

# Integration of Multi-Modal Public Transportation Systems



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# **INTEGRATION OF MULTI-MODAL PUBLIC TRANSPORTATION SYSTEMS**

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## **Executive Summary**

Transit ridership may be sensitive to fares, travel times, waiting times, and access times, among other factors. Thus, elastic demands are considered in formulations for maximizing the system welfare for conventional and flexible bus services. Two constrained nonlinear mixed integer optimization problems are solved with a genetic algorithm: 1) welfare maximization (for conventional and flexible services) with service capacity constraints and 2) welfare maximizations with the service capacity and subsidy constraints. Numerical examples find that with the input parameters assumed here, conventional services produce greater system welfare (consumer surplus + producer surplus) than flexible services. Numerical analysis also finds that if the operating cost is fully subsidized, flexible services generate more actual trips than conventional services. For comparing actual trips between the zero subsidy and the fully subsidized cases, the actual trips for conventional services is increased 10.5% while the actual trips for flexible services is increased 15.6%.

*Key Words: Social Welfare, Consumer Surplus, Producer Surplus, Conventional Bus, Flexible Bus, Genetic Algorithm, Constrained Optimization*

## 1. Introduction

In public transportation systems, feeder services serve a very important purpose for transit users because they start and end their journeys with feeder services. Thus trips may be concentrated into sufficient densities for economical use of mass transportation. Feeder services typically consist of two types: conventional services, which are also known as fixed-route bus services, and flexible services, which are often called demand-responsive services. Bus services with both conventional and flexible services have been studied extensively. Kocur and Hendrickson (1982) designed conventional services connecting the terminal and a local region. Since then, various studies analyzed conventional services (Chang, 1990; Chang and Schonfeld, 1991a & 1991b & 1991c & 1993; Lee et al., 1995; Kim and Schonfeld, 2012 & 2013; Diana et al., 2009). Thus, Chang and Schonfeld (1991a) compared conventional and flexible bus services for serving a local region. They provided closed form solutions with an analytic approach. Chang and Schonfeld (1993) analyzed the social system welfare for conventional services. Their solutions were found by analytic optimization (with approximations), but the proposed method may not be feasible for a multiple region analysis. Lee et al. (1995) considered mixed bus fleets for conventional services and found that mixed fleets conventional services are beneficial for single fleet conventional services when the demand fluctuates over regions.

Flexible feeder services were also actively explored since 1970's. Stein (1978) estimated the optimal tour distance for flexible services. Daganzo (1984) compared demand responsive services for Rectilinear and Euclidean distances. Various research questions for flexible services were addressed and explored (Chang and Schonfeld, 1991a & 1993b; Chandra and Quadrioglio, 2013a & 2013b; Chandra et al., 2011; Diana et al., 2007; Li and Quadrioglio, 2009 & 2011; Quadrioglio and Dessouky, 2007; Quadrioglio et al., 2007 & 2008; Shen and Quadrioglio, 2012; Horn, 2002; Luo and Schonfeld, 2011a & 2011b; Zhou et al., 2008). Luo and Schonfeld (2011a) proposed an online rejected-reinsertion heuristics for a dynamic dial-a-ride problem. Luo and Schonfeld (2011b) also developed metamodels for dial-a-ride services. Chandra and Quadrioglio (2013a) explored demand responsive services for estimating the tour length with an analytic queuing model.

Conventional services are generally favorable (with large bus sizes) at high demand densities. Conversely, flexible services are usually preferable when demand densities are low (Chang and Schonfeld, 1991a; Kim and Schonfeld, 2012 & 2013). Thus, if conventional and

flexible services are jointly provided, it may be possible to provide more efficient feeder services than either conventional or flexible services. To address such bus transit integration problems, Chang and Schonfeld (1991c) consider a temporal integration of conventional and flexible services. Kim and Schonfeld (2012) propose a variable-type bus service, which operates conventional services with higher demand periods and change to flexible services to low demand periods. In Kim and Schonfeld (2012), flexible services re-optimize headways, fleet size, and service area with given vehicle size. Kim and Schonfeld (2013) consider conventional and flexible services as well as a mixture of bus fleets. Their results show that when demand varies over time and over regions, the joint provision of conventional services and flexible services a mix of large and small buses reduces total costs. Quadrifoglio and his colleagues integrate bus feeder services using fixed-route and demand responsive bus services (Aldaihani et al., 2004; Diana et al., 2009; Quadrifoglio and Li, 2009; Li and Quadrifoglio, 2010).

Transit ridership may be sensitive to the elasticity of fares and other time factors such as in-vehicle times, waiting times and access times. However, most of research on feeder transit services mentioned above does not consider demand elasticity. A few studies explore demand elasticity in public transportation services, especially bus transit systems (Kocur and Hendrickson, 1982; Imam, 1998; Chang and Schonfeld, 1993; Zhou et al., 2008; Chien and Spasovic, 2002). When considering the demand elasticity, formulations typically become maximization problems, presumably because it makes little sense to minimize costs if demand is elastic (and may be driven to zero). Kocur and Hendrickson (1982) optimize transit decision variables, namely route spacing, headway, and fare, with demand elasticity. They assume a linear transit utility function rather than a logit form. Their justifications for the linear utility approximation are that it is analytically tractable, it is easily differentiated and manipulated, and it is convex within its upper and lower bounds. They consider wait time, walk time, in-vehicle time, fare, and auto time and cost in the demand model. They provide analytic closed form solutions, but this study is limited to a conventional bus service for one local region. Later, Imam (1998) extends Kocur and Hendrickson (1982)'s study by relaxing the linear demand function. Imam (1998) applies a log-additive demand function.

Chang and Schonfeld (1993) consider time-dependent supply and demand characteristics for a transit welfare maximization problem. They use a linear demand function as in Kocur and Hendrickson (1982). Decision variables are route spacing, headways, and fare. Since this study

considers multiple time periods, they optimize headways for multiple time periods. Their objective is to maximize consumer surplus and producer surplus. They solve this maximum welfare problem with alternative financial constraints, namely without any constraint, with a break-even constraint, and with subsidy. Their problem size extends to one local region and multiple periods. Solutions are obtained analytically with approximations. For the formulations with constraints, a Lagrange multipliers method is applied. The vehicle size is considered as an input, rather than a decision variable.

Zhou et al (2008) formulate welfare for conventional bus services and flexible bus services, but only for a system connecting a terminal to one local region in one period. They find solutions analytically because the formulation of a system that connects a terminal to one local region in one period is analytically tractable. Analyses of system welfare with larger problem sizes (i.e., multiple regions and multiple periods) for both conventional and flexible services are desirable. They analyze tradeoffs between subsidies and welfare, but do not provide detailed enough methods to duplicate their results.

Chien and Spasovic (2002) study a grid bus transit system with an elastic demand pattern. They optimize route spacings, station spacings, headways, and fare with the objective of maximum total operator profit and social welfare. The elastic demand is subtracted from the potential demand as in Chang and Schonfeld (1993), and the optimal solutions are found analytically. This work is applicable to conventional bus services.

Tsai et al (2013) find headway and fare solutions for a Taiwan High Speed Rail (THSR) line, with a maximum welfare objective. They consider elastic demand for the study, and apply a GA to obtain solutions. They compare solutions from a GA and solutions from a SSM (Successive Substitution Method).

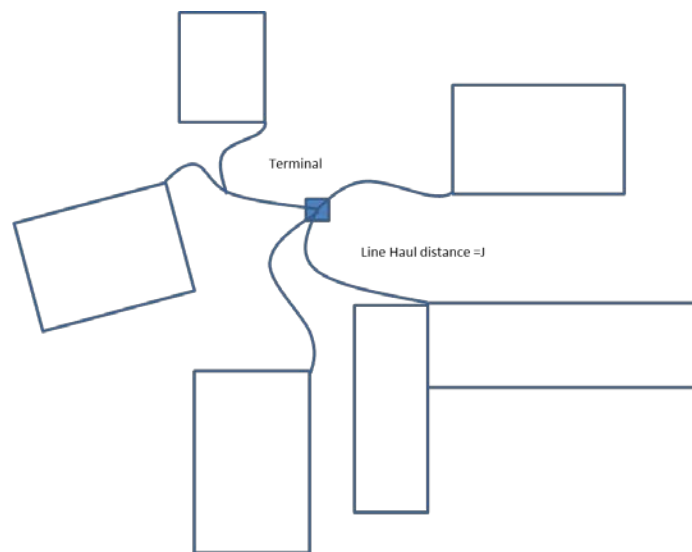
For the system welfare problems in bus transit systems, most of the literature covers conventional services. Most existing transit welfare problems are solved with analytic optimization. Analytic optimization can find solutions quickly with the possibility of the closed form solutions, but it is unable to solve more complex (e.g., multiple region analysis). For conventional services, the solved problem size encompasses a local region with multiple periods. For flexible services, the solved problem size encompasses a local region and one period. With numerical solutions it seems desirable to consider problems with multiple regions as well as multiple periods for both conventional and flexible services.

In this paper, different service qualities and demand elasticity are considered in conventional and flexible service formulations. Total cost minimization is not a reasonable objective when the demand is elastic, since the demand can be driven toward zero in minimizing costs. Instead of minimizing total system costs, the objective in this paper is to maximize the social welfare, which is the sum of consumer surplus (i.e. net user benefit) and producer surplus (i.e. profit).

A linear elastic demand function is applied for both conventional and flexible services. Using elastic demand functions, various decision variables, which are fares on conventional and flexible services, bus sizes, headways and fleet sizes for both service types, route spacings for conventional services, and service areas for flexible services, are optimized here. The optimization problems that are solved in this paper are suitable for the planning stage.

## 2. System Specifications and Assumptions

This section addresses assumptions for analyzing a general system (shown in Figure 1) with multiple local regions as well as multiple periods. Assumptions from Kim and Schonfeld (2013) are still applicable, and additional assumptions (for the welfare analysis) are introduced in the following sections when they are required.



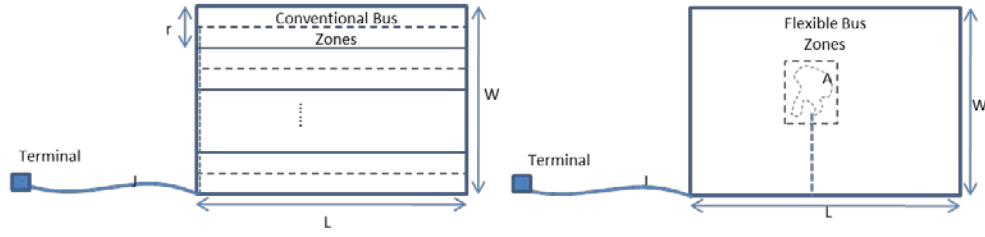


Figure 1 Local Regions and Bus Operations

Henceforth, superscripts  $k$  and  $i$  correspond to region and time period, respectively, while subscripts  $c$  and  $f$  represent conventional and flexible services, respectively. The definitions, units and default values of variables used in this paper are presented in Table 1.

