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ABSTRACT

Under nonlinear road pricing (or tolling), the price charged is not strictly proportional to the distance travelled inside a tolling area, the generalized travel cost is not link-wise additive, and finding a user equilibrium distribution is typically formulated as a complementarity problem. The latter is a difficult problem to solve in mathematical programming. In this report, we use piecewise linear functions to determine tolls and show that finding a user equilibrium distribution with such functions can be formulated as a convex optimization problem that is based on path flows and solvable by traditional algorithms such as simplicial decomposition. For area-based and two-part pricing schemes, the tolling function consists of only one linear piece and finding a user equilibrium distribution reduces to a convex optimization problem formulated in terms of link flows and solvable by any software for linearly constrained convex programs.

To find an optimal pricing scheme, e.g., one that maximizes the social benefit, we formulate the problem as a mathematical program with equilibrium constraints, an optimization problem that is generally non-convex and difficult to solve. However, it is possible to use search algorithms to find an optimal scheme because the number of parameters in our piecewise linear function is few. To illustrate, we use a coordinate search algorithm to find an optimal two-part pricing scheme for a small network with randomly generated data.

EXECUTIVE SUMMARY

This report describes the following:

- Procedures/algorithms for predicting traffic patterns in response to charging usage fees for particular areas (or tolling areas) in a road network. The literature on transportation contains procedures for similar tasks under the assumption that the usage fee is either based on a constant rate (e.g., per vehicle-mile traveled or VMT) or a one time (or access) fee. The procedures in the report allow the fee to be nonlinear, a topic not well-addressed in the literature. For example,

  - The fee can consist of two components, access and VMT fee. (In economics, this is often refers to as *two-part pricing*.

  - There can be two different fees, one of low road-usage and the other for high. For example, the fee for this first 10 miles can be $0.50 per mile and it is $0.75 for each additional mile in excess of 10.

- A procedure/algorithm for determining optimal nonlinear fees. Again, this topic is not well addressed in the literature. This procedure relies on the procedures for predicting traffic discussed above to evaluate congestion or pollution levels inside the tolling areas.

CHAPTER 1: INTRODUCTION

Nonlinear pricing generally refers to a case in which the price or tariff is not strictly proportional to the quantity purchased. Economists have been studying such pricing since the discussion of its manifestations in Dupuit (1894) and the later categorization of the phenomenon in Pigou (1920). Today, nonlinear pricing is prevalent in many industries. For example, railroad tariffs generally depend on the weight, volume, and distance of each shipment. However, those using full-cars and/or over long distances often receive discounts. The price per kilowatt-hour of electricity is different for different types of users. Heavy users during peak hours generally pay higher rates. Airlines routinely offer discount tickets for advance purchase, with non-cancellation restriction, and in competitive markets. In each of these examples, the average price paid per unit varies depending on characteristics of the purchase such as its size, time of usage, and restrictions.

In practice, road pricing is often nonlinear. The tolls in, e.g., Singapore (Menon et al., 1993), London (Santos and Shaffer, 2004), and Stockholm (Stockholmsforsoket, 2006) are not proportional to the distance travelled inside the tolling areas. In Stockholm, tolls are also not proportional to the number of times a user enters the tolling area. The amount of tolls paid on a given day is limited to SEK 60. After paying this maximum amount, users can freely enter the tolling area for the rest of the day. For its congestion charge, London offers monthly and annual passes to frequent users at an approximately 15% discount. Similarly, the Dulles Greenway's VIP Frequent Rider Program gives rebates to users with high mileage. During phase I of its Value Pricing Project on Interstate 15, San Diego sold $50 monthly permits that allow single occupancy vehicles to use lanes reserved for high occupancy vehicles. (During phase II, the permits were replaced by tolls.)

Despite its widespread use, the literature on nonlinear road pricing is limited. De Borger (2001) proposes a discrete choice model to study optimal two-part tariffs in the presence of externalities. In their nonlinear pricing study, Wang et al. (2011) consider three questions: which nonlinear pricing scheme (among the five they consider) is most profitable, how does the most profitable choice depend on congestion, and does usage-only pricing necessarily denominate other nonlinear schemes if congestion is severe? Both De Borger (2001) and Wang et al. (2011) opine that nonlinear pricing has been largely overlooked in the literature. Separate from the previous two papers, Gabriel and Bernstein (1997a) formulate the problem of finding a user equilibrium (UE) distribution on general road networks (or, more simply, the UE problem) when travel costs are not link-wise additive as a nonlinear complementarity problem or NCP. In their formulation, one component of the path travel cost is a nonlinear function of its travel distance. To solve their UE problem, Gabriel and Bernstein (1997a) propose an algorithm based on nonsmooth equations and sequential quadratic programming (see also Gabriel and Bernstein, 1997b). Lo and Chen (2000) consider a similar problem and convert their NCP into an unconstrained optimization problem based on a merit function. More recently, Agdeppa et al.

(2007) modify the model in Gabriel and Bernstein (1997a) by introducing a disutility function and formulate the problem as a monotone mixed complementarity problem instead. Maruyama and Harata (2006) and Maruyama and Sumalee (2007) propose an algorithm for area-based pricing, one form of nonlinear pricing. The authors of the last two papers observe that area-based pricing is not link-wise additive and it may be intuitive to conclude that there exists no equilibrium condition based on link flows. As demonstrated below, this intuition is incorrect.

This report considers nonlinear pricing in the context of managing travel demand, reducing congestion, and, perhaps, lessening the environmental impact in a tolling area. Although it is common to assume that a tolling area consists of connected roads or roads in a connected geographical area, such an assumption is unnecessary. For example, a tolling area can consist of not necessarily connected roads or highways that are under the jurisdiction of a single entity (a government agency or private company). It is also possible to let the tolling area be the entire road network and every road user must pay tolls. Doing so reduces our problem to the one addressed in Gabriel and Bernstein (1977a).

In this report, the amount of toll that users pay, $T(\ell)$, varies nonlinearly with ℓ, the distance travelled inside the tolling area. (Henceforth, $T(\ell)$ is also referred to as the tolling or pricing function.) We assume that $T(\ell)$ is piecewise linear and the number of linear pieces is two or less. As observed in Wilson (1993), a piecewise linear function with a small number of linear pieces is easier to understand, thus more practical, and can realize most of the advantages of general nonlinear pricing functions. As demonstrated below, the UE problem with piecewise linear pricing functions reduces to an optimization problem that is similar to the standard UE problem (see, e.g., Florian and Hearn, 2003) and solvable by well-known algorithms such as simplicial decomposition. For area-based and two-part pricing schemes, both user equilibrium conditions and the UE problem can be formulated in term of link flows despite the fact that the generalized cost is not link-wise additive. Solving the link-based UE problem eliminates the need to maintain information about individual paths and typically requires less computational resources. In fact, the UE problem with area-based and two-part pricing schemes can be solved by any software for linearly constrained convex programs.

