the $RSSP^r$.

Constraints 2.50 and 2.51 suggest that $\gamma_{rs}^t \leq \beta_s^{t+1} - \beta_r^{t+1} \forall t = 1, 2, \dots, T-1$, and $\gamma_{rs}^T \leq 0$. For each tuple of indices (i, r, j, s, t) , let us examine the following cases.

- **Case 1** $a_{ir}^{*t} = 1, a_{js}^{*t} = 1 \Rightarrow u_{irjs}^t = 1$

Since u_{irjs}^t is non-zero, C.S. ensures that constraint 2.52 must be tight. Therefore, $-\delta_{1irjs}^t - \delta_{2irjs}^t + \delta_{3irjs}^t = d_{ij}^t \gamma_{rs}^t$. It is to be noted that the L.H.S of this equation is the second term in the objective function of DSP. We consider 2 sub-cases. If $\gamma_{rs}^t \geq 0$, since the DSP is a maximization problem, we set $\delta_{1irjs}^t = \delta_{2irjs}^t = 0$ and $\delta_{3irjs}^t = d_{ij}^t \gamma_{rs}^t$. If $\gamma_{rs}^t < 0$, for the same reason, we set $\delta_{1irjs}^t = \delta_{2irjs}^t = \frac{d_{ij}^t \gamma_{rs}^t}{2}$ and $\delta_{3irjs}^t = 0$.

- **Case 2** $a_{ir}^{*t} = 1, a_{js}^{*t} = 0 \Rightarrow u_{irjs}^t = 0$

Constraint 2.46 translates to $0 \geq -1$, a weak constraint. From C.S., we know that $\delta_{1irjs}^t = 0$. Based on the value of γ_{rs}^t , we further discuss two sub cases. If $\gamma_{rs}^t \geq 0$, we set $\delta_{2irjs}^t = \delta_{3irjs}^t = 0$ and if $\gamma_{rs}^t < 0$, we set $\delta_{2irjs}^t = \delta_{3irjs}^t = -\frac{d_{ij}^t \gamma_{rs}^t}{2}$.

- **Case 3** $a_{ir}^{*t} = 0, a_{js}^{*t} = 1 \Rightarrow u_{irjs}^t = 0$

Same as Case 2, except that the values assigned to δ_{1irjs}^t and δ_{2irjs}^t are interchanged.

- **Case 4** $a_{ir}^{*t} = 0, a_{js}^{*t} = 0 \Rightarrow u_{irjs}^t = 0$

Constraint 2.48 turns into $0 \geq -1$, a weak constraint. From C.S., we can set $\delta_{3irjs}^t = 0$. Similar to earlier cases, using constraint 2.52 to discuss the two subcases. When $\gamma_{rs}^t \geq 0$, we set $\delta_{1irjs}^t = \delta_{2irjs}^t = 0$ and if $\gamma_{rs}^t < 0$, we set $\delta_{1irjs}^t = \delta_{2irjs}^t = -\frac{d_{ij}^t \gamma_{rs}^t}{2}$.

□

Lemma 2.2 provides an exact procedure to compute the Bender's optimality cut from integral solutions from the Master problem. Based on this, we describe the recursive heuristic in Algorithm 1. The idea is to solve the Master problem iteratively while adding a new Bender's cut at each iteration, and terminate when the Master problem's optimal solution is equal to the sub problem solution's optimal solution obtained.

```

Generate  $(x, z, a)$  from Master problem without Benders cut;
while  $B^* \neq \text{subproblemOBJ}^*$  do
    Solve separable subproblems  $RSSP^r \forall r \in R$  and obtain  $\text{subproblemOBJ}^*, \pi, \beta$ ;
    Compute  $\delta$  using Complimentary Slackness;
    Derive  $\phi(x, z, a)$  and add Benders cut to  $MP$ ;
    Resolve  $MP$ ;
end

```

Algorithm 1: Iterative Benders algorithm

3 Portable Station Routing Problem (PSRP)

In this Section, the idea of stations being physically fixed at certain locations is extended to a more general idea of portable stations. A portable station is a station with the dual capability of transporting bikes as well as serving as a station itself.