Table 1 Notation

Variable	Definition	Baseline Value
$a$	hourly fixed cost coefficient for operating bus (\$/bus hr)	30.0
$A^k$	service zone area(mile <sup>2</sup> )= $L^k W^k / N'$	-
$b$	hourly variable cost coefficient for bus operation (\$/seat hr)	0.2
$d$	bus stop spacing (miles)	0.2
$D_c^{ki}$	distance of one flexible bus tour in local region $k$ and period $i$ (miles)	-
$D_f^k$	equivalent line haul distance for flexible bus on region $k$ (= $(L^k + W^k)/z + 2J^k/y$ ), (miles)	-
$D^k$	equivalent average bus round trip distance for conventional bus on region $k$ (= $2J^k/y + W^k/z + 2L^k$ ), (miles)	-
$d_s^{ki}$	directional demand split factor	1.0
$F^{ki}$	fleet size for region $k$ and period $i$ (buses) subscript corresponds to ( $c$ = conventional, $f$ =flexible)	-
$h_c, h_c^{ki}$	headway for conventional bus; for region $k$ and period $i$ (hours/bus)	-
$h_f, h_f^{ki}$	headway for flexible bus; for region $k$ period $i$ (hours/bus)	-
$k, i$	index ( $k$ : region, $i$ : period)	-
$J^k$	line haul distance of region $k$ (miles)	-
$l_c, l_f$	load factor for conventional and flexible bus (passengers/seat)	1.0
$L^k, W^k$	length and width of local region $k$ (miles)	-
$M^k$	equivalent average trip distance for region $k$ (= $(J^k/y_c + W^k/2z_c + L^k/2)$ )	-
$n$	number of passengers in one flexible bus tour	-
$N_c^k, N_f^k$	number of zones in local region for conventional and flexible bus	-
$Q^{ki}$	actual demand density (trips/hr)	-
$q^{ki}$	potential demand density (trips/mile <sup>2</sup> /hr)	-
$r^k$	route spacing for conventional bus at region $k$ (miles)	-
$R_c^{ki}$	round trip time of conventional bus for region $k$ and period $i$ (hours)	-
$R_f^{ki}$	round trip time of flexible bus for region $k$ and period $i$ (hours)	-
$S_c, S_f$	sizes for conventional and flexible bus (seats/bus)	-
$t^{ki}$	time duration for region $k$ and period $i$	-
$u$	average number of passengers per stop for flexible bus	1.2
$V_c^i$	local service speed fo/r conventional bus in period $i$ (miles/hr)	30
$V_f^i$	local service speed for flexible bus in period $i$ (miles/hr)	25
$V_x$	average passenger access speed (mile/hr)	2.5
$v_v, v_w, v_x$	value of in-vehicle time, wait time and access time (\$/passenger hr)	5, 12, 12
$y$	express speed/local speed ratio for conventional bus	conventional bus = 1.8

		flexible bus = 2.0
$\emptyset$	constant in the flexible bus tour equation (Daganzo, 1984) for flexible bus	1.15
*	superscript indicating optimal value; subscript: c = conventional, f=flexible	-
$Y_c^{ki}, Y_f^{ki}$	total social welfare in region k and period i subscript: c = conventional, f=flexible	-
$P_c^{ki}, P_f^{ki}$	producer surplus (revenue – cost) in region k and period i subscript: c = conventional, f=flexible	-
$R_c^{ki}, R_f^{ki}$	revenue in region k and period i subscript: c = conventional, f=flexible	-
$C_c^{ki}, C_f^{ki}$	operating cost in region k and period i subscript: c = conventional, f=flexible	-
$f_c, f_f$	fares on the system ;subscript: c = conventional, f=flexible	-
$G_c^{ki}, G_f^{ki}$	consumer surplus in region k and period i subscript: c = conventional, f=flexible	
$e_v, e_w, e_x, e_p$	elasticity factors	0.35, 0.7, 0.7, 0.07
$TUB_c^{ki}, TUB_f^{ki}$	the total user benefit in region k and period i subscript: c = conventional, f=flexible	
$Y_c, Y_f$	the total welfare of system subscript: c = conventional, f=flexible	
$FS^{ki}$	the amount of subsidy for region k and period i	

## 2.1 Common assumptions for conventional and flexible services

All service regions, 1... k, are rectangular, with lengths  $L^k$  and widths  $W^k$ . These regions may have different line haul distances  $J^k$  (miles, in region k) connecting a terminal and each region's nearest corner.

- The demand is uniformly distributed over space within each region and over time within each specified period.
- The optimized bus sizes ( $S_c$  for conventional,  $S_f$  for flexible) are uniform throughout regions and time periods.
- The average waiting time of passengers is approximated as a constant fraction alpha of the headway ( $h_c$  for conventional,  $h_f$  for flexible). Alpha is usually assumed to be 0.5.
- Bus layover time is negligible.
- Within each local region k, the average speed ( $V_c^i$  for conventional bus,  $V_f^i$  for flexible bus) includes stopping times.
- External costs are assumed to be negligible.

## 2.2 Assumptions for conventional bus only (adopted from Kim and Schonfeld 2013, TR-B)

- The region k is divided into  $N^k$  parallel zones with a width  $r^k = W^k/N^k$  for conventional bus, as shown in Figure 1. Local routes branch from the line haul route segment to run along the middle of each zone, at a route spacing  $r^k = W^k/N^k$ .

- b)  $Q^{ki}$  trips/mile<sup>2</sup>/hour, entirely channeled to (or through) the single terminal, are uniformly distributed over the service area.
- c) In each round trip, as shown in Figure 1, buses travel from the terminal a line haul distance  $J^k$  at non-stop speed  $yV_c^i$  to a corner of the local regions, then travel an average of  $W^k/2$  miles at local non-stop speed  $zV_c^i$  from the corner to the assigned zone, then run a local route of length  $L^k$  at local speed  $V_c^i$  along the central axis of the zone while stopping for passengers every  $d$  miles, and then reverse the above process in returning to the terminal.

### 2.1.3 Assumptions for flexible bus only (*adopted from Kim and Schonfeld 2013, TR-B*)

- a) To simplify the flexible bus formulation, region  $k$  is divided into  $N^k$  equal zones, each having an optimizable zone area  $A^k = L^k W^k / N^k$ . The zones should be “fairly compact and fairly convex” (Stein, 1978).
- b) Buses travel from the terminal line haul distance  $J^k$  at non-stop speed  $yV_f^i$  and an average distance  $(L^k + W^k)/2$  miles at local non-stop speed  $zV_f^i$  to the center of each zone. They collect (or distribute) passengers at their door steps through an efficiently routed tour of  $n$  stops and length  $D_c^{ki}$  at local speed  $V_f^i$ .  $D_c^{ki}$  is approximated according to Stein (1978), in which  $D_c^{ki} = \phi \sqrt{n A^k}$ , and  $\phi = 1.15$  for the rectilinear space assumed here (Daganzo, 1984). The values of  $n$  and  $D_c^{ki}$  are endogenously determined. To return to their starting point the buses retrace an average of  $(L^k + W^k)/2$  miles at  $zV_f^i$  miles per hour and  $J^k$  miles at  $yV_f^i$  miles per hour.
- c) Buses operate on schedules with preset headways and with flexible routing designed to minimize each tour distance  $D_c^{ki}$ .
- d) Tour departure headways are equal for all zones in each region and uniform within each period.

## 3. Elastic Demand Functions and Operating Costs

### 3.1. Conventional Bus Services

In this section, the linear elastic demand function and the operating cost for conventional services are formulated. Chang and Schonfeld (1993) consider elastic demand for conventional

bus services for one region and multiple periods. Their elastic demand function for conventional services is modified here to accommodate multiple regions as well as multiple periods.

### 3.1.1. Elastic Demand Function for Conventional Bus Services

The demand density may be sensitive to in-vehicle time, waiting time, access time, and the fare of the system. A linear demand function,  $Q_c^{ki}$ , in region k and period i is formulated as follows.

$$Q_c^{ki} = L^k W^k q^{ki} \left\{ 1 - e_w z_1 h_c^{ki} - e_x z_2 \frac{(r^k + d)}{v_x} - e_v \frac{M_c^k}{v_c^i} - e_p f_c \right\} \quad (1)$$

where  $z_1$  = usually 0.5 for uniform passenger arrivals, uniform bus arrivals and sufficient bus capacity;  $z_2$  = usually 0.25 for rectilinear network. The elastic demand function in equation (1) can be rewritten as:

$$Q_c^{ki} = L^k W^k q^{ki} \left\{ K_c^k - e_w z_1 h_c^{ki} - e_x z_2 \frac{r^k}{v_x} - e_p f_c \right\} \quad (2)$$

where  $K_c^k = 1 - e_x z_2 \frac{d}{v_x} - e_v \frac{M_c^k}{v_c^i}$

### 3.1.2. Conventional Bus Operating Cost

The conventional bus operating cost in region k and time period i is formulated below. Unit operating cost,  $B_c$ , is assumed to be a function of vehicle size (i.e.,  $B_c = a + bS_c$ ):

$$C_c^{ki} = B_c N_c^k \frac{D^k t^{ki}}{v_c^i h_c^{ki}} \quad (3)$$

## 3.2. Flexible Bus Services

### 3.2.1. Elastic Demand Function for Flexible Bus Services

The demand density of flexible bus services is affected by the in-vehicle time, waiting time and fare. The access time factor is not considered for flexible services because we assume flexible services provide door-to-door services. Zhou et al (2008) considered a flexible service with elastic demand for only one time period and one region. Their solutions were obtainable with simple calculus since the problem was small. Here, the elastic demand function for flexible services is modified for multiple regions as well as multiple periods. The actual demand in region k and period i is formulated as:

$$Q_f^{ki} = L^k W^k q^{ki} \{1 - e_w z_1 h_f^{ki} - e_v M_f^{ki} - e_p f_f\} \quad (4)$$

where  $z_1$  = usually 0.5 uniform passenger arrivals, uniform bus arrivals and sufficient bus capacity. Equation (4) can be rewritten as

$$Q_f^{ki} = L^k W^k q^{ki} \left\{ K_f^{ki} - e_w z_1 h_f^{ki} - e_v \frac{(\emptyset A^k \sqrt{\frac{q^{ki} h_f^{ki}}{u}})}{2V_f^i} - e_p f_f \right\} \quad (5)$$

where  $K_f^{ki} = 1 - e_v \left( \frac{L^k + W^k}{2V_f^i y_f} + \frac{J^k}{V_f^i y_f} \right)$ .

### 3.2.2. Flexible Bus Operating Cost

Flexible bus operating cost,  $C_f^{ki}$ , is formulated by multiplying unit bus operating cost, the number of zones in region k, and round travel time:

$$C_f^{ki} = B_f N_f^k \frac{(D_f^k + D_c^{ki}) t^{ki}}{V_f^i h_f^{ki}} \quad (6)$$

$D_c^{ki}$  is the approximated flexible bus tour distance according to Stein (1978), in which  $D_c^{ki} = \emptyset \sqrt{n A^k}$ , and  $\emptyset = 1.15$  for the rectilinear space assumed here (Daganzo, 1984). The service area,  $A^k$ , of flexible bus in region k is equal to  $\frac{L^k W^k}{N_f^k}$ . Thus, by substituting average tour distance  $D_c^{ki}$  into equation (6), the flexible bus operating cost in region k and time period i is estimated as:

$$C_f^{ki} \approx B_f N_f^k \frac{(D_f^k + \emptyset \sqrt{n A^k}) t^{ki}}{V_f^i h_f^{ki}} = \frac{B_f N_f^k D_f^k t^{ki}}{V_f^i h_f^{ki}} + \frac{\emptyset B_f L^k W^k t^{ki} \sqrt{q^{ki} h_f^{ki} / u}}{V_f^i h_f^{ki}} \quad (7)$$

## 4. Welfare Maximization without Financial Constraints

For public transit services and in general, the social welfare is the sum of the consumer surplus and the producer surplus. In this section social welfare functions are formulated for both conventional and flexible bus services.

