INTEGRATION OF FIXED AND FLEXIBLE ROUTE PUBLIC TRANSPORTATION SYSTEM

PHASE II

The Pennsylvania State University ❖ University of Maryland
University of Virginia ❖ Virginia Polytechnic Institute and State University
West Virginia University

The Pennsylvania State University ❖ The Thomas D. Larson Pennsylvania Transportation Institute
Transportation Research Building ❖ University Park, Pennsylvania 16802-4710
Phone: 814-865-1891 ❖ Fax: 814-863-3707
www.pti.psu.edu/mautc

INTEGRATION OF FIXED AND FLEXIBLE ROUTE PUBLIC TRANSPORTATION SYSTEMS

PHASE II

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By

Dr. Paul M. Schonfeld, Principal Investigator
Myungseob (Edward) Kim, Graduate Research Assistant

University of Maryland
College Park, MD

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16. Abstract

Conventional bus service (with fixed routes and schedules) has lower average cost than flexible bus service (with demand-responsive routes) at high demand densities. At low demand densities flexible bus service has lower average costs and provides user-friendly door-to-door service. Bus size is interrelated with service type since large buses have lower average cost per passenger than small buses at high demand densities, and vice versa. The service type and vehicle size decisions are jointly optimized here for a bus transit system connecting a major terminal to local regions. Bus sizes, conventional route spacing and flexible service area are the decision variables in a proposed algorithm which optimizes a multi-dimensional nonlinear integer optimization problem. Numerical analysis shows how the proposed Variable Mode & Multiple Fleets Operation can reduce total cost compared to a Single Fleet & Single Mode operation. The sensitivity of results to important parameters is explored.

Additional details on the problem and its formulation, the solution methods employed, and the results obtained are provided in the following report. This report has been submitted for publication by a transportation journal under the title "Mixed-Fleet Variable-Type Bus Operation" and is currently under review.

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Summary

Conventional bus service (with fixed routes and schedules) has lower average cost than flexible bus service (with demand-responsive routes) at high demand densities. At low demand densities flexible bus service has lower average costs and provides user-friendly door-to-door service. Bus size is interrelated with service type since large buses have lower average cost per passenger than small buses at high demand densities, and vice versa. The service type and vehicle size decisions are jointly optimized here for a bus transit system connecting a major terminal to local regions. Bus sizes, conventional route spacing and flexible service area are the decision variables in a proposed algorithm which optimizes a multi-dimensional nonlinear integer optimization problem. Numerical analysis shows how the proposed Variable Mode & Multiple Fleets Operation can reduce total cost compared to a Single Fleet & Single Mode operation. The sensitivity of results to important parameters is explored.

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INTRODUCTION

In the literature on bus transit systems several studies focus on service type decisions (such as conventional service and/or subscription service) or fleet assignment problems (e.g. multiple fleet assignment for conventional and flexible bus operation) (1~4). Chang and Schonfeld developed optimization models for conventional and subscription bus services (2, 6). They confirmed that conventional bus (with fixed routes and fixed schedule) is preferable to subscription bus (which has demand responsive routes and flexible schedule) when demand density is high, and vice versa. They also studied optimal bus service dimensions (4), multiple period optimization of public bus systems (7), and temporal integration of fixed and flexible bus systems (3). Zhou et al (5) recently maximized welfare under financial constraints for various bus transit service types and determined conditions under which subsidies may be justifiable. The above studies provide various approaches for improving the performance of bus transit systems. However, most of them consider only one local service area.

Public bus services with flexible routes and schedules, have attracted considerable interest from researchers, especially in recent years (8~21). Lee et al (1) and Fu and Ishkhanov (10) analyzed the assignment of buses with dissimilar sizes (i.e. “mixed fleets”) to public transit operations. Lee et al (1) studied mixed bus fleet operations in conventional urban public transit systems. Fu and Ishkhanov (10) studied mixed fleet bus operation for paratransit services. Mixed fleets can reduce total system cost compared to single fleets when demand densities differ considerably over time or space because vehicles of different sizes may be matched to the operations for which they are most suited.

Kim and Schonfeld (22, 23) considered bus size and service type jointly, confirming that variable-type service can reduce system cost by changing the service type (or “mode”) as demand density changes. However, their study optimized decision variables and minimized total cost between one terminal and only one local region. The potential benefits of using variable service types (or “modes”) and multiple fleets should theoretically increase when multiple regions are considered, due to the increased variability of demand densities. To explore these potential benefits we analyze in this paper the concept of variable-mode bus operation with multiple fleets (VMBOMF) in multiple local regions. To provide efficient service, we optimize decision variables for bus sizes (i.e. large bus size and small bus size) and bus service (i.e. route spacing in local region for conventional bus, service area in local region for flexible bus). The remaining contents are as follows. In the next section we provide the system description, assumptions, and cost formulations. Then, we present an optimization algorithm for solving a nonlinear integer multi-dimensional minimization problem with four decision variables. Using this methodology, we provide a numerical case and sensitivity analysis.

BUS SERVICE DESCRIPTION AND ITS COST

The analyzed bus system provides service from a major terminal (or CBD) to multiple local regions. In Figure 1, a public bus system serves local regions connected to a central terminal. For local regions, either conventional bus or flexible bus can be provided. To analyze this local bus system, we specify some simplifying assumptions before explaining the cost functions and their optimization.

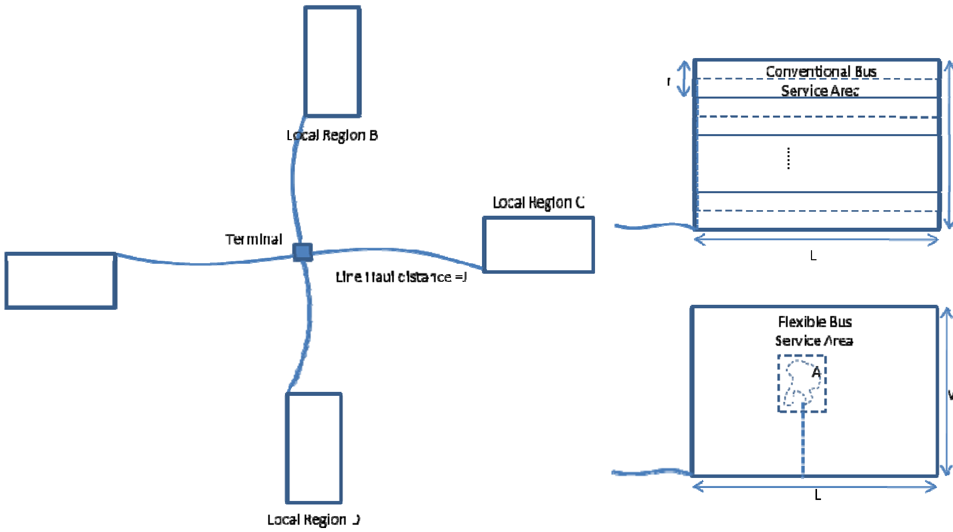

FIGURE 1 Local Service Regions and Bus Services

Assumptions

A previous study (22, 23) addressed assumptions for analyzing a one route service (i.e. connecting one terminal to one local region). Here, we modify some assumptions and notation to analyze a more general system with multiple local regions as well as multiple periods. Henceforth, superscripts k and i correspond to route and time period, respectively, while subscripts c and f represent conventional and flexible service, respectively. Definitions, units and default values of variables are presented in Table 1.