To our knowledge, there has been little or no attempt to find an optimal nonlinear pricing scheme for a general road network. To find an optimal scheme, De Borger (2001) assumes that the travel demand is measured in kilometres without an explicit road network. Similarly, Wang et al. (2011) consider a network with only one link. In this report, we formulate the problem of finding a nonlinear pricing scheme that, e.g., maximizes the social benefit as a mathematical program with equilibrium constraints. We demonstrate that such a problem can be solved using a search algorithm when the tolling function is piecewise linear.

For the remainder, Chapter 2 describes the pricing functions considered in this report. Chapter 3 defines our notation and states path-based UE conditions for later reference. Chapter 4 formulates the UE problem in terms of path flows and modifies simplicial decomposition to find a UE flow-demand pair under our nonlinear pricing functions. Chapter 5 states link-based UE conditions and discusses when these conditions are equivalent to those based on paths. Chapter 6 presents a search algorithm for finding optimal pricing parameters, e.g., those that maximize the social benefit. Finally, Chapter 7 studies numerical results from a small road network with randomly generated data and Chapter 8 concludes the report. To illustrate the simplicity of using link-based conditions and problems, the Appendix gives a version of the Frank-Wolfe algorithm (a well-known algorithm for linearly constrained convex programs) for solving the UE problem with two-part pricing.

CHAPTER 2: NONLINEAR PRICING FUNCTIONS

The tolling function, $T(\ell)$, in this report is of the form:

$$T(\ell) = \begin{cases} T^{min}(\ell) \text{ or } T^{max}(\ell), & \ell > 0 \\ 0, & \ell \leq 0 \end{cases}$$

where $T^{min}(\ell) = \min\{\beta_1 + \mu_1\ell, \beta_2 + \mu_2\ell\}$ and $T^{\max}(\ell) = \max\{\beta_1 + \mu_1\ell, \beta_2 + \mu_2\ell\}$. Recall that ℓ is the distance travelled inside the tolling area. (Herein, distances are measured in miles and we refer to a rate or fee based on miles travelled as a "VMT fee", where VMT is an abbreviation for "vehicle-mile travelled.") In both $T^{min}(\ell)$ and $T^{max}(\ell)$, μ_1 and μ_2 are nonnegative VMT fees. Typically, β_1 and β_2 are nonnegative. However, one may be negative to reproduce some tolling functions in practice more accurately. (See the discussion about three-part tariffs below.)

Both $T^{min}(\ell)$ and $T^{max}(\ell)$ are piecewise linear functions with two linear pieces. Although the number of linear pieces can be larger, i.e., $T^{min}(\ell) = \min\{\beta_1 + \mu_1\ell, \cdots, \beta_n + \mu_n\ell\}$ and $T^{max}(\ell) = \max\{\beta_1 + \mu_1\ell, \cdots, \beta_n + \mu_n\ell\}$, where $n \geq 2$, we set $n = 2$ in this report for two reasons. First, the results for $n = 2$ can be extended to the cases with larger n without much difficulty. As cautioned in Wilson (1993), the second reason is that large n is often *not* practical. Pricing functions with many linear pieces generally result in tolling schemes too complex for motorists to understand and respond properly. Moreover, pricing functions with only a few linear pieces can typically capture most of the benefits offered by those with many.

When β_1, μ_1, β_2 and μ_2 are chosen appropriately, $T^{\min}(\ell)$ and $T^{\max}(\ell)$ capture common nonlinear pricing functions in the economics and road pricing literature (see, e.g., Wilson, 1993 and Wang et al., 2011). Figure 2.1 displays tolling functions based on $T^{min}(\ell)$.

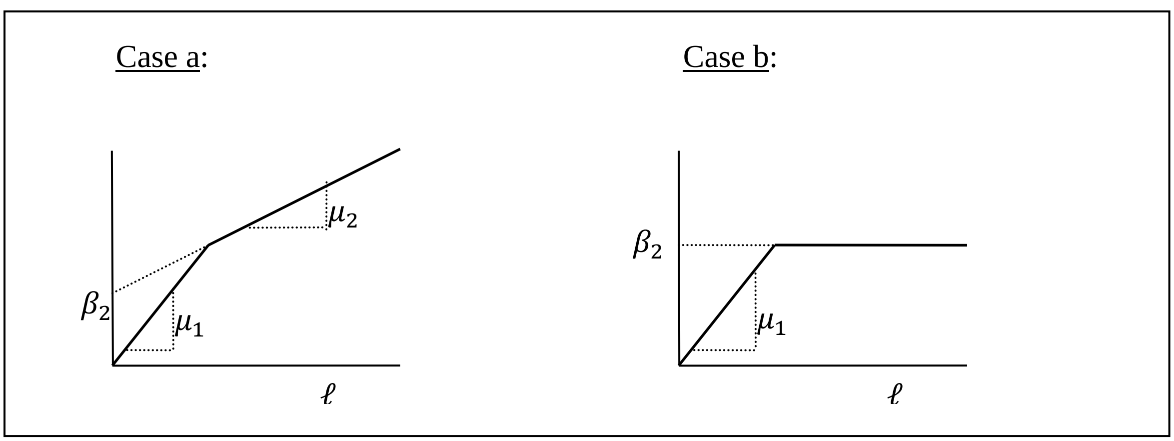

Figure 2.1: Pricing functions based on $\boldsymbol{T^{\min}(\ell)}$

In case (a), the VMT fee for a longer distance (μ_2) is smaller than the one for a shorter distance (μ_1), i.e., heavy road users receive discounts. Case (b) allows users to either pay a VMT fee at a rate μ_1 or a fixed fee, β_2, for unlimited travel inside the tolling area. The former is more economical when the travel distance is sufficiently short, i.e., less than the point where $\mu_1 \ell = \beta_2$. Although both cases may be suitable for many industries, it is not clear that they would be adopted for congestion mitigation.

