INTEGRATION OF FIXED AND FLEXIBLE ROUTE PUBLIC TRANSPORTATION SYSTEMS

PHASE I

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PHASE I

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16. Abstract

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SUMMARY

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CONVENTIONAL, FLEXIBLE AND VARIABLE-TYPE BUS SERVICES

INTRODUCTION

Conventional transit services (defined as having fixed routes and fixed schedules) are most effective at high demand densities, which justify frequent services and high densities of stations, which in turn reduce wait times and access times for passengers. Unconventional services (also called "demand responsive" or "paratransit" services) are interesting to researchers and transit operators because they can provide high service quality (including "doorstop" pickups and drop-offs of passengers) even when demand densities are low. The recent literature (such as Diana et al 2009; Horn 2002; Fu and Ishkhanov 2004; Quadrifoglio et al 2006, 2008, 2009; Luo and Schonfeld 2011a, 2011b; Shen et al 2011; Jung et al 2011; Becker et al 2011; Kim and Haghani 2011; Baumgartner and Schofer 2011; Nourbakhsh and Ouyang 2011) confirms the continuing interest in various paratransit concepts.

Baumgartner and Schofer (2011) introduce the concept of Call-n-Ride which is an operation without a central dispatcher, in which drivers take requests for service directly on their cell phones and make all routing and scheduling decisions. They present the results of current Call-n-Ride service in US, and provide a model for predicting the productivity of Call-n-Ride services. Becker and Teal (2011) study the service configuration aspects of next generation DRT (Demand Responsive Transit) by focusing on the experiences of the Denver transit agency. Kim and Haghani (2011) mainly focus on developing algorithms to solve a static multi-depot Dial-a-Ride problem with time-varying travel times and soft time windows. Jung and Jayakrishnan (2011) study an alternative transportation concept called High Coverage Point-to-Point Transit (HCPPT) that can reduce the number of transfers in urban transit systems, and note that HCPPT can be a good alternative to a conventional fixed route and conventional

DRT service. Additionally, Shen and Quadrifoglio (2011) study a realistic coordinated decentralized paratransit system. Luo and Schonfeld (2011a, 2011b) develop performance models for demand responsive many-to-many dial-a-ride services and a rejected-reinsertion heuristics for dynamic multi-vehicle DARP (Dial-a-Ride Problem). Aside from Chang and Schonfeld's (1991b) analysis of temporally integrated bus systems, it is difficult to find studies that consider variations in service type as demand changes. Thus, it seems worthwhile to examine not only the relative advantages and disadvantages of conventional and paratransit services, but also to explore variable-type bus alternatives in which the service type changes in response to demand changes while using the same pool of resources (i.e. buses and drivers).

Flexible bus services typically provide Many-to-One and/or One-to-Many service with flexible route tours that operate on semi-fixed schedules. (The departure times from or arrival times at the One major trip generator are usually pre-determined and the tours may have cyclical schedules.) The relative advantages of conventional and flexible bus services have been compared using analytic optimization models in Chang and Schonfeld (1991a, cited henceforth as CS). The models for conventional and flexible services used in the present study are adapted from those developed in CS. To compare the costs of conventional bus and flexible bus services, CS assume that both conventional bus and flexible bus either collect passengers from a local service area *OR* distribute passengers to a local area. In the present study we introduce a controllable directional split factor, which enables us to consider 2-directional demands in various proportions.

A different approach to reducing bus transit cost is to use different fleets of buses as the demand varies, with larger buses used at higher demand densities. Based on this idea, Fu and Ishkhanov (2004) consider mixed bus fleet operation for paratransit services. Similarly, Lee et al (1995) consider mixed bus fleets for urban conventional bus services. However, the potential advantages of variable-type bus for integrated conventional and flexible bus operations have not been sufficiently explored. Those potential advantages are the subject of this paper, in which we seek to quantify them. In this paper we (1) modify the cost functions to reflect two-directional demands in round trip times, (2) develop an integrated model for variable-type bus services and (3) compare conventional, flexible and variable-type bus services under various assumed conditions. This model is intended for conceptual comparisons of services rather than detailed planning and operations.

COST FUNCTIONS

Here, we briefly present and modify the CS cost functions upon which the present analysis relies. Although we have changed some notation, the detailed formulations can be found in Chang (1990), CS, and Schonfeld et al (2010). This section explains how CS formulate the costs of conventional and flexible bus services. The notation used throughout the paper and the baseline input values used in our numerical analysis are shown in Table 1.

Table 1 Notation

Variable	Definition	Baseline Value
a	hourly fixed cost coefficient for operating bus service (\$/bus hr)	30.0
a_c	fixed cost coefficient for bus ownership (capital cost) (\$/bus day)	100.0
$\bar{a}$	weighted fixed cost coefficient defined in Table 3	-
A	service zone area(sq.miles)=LW/N'	-
b	hourly variable cost coefficient for operating bus service (\$/seat hr)	0.2
b_c	variable cost coefficient for owning bus (capital cost) (\$/day)	0.5
$\bar{b}$	weighted fixed cost coefficient defined in Table 3	-
B	bus operating cost (\$/bus hr), for conventional and flexible service (=$a+bS_c$, $a+bS_f$)	-
B_c	bus operator cost for owning bus(capital cost) (\$/bus day)	-
C_c , C_f	service cost, for conventional and flexible service (\$/hr)	-
C_{ci} , C_{fi}	service cost in period i, for conventional and flexible service (\$/hr)	-
C_o , C_{oc} , C_{of}	operating cost; for conventional and flexible service (\$/hr)	-
C_p , C_{pc} , C_{pf}	capital cost; for conventional and flexible service (\$/day)	-
C_t , C_{tc} , C_{tf}	total cost; for conventional and flexible service (\$/day)	-
C_u , C_{uc} , C_{uf}	user cost; for conventional and flexible service (\$/hr)	-
C_v , C_{vc} , C_{vf}	in-vehicle cost; for conventional and flexible service (\$/hr)	-
C_w , C_{wc} , C_{wf}	waiting cost; for conventional and flexible service (\$/hr)	-
C_x , C_{xc}	access cost; for conventional service (\$/hr)	-
d	bus stop spacing (miles)	0.2
D	equivalent average bus round trip distance for conventional bus service (= $2J/y+W/z+2L$),(miles)	-
D_c	distance of one tour of flexible bus service at local area (miles)	-

D_f	equivalent line haul distance for flexible bus service *(=(L+W)/z+2J/y)*, (miles)	-
F_c , F_f	fleet size for conventional and flexible service (buses)	-
F_{ci} , F_{fi}	fleet size at period i for conventional and flexible bus service (buses)	-
f	directional demand split factor	1.0
h_c , h_f	headway for conventional and flexible service (hrs/bus)	-
h_{ci}^{max} , h_{fi}^{max}	maximum allowable headway in period i, for conventional and flexible service (hrs/bus)	-
h_{ci}^{opt} , h_{fi}^{opt}	optimized headway in period i, for conventional and flexible service (hrs/bus)	-
h_{ci} , h_{fi}	headway in period i for conventional and flexible service (hrs/bus)	-
i, k	period index	-
J	line haul distance (miles)	10.0
l_c , l_f	load factor for conventional service and flexible service (passengers/seat)	1.0
L, W	length and width of service area (miles)	5.0, 4.0
M	equivalent average trip distance ($=J/y_c+W/2z_c+L/2$)	-
n	number of passengers in one collection tour	-
N, N'	number of zones in service area for conventional and flexible service	-
Q	round trip demand density (trips/sq.mile/hr)	-
Q_i	round trip demand density in period i (trips/ sq.mile/hr)	-
Q_p	demand density at peak time (trips/sq.mile/hr)	-
$\bar{Q}$	average round trip demand density, as defined in Table 3	-
R_c	round travel time for conventional service (hours)	-
r	route spacing (miles)	-
S_c , S_f	bus size for conventional and flexible service (seats/bus)	-
t_i	duration of period i	-
u	average number of passengers per stop for flexible service	1.2
V_c , V_f	local service speed for conventional and flexible bus (miles/hr)	20, 18
V_x	average access speed (mile/hr)	2.5
v_v , v_w , v_x	value of in-vehicle time, wait time and access time ($/passenger hr)	5, 12, 12
y	express speed/local speed ratio for conventional service	conventional bus = 1.8 flexible bus = 2.0
Y	term used in Tables 2 and 3	-
z	non-stop ratio = local non-stop speed/local speed; same values as y	-
$Ø$	constant in the collection distance equation (Daganzo, 1984) for flexible bus service	1.15
*	superscript indicating optimal value	-

Assumptions and Analytic Result

The assumptions for both conventional bus and flexible bus are listed below. Definitions and baseline values of variables are provided in table 1.