TABLE 1 Notation

Variable	Definition	Baseline Value
a	hourly fixed cost coefficient for operating bus ($/bus hr)	30.0
a_c	fixed cost coefficient for bus ownership (capital cost) ($/bus day)	100.0
A	service zone area(mile^2)=LW/N'	-
b	hourly variable cost coefficient for bus operation ($/seat hr)	0.2
b_c	variable cost coefficient for owning bus (capital cost) ($/day)	0.5
d	bus stop spacing (miles)	0.2
	distance of one flexible bus tour in local region k and period i (miles)	-
	equivalent line haul distance for flexible bus on route k $(=(L+W)/z+2J^k/y)$, (miles)	-
	equivalent average bus round trip distance for conventional bus on route k (= $2J^k/y+W/z+2L$),(miles)	-
	directional demand split factor	1.0
	fleet size for route k and period i (buses) subscript corresponds to (c = conventional, f=flexible)	-
	headway for conventional bus; for route k and period i (hours/bus)	-
	headway for flexible bus; for route k period i (hours/bus)	-
	maximum allowable headway for route k and period i subscript: c = conventional, f=flexible	-
	minimum cost headway for route k and period i subscript: c = conventional, f=flexible	-
	Optimized headway for route k and period i subscript: c = conventional, f=flexible	-
k ,i	Index(k: route, i : period)	-
J^k	line haul distance of route k (miles)	-
	load factor for conventional and flexible bus (passengers/seat)	1.0

L, W	length and width of service area (miles)	5.0, 4.0
M^k	equivalent average trip distance for route k (=$J^k/y_c+W/2z_c+L/2$)	-
n	number of passengers in one flexible bus tour	-
N, N'	number of zones in local region for conventional and flexible bus	-
Q^{ki}	round trip demand density (trips/mile^2/hr)	-
Q_t^{ki}	threshold demand density between conventional and flexible service (trips/mile^2/hr)	-
r	route spacing for conventional bus (miles)	-
R_c^{ki}	round trip time of conventional bus for route k and period i (hours)	-
R_f^{ki}	round trip time of flexible bus for route k and period i (hours)	-
S_c , S_f	sizes for conventional and flexible bus (seats/bus)	-
S_c^{ki}, S_f^{ki}	conventional and flexible bus sizes for route k and period i (seats/bus)	-
S_c^{upper}, S_f^{upper}	upper bound of bus size arrays subscript: c = conventional, f=flexible	-
S_c^{lower}, S_f^{lower}	lower bound of bus size arrays subscript: c = conventional, f=flexible	-
SC_c^{ki}, SC_f^{ki}	service cost for route k and period i subscript: c = conventional, f=flexible	-
$SC_{co}^{ki}, SC_{fo}^{ki}$	operator cost for route k and period i subscript: c = conventional, f=flexible	-
$SC_{cu}^{ki}, SC_{fu}^{ki}$	user cost for route k and period i subscript: c = conventional, f=flexible	-
$SC_{cv}^{ki}, SC_{fv}^{ki}$	user in-vehicle cost for route k and period i subscript: c = conventional, f=flexible	-
$SC_{cw}^{ki}, SC_{fw}^{ki}$	user waiting cost for route k and period i subscript: c = conventional, f=flexible	-
SC_{cx}^{ki}	user access cost for route k and period i (conventional only)	-
TSC_c , TSC_f	total service cost over all routes and periods subscript: c = conventional, f=flexible	-
t^{ki}	time duration for route k and period i	-
u	average number of passengers per stop for flexible bus	1.2
V_c^i	local service speed for conventional bus in period i (miles/hr)	20 at i =1 30 at i = 2,3,4
V_f^i	local service speed for flexible bus in period i (miles/hr)	18 at i =1 25 at i = 2,3,4
V_x	average passenger access speed (mile/hr)	2.5
v_v, v_w, v_x	value of in-vehicle time, wait time and access time ($/passenger hr)	5, 12, 12
y	express speed/local speed ratio for conventional bus	conventional bus = 1.8 flexible bus = 2.0
z	non-stop ratio = local non-stop speed/local speed; same values as y	-
$Ø$	constant in the flexible bus tour equation (Daganzo, 25) for flexible bus	1.15
*	superscript indicating optimal value subscript: c = conventional, f=flexible	-

For both conventional and flexible buses

All regional areas are assumed to be similar (i.e. rectangular shaped, with length L and width W). However, local regions may have different line haul distances J^k (miles, in route k) connecting a terminal and local regions' nearest corner. .

a) The demand is fixed with respect to service quality and price.
b) The demand is uniformly distributed over space within the local regions and over time within each specified period.
c) The bus size (S_c *for conventional,* S_f *for flexible)* is uniform throughout a system.

d) The estimated average waiting time of passengers is equal to half the headway (h_c *for conventional,* h_f *for flexible*).
e) Bus layover time is negligible.
f) Within the local region, the average speed (V_c^i *for conventional bus,* V_f^i *for flexible bus*) includes stopping times.
g) External costs are assumed to be negligible.

For conventional bus only

a) The local region is divided into N parallel zones with a width r=*W/N* for conventional bus, as shown in Figure 1. Local routes branch from the line haul route segment to run along the middle of each zone, at a route spacing *r=W/N.*
b) Q^{ki} trips/mile^2/hour, entirely channeled to (or through) the single terminal, are uniformly distributed over the service area.
c) In each round trip, as shown in Figure 1, buses travel from the terminal a line haul distance J^k at non-stop speed yV_c^i to a corner of the local regions, then travel an average of *W/2* miles at local non-stop speed zV_c^i from the corner to the assigned zone, then run a local route of length *L* at local speed V_c^i along the central axis of the zone while stopping for passengers every *d* miles, and then reverse the above process in returning to the terminal.