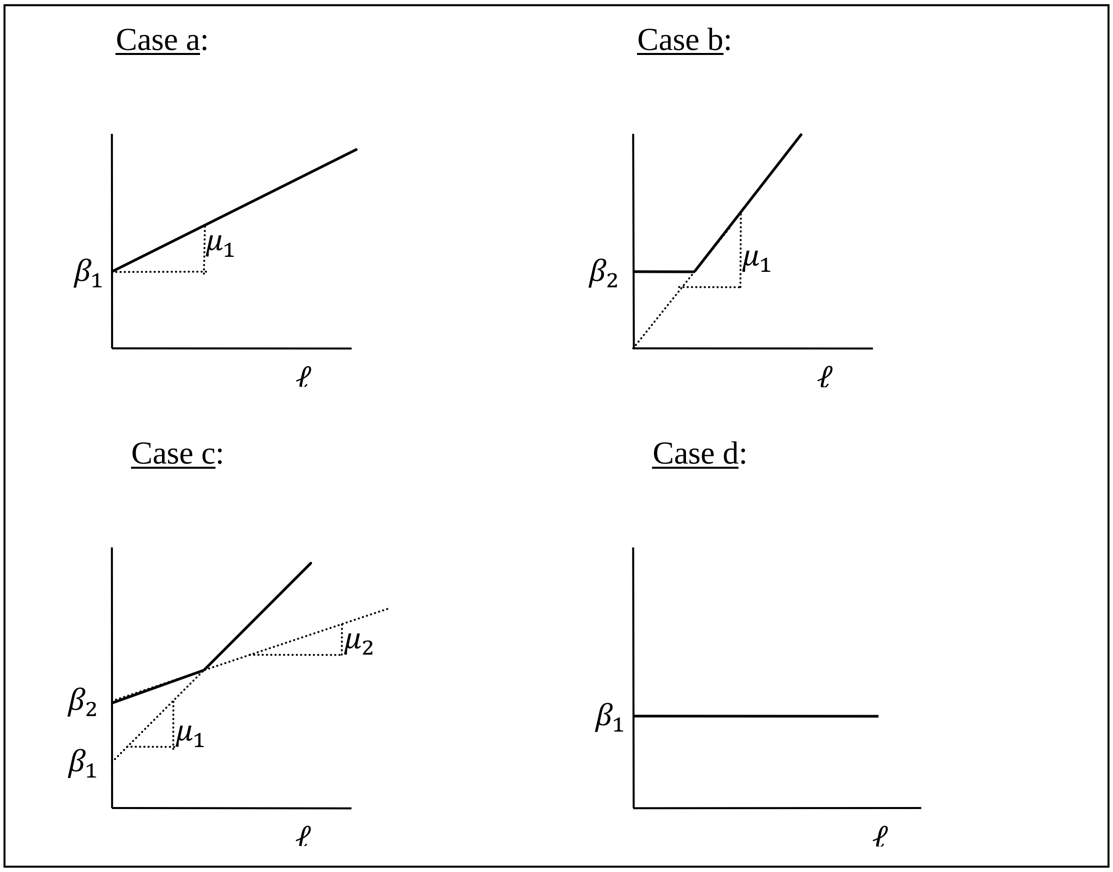

Figure 2.2: Pricing functions based on $\boldsymbol{T}^{\mathbf{max}}(\boldsymbol{\ell})$

For the pricing functions based on $T^{\max}(\ell)$ in Figure 2.2, case (a) requires users to pay two fees. One is an access fee (β_1) and the other is a VMT fee (μ_1). Economists commonly refer to this form of pricing as a two-part tariff or pricing scheme. Similarly, the function in case (b) also consists of an access and VMT fee. However, the latter only applies when the travel distance exceeds a threshold, a point where $\beta_1 + \mu_1 \ell = \beta_2$. (When β_2 and μ_1 are fixed, β_1 may need to be negative to achieve a desired threshold value.) In economics, some refer to case (b) as a three-part tariff. Instead of giving discounts to heavy users, case (c) discourages heavy road usage by charging a higher VMT fee (μ_1) when the travel distance exceeds a threshold, a point where $\beta_1 + \mu_1 \ell = \beta_2 + \mu_2 \ell$. Finally, the pricing function for case (d) is suitable for area-based pricing (see, e.g., Maruyama and Sumalee, 2007), a tolling scheme under which users can enter and use the tolling area as often and as much as they like during a specified period after paying an access fee, β_1. (Area-based pricing is different from cordon pricing. For the latter, users generally pay a fee each time they enter the tolling area.) In addition to those shown in the two figures, setting β_1, β_2, and μ_2 to zero reduces $T(\ell)$ to linear pricing, i.e., $T(\ell) = \mu_1 \ell$.

CHAPTER 3: PATH-BASED USER EQUILIBRIUM CONDITIONS UNDER NONLINEAR PRICING

This chapter states UE conditions under nonlinear pricing using path flows. Doing so allows us to define our notation and provide information for discussion in subsequent chapters.

Let Ω be the set of links (or arcs) in the road network. A link in Ω is denoted as a or a pair (i, j), where i and j are nodes corresponding to the start and end of a road segment. For travel demands, K denotes the set of origin-destination (OD) pairs and d_k is the demand for OD pair $k \in K$. Associated with each OD pair, there is an inverse demand function $D_k^{-1}(\cdot)$. Additionally, $\boldsymbol{d} \in R_+^{|K|}$ and $\boldsymbol{D}^{-1}(\cdot) \in R_+^{|K|}$ are vectors of these demands and their inverse functions, respectively. (Herein, the bold typeface indicates vectors of variables or functions and the plus sign in the subscript indicates that each component of the vector is nonnegative.)

To satisfy demands, P^k denotes the set of all possible paths for OD pair k. Then, f_r^k represents the number of travellers using path $r \in P^k$ and $\boldsymbol{f}$ is a vector of these path flows. Then, the set of all feasible flow-demand pairs, $(\boldsymbol{f}, \boldsymbol{d})$, can be described as follows:

$$V^f = \left\{(\boldsymbol{f}, \boldsymbol{d}): \sum_{r \in P^k} f_r^k = d_k, f_r^k \geq 0, d_k \geq 0, \forall k \in K, r \in P^k\right\}.$$

In words, $(\boldsymbol{f}, \boldsymbol{d})$ is a feasible flow-demand pair if the sum of the flows on all paths connecting the origin of OD pair k to its destination equals d_k and both $\boldsymbol{f}$ and $\boldsymbol{d}$ are nonnegative. It is also convenient to refer to a flow-demand pair as $(\boldsymbol{v}, \boldsymbol{d})$, where $\boldsymbol{v}$ a vector of the aggregate link flows, v_a. By letting $\delta_{ar} = 1$ if arc a is on path r and $\delta_{ar} = 0$ otherwise, it is possible to describe V^f as follows:

$$V^f = \left\{(\boldsymbol{v}, \boldsymbol{d}): v_a = \sum_{k} \sum_{r \in P^k} \delta_{ar} f_r^k, \sum_{r \in P^k} f_r^k = d_k, f_r^k \geq 0, d_k \geq 0, \forall k \in K, r \in P^k\right\}.$$

We use both definitions of V^f interchangeably throughout this report and refer the elements of V^f either as $(\boldsymbol{v}, \boldsymbol{d})$ or $(\boldsymbol{f}, \boldsymbol{d})$.