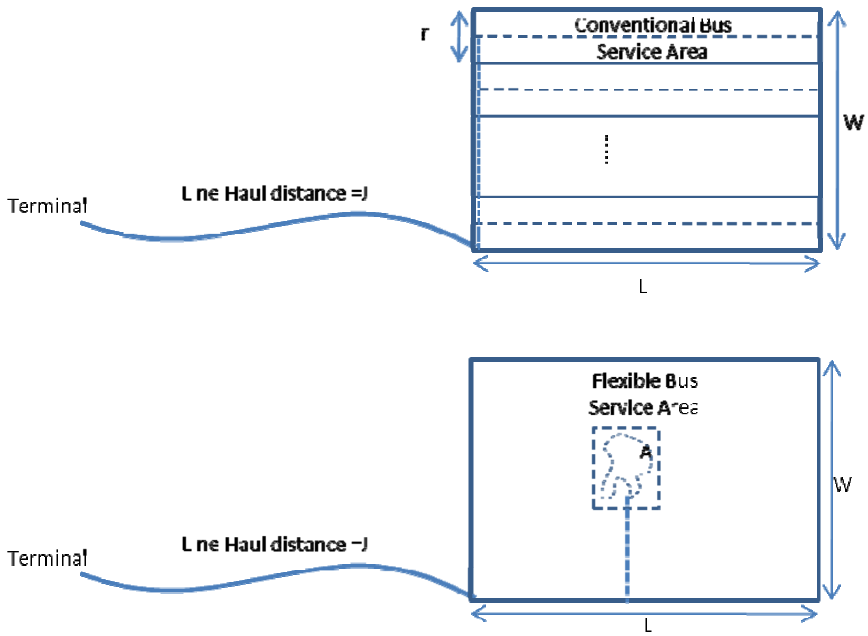

Figure 1 Conventional and Flexible Bus Services

<u>*For both conventional and flexible bus*</u>

a) A rectangular service area of length L and width W (as shown in Figure 1) is J miles away from a transportation terminal at its nearest corner.

b) The demand is fixed with respect to service quality and price.

c) The demand is uniformly distributed over space within the service area and over time within each specified period.

d) The vehicle size (S_c *for conventional bus,* S_f *for flexible bus)* is uniform throughout a system.

e) The estimated average waiting time of passengers is equal to half the headway (h_c *for conventional bus,* h_f *for flexible bus*).

f) Vehicle layover time is negligible.

g) Within the service area, the average speed (V_c for conventional bus, V_f for flexible bus) includes stopping times.

h) External costs are assumed to be negligible.

For conventional bus service only

a) The service area is divided into N parallel zones with a width r=W/N for conventional bus service, as shown in Figure 1. Local routes branch from the line haul route segment to run along the middle of each zone, at a route spacing $r=W/N$.

b) A demand of Q trips/mile^2/hour, which is entirely channeled to (or through) the single terminal, is uniformly distributed over the service area.

c) In each round trip, as shown in Figure 1, buses travel from the terminal a line haul distance J at non-stop speed yV_c to a corner of the service area, then travel an average of $W/2$ miles at local non-stop speed zV_c from the corner to the assigned zone, then run a local route of length L at local speed V_c along the central axis of the zone while stopping for passengers every d miles, and then reverse the above process in returning to the terminal..

For flexible bus service only

a) The service area is divided into N' equal zones, each having an optimizable zone area $A=LW/N'$. The zones should be "fairly compact and fairly convex" (Stein, 1978).

b) Buses travel from the terminal a line haul distance J at non-stop speed yV_f and an average distance $(L+W)/2$ miles at local non-stop speed zV_f to the center of each zone. They collect (or distribute) passengers at their door steps through a tour of n stops and length D_c at local speed V_f. The values of n and D_c are endogenously determined. To return to their starting point the buses retrace an average of $(L+W)/2$ miles at zV_f miles per hour and J miles at yV_f miles per hour.[1]

c) Buses operate on preset schedules with flexible routing designed to minimize each tour distance D_c.

d) The tours are routed on the rectilinear street network.

e) Tour departure headways are equal for all zones in the service area and uniform within each period.

[1] D_c is approximated by Stein (1978),in which $D_c = \emptyset\sqrt{nA}$, and $\emptyset = 1.15$ for rectilinear space according to Daganzo(1984)

The formulation proposed by CS considered one-way service (i.e. only collecting passengers OR distributing passengers) in which total demand density is Q trips/sq. mile. Based on these assumptions, the analytic optimization results CS obtained for conventional bus and flexible bus services are presented in Table 2. For bus operating cost, a linear (i.e. $B=a+bS$) cost function was used (Jansson, 1980; Oldfield and Bly, 1988).

Table 2 Analytic Results from Chang and Schonfeld (1991a)

Conventional bus service		Flexible bus service	
Vehicle Size S_c	$\sqrt[3]{\frac{8a^2D^2LQV_x}{v_w v_x V_c^2 l_c^3}}$	Vehicle Size S_f	$\left(\frac{a^3 D_f^3 Qu}{v_w Ø^2 V_f l_f^3 \left(b+\frac{v_v l_f}{2}\right)^2}\right)^{1/5}$
Route Spacing r	$\sqrt[3]{\frac{8aDv_w V_x^2}{v_x^2 LQV_c}}$	Service Area A	$\left(\frac{a v_w^3 V_f^3 D_f^3 u^{8/3} l_f^4}{Ø^4 Q^{7/3} Y^{10/3} \left(b+\frac{v_v l_f}{2}\right)^2}\right)^{1/5}$
Service Cost (Conventional Bus)	$3LWQ\left(\frac{v_w v_x D}{8LQV_c V_x}\right)^{\frac{1}{3}}+\frac{bDLWQ}{l_c V_c}+\frac{v_v LWQM}{V_c}+\frac{v_x LWQd}{4V_x}$		
Service Cost (Flexible Bus)	$LWQ\left[\frac{v_w a^2 D_f^2 Ø^2 \left(b+\frac{v_v l_f}{2}\right)^2}{u l_f^2 V_f^4 Q}\right]^{1/5}+1.5LWQ\left[\frac{v_w a^2 Ø^2}{l_f^2 V_f^4}\right]^{1/5}\left(\frac{Y^2}{uQ}\right)^{1/3}+\frac{LWQD_f\left(b+\frac{v_v l_f}{2}\right)}{V_f l_f}$		
Note	$Y=\left[a^2 v_w Ø^2 V_f l_f^3\right]^{1/5}+\left[uD_f^3 Q\left(b+v_v l_f/2\right)^3\right]^{1/5}$		

Total Cost including Capital Cost

When computing the total system cost for bus service, capital cost should be treated as another fixed cost. The capital cost C_p, is the cost to satisfy the peak period vehicle requirement. In equation (1), bus service cost is defined as the sum of bus operating cost C_o, user in vehicle cost C_v, user waiting cost C_w, and user access cost C_x:

Total cost = Capital cost + Bus operating cost + User cost (1)

Relation (1) can be rewritten as:

$$C_t = C_p + C_o + C_u = C_p + C_o + C_v + C_w + C_x \quad (2)$$

Analytic results with capital cost for conventional and flexible bus services are summarized in Table 3.