For flexible bus only

a) The local region is divided into *N'* equal zones, each having an optimizable zone area *A=LW/N'*. The zones should be "fairly compact and fairly convex".
b) Buses travel from the terminal line haul distance J^k at non-stop speed yV_f^i and an average distance *(L+W)/2* miles at local non-stop speed zV_f^i to the center of each zone. They collect (or distribute) passengers at their door steps through an efficiently routed tour of *n* stops and length D_c^{ki} at local speed V_f^i. D_c^{ki} is approximated by (24), in which $D_c^{ki} = \emptyset\sqrt{\text{nA}}$, and $\emptyset = 1.15$ for the rectilinear space assumed here (25). The values of *n* and D_c^{ki} are endogenously determined. To return to their starting point the buses retrace an average of *(L+W)/2* miles at zV_f^i miles per hour and J^k miles at yV_f^i miles per hour.
c) Buses operate on preset schedules with flexible routing designed to minimize each tour distance D_c^{ki} .
d) Tour departure headways are equal for all zones in the local region and uniform within each period.

Service Cost

In terms of service cost for conventional and flexible bus, we consider bus operating cost, user in-vehicle cost, user waiting cost, and user access cost. Since flexible bus provides door-to-door service, its user access cost is negligible.

Conventional Bus Cost

Conventional bus cost for route k and period i, SC_c^{ki}, includes operating cost, user in-vehicle cost, user waiting cost, and user access cost, as shown in Equation (1).

$$SC_c^{ki} = \frac{D^k W\left(a+bS_c\right)}{r \quad V_c^i h_c^{ki}} + \frac{v_v LWQ^{ki} M^k}{V_c^i} + \frac{v_w LWQ^{ki} h_c^{ki}}{2} + \frac{v_x LWQ^{ki}(r \quad +d)}{4V_x} \tag{1}$$

Detailed derivations and explanations for equation (1) are placed in Appendix 1. Total conventional bus cost over all routes and all periods, TSC_c , can be expressed as:

$$\begin{aligned} TSC_c &= \sum_{k=1}^{K} \sum_{i=1}^{I} \{SC_c^{ki} \times t^{ki}\} \\ &= \sum_{k=1}^{K} \sum_{i=1}^{I} \left\{ \left(\frac{D^k W\left(a+bS_c\right)}{r \quad V_c^i h_c^{ki}} + \frac{v_v LWQ^{ki} M^k}{V_c^i} + \frac{v_w LWQ^{ki} h_c^{ki}}{2} + \frac{v_x LWQ^{ki}(r \quad +d)}{4V_x} \right) \times t^{ki} \right\} \end{aligned} \tag{2}$$

Flexible Bus Cost

Similarly, flexible bus cost consists of bus operating cost, user in-vehicle cost and user waiting cost. Service cost for route k, in period i, SC_f^{ki}, is formulated as follows:

$$SC_f^{ki} = \frac{LW\left(a+bS_f\right)(D_f + \emptyset A^* \sqrt{\frac{Q^{ki} h_f^{ki}}{u}})}{A \quad V_f^i h_f^{ki}} + \frac{v_v LWQ^{ki}(D_f + \emptyset A^* \sqrt{\frac{Q^{ki} h_f^{ki}}{u}})}{2V_f^i} + \frac{v_w LWQ^{ki} h_f^{ki}}{2} \tag{3}$$

Details for equation (3) are provided in Appendix 2. Total service cost over all routes and periods, TSC_f , is then:

$$\begin{aligned} TSC_f &= \sum_{k=1}^{K} \sum_{i=1}^{I} \{SC_f^{ki} \times t^{ki}\} \\ &= \sum_{k=1}^{K} \sum_{i=1}^{I} \left\{ \left(\frac{LW\left(a+bS_f\right)(D_f + \emptyset A \quad \sqrt{\frac{Q^{ki} h_f^{ki}}{u}})}{A \quad V_f^i h_f^{ki}} + \frac{v_v LWQ^{ki}(D_f + \emptyset A \quad \sqrt{\frac{Q^{ki} h_f^{ki}}{u}})}{2V_f^i} + \frac{v_w LWQ^{ki} h_f^{ki}}{2} \right) \times t^{ki} \right\} \end{aligned} \tag{4}$$

Capital Cost

After headways are optimized for each period, they and the round trip times determine fleet size. Thus, with optimized bus sizes, we have the required fleet matrix for each route and period. For capital cost, which is our fixed cost component, the required fleet size is the largest of the fleet sizes that are required to provide service in any periods for all local regions (i.e. largest value among $\sum_{k=1}^{K} F^{k1}$, $\sum_{k=1}^{K} F^{k2}$, …, $\sum_{k=1}^{K} F^{kI}$) over all periods. Here, the capital cost units are \$/day.

MULTI-DIMENSIONAL OPTIMIZATION PROBLEM FOR VARIABLE-MODE BUS OPERATIONS WITH MIXED FLEETS

In this section, we present a solution method for this multi-dimensional optimization problem. This is a nonlinear integer optimization problem with integer bus sizes (i.e. integer number of seats/bus) and integer fleets. To ensure an integer fleet size in any period, we should have integer numbers of zones for both conventional and flexible bus. (These are necessary conditions for having integer fleets). Then, we take into account the route spacing (for conventional bus) and service area (for flexible bus) because these two variables affect the number of zones in local regions. Thus, in this problem, we have four decision variables, *Conventional Bus Size, Flexible Bus Size, Route Spacing (for conventional bus) and Service Area (for flexible bus)*.

Preliminary Analysis

Upper/Lower Bounds for Conventional and Flexible Bus Sizes

Maximum and minimum bus sizes can be estimated with the cost formulation. In any given period, maximum allowable headways for conventional and flexible bus are as follows:

$$h_{c\,max}^{ki} = \frac{S_c l_c}{rLfQ^{ki}} \text{ (for conventional bus)} \tag{5}$$
$$h_{f\,max}^{ki} = \frac{S_f l_f}{AQ^{ki}} \text{ (for flexible bus)} \tag{6}$$

By substituting equations (5) and (6) into (1) and (3), respectively, we obtain equations (7) and (8).

$$SC_c^{ki} = \frac{D^k(a+bS_c^{ki})LWfQ^{ki}}{V_c^i S_c^{ki} l_c} + \frac{v_v LWQ^{ki}M^k}{V_c^i} + \frac{v_w WS_c^{ki} l_c}{2rf} + \frac{v_x LWQ^{ki}(r+d)}{4V_x} \tag{7}$$
$$SC_f^{ki} = \frac{LWQ^{ki}D_f(a+bS_f^{ki})}{V_f^i S_f^{ki} l_f} + \frac{\emptyset LWQ^{ki}(a+bS_f^{ki})\sqrt{A/uS_f^{ki} l_f}}{V_f^i} + \frac{v_v LWQ^{ki}D_f}{2V_f^i} + \frac{v_v LWQ^{ki}\emptyset\sqrt{AS_f^{ki} l_f/u}}{2V_f^i} + \frac{v_w LWS_f^{ki} l_f}{2A} \tag{8}$$