3.1 Model

The FPSLP is solved by setting $K_{portable}$ to be equal to the number of portable stations available for service. It chooses subsets of candidate station locations to serve as fixed station locations (S_{fixed}) and portable station locations ($S_{portable}^t$). For each station $r \in S_{fixed} \cup S_{portable}^t$, the FPSLP also provides the inventory of bikes (I_r^t) at the start of t , the net number of bikes added and removed during t ($\sum_{s \in R \setminus \{r\}} (b_{sr}^t - b_{rs}^t)$), and the number of bikes to be externally added (\bar{w}_r^t) or removed (\underline{w}_r^t) at the end of t . Note that for any given r, t , only one of \bar{w}_r^t or \underline{w}_r^t will be positive when the FPSLP solution is optimal. A depot is considered as a station location denoted by 0 and its corresponding inventory and bike requirements are trivially set as: $I_0^t = \infty$, $b_{s0}^t = b_{0s}^t = 0$ for all $s \in R$ and $\bar{w}_0^t = \underline{w}_0^t = 0$.

This model will focus on routing portable stations at the end of each time period $t \in \{1, 2, \dots, T-1\}$. It assumes that a portable station serves as a bike station at its assigned location during the time interval $[t, t+1-\Delta]$ and moves to the next location within the time interval $(t+1-\Delta, t+1)$, where $\Delta \in [0, 1]$ is the fraction of time period corresponding to the time taken to move the portable stations between successive time periods t and $t+1$. The *end* of time period t is now set to be time $t+1-\Delta$. The idea is to ensure that every portable station location $r \in S_{portable}^{t+1}$ has a portable station positioned there by the start of $t+1$ with the required amount of inventory, I_r^{t+1} . Note that the value of Δ has to be chosen carefully, as choosing a value close to 0 will limit the time available to move the portable stations, making the model infeasible, while choosing a value close to 1 will result in less time spent serving as a bike station. More on choosing an appropriate Δ is discussed later in the model.

Consider a portable station location located at portable station location r and it travels to portable location r' before time period $t-1$. Then such travel must take place within the travel time available (Δ). Let the travel time to move from r to r' be denoted by $traveltime(r, r')$. Further, the number of bikes in such a portable station at the end of t must match the number of bikes required at portable station location r' . Since this model doesn't consider externally adding/removing bikes from portable stations, it is understandable that it is not possible it exactly satisfy the number of bikes required at r' . Based on the travel time constraint, the idea of a feasible pair of portable station locations is more formally defined to be a portable station pair.

Definition 3.1. A pair of station locations (r, r') is called a *portable station pair* for time period t if the following conditions are satisfied.

- (i) $r \in S_{portable}^t \cup \{0\}$, $r' \in S_{portable}^{t+1} \cup \{0\}$, but not both r and $r' = 0$
- (ii) $traveltime(r, r') \leq \Delta$

It should be noted that for certain value of Δ , it is possible that such a portable station pair does not exist, as condition (ii) in Def.3.1 is not satisfied for a portable station $r' \in S_{portable}^{t+1}$ with respect to any portable station location in $S_{portable}^t$ in order to form a portable station pair. In such cases, the decision maker would increase the value of Δ till a feasible plan can be generated.

It is useful to introduce a service level parameter ϵ_r , which is defined to be the threshold for the deviation in the number of bikes in the portable station positioned at r from the number of bikes required at location r at the start of a time period as obtained from the FTSLP.

In addition to finding feasible origin destination pairs for all the portable stations, the primary goal of the PSRP is to potentially assist in redistributing bikes in the fixed stations along the way. But it is important to note that if the available transit time is small, or the origin is far from the destination of a portable station, it wouldn't be feasible for it to visit many fixed stations along the way. Hence, the model explicitly places a limit on the number of fixed stations that a portable station can visit in one traversal. Let p denote the maximum number of fixed stations that any portable station is allowed to visit in a single traversal from its origin to its destination.

Every fixed station $s \in S_{fixed}$ needs \bar{w}_s^t bikes to be added or has \underline{w}_s^t bikes to be removed. This model assumes that when a portable station visits a fixed station along its path, it satisfies the entire addition/removal requirements of the fixed station. If a portable station is capable of only partially fulfilling a fixed station's requirements, then it will not visit that fixed station. Let the set of all portable station pairs be P . Based on these constraints, a feasible portable station path is more formally defined.