Table 3 Analytic Results with Capital Cost

Conventional bus service		Flexible bus service	
Vehicle Size S_c	$\sqrt[3]{\frac{8\bar{a}^2D^2L\bar{Q}V_x}{v_w v_x V_c^2 l_c^3}}$	Vehicle Size S_f	$\left(\frac{\bar{a}^3D_f^3\bar{Q}u}{v_w\emptyset^2V_f l_f^3\left(\bar{b}+\frac{v_v l_f}{2}\right)^2}\right)^{1/5}$
Routing Space r	$\sqrt[3]{\frac{8\bar{a}Dv_wV_x^2}{v_x^2L\bar{Q}V_c}}$	Service Area A	$\left(\frac{\bar{a}v_w^3V_f^3D_f^3u^{8/3}l_f^4}{\emptyset^4\bar{Q}^{7/3}Y^{10/3}\left(\bar{b}+\frac{v_v l_f}{2}\right)^2}\right)^{1/5}$
Total Cost (Conv. Bus)	$\frac{a_cDLWQ_p}{S_c{}^*V_cl_c}+\frac{b_cDLWQ_p}{V_cl_c}+\frac{D(a+bS_c{}^*)LW\sum_i^I Q_it_i}{V_cS_c{}^*l_c}+\frac{v_vMLW\sum_i^I Q_it_i}{V_c}+\frac{v_wlWS_c{}^*\sum_i^I t_i}{2r}+\frac{v_x(r+d)LW\sum_i^I Q_it_i}{4V_x}$		
Total Cost (Flex. Bus)	$\frac{LWQ_pD_f(a_c+b_cS_f)}{V_fS_fl_f}+\frac{\emptyset LWQ_p\,(a_c+b_cS_f)\sqrt{A/uS_fl_f)}}{V_f}+\sum_i^I\left\{\frac{LWQ_it_iD_f(a+bS_f)}{V_fS_fl_f}+\frac{\emptyset LWQ_it_i(a+bS_f)\sqrt{A/uS_fl_f}}{V_f}+\frac{v_vLWQ_it_iD_f}{2V_f}+\frac{v_vLWQ_it_i\emptyset\sqrt{AS_fl_f/u)}}{2V_f}+\frac{v_wLWS_fl_ft_i}{2A}\right\}$		
Note	$\bar{Q}=\frac{\sum_i^I Q_it_i}{\sum_i^I t_i},\bar{a}=\frac{a_cQ_p\sum_i^I a_iQ_it_i}{\sum_i^I Q_it_i},\bar{b}=\frac{b_cQ_p\sum_i^I b_iQ_it_i}{\sum_i^I Q_it_i},Y=\left[\bar{a}^2v_w\emptyset^2V_fl_f^3\right]^{1/5}+\left[uD_f^3\bar{Q}\left(\bar{b}+v_vl_f/2\right)^3\right]^{1/5}$		

Limitations of Previous CS Study

Here, we seek to overcome two main limitations in the CS bus service cost formulations. First, CS assume that trip demand for bus services is always one directional (i.e. either all demand from terminal to local or local to terminal). We modify that here by introducing a directional demand split factor, *f*. Second, CS only consider the maximum allowable headway (required to satisfy demand) rather than an optimized headway. It seems preferable to optimize the headway for each period, which should be the minimum of (1) the maximum feasible headway which satisfies the demand and (2) the headway that minimizes total costs.

COST FUNCTION MODIFICATION AND HEADWAY OPTIMIZATION

Here we introduce a directional demand split factor, *f,* for conventional bus service only (because flexible service does not need a directional demand split factor unless passengers are collected and distributed in different tours) as well as provide optimized headway solutions for both conventional bus and flexible bus services. If f=1.0 all demand is one-directional. In other words, buses return without any passengers. Similarly, if f=0.5, then demand is equal in the two directions. In flexible service, since passengers are collected and distributed within the same tours, no directional split factor is needed.

Therefore, if we assume the demand density Q is the sum of both collected passengers and distributed passengers, we can still use the CS flexible service cost functions.

Conventional Bus System Cost

When computing total system cost for conventional bus service, the capital cost C_p should satisfy the peak period fleet size requirement. In equation (3), the bus service cost is the sum of bus operating cost C_o, user in vehicle cost C_v, user waiting cost C_w, and user access cost C_x.

$$\begin{aligned} \textit{Total cost} &= \textit{Capital cost} + \textit{Bus service cost} \\ &= \textit{Capital cost} + \textit{Bus operating cost} + \textit{User cost} \end{aligned} \quad (3)$$

Detailed cost component derivations for operator and user costs are provided in Appendix 1. Equation (3) can be expressed as:

$$C_t = C_p + C_o + C_u = C_p + C_o + C_v + C_w + C_x \quad (4)$$

Although we reformulate the conventional bus cost, the overall procedure for computing total cost with capital cost is basically similar to that in CS. The capital cost for conventional bus system should be computed based on peak-period demand. Therefore, capital cost C_{pc} for conventional bus service is:

$$C_{pc} = \frac{D}{V_c}\frac{W}{r}\frac{1}{h_p}B_c = \frac{D}{V_c}\frac{W}{r}\frac{rLfQ_p}{S_c l_c}B_c = \frac{D}{V_c}\frac{W}{r}\frac{rLfQ_p}{S_c l_c}(a_c + b_c S_c) \quad (5)$$

The total daily service cost for conventional bus service C_{tc} is formulated below. Subscript i denotes time periods in the following equations and t_i represents the number of hours in period i.

$$C_{tc} = C_{pc} + \sum_i^I \{C_{oci} + C_{vci} + C_{wci} + C_{xci}\} \quad (6)$$

Equation (6) can be rewritten as follows:

$$C_{tc} = \frac{a_c DLWfQ_p}{S_c V_c l_c} + \frac{b_c DLWQ_p}{V_c l_c} + \frac{D(a+bS_c)LWf\sum_i^I Q_i t_i}{l_c V_c S_c} + \frac{v_v MLW\sum_i^I Q_i t_i}{V_c} + \frac{v_w WS_c l_c \sum_i^I t_i}{2rf} + \frac{v_x(r+d)LW\sum_i^I Q_i t_i}{4V_x} \quad (7)$$

By simultaneously solving the derivatives of C_{tc} in equation (7) with respect to route space r and vehicle size S_c we find the optimal values of r^* and ${S_c}^*$:

$$r^* = \sqrt[3]{\frac{8\bar{a}Dv_wV_x^2}{v_x^2L\bar{Q}V_c}} \tag{8}$$

$$S_c{}^* = \frac{2f}{l_c}\sqrt[3]{\frac{\bar{a}^2D^2L\bar{Q}V_x}{v_wv_xV_c^2}} \quad , where\ \bar{Q} = \frac{\sum_i^I Q_it_i}{\sum_i^I t_i}, \bar{a} = \frac{a_cQ_p + \sum_i^I a_iQ_it_i}{\sum_i^I Q_it_i}, \bar{b} = \frac{b_cQ_p + \sum_i^I b_iQ_it_i}{\sum_i^I Q_it_i} \tag{9}$$