In equation (7), conventional bus cost is a function of one decision variable, namely the bus size S_c . In other words, since we can assume that we know all the other input values except bus size, we optimize bus size with the given information by taking the first derivative of service cost SC_c^{ki} with respect to bus size S_c . Then, the optimized bus size is:

$$S_c^{ki} = \frac{f}{l_c}\sqrt{\frac{2arD^kLQ^{ki}}{v_w V_c^i}} \tag{9}$$

In equation (9), the optimized bus size for minimizing service cost is determined by round travel distance D^k, demand density Q^{ki}, and speed V_c^i. Equation (9) can be rewritten as:

$$S_c^{ki} = \left\{\frac{f}{l_c}\sqrt{\frac{2arL}{v_w}}\right\} \sqrt{\frac{D^kQ^{ki}}{V_c^i}} \tag{10}$$

Therefore, using input parameters such as demand, bus speed, and round travel distance, we can determine upper and lower bounds for optimized bus sizes.

$$S_c^{upper} = max \begin{Bmatrix} S_c^{A1} & S_c^{B1} & S_c^{C1} & S_c^{D1} \\ S_c^{A2} & S_c^{B2} & S_c^{C2} & S_c^{D2} \\ S_c^{A3} & S_c^{B3} & S_c^{C3} & S_c^{D3} \\ S_c^{A4} & S_c^{B4} & S_c^{C4} & S_c^{D4} \end{Bmatrix} \tag{11}$$
$$S_c^{lower} = min \begin{Bmatrix} S_c^{A1} & S_c^{B1} & S_c^{C1} & S_c^{D1} \\ S_c^{A2} & S_c^{B2} & S_c^{C2} & S_c^{D2} \\ S_c^{A3} & S_c^{B3} & S_c^{C3} & S_c^{D3} \\ S_c^{A4} & S_c^{B4} & S_c^{C4} & S_c^{D4} \end{Bmatrix} \tag{12}$$

In optimizing bus size for flexible bus in a single period, we use the first derivative of the flexible bus cost function (equation (8)) with respect to bus size S_f^{ki} . Unfortunately, that derivative is difficult to solve analytically since it leads to a 4^{th} order equation. Thus, we incorporate here a built-in solver from MATLAB Version 7.9. A function named *fminbnd* finds the minimum value in a nonlinear minimization problem within given search boundaries. Thus, we optimize the bus size for any period by solving equation (8) with the nonlinear optimization solver. Similarly to the conventional bus size matrices in equation (11, 12), the upper and lower bounds of bus size for flexible bus are given in equations (13, 14).

$$S_f^{upper} = max \begin{Bmatrix} S_f^{A1} & S_f^{B1} & S_f^{C1} & S_f^{D1} \\ S_f^{A2} & S_f^{B2} & S_f^{C2} & S_f^{D2} \\ S_f^{A3} & S_f^{B3} & S_f^{C3} & S_f^{D3} \\ S_f^{A4} & S_f^{B4} & S_f^{C4} & S_f^{D4} \end{Bmatrix} \tag{13}$$

$$S_f^{lower} = min \begin{Bmatrix} S_f^{A1} & S_f^{B1} & S_f^{C1} & S_f^{D1} \\ S_f^{A2} & S_f^{B2} & S_f^{C2} & S_f^{D2} \\ S_f^{A3} & S_f^{B3} & S_f^{C3} & S_f^{D3} \\ S_f^{A4} & S_f^{B4} & S_f^{C4} & S_f^{D4} \end{Bmatrix} \tag{14}$$

From bus size optimization for a single period, we obtain upper bound and lower bounds for conventional and flexible buses. These boundaries give us the bus size arrays:

$$\text{Conventional Bus Size Array} = \{S_c^{upper}, S_c^{upper} - 1, \ldots, S_c^{lower} + 1,\ S_c^{lower}\} \tag{15}$$
$$\text{Flexible Bus Size Array} = \{S_f^{upper}, S_f^{upper} - 1, \ldots, S_f^{lower} + 1, S_f^{lower}\} \tag{16}$$

These arrays in equations (15, 16) will be used by the solution algorithm to find the total cost.

Upper/Lower Bounds for Route Spacing and Service Area

Similarly, we seek integer numbers of zones for both modes and integer fleets for each zone. In other words, the number of zones in each local region helps determine route spacing and service area. For instance, the minimum number of zones for both modes is one. In this case, route spacing and service area per bus are automatically determined (i.e. from equations r = W/N and A=LW/N'). Conversely, if we have too many zones (i.e. requiring too many buses), bus service would be inefficient. This gives us the minimum route spacing and minimum service area (and the corresponding maximum number of zones for services) for avoiding unnecessary search iterations.

For instance, in our numerical analysis, we set the minimum route spacing and minimum service area (i.e. size of zone) as 0.5 mile and 2 $mile^2$, respectively. Thus, a local region may be divided into one (when route spacing is equal to width of region) to 8 (when route spacing is 0.5 mile) equal zones. Similarly, service area 2 $mile^2$ says that the number of zones for flexible service is from 1 zone to 10 zones. These values for route spacing and service area are used to optimize service cost with integer fleets.

Therefore, route spacing and service area combinations are presented below. In parentheses, the first number represents the number of zones and the second specifies route spacing in miles (or service area in $mile^2$).

Conventional Mode Route Spacing Combinations

= {(1, 4.0), (2, 2.0), (3, 1.33), (4, 1.0), (5, 0.8), (6, 0.67), (7, 0.57), (8, 0.5)} (17)

Flexible Mode Service Area Combinations
= {(1, 20.0), (2, 10.0), (3, 6.67), (4, 5), (5, 4), (6, 3.33), (7, 2.86), (8, 2.5), (9, 2.22), (10, 2.0)} (18)

Route spacing and service area combinations will be used with bus size arrays to optimize total cost.

Demand Threshold Matrix over Routes and Periods

Anticipating that conventional bus has lower average cost than flexible bus at high demand densities, and vice versa, we must find the threshold demand for route k in period i, above which conventional service is preferable and below which flexible service is preferable. This threshold is obtained by setting equations (7) and (8) to be equal:

$$Q_t^{ki} = \frac{\frac{v_w}{2}\left\{\frac{S_f\, l_f}{A} - \frac{S_c\, l_c}{rfL}\right\}}{\left\{\frac{D^k f\left(a+bS_c\right)}{V_c^i S_c\, l_c} - \frac{\left(a+bS_f\right)\left(D_f + \varnothing\sqrt{AS_f\, l_f/u}\right)}{V_f^i S_f\, l_f} + \frac{v_v M^k}{V_c^i} - \frac{v_v\left(D_f + \varnothing\sqrt{AS_f\, l_f/u}\right)}{2V_f^i} + \frac{v_x(r+d)}{4V_x}\right\}} \tag{19}$$

Using equation (19), for any combination of decision variables (S_c, S_f, *r and A*), we determine the demand threshold matrix for selecting the conventional or flexible mode.