### 4.1. Welfare Formulation for Conventional Services

The welfare of conventional bus services,  $Y_c^{ki}$ , in region k and period i is the sum of the producer surplus,  $P_c^{ki}$  and the consumer surplus,  $G_c^{ki}$ :

$$Y_c^{ki} = P_c^{ki} + G_c^{ki} \quad (8)$$

The producer surplus  $P_c^{ki}$  is the total revenue  $R_c^{ki}$  minus the operating cost  $C_c^{ki}$  of the conventional bus service:

$$P_c^{ki} = R_c^{ki} - C_c^{ki} \quad (9)$$

The total revenue  $R_c^{ki}$  in region k and period i is the fare multiplied by the total demand density in region k and time period i:

$$R_c^{ki} = f_c Q_c^{ki} t^{ki} = f_c L^k W^k q^{ki} t^{ki} \left\{ K_c^k - e_w z_1 h_c^{ki} - e_x z_2 \frac{r^k}{V_x} - e_p f_c \right\} \quad (10)$$

where  $K_c^k = 1 - e_x z_2 \frac{d}{V_x} - e_v \frac{M_c^k}{V_c^k}$

The producer surplus in equation (8) can be now rewritten as:

$$P_c^{ki} = f_c L^k W^k q^{ki} t^{ki} \left\{ K_c^k - e_w z_1 h_c^{ki} - e_x z_2 \frac{r^k}{V_x} - e_p f_c \right\} - B_c N_c^k \frac{D^k t^{ki}}{V_c^k h_c^{ki}} \quad (11)$$

The consumer surplus  $G_c^{ki}$  is formulated for region k and time period i. The consumer surplus is the total user benefit minus the prices that transit users actually pay. The total social benefit function can be obtained by using the willingness to pay function in equation (2). The fare in equation (2) is formulated as a function of the demand density,  $Q_c^{ki}$ :

$$f_c = \frac{1}{e_p} \left\{ K_c^k - e_w z_1 h_c^{ki} - e_x z_2 \frac{r^k}{V_x} \right\} - \frac{Q_c^{ki}}{e_p L^k W^k q^{ki}} \quad (12)$$

The total user benefit is then obtained by integrating equation (12) over the demand density,  $Q_c^{ki}$ , which is expressed as:

$$\int f_c dQ_c^{ki} = \frac{1}{e_p} \left\{ K_c^k - e_w z_1 h_c^{ki} - e_x z_2 \frac{r^k}{V_x} \right\} Q_c^{ki} - \frac{(Q_c^{ki})^2}{2 e_p L^k W^k q^{ki}} \quad (13)$$

Equation (13) is rearranged by substituting the potential demand density  $q^{ki}$  from equation (2).

The total user benefit  $TUB_c^{ki}$  is formulated as follows:

$$TUB_c^{ki} = \frac{Q_c^{ki} t^{ki}}{2 e_p} \left\{ K_c^k - e_w z_1 h_c^{ki} - e_x z_2 \frac{r^k}{V_x} + e_p f_c \right\} \quad (14)$$

The consumer surplus is formulated as the total user benefit minus the fares that users actually pay to the conventional bus providers:

$$G_c^{ki} = \frac{L^k W^k q^{ki} t^{ki}}{2 e_p} \left\{ K_c^k - e_w z_1 h_c^{ki} - e_x z_2 \frac{r^k}{V_x} - e_p f_c \right\}^2 \quad (15)$$

The total welfare in equation (8) that sums the producer surplus and consumer surplus in region k and period i is then expressed as:

$$Y_c^{ki} = f_c L^k W^k q^{ki} t^{ki} \left( K_c^k - e_w z_1 h_c^{ki} - e_x z_2 \frac{r^k}{V_x} - e_p f_c \right) - B_c N_c^k \frac{D^k t^{ki}}{V_c^i h_c^{ki}} + \frac{L^k W^k q^{ki} t^{ki}}{2e_p} \left( K_c^k - e_w z_1 h_c^{ki} - e_x z_2 \frac{r^k}{V_x} - e_p f_c \right)^2 \quad (16)$$

The total welfare for the entire system is formulated as follows:

$$\begin{aligned} Y_c &= \sum_{k=1}^K \sum_{i=1}^I Y_c^{ki} = \sum_{k=1}^K \sum_{i=1}^I (P_c^{ki} + G_c^{ki}) \\ Y_c &= \sum_{k=1}^K \sum_{i=1}^I \left\{ f_c L^k W^k q^{ki} t^{ki} \left( K_c^k - e_w z_1 h_c^{ki} - e_x z_2 \frac{r^k}{V_x} - e_p f_c \right) - B_c N_c^k \frac{D^k t^{ki}}{V_c^i h_c^{ki}} + \frac{L^k W^k q^{ki} t^{ki}}{2e_p} \left( K_c^k - e_w z_1 h_c^{ki} - e_x z_2 \frac{r^k}{V_x} - e_p f_c \right)^2 \right\} \end{aligned} \quad (17)$$

Equation (17) can be written as:

$$\begin{aligned} Y_c &= \sum_{k=1}^K \sum_{i=1}^I \left\{ f_c L^k W^k q^{ki} t^{ki} \left( K_c^k - e_w z_1 h_c^{ki} - e_x z_2 \frac{r^k}{V_x} - e_p f_c \right) \right\} - \sum_{k=1}^K \sum_{i=1}^I \left( B_c N_c^k \frac{D^k t^{ki}}{V_c^i h_c^{ki}} \right) + \\ &\sum_{k=1}^K \sum_{i=1}^I \left\{ \frac{L^k W^k q^{ki} t^{ki}}{2e_p} \left( K_c^k - e_w z_1 h_c^{ki} - e_x z_2 \frac{r^k}{V_x} - e_p f_c \right)^2 \right\} \end{aligned} \quad (18)$$

The social welfare in equation (18) is maximized by optimizing the decision variables of vehicle size, fares, headways, fleet sizes, and route spacings (or the numbers of zones).

#### 4.2. Welfare Formulation for Flexible Services

The welfare of flexible bus services in region k and period i,  $Y_f^{ki}$ , is formulated as the sum of producer surplus  $P_f^{ki}$  and consumer surplus  $G_f^{ki}$ :

$$Y_f^{ki} = P_f^{ki} + G_f^{ki} \quad (19)$$

The producer surplus  $P_f^{ki}$  is computed by subtracting the flexible bus operating cost from the revenue of the flexible bus service:

$$P_f^{ki} = R_f^{ki} - C_f^{ki} \quad (20)$$

The total revenue of the flexible bus service in region k and period i,  $R_f^{ki}$ , is the flexible bus service fare multiplied by total demand density:

$$R_f^{ki} = f_f Q_f^{ki} t^{ki} = f_f L^k W^k q_f^{ki} t^{ki} (1 - e_w z_1 h_f^{ki} - e_v M_f^{ki} - e_p f_f) \quad (21)$$

Then, the producer surplus in region k and period i  $P_f^{ki}$  is:

$$P_f^{ki} = f_f L^k W^k q_f^{ki} t^{ki} (1 - e_w z_1 h_f^{ki} - e_v M_f^{ki} - e_p f_f) - \left( \frac{B_f N_f^k D_f^k t^{ki}}{V_f^i h_f^{ki}} + \frac{\emptyset B_f L^k W^k t^{ki} \sqrt{q_f^{ki} h_f^{ki} / u}}{V_f^i h_f^{ki}} \right) \quad (22)$$

The consumer surplus  $G_f^{ki}$  in region k and period i the total social benefit of the flexible bus services minus the price that flexible bus users actually pay. The total social benefit of the flexible bus service  $TSB_f^{ki}$  can be found by integrating the willingness to pay function:

$$TSB_f^{ki} = \int f_f dQ_f^{ki} = \int \left\{ \frac{1}{e_p} \left( K_f^k - e_w z_1 h_f^{ki} - e_v \frac{D_c^{ki}}{2V_f^i} \right) - \frac{Q_f^{ki}}{e_p L^k W^k q_f^{ki}} \right\} dQ_f^{ki}$$

$$TSB_f^{ki} = \frac{Q_f^{ki}}{e_p} \left( K_f^k - e_w z_1 h_f^{ki} - e_v \frac{D_c^{ki}}{2V_f^i} \right) - \frac{(Q_f^{ki})^2}{2e_p L^k W^k q_f^{ki}} \quad (23)$$

By substituting the potential demand density  $q_f^{ki}$  from equation (4), the total social benefit of the flexible bus in region k and period i  $TSB_f^{ki}$  becomes:

$$TSB_f^{ki} = \frac{Q_f^{ki} t^{ki}}{2e_p} \left( K_f^k - e_w z_1 h_f^{ki} - e_v \frac{D_c^{ki}}{2V_f^i} + e_p f_f \right) \quad (24)$$

The consumer surplus of the flexible bus service is now formulated as the total social benefit minus the price that users actually pay to the flexible bus providers:

$$G_f^{ki} = \frac{L^k W^k q_f^{ki} t^{ki}}{2e_p} \left( K_f^k - e_w z_1 h_f^{ki} - e_v \frac{D_c^{ki}}{2V_f^i} - e_p f_f \right)^2 \quad (25)$$

The total welfare of the flexible bus service in region k and period i is now expressed as:

$$Y_f^{ki} = G_f^{ki} + P_f^{ki}$$

$$Y_f^{ki} = \frac{L^k W^k q_f^{ki} t^{ki}}{2e_p} \left( K_f^k - e_w z_1 h_f^{ki} - e_v \frac{D_c^{ki}}{2V_f^i} - e_p f_f \right)^2 + f_f L^k W^k q_f^{ki} t^{ki} \{ 1 - e_w z_1 h_f^{ki} - e_v M_f^{ki} - e_p f_f \} - \left( \frac{B_f N_f^k D_f^k t^{ki}}{V_f^i h_f^{ki}} + \frac{\emptyset B_f L^k W^k t^{ki} \sqrt{q_f^{ki} h_f^{ki} / u}}{V_f^i h_f^{ki}} \right) \quad (26)$$

The social welfare for the entire flexible bus services is formulated as follows:

$$Y_f = \sum_{k=1}^K \sum_{i=1}^I Y_f^{ki} = \sum_{k=1}^K \sum_{i=1}^I (G_f^{ki} + P_f^{ki})$$

$$Y_c = \sum_{k=1}^K \sum_{i=1}^I \left\{ \frac{L^k W^k q^{ki} t^{ki}}{2e_p} \left( K_f^k - e_w z_1 h_f^{ki} - e_v \frac{D_c^{ki}}{2V_f^i} - e_p f_f \right)^2 + f_f L^k W^k q^{ki} t^{ki} \{ 1 - e_w z_1 h_f^{ki} - e_v M_f^{ki} - e_p f_f \} - \left( \frac{B_f N_f^k D_f^k t^{ki}}{V_f^i h_f^{ki}} + \frac{\emptyset B_f L^k W^k t^{ki} \sqrt{q^{ki} h_f^{ki} / u}}{V_f^i h_f^{ki}} \right) \right\} \quad (27)$$

Equation (27) can be written as:

$$Y_f = \sum_{k=1}^K \sum_{i=1}^I \{ f_f L^k W^k q^{ki} t^{ki} (1 - e_w z_1 h_f^{ki} - e_v M_f^{ki} - e_p f_f) \} - \sum_{k=1}^K \sum_{i=1}^I \left\{ \frac{B_f N_f^k D_f^k t^{ki}}{V_f^i h_f^{ki}} + \frac{\emptyset B_f L^k W^k t^{ki} \sqrt{q^{ki} h_f^{ki} / u}}{V_f^i h_f^{ki}} \right\} + \sum_{k=1}^K \sum_{i=1}^I \left\{ \frac{L^k W^k q^{ki} t^{ki}}{2e_p} \left( K_f^k - e_w z_1 h_f^{ki} - e_v \frac{D_c^{ki}}{2V_f^i} - e_p f_f \right)^2 \right\} \quad (28)$$

Equation (28) must be maximized by optimizing the decision variables of flexible bus size, fares, headways, fleet sizes, and service areas.