Based on the optimized vehicle size $S_c{}^*$ and route spacing r^*, bus service cost for period i can be expressed as follows:

$$C_{ci} = \frac{DW(a + bS_c^*)}{r^*V_ch_{ci}} + \frac{v_vLWQ_iM}{V_c} + \frac{v_wLWQ_ih_{ci}}{2} + \frac{v_xLWQ_i(r^* + d)}{4V_x} \tag{10}$$

The optimized headway h_{ci}^{opt} for period i can be obtained by setting the first derivative of conventional bus service cost C_{ci} to zero.

$$h_{ci}^{opt} = \sqrt{\frac{2D(a + bS_c^*)}{v_wr^*V_cLQ_i}} \tag{11}$$

Therefore, the optimal headway h_{ci}^* for each period i is the minimum of h_{ci}^{max} and h_{ci}^{opt}:

$$h_{ci}^* = \min\{h_{ci}^{max}, h_{ci}^{opt}\} = min\left\{\frac{S_c^*l_c}{r^*LfQ_i}, \sqrt{\frac{2D(a + bS_c^*)}{v_wr^*V_cLQ_i}}\right\} \tag{12}$$

The optimal fleet size F_{ci}^* for each period depends on the optimal headway of that period:

$$F_{ci}^* = \frac{DW}{r^*V_ch_{ci}^*} \tag{13}$$

The capital cost should be determined from the peak period demand, which we define to be period 1, as follows:

$$C_{pc}^* = \frac{DW}{r^*V_ch_{c1}^*}B_c = \frac{DW(a_c + b_cS_c^*)}{r^*V_ch_{c1}^*} \tag{14}$$

The bus service cost C_{ci} for each period i can be formulated using the optimal headway of that period. Therefore, the conventional bus service cost C_c for all periods can be expressed as:

$$C_c^* = \sum_i \left\{\frac{DW(a + bS_c^*)}{r^*V_ch_{ci}^*} + \frac{v_vLWQ_iM}{V_c} + \frac{v_wLWQ_ih_{ci}^*}{2} + \frac{v_xLWQ_i(r^* + d)}{4V_x}\right\}t_i \tag{15}$$

The total cost including capital cost can be found by substituting optimal route spacing r^* and optimal vehicle size $S_c{}^*$ into equation (7):

$$C_{tc}^* = \frac{DW(a_c+b_cS_c^*)}{r^*V_ch_{c1}^*} + \sum_i \left\{\frac{DW(a+bS_c^*)}{r^*V_ch_{ci}^*} + \frac{v_vLWQ_iM}{V_c} + \frac{v_wLWQ_ih_{ci}^*}{2} + \frac{v_xLWQ_i(r^*+d)}{4V_x}\right\} t_i \quad (16)$$

Flexible Bus System Cost

When considering capital cost for Flexible Bus service, the optimized vehicle size S_f^* and vehicle service area A^* are provided in Table 2 from CS. In this section, we optimize headways for flexible bus service, unlike CS which only used the maximum allowable headway. The optimal headway should be the minimum of (1) the maximum allowable headway and (2) the minimum cost headway.

The maximum allowable headway h_{fi}^{max} for demand period i is a function of optimized vehicle size S_f^*, load factor l_f, service area A^*, and demand density Q_i:

$$h_{fi}^{max} = \frac{S_f^* l_f}{A^* Q_i} \quad (17)$$

From Table 2, the flexible bus service cost for period i C_{fi} can be rewritten as:

$$C_{fi} = \frac{LW\left(a+bS_f^*\right)(D_f+ØA^*\sqrt{\frac{Q_ih_{fi}}{u}})}{A^*V_fh_{fi}} + \frac{v_vLWQ_i(D_f+ØA^*\sqrt{\frac{Q_ih_{fi}}{u}})}{2V_f} + \frac{v_wLWQ_ih_{fi}}{2} \quad (18)$$

The optimized service headway h_{fi}^{opt}can be obtained by setting the first derivative equal to zero:

$$\frac{\partial C_{fi}}{\partial h_{fi}} = -\frac{LW\left(a+bS_f^*\right)D_f}{A^*V_f}\frac{1}{h_{fi}^2} - \frac{LW\left(a+bS_f^*\right)ØA^*\sqrt{\frac{Q_i}{u}}}{2A^*V_f}\frac{1}{\sqrt{h_{fi}^3}} + \frac{v_vLWQ_iØA^*\sqrt{\frac{Q_i}{u}}}{4V_f}\frac{1}{\sqrt{h_{fi}}} + \frac{v_wLWQ_i}{2} = 0 \quad (19)$$

Equation (19) is a quartic equation with respect to headway. We will denote optimized headway as h_{fi}^{opt}. The solution of equation (19) is presented in Appendix 2. Only one of the four solutions to this equation is feasible.

We now have two headway solutions. Therefore, the optimal headway h_{fi}^* is the minimum of the (1) maximum allowable headway and (2) optimized headway obtained by solving equation (19):

$$h_{fi}^* = min\left\{\frac{S_f^*l_f}{A^*Q_i}, h_{fi}^{opt}\right\} \quad (20)$$

Based on the optimal headway for period i, the required fleet size F_{fi}^* is

$$F_{fi}^{*} = \frac{LW(D_f + \O A^{*}\sqrt{\frac{Q_i h_{fi}^{*}}{u}})}{A^{*} V_f h_{fi}^{*}} \quad (21)$$

Finally, the total cost C_{tf}^{*} which is the sum of capital cost and bus service cost for all periods can be expressed as:

$$C_{tf}^{*} = \frac{LW(a_c + b_c S_f^{*})(D_f + \O A^{*}\sqrt{\frac{Q_i h_{f1}^{*}}{u}})}{A^{*} V_f h_{f1}^{*}} + \sum_i \left\{ \frac{LW(a + bS_f^{*})(D_f + \O A^{*}\sqrt{\frac{Q_i h_{fi}^{*}}{u}})}{A^{*} V_f h_{fi}^{*}} + \frac{v_v LW Q_i (D_f + \O A^{*}\sqrt{\frac{Q_i h_{fi}^{*}}{u}})}{2V_f} + \frac{v_w LW Q_i h_{fi}^{*}}{2} \right\} t_i \quad (22)$$

VARIABLE-TYPE BUS OPERATION USING CONVENTIONAL AND FLEXIBLE BUSES

Conceptually, conventional services using relatively large buses are expected to have lower cost per passenger trip than flexible services at higher demand densities, and vice versa. In this section, we investigate the demand boundary between conventional and flexible bus services. Below this boundary, we switch service type from conventional to flexible. Pure conventional and pure flexible service costs are also compared to variable-type services.

Integer Solution for Variable-type Bus Service

In the objective function shown in equation (23), only one service type (either conventional or flexible bus) is used in each period. The constraints in equations (23.1~23.4) are required to obtain integer values for the number of routes and fleet sizes per route.

$$C_t = \min\{C_{pc} + \sum_i (C_{ci} + C_{fi}) t_i\} \quad (23)$$

Subject to

$$S_c^{*} = S_f^{*} = \text{integer} \quad (23.1)$$

$$\frac{W}{r}, \frac{LW}{A} = \text{integer} \quad (23.2)$$

$$F_{ci}^{*}, F_{fi}^{*} = \text{integer} \quad (23.3)$$

$$\frac{F_{ci}^{*}}{N}, \frac{F_{fi}^{*}}{N'} = \text{integer} \quad (23.4)$$

To obtain total cost with integer solutions, we first optimize the decision variables, and then compare their neighboring integer solutions to satisfy such constraints.