Solution Search Algorithm

We minimize total cost by optimizing four decision variables, namely conventional bus size, flexible bus size, conventional bus route spacing, and flexible bus service area. The flow chart for the solution search algorithm is shown in Figure 2.

From equations (15 ~18), we have feasible search boundaries for four decision variables. With any combinations of these four variable values, we compute service cost and capital cost in internal loop. We keep searching numerically for the minimum cost solution until we check all feasible search boundaries of decision variables while updating results when finding a lower total cost.

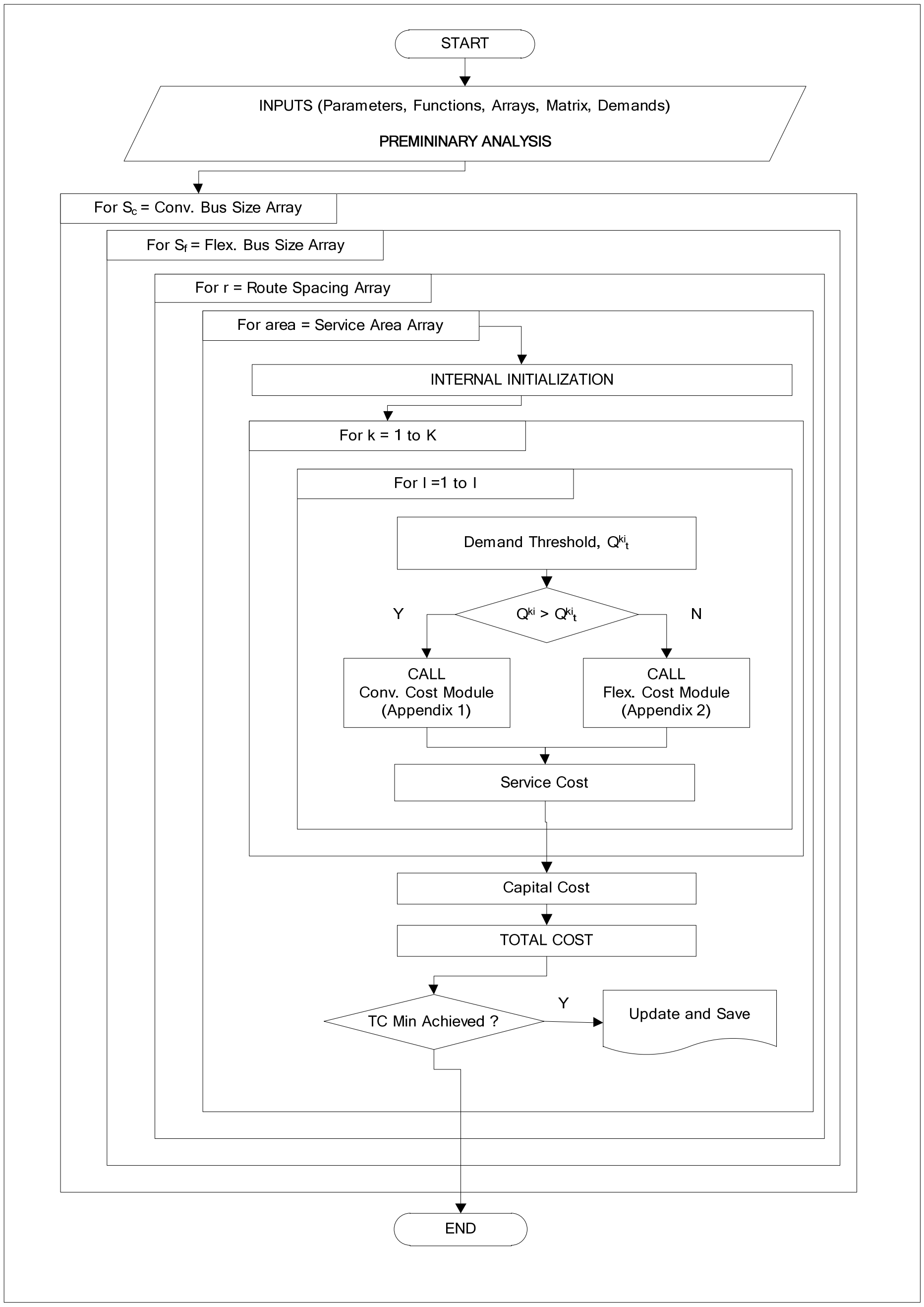

FIGURE 2 Search Algorithm

MUMERICAL ANALYSIS

To confirm that the proposed method minimizes cost effectively, we analyze a numerical case and compare VMBOMF to a single fleet conventional bus and a single fleet flexible bus. Furthermore, we conduct sensitivity analyses for important input parameters. In the following sections, a numerical case study and sensitivity analyses are presented.

Baseline Case Study

Inputs for Baseline Case Study

In the baseline numerical case, we consider four rectangular local regions, each 4 x 5 miles, and each with four periods (i.e. K = 4 and I = 4). Demand, service time and line-haul distance are presented in Table 2. All other required input parameters are presented in Table 1.

TABLE 2 Demand, Service Time, and Line-haul Distance

Demand (trips/mile²/hour)				
Period \ Region	A	B	C	D
1	50	45	60	55
2	30	35	40	40
3	10	15	30	15
4	5	7.5	10	5
Time(hours)				
Period \ Region	A	B	C	D
1	4	4	4	4
2	6	6	6	6
3	8	8	8	8
4	6	6	6	6
Region	A	B	C	D
Line-haul Distance (miles)	10	15	20	15

Results of Baseline Case Study

The optimized headways and fleet assignments used to minimize total cost for VMBOMF are presented in Table 3. In Region A, conventional bus operates only during Period 1. Flexible bus covers Periods 2 ~ 4. In Region C, conventional bus covers three periods, 1 to 3. Only Period 4 is served by flexible bus. For Regions B and D, conventional bus covers periods 1 and 2 while flexible bus serves Periods 3 and 4.