#### 4.3. Solution Method

The welfare formulations designed for both conventional and flexible services are nonlinear and they have both integer and continuous variables. Such nonlinear mixed integer formulations are known to be NP-hard, and thus no exact solution methods are available for them. In the bus transit welfare literature, analytic optimization is applicable to small problems. Thus, the problem of multiple regions as well as multiple periods has not been solved by analytic optimization. A numerical method (i.e., a real coded genetic algorithm (RCGA)) is chosen to solve the proposed formulations because it can overcome local optima as well as handling integer variables efficiently. Details of the real coded genetic algorithm we used can be found elsewhere (Deb, 2000; Deep et al, 2009).

#### 4.4. Numerical Example

In this section, a numerical case study is designed to check formulations without financial constraints. For this case study, the maximum allowable headway constraints are enforced. The vehicle size (seats/bus) is one of the input values, and its sensitivity to the system welfare is also analyzed.

#### 4.4.1. Input Values

For a numerical example, three local regions and four time periods are considered. The baseline input values are shown in Table 1. The potential demand densities, sizes of regions, and time periods are shown in Table 2. The minimum and maximum headways are assumed to be 3 and 60 minutes, respectively. The minimum and maximum fleet sizes can be obtained with headway boundaries. For the vehicle size inputs, 7, 10, 16, 20, 25, 35, and 45 seats are the acceptable values.

Regions A, B, and C have the same demand densities. However, the regional characteristics are different. Region A is 4 mile<sup>2</sup>, region B is 12.25 mile<sup>2</sup>, and region C is 25 mile<sup>2</sup>. Therefore, the total demand in region C exceeds those in regions A or B, although the demand densities are the same. The same demand density inputs are assumed initially for all regions, in order to identify the effects of region size.

Table 2 Potential Demand, Service time, Line-haul Distance, and Sizes of Regions

		Demand (trips/mile <sup>2</sup> /hour)		
Period	Region	A	B	C
1		90	90	90
2		40	40	40
3		20	20	20
4		10	10	10
		Time(hours)		
Period	Region	A	B	C
1		4	4	4
2		6	6	6
3		8	8	8
4		6	6	6
		A	B	C
Line-haul Distance (miles)		6	6	6
Length of Region (miles)		2	3.5	5
Width of Region (miles)		2	3.5	5
Regional Area (mile <sup>2</sup> )		4	12.25	25

#### 4.4.2. Discussion of Results

Figure 2 shows that fares optimized by RCGA for conventional services with 7 and 10 seated vehicles are non-zero. We constrained the minimum headway at three minutes. However, with these smaller vehicle sizes, resulting headways are less than three minutes, thereby violating the low boundary of the headway. In other words, the entire demand cannot be served with the minimum 3 minutes headways. Thus, it is confirmed that the optimization model cannot find any feasible solutions with the conventional vehicle sizes of 7 or 10 seats because the input demand

is too high. It is possible that if demand inputs were lower, conventional services with 7 or 10 seats buses could have solutions with optimized fares of zero. The flexible services also cannot find any feasible solutions for vehicle sizes of 7, 10, and 16 seats.

The solution quality of RCGA can be checked with the fare result comparison between analytic optimization and RCGA. For larger vehicle sizes such as 16 seats for conventional services and 20 or more seats for both conventional and flexible services, the optimized fares obtained by RCGA are zero. The fares obtained through analytic optimization using equations (18) and (20) are also zero. Thus, it is possible to confirm that RCGA finds very good (i.e., at least nearly optimal) solutions, although solutions for other decision variables cannot be compared with analytic optimization.

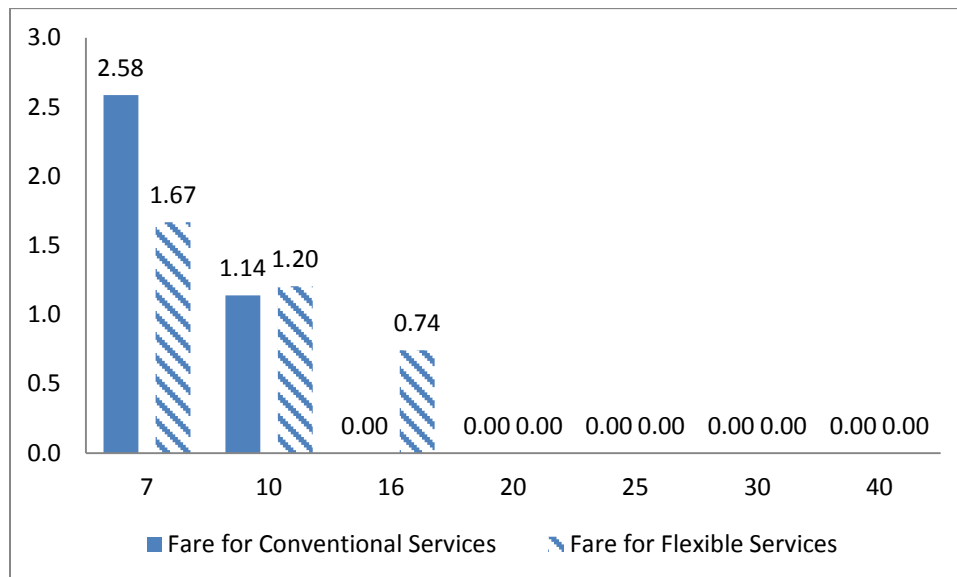


Figure 2 Optimized Fares with Vehicle Size Inputs

Figure 3 shows that the welfare is maximized (at 120219 \$/day) when the conventional bus size is 25 seats. For the flexible services, the maximum welfare of flexible services is 113999 \$/day with 20 seat vehicles.

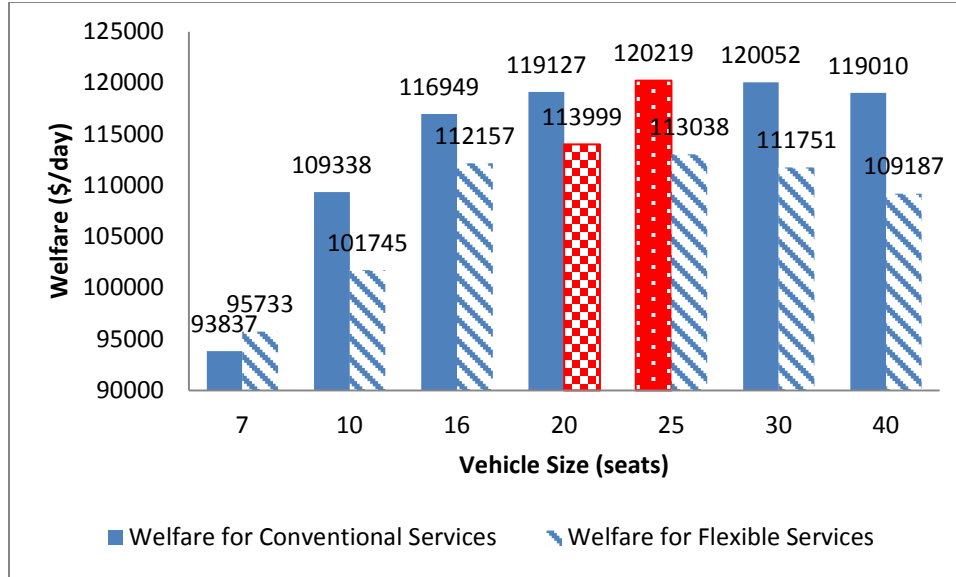


Figure 3 Welfare vs. Vehicle Sizes

Table 3 shows the actual trip density results of conventional and flexible services. The actual trip densities of conventional services among regions A, B and C are very close (less than one trip/mile<sup>2</sup>/hour). The sizes of the region A, B, and C are 4, 12.25, and 25 mile<sup>2</sup>, respectively as shown in Table 2. These results confirm that the size of regions does not significantly affect the density of actual trips. However, it is found that the actual trips for flexible services are higher than the actual trips for conventional services.

The main reason for more actual trips in flexible services is that flexible services have door-to-door services, and hence zero access costs. However, conventional services include access times in the elastic demand function. Thus, it is noted that when demands are sensitive to the in-vehicle time, waiting time, and access time, flexible services are preferable to conventional services in terms of the total actual trips served.

Table 3 Demand with Elasticity

Demand (trips/mile <sup>2</sup> /hour)						
Period \ Region	Conventional Services			Flexible Services		
	A	B	C	A	B	C
1	74.21	73.59	73.35	77.01	75.86	74.20
2	32.53	31.74	31.47	33.68	32.83	32.30
3	15.81	15.26	15.36	16.05	16.32	15.98
4	7.22	6.72	7.30	7.85	7.55	8.10

Table 4 shows the optimized number of zones for conventional and flexible services. The numbers of zones for conventional services are two, four, and six for regions A, B, and C, respectively. The route spacings (which can be obtained by Width of region / Number of zones) are then 1.0, 0.875, and 0.833 miles for regions A, B, and C, respectively. The number of zones increases as the width of a region increases.

The sizes of regions A, B, and C are 4, 12.25, and 25 mile<sup>2</sup>, respectively, as shown in Table 2. The numbers of zones for flexible services are one, three, and five for regions A, B, and C, respectively. Hence, optimized service areas for flexible services are 4.0, 4.08, and 5.0 mile<sup>2</sup> for regions A, B, and C, respectively. Additional zones increase operating costs. Thus, it is concluded that the optimal areas of flexible services with given inputs range between four and five square miles.

Table 4 Optimized Number of Zones

Period \ Region	Conventional Services			Flexible Services		
	A	B	C	A	B	C
Number of Zones	2	4	6	1	3	5
Route Spacings (mile)	1.0	0.875	0.833	-	-	-
Service Areas (mile <sup>2</sup> )	-	-	-	4.0	4.08	5.0

The optimized headways for both conventional and flexible services are provided in Table 5. For conventional services, period 1, which has higher demand densities than other periods, has optimized headways of about four to six minutes. Optimized headways increase as demand densities decrease. It is also found that headways of flexible services are generally lower than headways of conventional services if they are compared in the same period and the same region. This indirectly explains why flexible services obtain more actual trips than conventional services. For period 1, flexible service headways are slightly above 3 minutes, which is the minimum headway boundary.

For conventional services, the longest headway, which is about 31 minutes, is used for period 4 in region B. For flexible services, period 4 in region B has headways of about 21 minutes.

Table 5 Optimized Headways in Minutes

Headways (minute)						
Region	Conventional Services			Flexible Services		
	A	B	C	A	B	C

Period						
1	5.89	6.24	4.86	3.53	3.59	3.17
2	7.85	10.41	9.72	6.77	7.68	6.60
3	11.78	15.61	12.96	13.54	10.52	9.84
4	23.56	31.22	19.44	18.43	21.03	10.83

Table 6 shows optimized fleet sizes for conventional and flexible services. As expected, period 1 requires larger fleet sizes than other periods. It is also noted that flexible services require much larger fleet sizes than conventional services. Conventional services require a total of 166 vehicles with 25 seats, while flexible services require 268 vehicles with 20 seats. Larger fleet sizes imply higher operating costs.