Numerical Analysis

In this section, we compute and compare bus operation costs (pure conventional bus service, pure flexible bus service, and variable-type bus service). In this numerical analysis the cumulative demand distribution over time has four values, as shown in Figure 2.

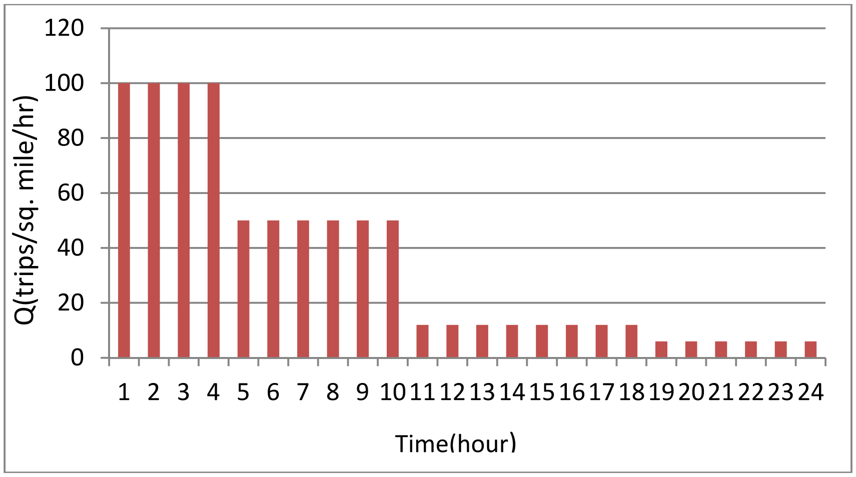

Figure 2 Demand Variation over Time

For pure conventional service cost, equations (3~38) are used in this numerical example. For flexible service cost, optimized decision variables (optimized vehicle size and service area) can be found from Tables 1 and 2. The previous CS study used the maximum allowable headway, without considering the minimum cost headway. Here, we analytically optimized the flexible bus headway in equations (17~20). More details can be found in Schonfeld et al (2010). Appendix 2 shows how flexible service headways for this formulation can be optimized.

Variable-Type Bus Service Boundary

For variable-type bus operation, the optimized bus size is usually determined by the conventional service requirements. As mentioned for constraint (23.2), integer values for both W/r and LW/A are required to obtain integer fleets. The resulting possible values of decision variables r and A are shown in Table 4.

Table 4 Possible Values of Decision Variables r and A

N, N'	r = W/N for conventional bus periods	A=LW/N' for flexible bus periods
1	4	20
2	2	10
3	1.333	6.667
4	1	5
5	0.8	4
6	0.667	3.333
…	…	…

These values in Table 4 will be used to search for the minimum cost route spacing r^* for conventional bus and minimum cost service area A^* for flexible bus services.

Procedure for finding minimum variable-type service cost

In general, if our demand distribution has k periods, then we have (k+1) possible boundaries between periods (i.e. boundary 1… k+1) when service switches from one type to another. Thus our numerical example has five possible boundaries because it has four cumulative demand periods. Variable-type service is provided when $1 < k < 5$, while k = 1 means that service is always purely flexible service and k=5 implies purely conventional service in every period.

The computation procedures for variable-type service are as follows:

1) Set up boundary k =1.
2) Based on boundary, optimize decision variables, namely vehicle sizes and route spacing for conventional operations, or service area for flexible service.
3) Optimize headway for variable-type service.
4) Compute total cost using results in 2) and 3).
5) Change boundary k to k+1 (i.e. One more period has conventional service and the remaining periods have flexible service).

6) Continue 2) ~ 4) until the total cost starts increasing. Then we have the optimal boundary which minimizes total cost for variable-type service.

Numerical Analysis Result

The results obtained with baseline inputs are provided in Table 1. The optimized pure conventional bus service costs $107,166/day, including capital costs and user time costs. To operate conventional bus service with given demand density, 60 buses are required. Vehicle size is optimized to satisfy all demand periods, at 40 seats/bus. The local area route spacing is jointly optimized (subject to a constraint requiring an integer number of zones) at one mile.

Pure flexible bus results show that the optimized total cost for serving this demand is about $118,377/day, which is much costlier than pure conventional bus service. The reason is that the optimized flexible services use many more zones (9 versus 4) and vehicles, but much smaller vehicles, than optimized conventional services. Moreover, pure flexible service requires more buses to cover peak demand since its optimized vehicles are smaller than for conventional bus. As shown in Table 5, flexible service requires 108 buses in the peak period, which increases capital cost.

For variable-type service, the pure conventional bus size of 40 seats is used in all periods for both conventional and flexible operations as well as for capital cost computation. In this numerical analysis, we find that variable-type service is preferable to pure bus services. Therefore, select conventional service in periods 1 and 2, and flexible service in periods 3 and 4, using the same bus size.

With variable-type service (using flexible service in period 3 and 4), we reduce cost compared to both pure conventional and pure flexible services. Compared to pure conventional service, variable-type service saves $1,382/day. Similarly, variable-type service costs about $12,600/day less than pure flexible service.

Table 5 Numerical Results with Baseline Inputs

	Pure Conventional Service	Pure Flexible Service	Variable-type Service	
S_c, S_f (seats/bus)	40	23	40	
r, A	1	2.222	0.8	6.667
N	4	9	5	3
h1(hrs)	0.078	0.089	0.097	
h2(hrs)	0.146	0.177	0.194	
h3(hrs)	0.389	0.510		0.269
h4(hrs)	0.583	0.476		0.340
F1(vehicles)	60	108	60	
F2(vehicles)	32	54	30	
F3(vehicles)	12	18		12
F4(vehicles)	8	18		9
C1($/hr)	10,676.7	11,175.1	10,430.0	
C2($/hr)	5,822.7	8,111.1	5,798.3	
C3($/hr)	1,911.6	5,085.4		1927.3
C4($/hr)	1,171.8	1,824.9		1109.3
t1(hrs)	4	4	4	
t2(hrs)	6	6	6	
t3(hrs)	8	8		8
t4(hrs)	6	6		6
Cp($/day)	7,200.0	12,042.0	7,200.0	
TC($/day)	107,166.3	118,376.8	105,784.3	
% Change	1.290 %	10.64 %		

SENSITIVITY ANALYSIS

Sensitivity analyses are conducted to explore the relative merits of conventional, flexible and variable-type bus services in different circumstances. Seven cases are presented below.

Case I –Directional Demand Split Factor (*f*) for Conventional Service.

The directional demand split was changed to 75% & 25% (vs. 100% & 0% in the baseline). In Table 6, the total costs of conventional and variable-type service in Case I decrease compared to the baseline results in Table 5. In this case, variable-type service reduces total cost by 1.39% from pure conventional and 11.67% from pure flexible service, respectively. In this case I with f=0.75, a directional demand split factor can slightly reduce costs below the baseline case.