Definition 3.2. For any $(r, r') \in P$, a finite sequence of station locations represented by $\langle r, (s_i)_{i \in \{1, 2, \dots, m\}}, r' \rangle$ where $m \in \{0, 1, \dots, p\}$ is called a portable station path if the following conditions are satisfied.

- (i) $s_i \in S_{fixed}$ for all $i \in \{1, 2, \dots, m\}$
- (ii) $I_r^t + \sum_{s \in R \setminus \{r\}} (b_{sr}^t - b_{rs}^t) + \sum_{j \in \{1, 2, \dots, i\}} (\bar{w}_{s_j}^t - \underline{w}_{s_j}^t) \leq M_{portable}$ for all $i \in \{1, 2, \dots, m\}$
- (iii) If $r \in S_{portable}^t$ and $r' \in S_{portable}^{t+1}$, $|I_r^t + \sum_{s \in R \setminus \{r\}} (b_{sr}^t - b_{rs}^t) + \sum_{j \in \{1, 2, \dots, m\}} (\bar{w}_{s_j}^t - \underline{w}_{s_j}^t) - I_{r'}^{t+1}| < \epsilon_{r'}$

In the above definition, it is to be noted that for $m = 0$, the it is a trivial path from r to r' visiting no fixed stations along the way. Let K be a collection of all possible portable station paths. Each path $k \in K$ originates at a portable station location at t , traverses through a set of fixed stations and terminates at a portable station location at $t + 1$. For each path $k \in K$, let n_k be the number of fixed stations that it passes through, and let y_{ik} be a binary indicator which is 1 if path k traverses through station $i \in S_{fixed} \cup S_{portable}^t \cup S_{portable}^{t+1}$, and 0 otherwise. The PSRP selects a subset of paths from K such that there is exactly one path that through an origin portable station location, exactly one path through a destination portable station location, and no more than one path visits a fixed station location. For each path $k \in K$, let the decision variable be η_k , which is a binary variable with value 1 when path k is chosen, and 0 otherwise. This is formulated as an integer program.

$$(PSRP) \quad \max \sum_{k \in K} n_k \eta_k \tag{3.1}$$

s.t.

$$\sum_{k \in K} y_{ik} \eta_k = 1 \quad i \in S_{portable}^t \cup S_{portable}^{t+1}, \tag{3.2}$$

$$\sum_{k \in K} y_{ik} \eta_k \leq 1 \quad i \in S_{fixed}, \tag{3.3}$$

$$\eta_k \in \{0, 1\} \quad k \in K. \tag{3.4}$$

The objective in (3.1) maximizes the number of fixed stations visited by all the portable stations. The constraints (3.2) ensure that a portable station location is visited by exactly one of the paths, while

constraints (3.3) ensure that a fixed station is not visited by more than one path. Constraints (3.4) set the binary nature of the decision variables.

The optimal solution for the PSRP at the end of time period t is a set of paths for the portable stations to take, while ensuring that bikes are redistributed in the fixed stations. It is to be observed that PSRP not only selects which fixed stations to visit, but also pairs portable station locations in such a way that the maximum number of fixed stations satisfy their addition/removal requirements along the way.

4 Conclusions

In this project, we proposed an integrated mathematical framework for operating a bike-sharing system using portable stations. The proposed models determine the optimal location of fixed and portable stations and the number of bikes to be added to or removed from each station every time period to satisfy the demand-supply needs. At the first level, the optimization model determines the positions of the bike stations over the entire time horizon. This includes the positions of fixed stations from a set of candidate locations, and the positions of the portable stations for each time period. The objective of this model is to minimize the number of bikes to be added or removed from each station over all the time periods. The model transforms the origin-destination pair flow for each time period into station-station flow for the chosen station locations. Also, the inventory level at each chosen station during each time period is reduced/increased by the net flow of bikes from/to such station, and the slack/surplus carried over to the next time period is the extra bikes to be added/removed. The objective is to minimize the slack/surplus over all time periods and chosen station locations, in order to ensure that the burden on the redistribution logistics is minimized.

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