Table 6 Optimized Fleet Sizes

Fleet Sizes (buses)						
Period	Conventional Services			Flexible Services		
	A	B	C	A	B	C
1	4	5	8	12	13	17
2	3	3	4	6	6	8
3	2	2	3	3	4	5
4	1	1	2	2	2	4
Total Fleet Size (buses)	166			268		

Table 7 provides costs and profits for conventional and flexible services. The analytically optimized fares using equations (18) and (28) are zero. Numerically optimized fares for both conventional and flexible services also show that fares are zero. Thus, without subsidy, revenues of conventional and flexible services in any periods are zero, and thus total revenues are also zero. Therefore, profits are simply negative values of costs in each period and each region. Costs and profits are shown per hour because each period has a different duration. As shown in Figure 7, the total cost of flexible services exceeds that of conventional services by about 52%..

Table 7 Costs and Profits for Conventional and Flexible Services

Cost (\$/hour)						
Region \ Period	Conventional Services			Flexible Services		
	A	B	C	A	B	C
1	280.0	700.0	1680.0	408.0	1326.0	2890.0
2	210.0	420.0	840.0	204.0	612.0	1360.0
3	140.0	280.0	630.0	102.0	408.0	850.0
4	70.0	140.0	420.0	68.0	204.0	680.0
Total Cost (\$/day)	31640			48144		
Profit (\$/hour)						
Region \ Period	Conventional Services			Flexible Services		
	A	B	C	A	B	C
1	-280.0	-700.0	-1680.0	-408.0	-1326.0	-2890.0

2	-210.0	-420.0	-840.0	-204.0	-612.0	-1360.0
3	-140.0	-280.0	-630.0	-102.0	-408.0	-850.0
4	-70.0	-140.0	-420.0	-68.0	-204.0	-680.0
Total Profit (\$/day)	-31640			-48144		

Table 8 shows the consumer surplus results from conventional and flexible services. Period 1 in region A has the highest consumer surplus, which is 42705\$/period for conventional services and 43692 \$/period for flexible services. It is found that the consumer surplus of flexible services exceeds that of conventional services. The main reason is that with elasticity, the actual trips for flexible services exceed those for conventional services. The main reason for the difference in actual trips is the access time factor, as already discussed. The total consumer surplus of flexible services is \$162143/day while the total consumer surplus of conventional services is \$151859/day.

Table 8 Consumer Surplus

Consumer Surplus (\$/period)						
Period \ Region	Conventional Services			Flexible Services		
	A	B	C	A	B	C
1	6994.1	21060.7	42705.5	7531.7	22381.2	43692.5
2	4534.1	13218.9	26523.8	4862.4	14148.1	27936.9
3	2854.9	8151.2	16842.9	2944.8	9321.6	18240.9
4	892.5	2370.6	5709.4	1056.3	2989.8	7037.4
Total (\$/day)	151859			162143		

From previously discussed results, it is found that flexible services have a larger consumer surplus than conventional services. Flexible services also have higher operating costs than conventional services, which explain why flexible services have a larger negative profit (i.e., loss) than conventional services.

Table 9 provides welfare results of conventional and flexible services for each period and region. It is noted that the total welfare of conventional services exceeds that of flexible services. For region A, the welfare of flexible services exceeds that of conventional services. For regions B and C, conventional services produce greater welfare than flexible services. As discussed, the higher cost of flexible service is the main reason why welfare is higher for conventional services than for flexible services. The total welfare difference between conventional and flexible services is about 5.45% (120219 vs. 113999).

Table 9 Social Welfare

Welfare (\$/period)						
Period \ Region	Conventional Services			Flexible Services		
	A	B	C	A	B	C
1	5874.1	18260.7	35985.5	5899.7	17077.2	32132.5
2	3274.1	10698.9	21483.8	3638.4	10476.1	19776.9
3	1734.9	5911.2	11802.9	2128.8	6057.6	11440.9
4	472.5	1530.6	3189.4	648.3	1765.8	2957.4
Total (\$/day)	120219			113999		

## 5. Welfare Maximization with Financial Constraints

In addition to vehicle capacity constraints, financial (i.e., subsidy) constraints are considered in this section. With various subsidy inputs, the resulting variations of fares, headways and fleet sizes are explored. To consider additional financial constraints, formulations for conventional and flexible services are modified.

### 5.1. Formulations and Solution Method

#### 5.1.1. Conventional Service Formulations

The total welfare  $Y_c$  is the sum of the welfare for all time and all regions, shown in equation (29). The financial constraint is expressed in equation (30). The amount of subsidies is an input value. If zero subsidies are provided, the financial constraint simply becomes that the profit should be non-negative. The maximum allowable headway (service capacity) constraints are also applied in equation (31):

$$\text{Maximize } Y_c = \sum_{k=1}^K \sum_{i=1}^I \{Y_c^{ki}\} \quad (29)$$

subject to

$$\sum_{k=1}^K \sum_{i=1}^I \{P_c^{ki}\} + \sum_{k=1}^K \sum_{i=1}^I \{FS^{ki}\} \geq 0 \quad (30)$$

$$h_c^{ki} \leq h_{c,max}^{ki} = \frac{S_c l_c}{r^k W^k d_s^{ki} Q_c^{ki}} \quad (31)$$

#### 5.1.2. Flexible Service Formulations

Flexible service formulations that consider financial constraints are provided in equations (32~34). The maximum allowable headway constraints for flexible services in equation (34) are different from those for conventional services in equation (31):

$$\text{Maximize } Y_f = \sum_{k=1}^K \sum_{i=1}^I \{Y_f^{ki}\} \quad (32)$$

subject to

$$\sum_{k=1}^K \sum_{i=1}^I \{P_f^{ki}\} + \sum_{k=1}^K \sum_{i=1}^I \{FS^{ki}\} \geq 0 \quad (33)$$

$$h_f^{ki} \leq h_{f,max}^{ki} = \frac{S_f^{lf}}{A^k Q_f^{ki}} \quad (34)$$

### 5.1.3. Solution Method

Welfare formulations for conventional and flexible services are highly nonlinear. In addition to the nonlinear objective functions, constraints are moved to the objective function with the Lagrange multiplier. Then, the objective function becomes more complex. Since objective functions are nonlinear and variables are continuous or integer, a real coded genetic algorithm (RCGA) is chosen to solve formulations. Fares for either conventional or flexible services are continuous variables, and fleet sizes are integer variables. Headways can be obtained from the optimized fleet sizes.

### 5.2. Numerical Examples

In this numerical example, financial (subsidy) constraints are enforced in addition to maximum allowable headway constraints. Different input values for subsidy are considered through sensitivity analysis. As explained below, the sum of the total revenue and the total subsidy should be larger or equal to the total cost. If the total subsidy is zero, the total revenue minus the total cost (i.e. the profit) should be non-negative. The total subsidy is an input value; unit subsidy (\$/potential trip) is used to calculate the total amount of subsidies in this numerical analysis.

It is possible to jointly optimize vehicle sizes, numbers of zones, headways, and fleet sizes with a financial constraint. However, computation times are much longer and optimized vehicle sizes and numbers of zones are not significantly different from the ones in the financially unconstrained case. Thus, by using the optimized vehicle sizes and numbers of zones from financially unconstrained results, the complexity of financially constrained welfare formulations is reduced and converged solutions are found relatively quickly. It is also reasonable to think that route spacings of conventional services, service areas of flexible services, and vehicle sizes can

be determined in an earlier planning level. The service providers (operators) may then want to re-optimize service frequencies and fares based on the subsidy.

Thus, headways, fleet sizes, and fares are optimized here with the various subsidy inputs. The value of route spacings for conventional services, service areas for flexible services, and vehicle sizes are adapted from the solution of the financially unconstrained optimization model. The main focus of this numerical analysis is on exploring how optimized fares are changed along other decision variables with different financial constraints (i.e., subsidy). Results of conventional services will be discussed first, and then results of flexible services will be discussed.

### 5.2.1. Results of Conventional Services

Subsidy inputs are applied from zero to 1.2\$/potential trip with 0.2\$/potential trip increment. Table 10 provides detailed results for conventional services with various subsidy inputs.

For conventional services, seven sensitivity cases are considered, as shown in Figure 4. The amount of subsidy increases linearly. The total number of potential trips for the system with given inputs is 33825. Thus, when the unit subsidy is 1.0\$/potential trip, the total subsidy is \$33825/day.

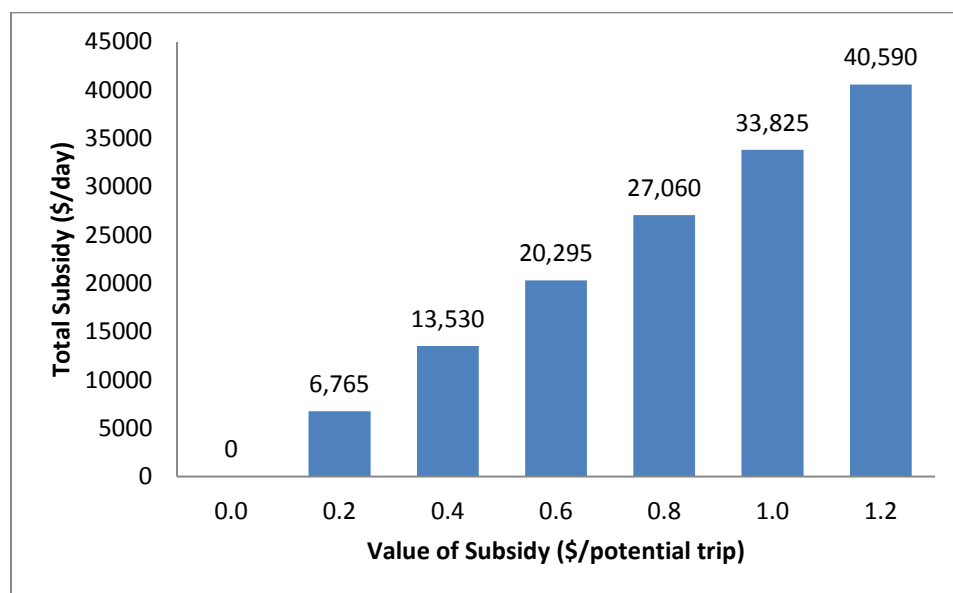


Figure 4 Total Amount of Subsidy

Figure 5 provides optimized fares of conventional services from various subsidy inputs. For the zero subsidy case, the conventional service fare is 1.3\$/actual trip. As the subsidies increases, the optimized fare decreases quite linearly. When the unit subsidy is about 1.0\$/potential trip, the fare becomes zero, which means the total revenue is zero, and all the costs of bus operations are covered by the subsidy.

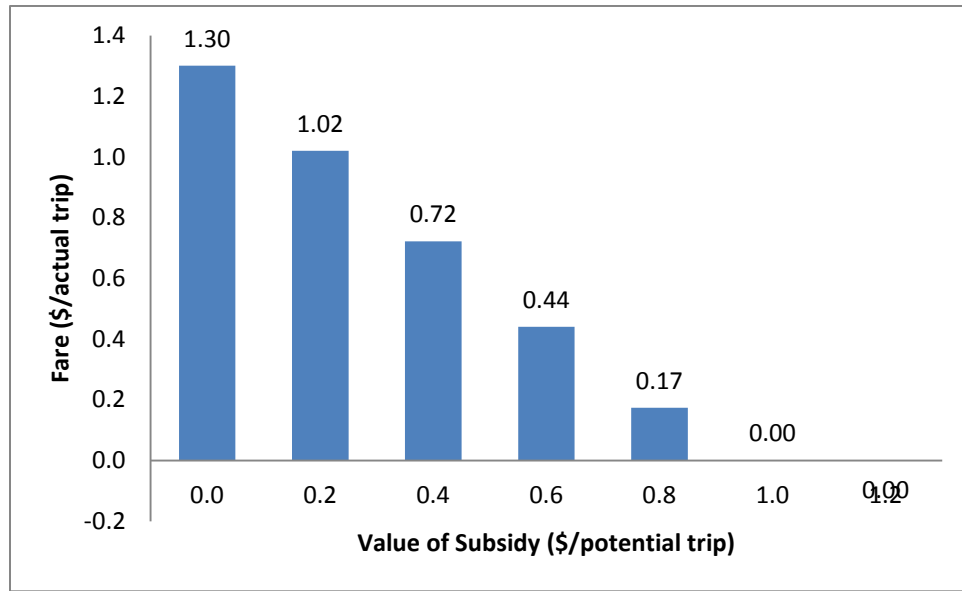


Figure 5 Fares for Conventional Services with Subsidies

Figure 6 provides profit results from various financial constraints (i.e., subsidy inputs). With the subsidy provision, the revenue decreases since the optimized fare decreases. Thus, the profit also decreases (as expected) because the revenue decreases. In the formulation shown in equation (30) the sum of the profit and subsidy can be either zero or positive. For the zero subsidy case result, the profit is positive, which means the optimized fare could have been slightly reduced to use this available budget.