Table 6 Sensitivity Analysis Results of Directional Demand Split Factor

	Pure Conventional Service	Pure Flexible Service	Variable Type Service	
S_c, S_f (seats/bus)	31	23	31	
r, A	1	2.857	0.8	6.667
N	4	7	5	3
h1(hrs)	0.078	0.066	0.097	
h2(hrs)	0.146	0.154	0.194	

h3(hrs)	0.389	0.334		0.269
h4(hrs)	0.583	0.487		0.340
F1(veh)	60	112	60	
F2(veh)	32	49	30	
F3(veh)	12	21		12
F4(veh)	8	14		9
C1($/hr)	10,568.7	11,481.7	10,322.0	
C2($/hr)	5,765.1	6,087.9	5,744.3	
C3($/hr)	1,890.0	1,990.4		1,897.1
C4($/hr)	1,157.4	1,220.0		1,087.7
t1(hrs)	4	4	4	
t2(hrs)	6	6	6	
t3(hrs)	8	8		8
t4(hrs)	6	6		6
Cp($/day)	6,930.0	12,488.0	6930.0	
TC($/day)	105,859.5	118,185.3	104,387.2	
% Change	1.39 %	11.67 %	-	

Case II – Load Factors

In case II, maximum load factors for both conventional and flexible service are increased from 1 to 1.25 (implying that some standees are allowed). Table 6 shows the resulting costs. We note that the costs of pure conventional service in Table 7 are below the baseline case (Table 5). Similarly, pure flexible and variable-type services benefit from higher load factors. However, similarly to Case I, the effect of variable-type service is saving about 1.41% and 9.71% savings compared to pure conventional and pure flexible service, respectively.

Table 7 Sensitivity Analysis Results of Load Factors

	Pure Conventional Service	Pure Flexible Service	Variable Type Service	
S_c, S_f (seats/bus)	32	23	32	
r, A	1	3.333	0.667	6.667
N	4	6	6	3
h1(hrs)	0.078	0.068	0.117	
h2(hrs)	0.146	0.118	0.194	
h3(hrs)	0.389	0.340		0.209
h4(hrs)	0.583	0.494		0.340
F1(veh)	60	96	60	
F2(veh)	32	54	36	
F3(veh)	12	18		15
F4(veh)	8	12		9
C1($/hr)	10,580.7	11,391.4	10,247.3	
C2($/hr)	5,771.5	6,149.2	5,808.7	
C3($/hr)	1,892.4	1,940.2		1,896.5
C4($/hr)	1,159.0	1,176.9		1,090.1
t1(hrs)	4	4	4	
t2(hrs)	6	6	6	

t3(hrs)	8	8		8
t4(hrs)	6	6		6
Cp($/day)	6,960.0	10,704.0	6960.0	
TC($/day)	106,004.7	115,747.8	104514.3	
% Change	1.41 %	9.71 %	-	

Case III – Service Period Demand Variation (Q1=10, Q2=5, Q3=1.2, Q4=0.6 trips/sq. mile)

This case explores the effect of very low demand density (i.e. 10% of baseline value). Here the costs of pure conventional and flexible operation are very close. With variable-type service, as shown in Table 8, we can save 3.19% and 4.06% from pure conventional and flexible services, respectively. It is interesting here that conventional service is only used during the highest demand period, leaving the other three periods to flexible service.

Table 8 Sensitivity Analysis Results of Demand Variation

	Pure Conventional Service	Pure Flexible Service	Variable-Type Service	
S_c, S_f (seats/bus)	17	16	17	
r, A	2.0	10.0	2.0	10.0
N	2	2	2	2
h1(hrs)	0.167	0.111	0.167	
h2(hrs)	0.292	0.198		0.253
h3(hrs)	1.167	0.472		0.472
h4(hrs)	1.167	0.944		0.944
F1(veh)	14	18	14	
F2(veh)	8	10		8
F3(veh)	2	4		4
F4(veh)	2	2		2
C1($/hr)	1,653.9	1,692.3	1,653.9	
C2($/hr)	935.4	930.2		903.0
C3($/hr)	353.2	317.7		318.7
C4($/hr)	210.0	192.8		193.4
t1(hrs)	4	4	4	
t2(hrs)	6	6		6
t3(hrs)	8	8		8
t4(hrs)	6	6		6
Cp($/day)	1,519.0	1,944.0	1519.00	
TC($/day)	17,832.1	17,992.9	17,262.8	
% Change	3.19 %	4.06 %		

Case IV - Service Period Time Variation (t1=2, t2=4, t3=4, t4=14)

In the baseline case (Table 5) there are 4, 6, 8, and 6 hours, respectively, in periods 1, 2, 3 and 4. In Case IV we explore the effect of higher demand variability by changing those four periods to 2, 4, 4 and

14 hours. The results in Table 9 show that variable-type bus service now achieves much greater savings compared to the baseline case (Table 5). These savings are about 3.41% and 13.08% compared to pure services, while in the baseline (Table 5), variable-type bus service cost savings from pure conventional service are about 1.29 %.

Based on the sensitivity of results in these cases, we find that significant advantages of variable-type bus service occur when we have long periods of low demand that is far below peak levels.

Table 9 Sensitivity Analysis Results of Service Time Variation

	Pure Conventional Service	Pure Flexible Service	Variable Type Service	
S_c, S_f (seats/bus)	40	21	40	
r, A	1.333	3.333	0.667	10.0
N	3	6	6	2
h1(hrs)	0.058	0.052	0.117	
h2(hrs)	0.117	0.105	0.194	
h3(hrs)	0.389	0.340		0.179
h4(hrs)	0.583	0.494		0.259
F1(veh)	60	120	60	
F2(veh)	30	60	36	
F3(veh)	9	18		12
F4(veh)	6	12		8
C1($/hr)	11,243.3	11,775.8	10,343.3	
C2($/hr)	5,971.7	6,202.7	5,866.3	
C3($/hr)	1,893.6	1,931.7		1,958.9
C4($/hr)	1,143.8	1,171.4		1,096.0
t1(hrs)	2	2	2	
t2(hrs)	4	4	4	
t3(hrs)	4	4		4
t4(hrs)	14	14		14
Cp($/day)	7,200.0	13,260.0	7,200.0	
TC($/day)	77,160.9	85,748.9	74531.6	
% Savings	3.41 %	13.08 %		

Case V – Operating Cost Parameters (a=45, b=0.3)

In this case, we explore the sensitivity of total cost and other results to bus operating cost that is a linear function of number of seats (i.e. B=a+bS). Here, we increase parameter a & b values by 50 %. The results in Table 10 show that we have the cheapest total cost by providing variable-type service When we operating variable-type service, we have 1.379% and 14.86% savings compared to pure conventional and flexible services, respectively.

Table 10 Sensitivity Analysis Results of Operating Cost Input Parameters

	Pure Conventional Service	Pure Flexible Service	Variable Type Service	
S_c, S_f (seats/bus)	50	27	50	
r, A	1	2.857	1	6.667
N	4	7	4	3
h1(hrs)	0.097	0.077	0.097	
h2(hrs)	0.194	0.154	0.194	
h3(hrs)	0.583	0.527		0.269
h4(hrs)	1.167	0.487		0.340
F1(veh)	48	98	48	
F2(veh)	24	49	24	
F3(veh)	12	14	12	
F4(veh)	4	14		3
C1($/hr)	11,510.0	13,381.0	11,510.0	
C2($/hr)	6,338.3	7,153.1	6,338.3	
C3($/hr)	2,215.6	2,357.3	2,175.6	
C4($/hr)	1,527.8	1,510.0		1,312.3
t1(hrs)	4	4	4	
t2(hrs)	6	6	6	
t3(hrs)	8	8	8	
t4(hrs)	6	6		6
Cp($/day)	6,000.0	11,123.0	6000.0	
TC($/day)	116,961.6	135,483.6	115348.9	
% Savings	1.379 %	14.86 %		

Case VI – Length of Service Region (L=6 miles)

In case VI we increase the service region length by 20% (from 5 to 6 miles). We note, for variable-type service, that conventional bus serves periods 1, 2, and 3; flexible bus only serves period 4. This result shows that as the local service region lengthens, the potential savings of variable-type service decrease because, region lengthens, demand also increases, thus favoring conventional service. In Table 11, Period 3 in variable-type service is served by conventional service, unlike in the baseline case (Table 5).