Associated with each arc, there is a travel time or link performance function, $s_a(\cdot)$, and $\boldsymbol{s}(\cdot) \in R_{++}^L$ is a vector of these functions, where L is the cardinality of Ω and the "++" sign in the subscript indicates that each component of the vector is positive. In addition, l_a denotes the length of arc a and $l_a > 0$ for all $a \in \Omega$. For tolling, Ω is partitioned into two subsets, Ω^1 and Ω^2, where the former contains links inside the tolling area and the later consists of those outside. By definition, $\Omega^1 \cap \Omega^2 = \emptyset$ and $\Omega = \Omega^1 \cup \Omega^2$. As mentioned previously, arcs in Ω_1 need not be connected. Similarly, P^k is divided into two subsets: TP^k and NP^k. The former, TP^k, consists of paths containing arcs in Ω^1 and using these paths requires paying tolls. In general, paths in TP^k contain links in both Ω^1 and Ω^2 to connect the origin of OD pair k to its destination. On the other hand, paths in NP^k contain no link in Ω^1 and are thus toll-free. Given a pricing function $T(\cdot)$, $(\boldsymbol{v}, \boldsymbol{d}) \in V^f$ is in tolled UE if the following conditions hold:

$$T\left(\sum_{a\in\Omega^1} \delta_{ar} l_a\right) + \sum_{a\in\Omega} \delta_{ar} s_a(\boldsymbol{v}) \quad = D_k^{-1}(d_k) \quad \forall r \in TP_{++}^k(\boldsymbol{v}, \boldsymbol{d}), k \in K, \tag{3.1}$$

$$T\left(\sum_{a\in\Omega^1} \delta_{ar} l_a\right) + \sum_{a\in\Omega} \delta_{ar} s_a(\boldsymbol{v}) \quad \geq D_k^{-1}(d_k) \quad \forall r \in TP_0^k(\boldsymbol{v}, \boldsymbol{d}), k \in K, \tag{3.2}$$

$$\sum_{a\in\Omega^2} \delta_{ar} s_a(\boldsymbol{v}) \quad = D_k^{-1}(d_k) \quad \forall r \in NP_{++}^k(\boldsymbol{v}, \boldsymbol{d}), k \in K, \tag{3.3}$$

$$\sum_{a\in\Omega^2} \delta_{ar} s_a(\boldsymbol{v}) \quad \geq D_k^{-1}(d_k) \quad \forall r \in NP_0^k(\boldsymbol{v}, \boldsymbol{d}), k \in K. \tag{3.4}$$

In (3.1), $TP_{++}^k(\boldsymbol{v}, \boldsymbol{d})$ denotes the set of utilized toll paths with respect to $(\boldsymbol{v}, \boldsymbol{d}) \in V^f$, i.e., $TP_{++}^k(\boldsymbol{v}, \boldsymbol{d}) = \{r \in TP^k : f_r^k > 0, r \in P^k\}$. Similarly, $TP_0^k(\boldsymbol{v}, \boldsymbol{d})$ in (3.2) is the set of paths not utilized and $TP_0^K(\boldsymbol{v}, \boldsymbol{d}) = \{r \in TP^k : f_r^k = 0, r \in P^k\}$. In (3.3) and (3.4), $NP_{++}^k(\boldsymbol{v}, \boldsymbol{d})$ and $NP_0^k(\boldsymbol{v}, \boldsymbol{d})$ are similarly defined for toll-free paths. The expression on the left hand side of (3.1) and (3.2) consists of the toll amount and travel time for path $r \in TP^k$. (In this report, tolls are measured in units of time.) Because paths in NP^k are toll free, their costs or the summations on the left hand side of (3.3) and (3.4) consist solely of travel times. In words, (3.1) and (3.3) state that, at equilibrium, all utilized paths (toll or not) must have the same generalized cost that equals to the value of the inverse demand function evaluated at the "realized" demand d_k. Conditions (3.2) and (3.4) imply that the costs of those not utilized cannot be lower than $D_k^{-1}(d_k)$.

When $T(\cdot)$ is nonlinear, the generalized cost expressions on the left hand side of conditions (3.1) and (3.2) are not link-wise additive and it may be intuitive to conclude that tolled UE conditions based on link flows do not exist (see, e.g., Maruyama and Sumalee, 2007). However, results in Chapter 5 show otherwise.

To simplify our presentation and highlight key ideas, assume that $s_a(\cdot)$ is a function only of v_a, i.e., the Jacobian of $\boldsymbol{s}(\boldsymbol{v})$ is diagonal. Under this assumption, finding a toll-free equilibrium flow-demand pair reduces to a convex optimization problem. An extension to, e.g., the case with an asymmetric and positive definite Jacobian is straightforward and generally involves finding solutions to variational inequalities or VIs (see, e.g., Florian and Hearn, 2003, and Patriksson, 1994). In addition, $D_k^{-1}(\cdot)$ is assumed to be non-increasing and $0 \leq D_k^{-1}(d) < \infty, \forall d \geq 0$.

Henceforth, we assume that $T(\ell)$ is based on $T^{max}(\ell)$, a function more suitable for managing travel demand, reducing congestion, and lessening the environmental impacts.

Although most discussion and many results herein extend in an obvious manner to the case with $T^{min}(\ell)$, the resulting optimization problems and VIs generally minimize a non-convex objective function and are defined with functions whose Jacobians are indefinite, respectively. Solving such optimization problems with, e.g., commercial software may not yield globally optimal solutions and VIs with indefinite Jacobians are not well solved (see, e.g., Facchinei and Pang, 2003).

CHAPTER 4: FINDING AN EQUILIBRIUM FLOW-DEMAND PAIR USING PATH FLOWS

This chapter assumes that $T(\ell)$ is based on $T^{max}(\ell)$ defined in Chapter 2 and modifies traditional algorithms such as simplicial decomposition (see, e.g., von Hohenbalken, 1977, Lawphongpanich and Hearn, 1984, Hearn et al., 1987, and Patriksson, 1994) to find a UE flow-demand pair. This is advantageous for two reasons. The underlying concepts in traditional algorithms are well understood and, as demonstrated below, they work well when $T(\ell)$ is defined with $T^{max}(\ell)$. The former also makes the software development easier because existing computer programs for traditional algorithms can be modified to include nonlinear pricing.

For each path $r \in P^k$, its travel distance inside the tolling area, $\sum_{a\in\Omega^1} \delta_{ar} l_a$, is fixed. Thus, the toll, τ_r, for path r is also fixed. Specifically, $\tau_r = 0, \forall r \in NP^k$, and $\tau_r = T(\sum_{a\in\Omega^1} \delta_{ar} l_a)$ is nonnegative for all $r \in TP^k$. Then, the tolled user equilibrium (TUE) problem, i.e., the problem of finding a UE flow-demand pair with a given pricing function $T(\cdot)$, can be formulated in terms of path flows as follows:

$$
\begin{aligned}
TUE: \quad min \quad & \sum_{a\in\Omega} \int_0^{v_a} s_a(\omega) d\omega - \sum_{k\in K} \int_0^{d_k} D_k^{-1}(\omega) d\omega + \sum_{k\in K} \sum_{r\in P^k} \tau_r f_r^k \\
s.t. \quad & \sum_{r\in P^k} f_r^k - d_k = 0, \forall k \in K \\
& v_a = \sum_k \sum_{r\in P^k} \delta_{ar} f_r^k, \forall a \in \Omega \\
& f_r^k \geq 0, \forall k \in K, r \in P^k
\end{aligned}
$$

Without the last term in the objective function, the above problem reduces to a problem for finding a (toll-free) UE flow-demand pair when demands are elastic (see, e.g., Florian and Hearn, 2003). Under the assumptions stated at the end of Chapter 3, the functions in the first and second summations in the objective are convex. The last summation calculates the toll revenue and is linear with respect to f_r^k, the path-flow variables. The two main sets of constraints ensure feasibility and convert path flows into aggregate link flows. Moreover, the Karush-Kuhn-Tucker or KKT conditions (see, e.g., Bazaraa et al., 2006) are both necessary and sufficient for TUE because it is a linearly constrained convex program. Using the fact that $\tau_r = T(\sum_{a\in\Omega^1} \delta_{ar} l_a)$ and $P^k = TP^k \cup NP^k, \forall k \in K$, it is relatively simple to demonstrate that the KKT conditions for TUE reduce to (3.1) – (3.4). Thus, an optimal solution to TUE is a UE flow-demand pair under the pricing function $T(\cdot)$.