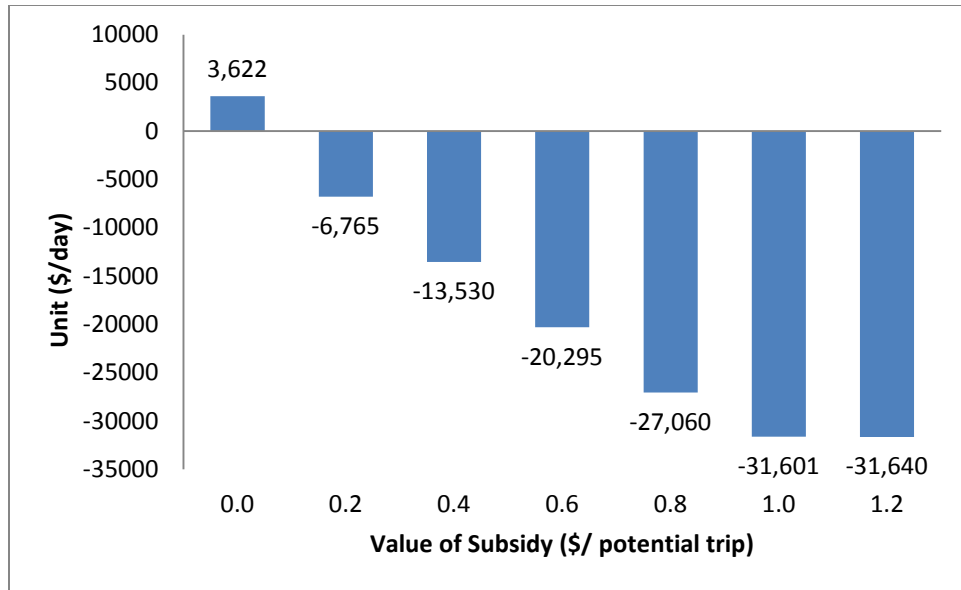


Figure 6 Profits of Conventional Services with Subsidies

Figure 7 shows total costs of conventional services for various subsidy inputs. As expected, the cost of the zero subsidy case is lower than other subsidy cases. From the 0.2 \$/potential trip to 1.2\$/potential trip, total costs are identical, which means, their resulting fleet sizes do not change over different subsidy inputs. It explains why fleet sizes and headways do not change significantly in conventional services with financial constraints while the fare decreases as the subsidy increases.

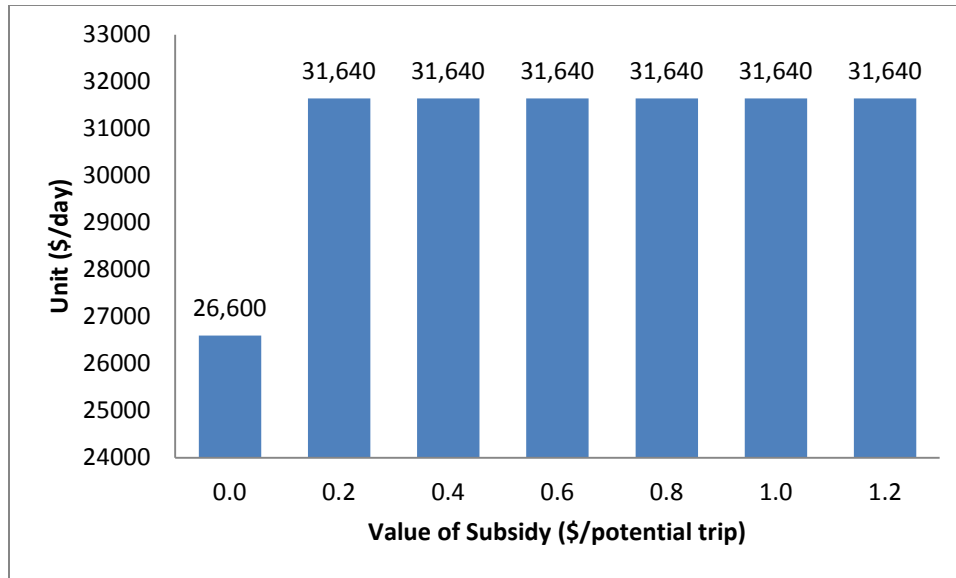


Figure 7 Total Costs of Conventional Services with Subsidies

The total consumer surplus in the zero subsidy case is \$114699/day, as shown in Figure 8. Consumer surplus results of other subsidy cases show that the consumer surplus increases until the unit subsidy is 1.0\$/potential trip. After that, the consumer surplus does not change significantly. When the unit subsidy is 1.2\$/potential trip, the consumer surplus for the conventional services is 151859\$/day. The consumer surplus difference between the subsidy inputs 1.0 and 1.2 \$/potential trip is 39\$/day, which is tiny if 39\$ is divided by the total actual trips served per day. Thus, it can be confirmed that the consumer surplus does not increase although a unit subsidy above 1.0\$/potential trip is provided.

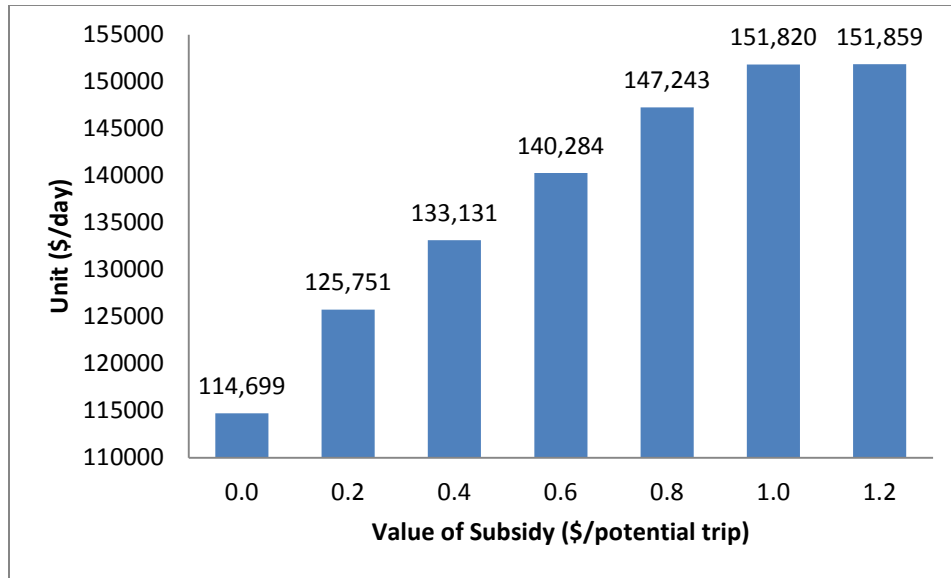


Figure 8 Consumer Surplus of Conventional Services with Subsidies

Figure 9 shows the social welfare results for conventional services. The welfare of the zero subsidy case is 118321\$/day, while the maximum system welfare is found as 120219\$/day from the unit subsidy of 1.0\$/potential trip or more. As expected, when the total cost is fully covered by subsidies, the system welfare becomes identical to the one without financial constraints (discussed in the previous section). There is no unusual observation among comprehensive sets of sensitivity analyses. Thus, numerical results confirm that RCGA used here finds good and consistent solutions although it does not guarantee the global optimality of solutions.

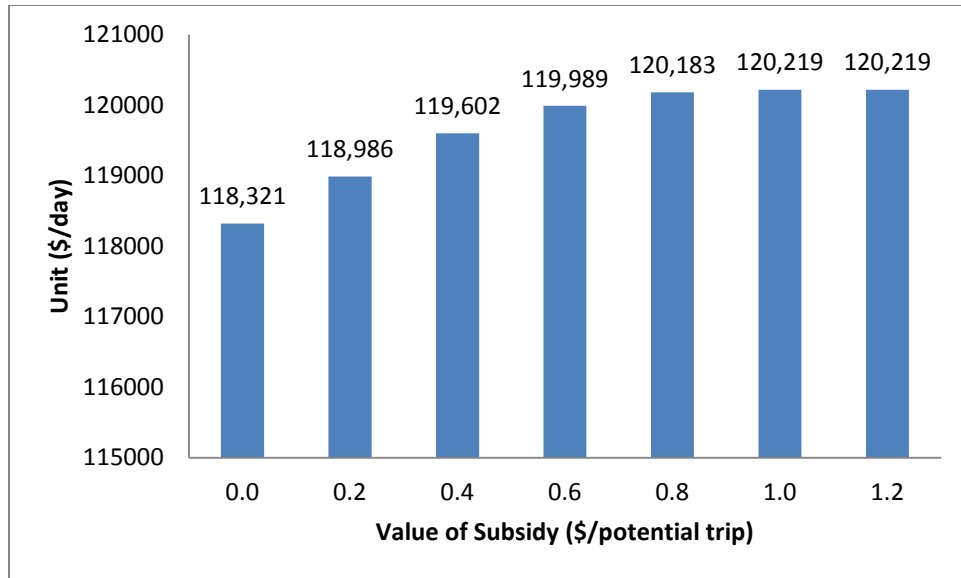


Figure 9 Total System Welfares of Conventional Services with Subsidies

Table 10 summarizes all results of conventional services with various subsidy input values. One further finding worth noting is that actual trips increase as the subsidy increases. For instance, the zero subsidy case serves about 68.7% of the total potential demand, but the fully subsidized case serves about 79.2% of the total potential demand.

Table 10 Results of Conventional Services with Financial Constraints

Unit Subsidy (\$/trip)	0.0	0.2	0.4	0.6	0.8	1.0	1.2
Fare	1.30	1.02	0.72	0.44	0.17	0.00	0.00
Revenue	30221.6	24875.0	18110.4	11345.0	4580.0	38.8	0.0
Cost	26600.0	31640.0	31640.0	31640.0	31640.0	31640.0	31640.0
Profit	3621.6	-6765.0	-13529.6	-20295.0	-27060.0	-31601.2	-31640.0
Subsidy	0.0	6765.0	13530.0	20295.0	27060.0	33825.0	40590.0
Profit + Subsidy	3621.6	0.0	0.4	0.0	0.0	2223.8	8950.0
Consumer Surplus	114699.0	125751.3	133131.3	140283.9	147242.9	151819.8	151858.6
Welfare	118321.0	118986.3	119601.7	119988.9	120183.0	120218.6	120218.6
Total Actual Trips	23247	24381	25087	25754	26386	26793	26797
Total Actual Trips / Total Potential Trips	68.7%	72.1%	74.2%	76.1%	78.0%	79.2%	79.2%

### 5.2.2. Results of Flexible Services

Figures 10~15 provide results for flexible services with financially constrained cases (i.e., sensitivity analyses of subsidies with respect to welfares). Table 11 also provides details on these results.