Table 11 Sensitivity Analysis Results of Service Region Length

	Pure Conventional Service	Pure Flexible Service	Variable Type Service	
S_c, S_f (seats/bus)	45	25	45	
r, A	1	3.429	1	8
N	4	7	4	3
h1(hrs)	0.075	0.057	0.075	
h2(hrs)	0.141	0.122	0.141	
h3(hrs)	0.422	0.352	0.422	
h4(hrs)	0.633	0.511		0.358
F1(veh)	68	133	68	
F2(veh)	36	63	36	

F3(veh)	12	21	12	
F4(veh)	8	14		9
C1($/hr)	12,980.9	14,299.2	12,980.9	
C2($/hr)	7,045.3	7,524.1	7,045.3	
C3($/hr)	2,308.3	2,370.2	2,308.3	
C4($/hr)	1,414.6	1,435.0		1,351.4
t1(hrs)	4	4	4	
t2(hrs)	6	6	6	
t3(hrs)	8	8	8	
t4(hrs)	6	6		6
Cp($/day)	8,330.0	14,962.5	8,330.00	
TC($/day)	129,479.7	144,875.7	129,100.6	
% Savings	0.29 %	10.89 %		

Case VII – Line-haul Distance (J=20 miles, J/L = 4)

In Case VII, we analyze sensitivity to line-haul distance (from 10miles to 20miles). Here the ratio of line-haul distance/length of local area (i.e. J/L) is increased from 2 to 4. Table 12 shows variable-type service reduces total cost by 0.704% and 11.12% compared to pure services. By increasing line-haul distance (without changing demand), round trip time increases for both conventional and flexible service, favoring larger vehicles because bus operator wants to carry more passengers in round trip time. Thus, in Table 12, vehicle size for variable-type service is 50 seats/bus, but only 40 seats/bus in the baseline case (Table 5) is 40 seats/bus. With variable-type service, we save service cost in Periods 1 & 4 compared to pure conventional service. These service cost savings and capital cost savings allow variable-type service favorable compared to pure services.

Table 12 Sensitivity Analysis Results of Line-haul Distance

	Pure Conventional Service	Pure Flexible Service	Variable Type Service	
S_c, S_f (seats/bus)	50	31	50	
r, A	1	4	0.8	10
N	4	5	5	2
h1(hrs)	0.096	0.066	0.123	
h2(hrs)	0.191	0.139	0.246	
h3(hrs)	0.431	0.405		0.239
h4(hrs)	0.574	0.523		0.324
F1(veh)	72	125	70	
F2(veh)	36	60	35	
F3(veh)	16	20		14
F4(veh)	12	15		10
C1($/hr)	14,269.3	15,954.5	14,037.3	
C2($/hr)	7,708.7	8,347.3	7,756.7	
C3($/hr)	2,488.9	2,546.6		2,539.7

C4($/hr)	1,507.8	1,523.0		1,422.5
t1(hrs)	4	4	4	
t2(hrs)	6	6	6	
t3(hrs)	8	8		8
t4(hrs)	6	6		6
Cp($/day)	9,000.0	14,437.5	8,750.0	
TC($/day)	141,287.5	157,850.3	140,292.1	
% Savings	0.704 %	11.12 %		

CONCLUSIONS

In this paper optimization models are developed for analyzing and integrating conventional services (having fixed routes and schedules) and flexible bus services. The optimization models are improved from those of Chang and Schonfeld (1991a), mainly by (1) optimizing the flexible service headways rather than just using maximum allowable headways and (2) introducing directional demand split factors. These models are used to compare pure conventional services, pure flexible services, and variable-type services which can switch between conventional and flexible service as the demand changes over time.

Our numerical analysis indicates that variable-type bus operation can reduce total cost compared to a pure conventional bus or pure flexible bus service. In our baseline case, variable-type service can reduce costs by about 1.29% compared to pure conventional service and about 10.64% compared to pure flexible service. Moreover, we present various sensitivity analyses to explore how major parameter changes affect the optimized results. In case IV (when service periods are adjusted to increase the variability of demand over time), we find that variable-type service can reduce costs by more than 3.41% and 13.08 %, respectively, compared to pure conventional and flexible services. These results confirm that such variable-type services are especially promising for systems whose demand (1) varies greatly over time and (2) straddles the threshold between conventional and flexible services.

To summarize, we confirm that conventional service with large buses is preferable when demand is high. Similarly, flexible service is less costly at relatively low demand. A public bus system alternating among these two service concepts based on demand variation and other conditions can be used to improve service efficiency. In extending this study, we should explore how service type can be best matched to demand in regions where the demand varies over space as well as time.

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APPENDIX 1 Conventional Service Cost Formulation

As we can see in Figure 1, buses travel from the terminal a line haul distance J at non-stop speed yV_c to a corner of the service area, then travel an average of $W/2$ miles at local non-stop speed zV_c from the corner to the assigned zone, run a distribution route of length L at local speed V_c along the central axis of the zone while stopping for passengers every d miles, and the reverse the process in returning. Therefore, the buses' average round trip time is:

$$R_c = \frac{2J}{yV_c} + \frac{W}{zV_c} + \frac{2L}{V_c} \qquad \text{(A-1)}$$

This round trip time can be re-written as:

$$R_c = \left\{\frac{2J}{y} + \frac{W}{z} + 2L\right\}/V_c \qquad \text{(A-2)}$$

In equation (A-2), the expression in parentheses represents an equivalent vehicle round trip distance, D.

The total cost of conventional bus service includes the operator cost C_{oc} and the user costs C_{uc}. To determine operator cost, we determine the fleet size N, which is the total vehicle round trip time divided by the headway. With the equivalent vehicle round travel distance D, a controllable directional split factor f, and conventional bus speed V_c, we obtain required fleet size F_c :

$$F_c = \frac{DW}{rh_cV_c} \quad ,where\ D = 2J/y + W/z + 2L \qquad \text{(A-3)}$$

The hourly conventional bus operator cost C_{oc} is the required fleet size multiplied by bus operating cost:

$$C_{oc} = F_cB \qquad \text{(A-4)}$$

The bus operating cost B is formulated as:

$$B = a + bS_c \qquad \text{(A-5)}$$

and the required service headway h_c is:

$$h_c = \frac{S_cl_c}{rLfQ} \qquad \text{(A-6)}$$

The operating cost C_{oc} can be reformulated by substituting equations (A-3), (A-5), and (A-6) into equation (A-4):

$$C_{oc} = \frac{D(a+bS_c)LWfQ}{l_cV_cS_c} \qquad \text{(A-7)}$$

The hourly user cost for the conventional bus system C_{uc} is the sum of in-vehicle cost C_{vc}, waiting cost C_{wc}, and access cost C_{xc}:

$$C_{uc} = C_{vc} + C_{wc} + C_{xc} \tag{A-8}$$

The hourly in-vehicle cost for the conventional system is then:

$$C_{vc} = v_v LWQt \tag{A-9}$$

The average travel time t per passenger trip is formulated as:

$$t = \frac{J}{yV_c} + \frac{W}{2zV_c} + \frac{L}{2V_c} = \frac{M}{V_c} \text{ ,where } M=J/y + W/2z + L/2 \tag{A-10}$$

Then equation (A-9) can be written as:

$$C_{vc} = v_v LWQ \frac{M}{V_c} \tag{A-11}$$

We assume the average waiting time is half the headway. Therefore, the hourly user waiting cost for conventional system C_{wc} is:

$$C_{wc} = v_w LWQ \frac{h_c}{2} = v_w LWQ \frac{S_c l_c}{2rLfQ} = \frac{v_w W S_c l_c}{2rf} \tag{A-12}$$