TABLE 3 Baseline Case Results

	Conventional Bus Headway (hours)					Flexible Bus Headway (hours)				
Period \ Region	A	B	C	D		A	B	C	D	
1	0.1795	0.2063	0.1566	0.1699		-	-	-	-	
2	-	0.1926	0.1997	0.1834		0.0889	-	-	-	
3	-	-	0.2296	-		0.2323	0.1765	-	0.1765	
4	-	-	-	-		0.3579	0.3000	0.2700	0.3904	
	Conventional Bus Fleet Assignment (buses)					**Flexible Bus Fleet Assignment (buses)**				
Period \ Region	A	B	C	D	Sum	A	B	C	D	Sum
1	26	28	44	34	132	0	0	0	0	0
2	0	20	23	21	64	35	0	0	0	35
3	0	0	20	0	20	13	21	0	21	55
4	0	0	0	0	0	8	12	16	9	45
	Conventional Bus Cost ($/hour)					**Flexible Bus Cost ($/hour)**				
Period \ Region	A	B	C	D		A	B	C	D	
1	5,833.0	6,201.0	9,005.5	7,329.9		0.0	0.0	0.0	0.0	
2	0.0	3,998.4	4,979.6	4,452.4		3,051.8	0.0	0.0	0.0	
3	0.0	0.0	3,950.9	0.0		1,218.9	1,949.9	0.0	1,949.9	
4	0.0	0.0	0.0	0.0		722.2	1,123.1	1,581.6	829.4	
	Service Cost × Time									
Period \ Region	A		B			C		D		

1	23,332.0	24,803.9	36,022.2	29,319.6
2	18,310.9	23,990.4	29,877.7	26,714.5
3	9,751.4	15,599.5	31,607.1	15,599.5
4	4,333.2	6,738.9	9,489.6	4,976.4
Total Service Cost ($/day) = 310,466.6				

From fleet assignment results in Table 3, we compute capital cost for conventional and flexible bus over all time periods. For conventional service, we need at least 132 buses. Similarly, flexible bus requires 55 buses. With the capital cost function ($B_c = a_c + b_c \times S$), we compute capital cost for both modes (i.e. $CC_c = (100+0.5\times47)$ *\$/bus day* $\times 132$ *buses*$= 16,302$ *\$/day*, $CC_f = (100+0.5\times18)$ *\$/bus day* $\times$ 55 *buses* $= 5,995$ *\$/day*). Total capital cost is then $22,297/day. Service cost for routes and periods are also provided in Table 3. After multiplying service cost by the number of hours in each period, we obtain the service cost matrix. Hence, the total service cost is 310,467$/day using VMBOMF. Total cost (including capital cost) is $332,764/day.

In Table 4, VMBOMF provides lower total cost than a single fleet conventional bus, or a single fleet flexible bus. Optimized bus sizes for VMBOMF are 47 seats/bus for conventional and 18 seats/bus for flexible bus. Additionally, route spacing is 1 mile for conventional bus and service area is 6.67 mile^2 for flexible bus service.

We also develop models that compute total cost for single-fleet conventional flexible bus services. In comparisons with single fleet, single mode services we note that VMBOMF reduces total cost. As shown in Table 4, VMBOMF, saves $537 (0.16%) and $23,483(6.59%) per day compared to single fleet conventional and flexible services, respectively. In this baseline case study, we confirm that VMBOMF can reduce total cost compared to single fleet & single mode services.

TABLE 4 Total Cost Comparisons with Single Fleet Operations

	VMBOMF	Single Fleet Conv. Bus	Single Fleet Flex. Bus
Bus Size (seats/bus)	47/18	47	29
Route Spacing / Service Area	1.0 / 6.67	1.0	2
Fleet Size (buses)	132 / 55	132	255
Capital Cost ($/day)	22,297	16,302	29,198
Total Cost ($/day)	332,764	333,301	356,247
Total Cost Saving (%)	-	0.16	6.59

Sensitivity Analysis

In this section, we explore how the relative advantages of VMBOMF can be affected by demand variation over time. Table 5 show input values for two sensitivity analysis cases.

TABLE 5 Input Values for Sensitivity Analysis	Case I				Case II			
	Demand (trips/mile^2/hour)				Time(hours)			
Period \ Region	A	B	C	D	A	B	C	D
1	10	9	12	11	2	2	2	2
2	6	7	8	8	4	4	4	4
3	2	3	6	3	4	4	4	4
4	1	2	2	1	14	14	14	14

Sensitivity Analysis Results

In Case 1, we check how an 80% reduction in demand density (to 20 % of baseline demand) affects total cost and the optimized decision variables. As shown in Table 6, VMBOMF decreases total costs by about 2.40% and 1.96% compared to single mode systems.

TABLE 6 Sensitivity Analysis Results

	Case 1 Result		
	VMBOMF	Single Fleet Conv. Bus	Single Fleet Flex. Bus
Bus Size (seats/bus)	38/18	32	23
Route Spacing / Service Area	2/10	2	10
Fleet Size (buses)	22/ 46	40	75
Capital Cost ($/day)	2499 / 5014	4640	8363
Total Service Cost ($/day)	83,626	88,737	84,596
Total Cost ($/day)	91,139	93,377	92,959
Total Cost Saving (%)	-	2.40	1.96
	Case 2 Result		
	VMBOMF	Single Fleet Conv. Bus	Single Fleet Flex. Bus
Bus Size (seats/bus)	53/21	53	30
Route Spacing / Service Area	1/10	1.33	5
Fleet Size (buses)	122/53	117	258
Capital Cost ($/day)	15,433 / 5,857	14801	29,670
Total Service Cost ($/day)	218,683	228,340	230,215
Total Cost ($/day)	239,973	243,141	259,885
Total Cost Saving (%)	-	1.30	7.66

Case 2 explores the sensitivity of results to the distribution of demand over time, considering relatively long periods with low demand density, as shown in Table 5. In results provided in Table 6, with longer periods of low demand, VMBOMF saves 1.30% and 7.66% in total cost compared to single fleet conventional and flexible services, respectively. The sensitivities of costs and other performance measure to other important parameters are provided in Kim and Schonfeld (26).

CONCLUSIONS

Summary and Contribution

In this study, we analyze VMBOMF (Variable Mode Bus Operation with Multiple Fleets) for multiple local regions. To solve our nonlinear integer problem, we propose a solution algorithm based on analytic optimization of variables and feasible search regions. Results confirm that VMBOMF can reduce costs compared to single fleet & single mode services (such as single fleet conventional or single fleet flexible bus). The baseline case shows that 0.16 % and 6.59 %

savings are achievable compared to single fleet conventional and flexible services, respectively. In the baseline results the total cost of VMBOMF is close to that of single fleet conventional service because the demand density is fairly high. Sensitivity analysis shows that the relative advantages of VMBOMF increase when demand densities decrease and last longer.