Below is a version of simplicial decomposition (SD) that generates the necessary paths between every OD pair. (For other variations, see, e.g., Lawphongpanich and Hearn, 1984, and Hearn et al., 1987.) Briefly, the algorithm starts with a zero flow-demand pair in Step 1 (i.e., there is no travel demand initially) and solves an optimization problem to generate new paths in Step 2 for all OD pairs. In Step 3, the algorithm stops when paths generated in Step 2 cannot further reduce the objective value of TUE. If the algorithm does not stop, Step 4 adds paths from Step 2 to Π^k, the set of indices associated with the generated paths for OD pair k. Typically, $\Pi^k \subset P^k, \forall k \in K$. In Step 5, the algorithm solves an approximate version of TUE in which P^k is replaced by Π^k and returns to Step 2 where the process repeats.

Simplicial Decomposition for TUE

Step 1: Set $(\boldsymbol{v}^1, \boldsymbol{d}^1) = (0, 0)$ and $n = 1$. For each OD pair k, set $\Pi^k = \emptyset$ and $r^k = 1$.

Step 2: For each OD pair k, let $(\boldsymbol{z}^k, w^k)$ solve the following (sub)problem and c^k denotes its optimal objective value:

$$
\begin{aligned}
c^k = \quad & min \quad \sum_{a\in\Omega} s_a(\boldsymbol{v}^n) z_a^k + w^k \\
& s.t. \quad A\boldsymbol{z}^k = \boldsymbol{E}_k \\
& \qquad \sum_{a\in\Omega^1} z_a^k \le M q^k \\
& \qquad \beta_1 q^k + \mu_1 \sum\nolimits_{a\in\Omega^1} l_a z_a^k \le w^k \\
& \qquad \beta_2 q^k + \mu_2 \sum\nolimits_{a\in\Omega^1} l_a z_a^k \le w^k \\
& \qquad q^k \in \{0,1\}, z_a^k \in \{0,1\}, \forall a \in \Omega
\end{aligned}
$$

Step 3: If $c^k - D_k^{-1}(d_k^n) \ge 0, \forall k \in K$, stop and the current solution $(\boldsymbol{v}^n, \boldsymbol{d}^n)$ is a tolled UE flow-demand pair. Otherwise, go to Step 4.

Step 4: For each OD pair k such that $c^k - D_k^{-1}(d_k^n) < 0$, set $\delta_{ar^k} = z_a^k$, $\tau_{r^k} = w^k$, $\Pi^k = \Pi^k \cup \{r^k\}$, and $r^k = r^k + 1$.

Step 5: Let $(\boldsymbol{v}^{n+1}, \boldsymbol{d}^{n+1})$ solve the (master) problem below, set $n = n + 1$, and return to Step 2.

$$
\begin{aligned}
min \quad & \sum_{a\in\Omega} \int_0^{v_a} s_a(\omega) d\omega - \sum_{k\in K} \int_0^{d_k} D_k^{-1}(\omega) d\omega + \sum_{k\in K} \sum_{r\in\Pi^k} \tau_r f_r^k \\
s.t. \quad & \sum_{r\in\Pi^k} f_r^k - d_k = 0, \forall k \in K \\
& v_a = \sum_k \sum_{r\in\Pi^k} \delta_{ar} f_r^k, \forall a \in \Omega \\
& f_r^k \ge 0, \forall k \in K, r \in \Pi^k
\end{aligned}
$$

In the above, Step 1 uses a zero flow-demand pair to initialize the algorithm. Subsequently, the link travel times, $s_a(\boldsymbol{v}^n)$, in the subproblem in Step 2 (or, more descriptively, the path-generation problem) are free-flow travel time during the first iteration, i.e., when $n = 1$.

For each OD pair, the subproblem finds a path with the least generalized cost. The first summation in the objective function computes the path travel time and w^k is the toll amount. In the first constraint, A is the node-arc incidence matrix of the road network and $\boldsymbol{E}_k \in R^N$ is an (input-output) vector with exactly two non-zero components. The component corresponding to the origin node of the OD pair k contains a "1" and the one for the destination contains a "-1." Thus, the first constraint balances the flows into and out of each node. The binary variable q^k in the second constraint indicates whether to pay tolls and M is a sufficiently large positive constant, e.g., $M = |\Omega^1| + 1$. Setting $q^k = 0$ forces z_a^k to be zero for all $a \in \Omega^1$, i.e., the path does not enter the tolling area. With $q^k = 0$ and $z_a^k = 0, \forall a \in \Omega^1$, the left-hand sides of the next two constraints (the 3rd and 4th constraints) reduce to zero. Consequently, w^k must be zero to minimize the objective function and the path associated with $\boldsymbol{z}^k$ is toll-free. When $q^k = 1$, z_a^k for $a \in \Omega^1$ are allowed to be one, i.e., the path can use links in the tolling area, and the combination of the 3rd and 4th constraints ensure that

$$\max\left\{\beta_1 + \mu_1 \sum_{a\in\Omega^1} l_a z_a^k, \beta_2 + \mu_2 \sum_{a\in\Omega^1} l_a z_a^k\right\} \leq w^k.$$

As before, the inequality "$\leq$" in the above expression must hold at equality to minimize the objective function, i.e., w^k is the toll amount associated with $\boldsymbol{z}^k$.

The stopping criterion in Step 3 ensures that all paths, i.e., those in the current set Π^k and otherwise, cost no less than $D_k^{-1}(d_k^n)$. This implies that no path can lead to a smaller objective value. Then, the fact that $(\boldsymbol{v}^n, \boldsymbol{d}^n)$ solves the master problem ensures, via its KKT conditions, that the solution satisfies the tolled UE conditions. Also, it is more practical to replace the stopping criterion in Step 3 with $c^k - D^{-1}(d_k^n) \geq -\epsilon$, where ϵ is a sufficiently small positive constant, e.g., $\epsilon = 10^{-6}$.

Step 4 adds an additional path to the set Π^k and performs the necessary updates. Finally, the master problem in Step 5 is a convex optimization problem with linear constraints, a class of problems relatively easy to solve. As mentioned previously, the master problem is also an approximation of the TUE problem.