Figure 10 shows optimized fares for flexible services with different subsidy inputs. In the zero subsidy case, the optimized fare is 1.91\$/actual trip, which exceeds the optimized fare (1.30\$/actual trip, Table 5) of conventional services with the zero subsidy case. The higher flexible service operating cost results in the higher flexible service fare. The optimized fares decrease as the subsidy increases. When the unit subsidy is 1.4\$/potential trip, the optimized fare for flexible services is close to zero (three cents per actual trip). When the unit subsidy is 1.4\$/potential trip, the total subsidy is 47355\$/day, as shown in Figure 11. For conventional services (shown in Figure 5), the optimized fare becomes zero when the subsidy reaches 1.0\$/potential trip. Thus, it is found that flexible services require larger subsidies than conventional services to fully cover the operating cost.

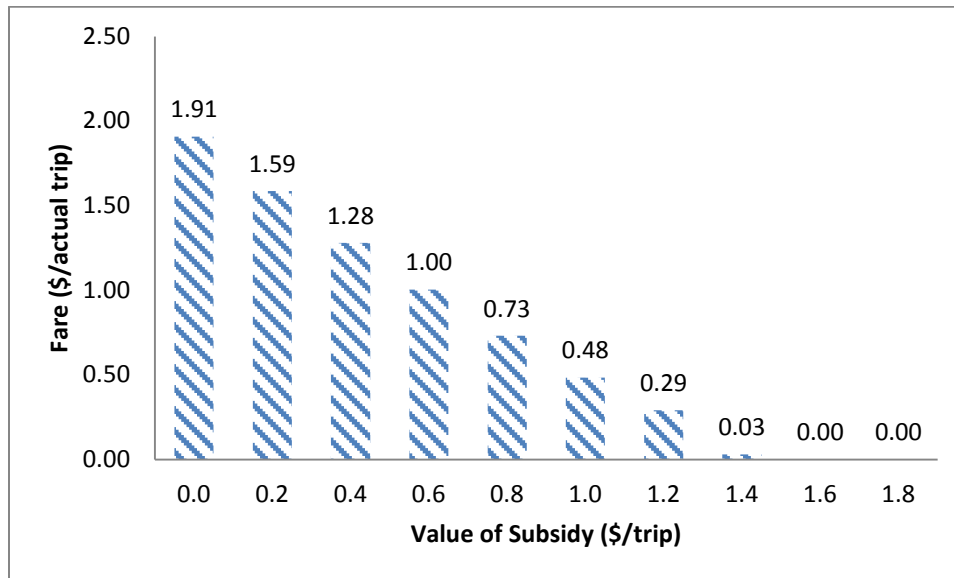


Figure 10 Fares for Flexible Services with Subsidies

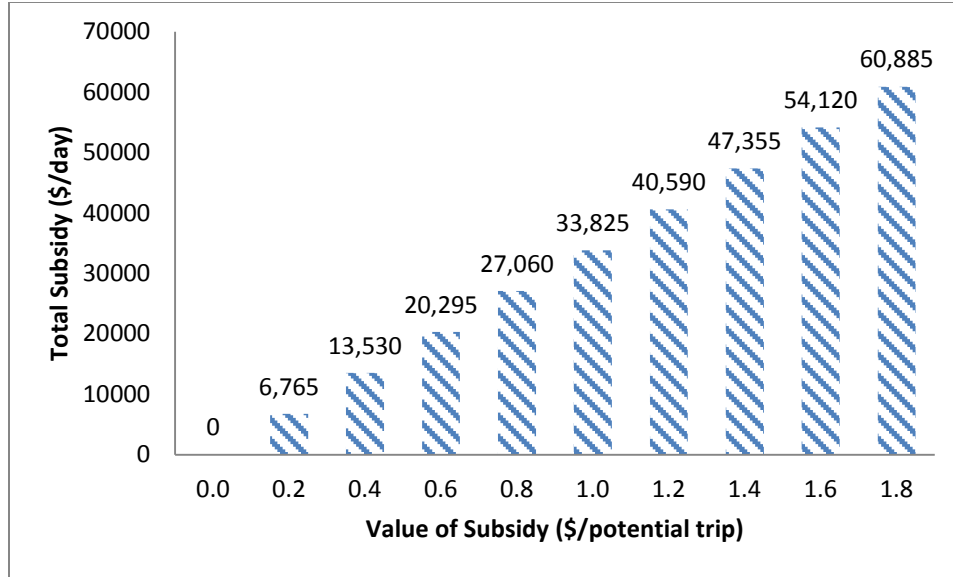


Figure 11 Total Amount of Subsidy

Figure 12 provides results of the profit. For the zero subsidy case, flexible services have zero profit, as expected; this means the operating cost is exactly equal to the revenue. The profit decreases as the subsidy increases because the optimized fare decreases.

The cost of flexible service operation increases with the provision of subsidies since the financial subsidy allows providing more service frequencies. As shown in Figure 13, the total operating cost increases with the larger subsidy. As the subsidy increases, the optimized fare decreases so that the revenue also decreases. When the operating cost is fully subsidized, the optimized fare is supposed to be zero so that the collected revenue equals to zero as well. Since the profit is the revenue minus the cost, the absolute value of the profit will be equal to the absolute value of the profit if the revenue is zero. The absolute value of the minimum profit in Figure 12 and the absolute value of the maximum cost in Figure 13 are identical (i.e., unit subsidy of \$1.8/actual trip). Thus, this finding explains why the optimized fare and revenue are zero for the fully subsidized case. This finding also confirms that the constraint in equation (33) is binding.

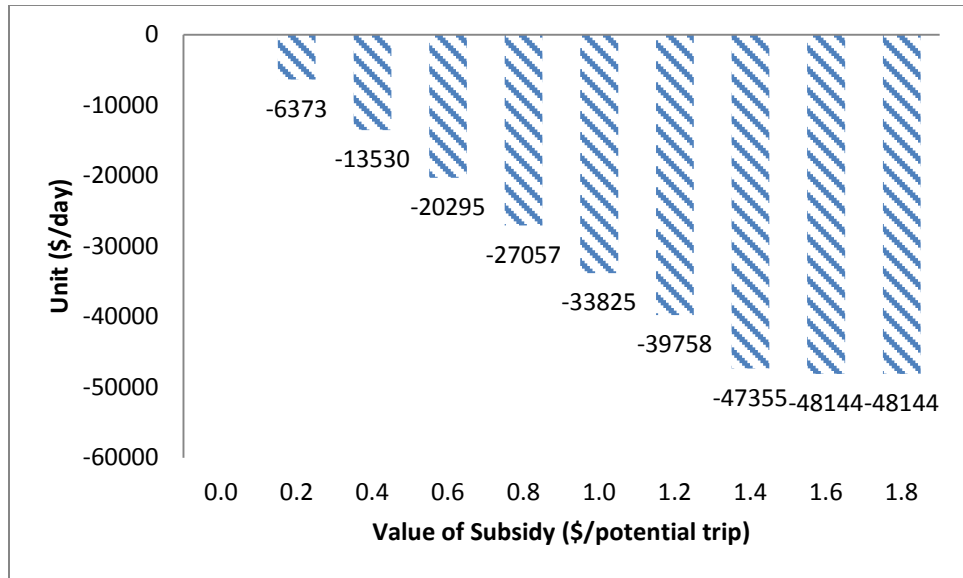


Figure 12 Profits of Flexible Services with Subsidies

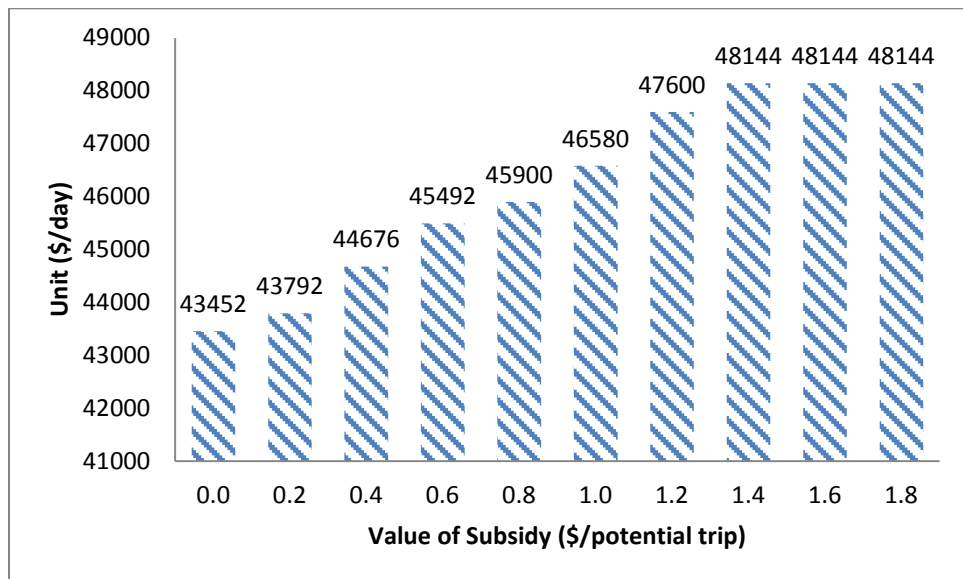


Figure 13 Costs of Flexible Services with Subsidies

The consumer surplus with the zero subsidy case is \$109,693/day, as shown in Figure 14. The higher subsidies result in the reduced fare and increase in actual trips. Therefore, the consumer surplus increases as the subsidy increases. The maximum consumer surplus is \$162,143/day when the unit subsidy is \$1.6/potential trip or higher.

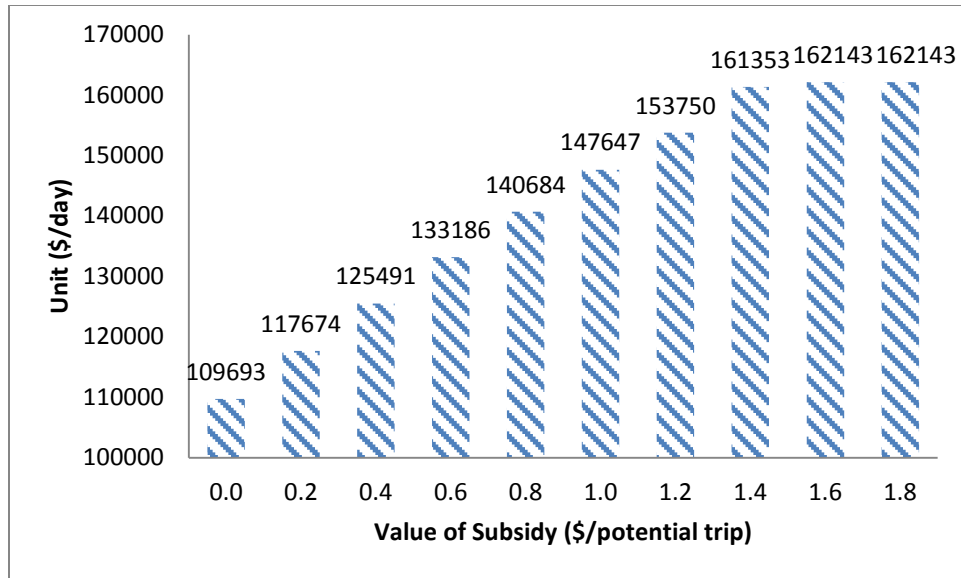


Figure 14 Consumer Surplus of Flexible Services with Subsidies

Figure 15 provides the system welfare results for flexible services. The welfare in the zero subsidy case is 109693\$/day, which is identical to the consumer surplus because the profit of the zero subsidy case is zero. For flexible services, the zero subsidy case is the break-even case. The system welfare of flexible services converges to \$113,999/day without exhausting the available subsidies.