Extensions

Various additional questions seem worth exploring. Here we only present two sensitivity analysis cases. Many other interesting input parameter variations such as region length, line haul distance, bus speed, walking speed, bus stop spacing (for conventional bus), unit costs and passenger time values might also be considered. In this paper, we assume that all trips go either from local regions to terminal or from terminal to local area. Comparisons of VMBOMF with variable-mode & single-fleet services and single-mode & multiple-fleet services are also desirable. For travel among the local regions an optimization model should also seek to coordinate headways and minimize transfer delays at the major terminal.

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APPENDIX 1 Conventional Bus Cost Formulations

Kim and Schonfeld (2011a) presents conventional service cost formulations for a one route bus system in analyzing the variable-type bus service concept. In this paper, we modify its formulations to extend them to multiple local regions.

As shown in Figure 1, buses travel from the terminal a line haul distance J^k at non-stop speed yV_c^i to a corner of the service area, then travel an average of $W/2$ miles at local non-stop speed zV_c^i from the corner to the assigned zone, run a distribution route of length L at local speed V_c^i along the central axis of the zone while stopping for passengers every d miles, and the reverse the process in returning. Therefore, the buses' average round trip time is:

$$R_c^{ki} = \frac{2J^k}{yV_c^i} + \frac{W}{zV_c^i} + \frac{2L}{V_c^i} \tag{A1-1}$$

This round trip time can differ among routes and periods. The previous study (Kim and Schonfeld, 2011a) does not consider travel speed variations over time. Equation (A1-1) can be re-written as:

$$R_c^{ki} = \left\{\frac{2J^k}{y} + \frac{W}{z} + 2L\right\}/V_c^i \tag{A1-2}$$

In equation (A1-2), the expression in parentheses represents an equivalent bus round trip distance, D^k.

The total cost of conventional bus service (in route k, at period i) includes the operator cost SC_{co}^{ki} and the user cost SC_{cu}^{ki}. To determine operator cost, we determine the fleet size N, which is the total bus round trip time divided by the headway. With the equivalent bus round travel distance D^k, a controllable directional split factor f, and conventional bus speed V_c, we obtain the required fleet size F_c^{ki}:

$$F_c^{ki} = \frac{D^k W}{r h_c^{ki} V_c^i} \qquad ,where\ D^k = 2J^k/y + W/z + 2L \tag{A1-3}$$

The hourly conventional bus operator cost SC_{co}^{ki} is the required fleet size multiplied by bus operating cost:

$$SC_{co}^{ki} = F_c^{ki} B = \frac{D^k W}{r h_c^{ki} V_c^i}(a + bS_c) \tag{A1-4}$$

The hourly user cost for the conventional bus service at route k & period i, SC_{cu}^{ki} is the sum of in-vehicle cost SC_{cv}^{ki}, waiting cost SC_{cw}^{ki}, and access cost SC_{cx}^{ki}:

$$SC_{cu}^{ki} = SC_{cv}^{ki} + SC_{cw}^{ki} + SC_{cx}^{ki} \tag{A1-5}$$

The hourly in-vehicle cost for the conventional service is then:

$$SC_{cv}^{ki} = v_v LWQ t_c^{ki} \tag{A1-6}$$

The average travel time t_c^{ki} for passenger trip is formulated as:

$$t_c^{ki} = \frac{J^k}{yV_c^i} + \frac{W}{2zV_c^i} + \frac{L}{2V_c^i} = \frac{M^k}{V_c^i} \text{ ,where } M^k{=}J^k/y + W/2z + L/2 \quad \text{(A1-7)}$$

Then equation (A1-7) can be written as:

$$SC_{cv}^{ki} = v_v LWQ^{ki} \frac{M^k}{V_c^i} \quad \text{(A1-8)}$$

We assume the average waiting time is half the headway. Therefore, the hourly user waiting cost for conventional system $SC_{cw}^{ki}$is:

$$SC_{cw}^{ki} = v_w LWQ^{ki} \frac{h_c^{ki}}{2} \quad \text{(A1-9)}$$

Since the spacing between adjacent branches of local bus service is *r*, and since service trip origins (or destinations) are uniformly distributed over the local region, the average access distance to the nearest route is one-fourth of route spacing, *r/4*. Similarly, the access distance alongside the route to the nearest transit stop is one-fourth of the bus stop spacing, i.e., *d/4*. Therefore, the hourly access cost for the conventional bus system SC_{cx}^{ki} is:

$$SC_{cx}^{ki} = \frac{v_x LWQ^{ki}(r+d)}{4V_x} \quad \text{(A1-10)}$$

The total service cost for the conventional system SC_{cu}^{ki} is the sum of operating cost and user cost:

$$SC_{cu}^{ki} = \frac{D^k W}{rh_c^{ki}V_c^i}(a + bS_c) + v_v LWQ^{ki}\frac{M^k}{V_c^i} + \; v_w LWQ^{ki}\frac{h_c^{ki}}{2} \; + \; \frac{v_x LWQ^{ki}(r+d)}{4V_x} \quad \text{(A1-11)}$$

Since we consider multiple periods for bus operations, the optimized headway should be smaller value between the maximum allowable headway and minimum cost headway. The maximum allowable headway for route k and period i is:

$$h_{c\,max}^{ki} = \frac{S_c l_c}{rLfQ^{ki}} \quad \text{(A1-12)}$$

The minimum cost headway can be obtained from the partial derivative of equation (A1-11) with respect to headway;

$$h_{c\,min}^{ki} = \sqrt{\frac{2D^k(a+bS_c)}{v_w LrQ^{ki}V_c^i}} \quad \text{(A1-13)}$$

Then, optimal headway is then:

$$h_{c\,opt}^{ki} = \min\left\{\frac{S_c l_c}{rLfQ^{ki}}, \sqrt{\frac{2D^k(a+bS_c)}{v_w LrQ^{ki}V_c^i}}\right\} \quad \text{(A1-14)}$$

The optimized headway obtained in equation (A1-14) applies to equation (A1-3) for optimizing the conventional service fleet size for route k and period i ($F_{c\,opt}^{ki} = \frac{D^k W}{rh_{c\,opt}^{ki}V_c^i}$). However, this fleet size must be rounded off to an integer value. The modified headway h_c^{ki*} can similarly obtained by using equation (A1-3). All such modified headways should be equal to or below the maximum allowable headway in equation (A1-12).