Since the spacing between adjacent branches of local bus service is *r*, and since service trip origins (or destinations) are uniformly distributed over the area, the average access distance to the nearest route is one-fourth of route spacing, *r/4*. Similarly, the access distance alongside the route to the nearest transit stop is one-fourth of the bus stop spacing, i.e., *d/4*. Therefore, the hourly access cost for the conventional bus system C_{xc} is:

$$C_{xc} = \frac{v_x LWQ(r+d)}{4V_x} \tag{A-13}$$

The total cost for the conventional system C_c is the sum of operating cost and user costs:

$$C_c = \frac{D(a+bS_c)LWfQ}{l_c V_c S_c} + \frac{v_v LWQM}{V_c} + \frac{v_w W S_c l_c}{2rf} + \frac{v_x LWQ(r+d)}{4V_x} \tag{A-14}$$

In equation (A-14), the optimizable variables are routing space *r* and vehicle size S_c, which we optimize by taking partial derivatives of C_c in equation (A-14). Setting the partial derivatives equal to zero and solving simultaneously, we obtain:

$$S_c^* = \frac{2f}{l_c}\sqrt[3]{\frac{a^2D^2LQV_x}{v_w v_x V_c^2}} \quad \text{(A-15)}$$

$$r^* = \sqrt[3]{\frac{8aDv_wV_x^2}{v_x^2LQV_c}} \quad \text{(A-16)}$$

The second derivatives of equation (A-14) with respect to vehicle size S_c and routing space r are positive for any reasonable inputs. Therefore, equations (A-15 and A-16) yield the globally minimal total cost. From equations (A-15 and A-16) we can observe that product of the optimized vehicle size and optimized route spacing is constant (i.e, $S_c^* \times r^* = (4faDV_x)/(l_c v_x V_c) = \text{constant}$).

After optimizing vehicle size S_c^* and route spacing r^*, we optimize the headway h_c^* which minimizes total cost C_c. Optimal headway h_c^* should be the minimum of the maximum allowable headway and minimum cost headway. The maximum allowable headway h_c^{max} can be found by substituting equations (A-15) and (A-16) into equation (A-6).

$$h_c^{max} = \frac{S_c^* l_c}{r^* LfQ} \quad \text{(A-17)}$$

The optimized headway h_c^{opt} can be found from the total cost function, which is provided in equation (A-18), by setting its first derivative equal to zero. The second derivative is positive. Therefore, the optimized headway will yield the globally minimal total cost.

$$C_c = \frac{DW(a+bS_c)}{rV_ch_c} + \frac{v_vLWQM}{V_c} + \frac{v_wLWQh_c}{2} + \frac{v_xLWQ(r+d)}{4V_x} \quad \text{(A-18)}$$

The resulting minimum cost headway is:

$$h_c^{opt} = \sqrt{\frac{2D(a+bS_c^*)}{v_w r^* V_c LQ}} \quad \text{(A-19)}$$

Overall, the optimal headway h_c^* is then:

$$h_c^* = min\left\{\frac{S_c^* l_c}{r^* LfQ}, \sqrt{\frac{2D(a+bS_c^*)}{v_w r^* V_c LQ}}\right\} \quad \text{(A-20)}$$

By substituting equations (A-16) and (A-17) into equation (A-3) we obtain the optimal fleet size F_c^* for the conventional bus system:

$$F_c{}^* = \frac{DW}{r^* h_c^* V_c} \quad \text{(A-21)}$$

Therefore, the bus service cost based on the jointly optimized vehicle size S_c^*, route spacing r^*, and optimal headway h_c^* is:

$$C_c = \frac{DW(a+bS_c^*)}{r^* V_c h_c^*} + \frac{v_v LWQM}{V_c} + \frac{v_w LWQ h_c^*}{2} + \frac{v_x LWQ(r^*+d)}{4V_x} \quad \text{(A-22)}$$

APPENDIX 2 Optimized Headway for Flexible Service

$$-\frac{LW\left(a+bS_f^*\right)D_f}{A^* V_f}\frac{1}{h_{fi}^2} - \frac{LW\left(a+bS_f^*\right)\emptyset A^* \sqrt{\frac{Q_i}{u}}}{2A^* V_f}\frac{1}{\sqrt{h_{fi}^3}} + \frac{v_v LWQ_i \emptyset A^* \sqrt{\frac{Q_i}{u}}}{4V_f}\frac{1}{\sqrt{h_{fi}}} + \frac{v_w LWQ_i}{2} = 0 \quad \text{(19, A-23)}$$

In equation (A-23), we can substitute $\frac{1}{\sqrt{h_{fi}}}$ into *t*. Therefore, equation (A-23) becomes

$$-\frac{LW\left(a+bS_f^*\right)D_f}{A^* V_f}t^4 - \frac{LW\left(a+bS_f^*\right)\emptyset A^* \sqrt{\frac{Q_i}{u}}}{2A^* V_f}t^3 + 0 \times t^2 + \frac{v_v LWQ_i \emptyset A^* \sqrt{\frac{Q_i}{u}}}{4V_f}t^1 + \frac{v_w LWQ_i}{2} = 0 \quad \text{(A-24)}$$

Equation (A-24) can be rewritten as:

$$At^4 + Bt^3 + Ct^2 + Dt^1 + E = 0 \quad \text{(A-25)}$$

where $A = -\frac{LW\left(a+bS_f^*\right)D_f}{A^* V_f}, B = -\frac{LW\left(a+bS_f^*\right)\emptyset A^* \sqrt{\frac{Q_i}{u}}}{2A^* V_f}, C = 0, D = \frac{v_v LWQ_i \emptyset A^* \sqrt{\frac{Q_i}{u}}}{4V_f}, and\ E = \frac{v_w LWQ_i}{2}$

To solve equation (A-25), we need to compute P, Q, R, S, T, V values using A, B, C, D and E.

$$P = \frac{B}{4A}, Q = \frac{2C}{3A}, R = C^2 - 3BD + 12AE, S = 2C^2 - 9BCD + 27AD^2 + 27EB^2 - 72ACE$$

$$T = -\frac{B^3}{A^3} + \frac{4BC}{A^2} - \frac{8D}{A}, V = \frac{\sqrt[3]{2}R}{3A\sqrt[3]{S+\sqrt{-4R^3+S^2}}} + \frac{\sqrt[3]{S+\sqrt{-4R^3+S^2}}}{3\sqrt[3]{2}A} \quad \text{(A-26)}$$

After finding the values of P, Q, R, S, T and V, we obtain the following results:

$$X1 = -P - \frac{1}{2}\sqrt{4P^2 - Q + V} - \frac{1}{2}\sqrt{8P^2 - 2Q - V - \frac{T}{4\sqrt{4P^2 - Q + V}}}$$

$$X2 = -P - \frac{1}{2}\sqrt{4P^2 - Q + V} + \frac{1}{2}\sqrt{8P^2 - 2Q - V - \frac{T}{4\sqrt{4P^2 - Q + V}}}$$

$$X3 = -P + \frac{1}{2}\sqrt{4P^2 - Q + V} - \frac{1}{2}\sqrt{8P^2 - 2Q - V + \frac{T}{4\sqrt{4P^2 - Q + V}}}$$

$$X4 = -P + \frac{1}{2}\sqrt{4P^2 - Q + V} + \frac{1}{2}\sqrt{8P^2 - 2Q - V + \frac{T}{4\sqrt{4P^2 - Q + V}}} \quad \text{(A-27)}$$

X1~X4 correspond to t (=$\frac{1}{\sqrt{h_{fi}}}$). Therefore, among the four solutions, the only one feasible solution satisfying both t>0 and h_{fi} >0 is the optimized headway h_{fi}^{opt}.