The above SD algorithm converges to an optimal solution in a finite number of iterations. The argument is similar to those in the literature (see, e.g., Lawphongpanich and Hearn, 1984) and follows from three facts. First, the number of paths without cycles is finite. (Recall that we assume that the link performance function $s_a(\cdot)$ is positive for all $a \in \Omega$. Thus, the solutions to the problem in Step 2 must correspond to paths without cycles.) Second, because SD never eliminates paths from Π^k, new paths generated in Step 2 must be distinct from those in the current Π^k. Finally, the optimal objective value of the master problem strictly decreases at the end of every iteration prior to termination because newly added paths in Step 4 satisfy $c^k - D_k^{-1}(d_k^n) < 0$, i.e., a condition that ensures a decrease in the objective value.

4.1. Solving the path generating problem in Step 2

Consider the path-generating problem (PG) in Step 2. Although it is possible to solve PG as a single problem, our numerical experiments indicate that it is more efficient to obtain an optimal solution to PG by solving two smaller problems for each OD pair, one contains binary variables and the other does not. Solving these two problems is akin to solving PG twice, once using the tolling area ($q^k = 1$) and another not using it ($q^k = 0$). Then, the better of the two optimal solutions is the solution to PG.

When $q^k = 1$, the third constraint in PG becomes $\sum_{a\in\Omega^1} z_a^k \le M$. When M is sufficiently large, the constraint is never binding and can be eliminated. Consequently, PG reduces to the following:

$$
\begin{aligned}
SUB1(\boldsymbol{v}^n):\quad & min \quad \sum_{a\in\Omega} s_a(\boldsymbol{v}^n) z_a^k + w^k \\
& s.t. \quad A\boldsymbol{z}^k = \boldsymbol{E}_k \\
& \qquad \beta_1 + \mu_1 \sum_{a\in\Omega^1} l_a z_a^k \le w^k \\
& \qquad \beta_2 + \mu_2 \sum_{a\in\Omega^1} l_a z_a^k \le w^k \\
& \qquad z_a^k \in \{0,1\}, \forall a \in \Omega
\end{aligned}
$$

The above problem can be viewed as a generalization of a shortest path problem with two side constraints (see, e.g., Ahuja et al., 1993), a NP-complete problem. When compared to other NP-complete problems, our numerical experiments indicate that commercial software such as CPLEX (IBM, 2009) can solve $SUB1(\boldsymbol{v}^n)$ efficiently because the 2nd and 3rd constraints can be satisfied easily. For any binary $\boldsymbol{z}^k$ feasible to the first constraint, setting $w^k = \max\{\beta_1 + \mu_1 \sum_{a\in\Omega^1} l_a z_a^k, \beta_2 + \mu_2 \sum_{a\in\Omega^1} l_a z_a^k\}$ yields a pair $(\boldsymbol{z}^k, w^k)$ feasible to $SUB1(\boldsymbol{v}^n)$.

For the other case ($q^k = 0$), we partition A into two submatrices, A_1 and A_2, where A_n is the node-arc incidence matrix for the network induced by arcs in Ω^n, where $n = 1, 2$. Thus, A can be written as $[A_1 \colon A_2]$. Similarly, we also partition $\boldsymbol{z}^k$ as follows:

$$
\boldsymbol{z}^k = \begin{bmatrix} \boldsymbol{z}_1^k \\ \boldsymbol{z}_2^k \end{bmatrix}.
$$

In the above, $\boldsymbol{z}_1^k$ is a (sub)vector consisting of variables z_a^k for $a \in \Omega^1$. The similar holds for $\boldsymbol{z}_2^k$. Under this partitioning, the flow-balance constraint becomes $A_1\boldsymbol{z}_1^k + A_2\boldsymbol{z}_2^k = \boldsymbol{E}_k$. When $q^k = 0$, the path cannot enter the tolling area. Thus, $\boldsymbol{z}_1^k = 0$ and the subproblem in Step 2 reduces to the following because the constraints involving arcs in Ω^1 are irrelevant and thus eliminated:

$$
\begin{aligned}
SUB2(\boldsymbol{v}^n): \quad & min \quad \sum_{a \in \Omega^2} s_a(\boldsymbol{v}^n) z_a^k \\
& s.t. \quad A_2 \boldsymbol{z}_2^k = \boldsymbol{E}_k \\
& \qquad\quad z_a^k \in \{0,1\}, \forall a \in \Omega^2
\end{aligned}
$$

Note that A_2 is totally unimodular because it is a submatrix of A, a totally unimodular matrix. Thus, basic solutions to $A_2 \boldsymbol{z}_2^k = \boldsymbol{E}_k$ are always integral and the binary restriction for z_a^k is unnecessary. In other words, $SUB2(\boldsymbol{v}^n)$ can be equivalently written as follows:

$$
\begin{aligned}
SUB2a(\boldsymbol{v}^n): \quad & min \quad \sum_{a \in \Omega^2} s_a(\boldsymbol{v}^n) z_a^k \\
& s.t. \quad A_2 \boldsymbol{z}_2^k = \boldsymbol{E}_k \\
& \qquad\quad z_a^k \geq 0, \forall a \in \Omega^2
\end{aligned}
$$

Observe that a unit upper bound on z_a^k is unnecessary in $SUB2a(\boldsymbol{v}^n)$ because $\boldsymbol{E}_k$ implies that there is only one unit of flow in the problem. Instead of solving PG directly, we solve $SUB1(\boldsymbol{v}^n)$ and $SUB2a(\boldsymbol{v}^n)$ and, between the two solutions, the one with a smaller objective value is optimal to PG.

CHAPTER 5: LINK-BASED USER EQUILIBRIUM CONDITIONS UNDER NONLINEAR PRICING

This chapter investigates properties under which equilibrium conditions and the UE problem can be formulated using link flows. Below, Section 5.1 discusses one such property that relies on the relationship between $SUB1(\boldsymbol{v}^n)$ and its dual problem. (Recall that $SUB1(v^n)$ is a problem associated with the PG problem in Step 2 of SD.) Then, Section 5.2 provides two sets of link-based UE conditions. One is equivalent to (3.1) – (3.4) when $SUB1(\boldsymbol{v}^n)$ has no duality gap and the other is only sufficient. In Section 5.3, we show that equilibrium conditions and the UE problem under area-based and two-part pricing schemes can be stated in terms of link flows.

5.1. Lagrangian Dual Problems

In this and subsequent sections, we remove the iteration index, n, from $SUB(\boldsymbol{v}^n)$ because it is irrelevant. The problem is well defined for any $\boldsymbol{v}$ such that, for some travel demand vector $\boldsymbol{d}$**,** $(\boldsymbol{v}, \boldsymbol{d}) \in V^f$.