The service cost for route k and in period i, is finally formulated by substituting modified headway into equation (A1-3).

$$SC_c^{ki*} = \frac{D^k W}{r h_c^{ki*} V_c^i}(a + bS_c) + v_v LWQ^{ki}\frac{M^k}{V_c^i} + v_w LWQ^{ki}\frac{h_c^{ki*}}{2} + \frac{v_x LWQ^{ki}(r+d)}{4V_x} \tag{A1-15}$$

APPENDIX 2 Flexible Bus Cost Formulations

The flexible cost formulation introduced by Chang and Schonfeld (2) considers only one-directional service (i.e. only collecting OR distributing passengers). Here, we consider 2-directional demand (i.e. pick up and drop off passengers within one tour). Since the formulation from Chang and Schonfeld (2) applies for only one local region, we modify some notation and improve equations to consider multiple local regions in the flexible service formulation even though most of them are similar to those by Chang and Schonfeld (2)

For flexible service such as Dial-A-Ride, the efficient tour distance D_c for visiting randomly and independently dispersed n points among area A is approximately (24, 25)):

$$D_c = \O\sqrt{nA} \tag{A2-1}$$

For a grid network (i.e. rectilinear space), a Ø value of 1.15 is appropriate for analyzing flexible service cost (25). In equation (A2-1), n is the number of stops per tour, which is the hourly round trip demand in each zone AQ^{ki} multiplied by service headway h_f^{ki} and divided by number of passengers *u* per stop:

$$n = \frac{AQ^{ki}h_f^{ki}}{u} \tag{A2-2}$$

Substituting equation (A2-2) into (A2-1), we optimize the tour distance in service area *A* as:

$$D_c^{ki} = \O\sqrt{\frac{AQ^{ki}h_f^{ki}}{u}A} = \O A\sqrt{\frac{Q^{ki}h_f^{ki}}{u}} \tag{A2-3}$$

Round travel time can be calculated as follows:

$$R_f^{ki} = 2\left(\frac{L+W}{2zV_f^i} + \frac{J^k}{yV_f^i}\right) + \frac{D_c^{ki}}{V_f^i} = \frac{(L+W)/z + 2J/y}{V_f^i} + \frac{D_c^{ki}}{V_f^i} = \frac{D_f^k + D_c^{ki}}{V_f^i} \tag{A2-4}$$

where, equivalent round travel distance D_f^k=*(L+W)/z +2*J^k*/y*

The fleet size is:

$$F_f^{ki} = \frac{LWR_f^{ki}}{Ah_f^{ki}} = \frac{LW}{Ah_f^{ki}}\frac{D_f^k + D_c^{ki}}{V_f^{ki}} = \frac{LW(D_f^k + \O A\sqrt{Q^{ki}h_f^{ki}/u})}{V_s A h_f^{ki}} \tag{A2-5}$$

The hourly flexible bus operator cost SC_{fo}^{ki} is the required fleet size multiplied by bus operating cost:

$$SC_{fo}^{ki} = F_f^{ki} B = \frac{LW(D_f^k + \emptyset A\sqrt{Q^{ki} h_f^{ki}/u})}{V_f^{ki} A h_f^{ki}} (a + bS_f) \quad \text{(A2-6)}$$

The hourly user cost for the flexible bus service for route k & period i, SC_{fu}^{ki} is the sum of in-vehicle cost SC_{fv}^{ki}, and waiting cost SC_{fw}^{ki}:

$$SC_{fu}^{ki} = SC_{fv}^{ki} + SC_{fw}^{ki} \quad \text{(A2-7)}$$

The hourly in-vehicle cost for the flexible service is then:

$$SC_{fv}^{ki} = v_v LWQ^{ki} \frac{R_f^{ki}}{2} = \frac{v_v LWQ^{ki}}{2} \frac{D_f^k + D_c^{ki}}{V_f^{ki}} = \frac{v_v LWQ^{ki}}{2} \frac{D_f^k + \emptyset A\sqrt{Q^{ki} h_f^{ki}/u)}}{V_f^{ki}} \quad \text{(A2-8)}$$

Assuming average waiting time is approximately half of the headway, the waiting cost SC_{fw}^{ki} is:

$$SC_{fw}^{ki} = v_w\, LWQ^{ki} \frac{h_f^{ki}}{2} \quad \text{(A2-9)}$$

The total service cost for the flexible bus operation SC_{cu}^{ki} is the sum of operating cost and user costs:

$$SC_f^{ki} = \frac{LW(a + bS_f)(D_f^k + \emptyset A\sqrt{Q^{ki} h_f^{ki}/u)}}{V_f^{ki} A h_f^{ki}} + \frac{v_v LWQ^{ki}}{2} \frac{D_f^k + \emptyset A\sqrt{Q^{ki} h_f^{ki}/u)}}{V_f^{ki}} + \; v_w LWQ^{ki} \frac{h_f^{ki}}{2} \quad \text{(A2-10)}$$

Since we consider multiple periods, the optimized headway should be (1) the maximum allowable headway OR (2) the minimum cost headway, whichever is smaller. The maximum allowable headway for route k and period i is:

$$h_{f\,max}^{ki} = \frac{S_f l_f}{AQ^{ki}} \quad \text{(A2-11)}$$

The minimum cost headway can be obtained from the partial derivative equation (A2-10) with respect to headway h_f^{ki}. An analytically optimized solution with respect to headway for a one route bus service is provided in Kim and Schonfeld (22). However, since our problem here is a nonlinear minimization problem, and difficult to solve analytically, we here apply the function *fminbnd* in MATLAB Version 7.9 to find the minimum value. This function allows us to easily obtain the minimum cost headway $h_{f\,min}^{ki}$. Thus, the optimized headway for flexible service is:

$$h_{f\,opt}^{ki} = \; min \left\{\frac{S_f l_f}{AQ^{ki}}, h_{f\,min}^{ki}\right\} \quad \text{(A2-12)}$$

The optimized fleet size for flexible service is found by substituting equation (A2-12) into (A2-5):

$$F_{f\,opt}^{ki} = \frac{LW(D_f^k + \emptyset A\sqrt{Q^{ki} h_{f\,opt}^{ki}/u)}}{V_s A h_{f\,opt}^{ki}} \quad \text{(A2-13)}$$

This equation (A2-13), similarly to conventional service fleet size, must yield an integer value. Therefore, we round off fleet size to an integer value, and then check if modified headway violates maximum allowable headway. If modified headway violates maximum allowable headway, we round up fleet size to have integer number of buses. Modified headway denoted as h_f^{ki*} provides minimum total service cost with integer fleet size.

Minimum service cost with integer fleet for flexible bus operation is obtained by applying modified headway into equation (A2-10):

$$SC_f^{ki*} = \frac{LW(a+bS_f)(D_f^k+\emptyset A\sqrt{Q^{ki}h_f^{ki*}/u})}{V_f^{ki}Ah_f^{ki*}} + \frac{v_v LWQ^{ki}}{2}\frac{D_f^k+\emptyset A\sqrt{Q^{ki}h_f^{ki*}/u)}}{V_f^{ki}} + v_w LWQ^{ki}\frac{h_f^{ki*}}{2} \qquad \text{(A2-14)}$$