Table 11 shows detailed results for flexible services. In Table 11 there is a row with “Profit + Subsidy”. When the unit subsidy is \$1.8/potential trip, the sum of the profit and the subsidy is positive, which means some budget is still available but unused. In the formulation, the sum of profit and subsidy is larger or equal to the cost. Therefore, the system does not have to use the entire budget. With this leftover amount of subsidies, the system welfare is not decreasing after reaching its maximum. Therefore, results confirm that unit subsidies beyond about \$1.2/potential trip yield no additional social benefits.

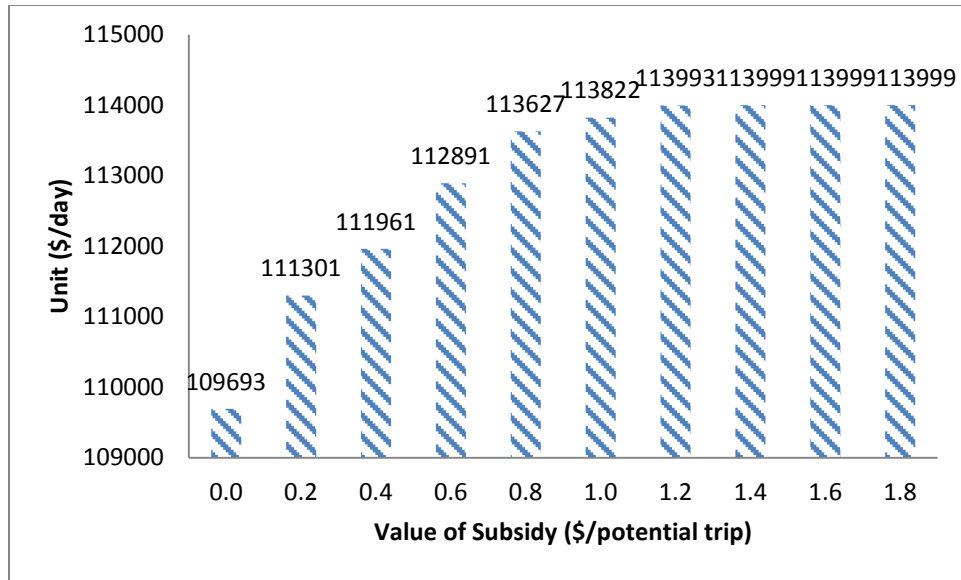


Figure 15 Total System Welfares of Flexible Services with Subsidies

In the zero subsidy case, 67.7% of the total potential demand yields actual trips. However, when the operating cost is fully subsidized, about 81.9% of the potential demand is served.

Table 11 Results of Flexible Services with Financial Constraints

Unit Subsidy (\$/trip)	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8
Fare	1.91	1.59	1.28	1.00	0.73	0.48	0.29	0.03	0.00	0.00
Revenue	43452.0	37419.1	31146.0	25197.0	18842.8	12755.2	7842.3	789.0	0.0	0.2
Cost	43452.0	43792.0	44676.0	45492.0	45900.0	46580.0	47600.0	48144.0	48144.0	48144.0
Profit	0.0	-6372.9	-13530.0	-20295.0	-27057.2	-33824.8	-39757.7	-47355.0	-48144.0	-48143.8
Subsidy	0.0	6765.0	13530.0	20295.0	27060.0	33825.0	40590.0	47355.0	54120.0	60885.0
Profit + Subsidy	0.0	392.1	0.0	0.0	2.8	0.2	832.3	0.0	5976.0	12741.2
Consumer Surplus	109692.6	117674.1	125490.7	133186.4	140684.4	147646.5	153750.4	161353.5	162143.5	162143.3
Welfare	109692.6	111301.2	111960.7	112891.4	113627.2	113821.7	113992.6	113998.5	113999.5	113999.5
Total Actual Trips	22774.8	23597.9	24359.6	25102.6	25803.4	26434.1	26977.1	27635.3	27702.9	27702.9
Total Actual Trips / Total Potential Trips	67.3%	69.8%	72.0%	74.2%	76.3%	78.1%	79.8%	81.7%	81.9%	81.9%

## 6. Conclusions

In this paper, conventional and flexible services are formulated with the demand elasticity. The actual ridership is formulated as a linear function using elastic factors of the fare,

in-vehicle time, waiting time, and access time. The welfare, which is sum of the consumer surplus and the producer surplus, is also formulated for multiple regions as well as multiple time periods for conventional and flexible services. Two constrained optimization models are analyzed. They have: 1) an objective of the maximum system welfare with the service capacity (maximum headway) constraints, and 2) an objective of the maximum system welfare with the service capacity and financial constraints, for both conventional and flexible services.

This study extends the maximization of welfare for conventional and flexible public transportation services to multiple regions and multiple periods. Objective functions (i.e., welfare functions) are highly nonlinear and decision variables include continuous and integer variables. Such nonlinear mixed integer formulations are known to be NP-hard problems, and have no proven method for finding their exact optimum solution. Commercial optimization programs such as GAMS or LINGO are excluded because they only guarantee a local solution. Thus, a genetic algorithm, which is a heuristic search technique, is chosen to solve the formulations developed here.

In numerical examples, the fares, route spacings for conventional services, service areas for flexible services, headways and fleet sizes are optimized. The numerical examples show that the welfare of conventional services exceeds those of flexible services, with given input values. Numerical examples also explore the sensitivity of vehicle sizes and the sensitivity of the subsidies with respect to the social welfare for conventional and flexible services. For both conventional and flexible services, the actual trips increase as the subsidies increase.

For conventional services, the problem of one local region with multiple periods has been solved in previous studies. For flexible services, a problem with one local region and one period has also been solved in the literature. These were all solved with analytic optimization (and with approximations).

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## References

1. Aldaihani, M. M., Quadrifoglio, L., Dessouky, M. M., Hall, R., 2004. Network design for grid hybrid transit service. *Transportation Research Part A*, 38, 511-530.
2. Chandra, S. and Quadrifoglio, L., 2013a. A model for estimating the optimal cycle length of demand responsive feeder transit services. *Transportation Research Part B* 51, 1-16.
3. Chandra, S. and Quadrifoglio, L., 2013b. A new street connectivity indicator to predict performance for feeder transit services. *Transportation Research Part C* 30, 67-80.
4. Chandra, S., Shen, C.-W. Quadrifoglio, L., 2011. A simulation study to derive the optimal cycle length for feeder transit services. *International Journal of Information Systems in the Service Sector* 3(3), 27-47.
5. Chang, S. K., Schonfeld, P., 1991b. Integration of fixed- and flexible-route bus systems. *Transportation Research Record* 1308, 51-57.
6. Chang, S. K., Schonfeld, P., 1991c. Multiple period optimization of bus transit systems. *Transportation Research Part B* 25(6), 453-478.
7. Chang, S. K., Schonfeld, P., 1993, Welfare maximization with financial constraints for bus transit systems. *Transportation Research Record* 1395, 48-57.
8. Chang, S. K., Schonfeld, P., 1993. Welfare maximization with financial constraints for bus transit systems. *Transportation Research Record* 1395, 48-57.
9. Chang, S.K., 1990. Analytic optimization of bus systems in heterogeneous environments. PhD Dissertation, University of Maryland, College Park, MD, United States.
10. Chang, S.K., Schonfeld, P., 1991a. Optimization models for comparing conventional and subscription bus feeder services. *Transportation Science* 25(4), 281-298.
11. Chien, S. I.-J., Spasovic, L. N., 2002. Optimization of grid bus transit systems with elastic demand. *Journal of Advanced Transportation* 36(1), 63-91.
12. Daganzo, C. F., 1984. The length of tour in zones of different shapes. *Transportation Research Part B* 18(2), 135-145.
13. Deb, K., 2000. An efficient constraint handling method for genetic algorithms. *Computer Methods in Applied Mechanics and Engineering* 186, 311-338.
14. Deep, K., Singh, K. P., Kansal, M. L., Mohan, C., 2009. A real coded genetic algorithm for solving integer and mixed integer optimization problems. *Applied Mathematics and Computation* 212(2), 505-518.

15. Diana, M. Quadrifoglio, L., Pronello, C., 2009. A methodology for comparing distances traveled by performance-equivalent fixed-route and demand responsive transit services. *Transportation Planning and Technology* 32(4), 377-399.
16. Diana, M., Quadrifoglio, L., Pronello, C., 2007. Emissions of demand responsive services as an alternative to conventional transit systems. *Transportation Research Part D* 12, 183-188.
17. Fu, L., Ishkhanov, G., 2004. Fleet size and mix optimization for paratransit services. *Transportation Research Record* 1884, 39-46.
18. Horn, M. E. T., 2002. Multi-modal and demand-responsive passenger transport systems: a modeling framework with embedded control systems. *Transportation Research Part A* 36(2), 167-188.
19. Imam, M. O., 1998. Optimal design of public bus service with demand equilibrium. *Journal of Transportation Engineering* 124(5), 431-436.
20. Kim, M., Schonfeld, P., 2012. Conventional, flexible and variable-type services. *Journal of Transportation Engineering* 138(3), 263\*-273.
21. Kim, M., Schonfeld, P., 2013. Integrating bus services with mixed fleets. *Transportation Research Part B*, accepted for publication, Apr. 2013.
22. Kocur, G., Hendrickson, C., 1982. Design of local bus service with demand equilibration. *Transportation Science* 16(2), 149-170.
23. Li, X. and Quadrifoglio, L., 2009. Optimal zone design for feeder transit services. *Transportation Research Record: Journal of the Transportation Research Board*, No. 2111, Transportation Research Board of the National Academies, Washington, D. C., 100-108.
24. Li, X. and Quadrifoglio, L., 2010. Feeder transit services: choosing between fixed and demand responsive policy. *Transportation Research Part C* 18, 770-780.
25. Li, X. and Quadrifoglio, L., 2011. 2-vehicle zone optimal design for feeder transit services. *Public Transportation* 3, 89-104.
26. Quadrifoglio, L. and Li, X., 2009. A methodology to derive the critical demand density for designing and operating feeder transit services. *Transportation Research Part B* 43, 922-935.

27. Quadrifoglio, L., Dessouky, M. M., 2007. Sensitivity analyses over the service area for MAST services. *Computer-Aided Systems in Public Transport, Lecture Notes in Economics and Mathematical Systems* 600, 419-432.
28. Quadrifoglio, L., Dessouky, M. M., Ordonez, F., 2008. A simulation study of demand responsive transit system design. *Transportation Research Part A* 42, 718-737.
29. Quadrifoglio, L., Dessouky, M. M., Palmer, K., 2007. An insertion heuristic for scheduling mobility allowance shuttle transit (MAST) services. *Journal of Scheduling* 10, 25-40.
30. Shen, C.-W., Quadrifoglio, L., 2012. Evaluation of zoning design with transfers for paratransit services. *Transportation Research Record: Journal of the Transportation Research Board*, No. 2277, Transportation Research Board of the National Academies, Washington, D. C., 82-89.
31. Stein, D. M., 1978. An asymptotic probabilistic analysis of a routing problem. *Mathematics of Operations Research* 3(2), 89-101.
32. Tsai, F.-M., Chien, S., Wei, C.-H., 2013. Joint optimization of temporal headway and differential fare for transit systems considering heterogeneous demand elasticity. *Journal of Transportation Engineering* 139(1), 30-39.
33. Zhou, Y., Kim, H.S., Schonfeld, P., Kim, E., 2008. Subsidies and welfare maximization tradeoffs in bus transit systems. *The Annals of Regional Science* 42, 643-660.