For a given OD pair k, the Lagrangian dual problem for $SUB1(\boldsymbol{v})$ can be written as follows (see, e.g., Bazaraa et al., 2006):

$$\begin{aligned} D1(\boldsymbol{v}): \quad & max \quad L_v^k(\alpha_1^k, \alpha_2^{\boldsymbol{k}}) \\ & s.t. \quad \alpha_1^k, \alpha_2^k \geq 0 \end{aligned}$$

where $L_v^k(\alpha_1^k, \alpha_2^{\boldsymbol{k}})$, the Lagrangian function associated with $SUB1(\boldsymbol{v})$, is defined as follows:

$$\begin{aligned} L_v^k\left(\alpha_1^k, \alpha_2^{\boldsymbol{k}}\right) = \quad & min \quad \sum_{a \in \Omega} s_a(\boldsymbol{v}) z_a^k + w^k + \sum_{m=1}^{2} \alpha_m^k \left(\beta_m + \mu_m \sum_{a \in \Omega_1} l_a z_a^k - w^k \right) \\ & s.t. \quad A\boldsymbol{z}^k = \boldsymbol{E}_k \\ & \qquad\quad z_a^k \in \{0,1\}, \forall a \in \Omega. \end{aligned}$$

The variables α_m^k, for $m = 1\ ,2$, are Lagrange multipliers constrained to be nonnegative. In literature, some refer to the above problem as a Lagrangian subproblem.

Let $(\hat{\boldsymbol{z}}^k, \widehat{w}^k)$ and $(\bar{\alpha}_1^k, \bar{\alpha}_2^k)$ solve $SUB1(\boldsymbol{v})$ and $D1(\boldsymbol{v})$, respectively. Then, it follows from the weak duality theorem (see, Bazaraa et al., 2006) that:

$$\sum_{a \in \Omega} s_a(\boldsymbol{v}) \hat{z}_a^k + \widehat{w}^k \geq L_v^k\left(\bar{\alpha}_1^k, \bar{\alpha}_2^k\right).$$

The result below assumes that the inequality in the above expression holds at equality, i.e., the strong duality condition holds or $SUB1(\boldsymbol{v})$ has no or zero duality gap.

Lemma 5.1: If $SUB1(\boldsymbol{v})$ has no duality gap, then its solution also solves the Lagrangian subproblem of $D1(\boldsymbol{v})$.

Proof: As discussed above, let $(\hat{\boldsymbol{z}}^k, \widehat{w}^k)$ and $(\bar{\alpha}_1^k, \bar{\alpha}_2^k)$ solve $SUB1(\boldsymbol{v})$ and $D1(\boldsymbol{v})$, respectively. Then, the following must hold:

$$\begin{aligned}
\sum_{a\in\Omega} s_a(\boldsymbol{v})\hat{z}_a^k + \widehat{w}^k \quad &= L_v^k\left(\bar{\alpha}_1^k, \bar{\alpha}_2^k\right) \\
&= \min\left\{\sum_{a\in\Omega} s_a(\boldsymbol{v})z_a^k + w^k + \sum_{m=1}^{2}\bar{\alpha}_m^k\left(\beta_m + \mu_m\sum_{a\in\Omega_1} l_a z_a^k - w^k\right): A\boldsymbol{z}^k = \boldsymbol{E}_k, z_a^k \in \{0,1\}\right\} \\
&\le \sum_{a\in\Omega} s_a(\boldsymbol{v})\hat{z}_a^k + \widehat{w}^k + \sum_{m=1}^{2}\bar{\alpha}_m^k\left(\beta_m + \mu_m\sum_{a\in\Omega_1} l_a \hat{z}_a^k - \widehat{w}^k\right) \\
&\le \sum_{a\in\Omega} s_a(\boldsymbol{v})\hat{z}_a^k + \widehat{w}^k.
\end{aligned}$$

In the above, the first two equalities follow from the zero duality gap assumption and the definition of the Lagrangian function at the optimal dual solution $\left(\bar{\alpha}_1^k, \bar{\alpha}_2^k\right)$, respectively. Next, the first inequality holds because $(\hat{\boldsymbol{z}}^k, \widehat{w}^k)$ is feasible to the minimization problem. When viewed as an optimal solution to $SUB1(\boldsymbol{v})$, $(\hat{\boldsymbol{z}}^k, \widehat{w}^k)$ satisfies $\left(\beta_m + \mu_m\sum_{a\in\Omega_1} l_a\hat{z}_a^k - \widehat{w}^k\right) \le 0$ for $m = 1{,}2$. Combining the latter with the fact that $\bar{\alpha}_m^k \ge 0$, for $m = 1, 2$, implies that $\sum_{m=1}^{2}\bar{\alpha}_m^k\left(\beta_m + \mu_m\sum_{a\in\Omega_1} l_a\hat{z}_a^k - \widehat{w}^k\right) \le 0$. Thus, the last inequality must hold.

The above sequence of equalities and inequalities begins and ends with the same expression. Thus, the two inequalities must be equalities, i.e., $(\hat{\boldsymbol{z}}^k, \widehat{w}^k)$ must be optimal to the minimization problem, i.e., the Lagrangian subproblem of $D1(\boldsymbol{v})$ associated with $\left(\bar{\alpha}_1^k, \bar{\alpha}_2^k\right)$. □

To make a problem structure more evident, observe that β_m and α_m^k, $m = 1, 2$, are constants with respect to the minimization and the Lagrangian subproblem can be written as

$$\begin{aligned}
L_v^k\left(\alpha_1^k, \alpha_2^k\right) = \left(\alpha_1^k\beta_1 + \alpha_2^k\beta_2\right) + \quad min \quad & \sum_{a\in\Omega} s_a(\boldsymbol{v})z_a^k + w^k(1 - \alpha_1^k - \alpha_2^k) + \left(\alpha_1^k\mu_1 + \alpha_2^k\mu_2\right)\sum_{a\in\Omega_1} l_a z_a^k \\
s.t. \quad & A\boldsymbol{z}^k = \boldsymbol{E}_k \\
& z_a^k \in \{0,1\}, \forall a \in \Omega.
\end{aligned}$$

In the above, w^k is unrestricted. When $\alpha_1^k + \alpha_2^k > 1$, $L_v^k\left(\alpha_1^k, \alpha_2^{\boldsymbol{k}}\right) = -\infty$ because setting $w^k = \infty$ is optimal. On the other hand, when $\alpha_1^k + \alpha_2^k \le 1$, the optimal value for w^k is zero and $L_v^k\left(\alpha_1^k, \alpha_2^{\boldsymbol{k}}\right)$ is finite. To maximize the value of $L_v^k\left(\alpha_1^k, \alpha_2^{\boldsymbol{k}}\right)$ in problem $D1(\boldsymbol{v})$, it makes sense to restrict α_1^k and α_2^k to the region where $\alpha_1^k + \alpha_2^k \leq 1$ and $\alpha_1^k, \alpha_2^{\boldsymbol{k}} \geq 0$. Thus, the Lagrangian dual problem for $SUB1(\boldsymbol{v})$ can be equivalently written as:

$$
\begin{aligned}
D2(\boldsymbol{v}):\quad & max \quad && \tilde{L}_v^k\left(\alpha_1^k, \alpha_2^{\boldsymbol{k}}\right) + \left(\alpha_1^k \beta_1 + \alpha_2^k \beta_2\right) \\
& s.t. && \alpha_1^k + \alpha_2^{\boldsymbol{k}} \leq 1 \\
& && \alpha_1^k, \alpha_2^{\boldsymbol{k}} \geq 0
\end{aligned}
$$

where $\tilde{L}_v^k\left(\alpha_1^k, \alpha_2^{\boldsymbol{k}}\right)$ is a modified Lagrangian function and, because A is totally unimodular, it can defined as follows:

$$
\begin{aligned}
\tilde{L}_v^k\left(\alpha_1^k, \alpha_2^{\boldsymbol{k}}\right) = \quad & min \quad && \sum_{a \in \Omega^1} \left(s_a(\boldsymbol{v}) + \left(\alpha_1^k \mu_1 + \alpha_2^k \mu_2\right) l_a\right) z_a^k + \sum_{a \in \Omega^2} s_a(\boldsymbol{v}) z_a^k \\
& s.t. && A\boldsymbol{z}^k = \boldsymbol{E}_k \\
& && z_a^k \geq 0, \forall a \in \Omega
\end{aligned}
$$

We also refer to the problem directly above as the modified Lagrangian subproblem. Because $D1(\boldsymbol{v})$ and $D2(\boldsymbol{v})$ are equivalent, it follows from Lemma 4.1 that, if $SUB1(\boldsymbol{v})$ has no duality gap, its solution also solves the modified Lagrangian subproblem and

$$
\sum_{a \in \Omega} s_a(\boldsymbol{v}) \hat{z}_a^k + \widehat{w}^k = \tilde{L}_v^k\left(\bar{\alpha}_1^k, \bar{\alpha}_2^k\right) + \left(\bar{\alpha}_1^k \beta_1 + \bar{\alpha}_2^k \beta_2\right).
$$

5.2. Link-based Equilibrium Conditions: General Case

For a given $(\boldsymbol{f}, \boldsymbol{d}) \in V^f$, define

$$
\begin{aligned}
x_a^k(\boldsymbol{f}) &= \sum\nolimits_{a \in \Omega} \sum\nolimits_{r \in TP^k} \delta_{ar} f_r^k \\
y_a^k(\boldsymbol{f}) &= \sum\nolimits_{a \in \Omega} \sum\nolimits_{r \in NP^k} \delta_{ar} f_r^k \\
\sigma_k &= \sum\nolimits_{r \in TP^k} f_r^k \\
\eta_k &= \sum\nolimits_{r \in NP^k} f_r^k
\end{aligned}
$$

In words, $\boldsymbol{x}^k(\boldsymbol{f})$ and $\boldsymbol{y}^k(\boldsymbol{f})$ are, respectively, vectors of link flows on toll and toll-free paths associated with $(\boldsymbol{f}, \boldsymbol{d})$. As constructed, $y_a^k(\boldsymbol{f}) = 0, \forall a \in \Omega^1, k \in K$, i.e., $\boldsymbol{y}(\boldsymbol{f})$ is the link-flow vector associated with toll-free paths. In the last two equations, σ_k and η_k are variables representing the numbers of users who pay and do not pay tolls for OD pair k, respectively. For every OD pair k, the above vectors and variables satisfy the following linear systems:

$$[A_1 \colon A_2]\boldsymbol{x}^k = \sigma_k \boldsymbol{E}_k$$
$$[0 \colon A_2]\boldsymbol{y}^k = \eta_k \boldsymbol{E}_k$$
$$\sigma_k + \eta_k = d_k$$

where, as previously defined, A_1 and A_2 are node-arc incidence matrices for subnetworks induced by arcs in the sets Ω^1 and Ω^2, respectively.

The above motivates a link-based representation of feasible flow-demand pairs based on $\boldsymbol{x}^k$ and $\boldsymbol{y}^k$. In particular, the set of all feasible flow-demand pair can be equilivalently written as

$$V^x = \left\{ \begin{matrix} (\boldsymbol{v}, \boldsymbol{d}) \colon \boldsymbol{v} = \sum_{k \in K} (\boldsymbol{x}^k + \boldsymbol{y}^k), d_k = \sigma_k + \eta_k, A\boldsymbol{x}^k = \sigma_k \boldsymbol{E}_k, [0 \colon A_2]\boldsymbol{y}^k = \eta_k \boldsymbol{E}_k, \\ \boldsymbol{x}^k \geq 0, \boldsymbol{y}^k \geq 0, \sigma_k \geq 0, \eta_k \geq 0, \forall k \in K \end{matrix} \right\}.$$

Because the value of y_a^k, $\forall a \in \Omega^1$, is unspecified in the above expression, it is assumed that they are always zero, i.e., flows associated with $\boldsymbol{y}$ do not enter the tolling area. Later, we also refer to elements of V^x in a disaggregate form or as a quadruplet $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\sigma}, \boldsymbol{\eta}) \in V^x$, i.e., we also define V^x as follows:

$$V^x = \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\sigma}, \boldsymbol{\eta}) \colon A\boldsymbol{x}^k = \sigma_k \boldsymbol{E}_k, [0 \colon A_2]\boldsymbol{y}^k = \eta_k \boldsymbol{E}_k, \boldsymbol{x}^k \geq 0, \boldsymbol{y}^k \geq 0, \sigma_k \geq 0, \eta_k \geq 0, \forall k \in K\}.$$

For every $(\boldsymbol{v}, \boldsymbol{d})$ in V^x, there must exist a pair $\big(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v})\big)$, not necessarily unique, such that $x_a^k = \sum_{a \in \Omega} \sum_{r \in TP^k} \delta_{ar} f_r^k(\boldsymbol{v})$, $y_a^k = \sum_{a \in \Omega} \sum_{r \in NP^k} \delta_{ar} f_r^k(\boldsymbol{v})$, and $d_k(\boldsymbol{v}) = \sum_{r \in TP^k} f_r^k(\boldsymbol{v}) + \sum_{r \in NP^k} f_r^k(\boldsymbol{v})$. Moreover, the pair $(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v}))$ also belongs to V^f and such a pair is said to be compatible with $(\boldsymbol{v}, \boldsymbol{d}) \in V^x$. The theorem below specifies conditions for equilibrium based on elements in V^x or link flows. Its proof relies on the zero duality gap assumption and the above relationship between V^x and V^f.

Theorem 5.2: Assume that $SUB1(\boldsymbol{v})$ has no duality gap. Then, $(\boldsymbol{v}, \boldsymbol{d}) \in V^x$ is in tolled UE if and only if, for each $k \in K$, there exist $\boldsymbol{\rho}^k \in R^N$, $\boldsymbol{\gamma}^k \in R^N$, α_1^k, and α_2^k such that the following link-based conditions hold:

$$\left(\alpha_1^k \mu_1 + \alpha_2^k \mu_2\right) l_{ij} + s_{ij}(\boldsymbol{v}) - \left(\rho_i^k - \rho_j^k\right) = 0, \quad \forall (i,j) \in \Omega^1, k \in K: x_{ij}^k > 0 \quad (5.1)$$
$$\left(\alpha_1^k \mu_1 + \alpha_2^k \mu_2\right) l_{ij} + s_{ij}(\boldsymbol{v}) - \left(\rho_i^k - \rho_j^k\right) \geq 0, \quad \forall (i,j) \in \Omega^1, k \in K: x_{ij}^k = 0 \quad (5.2)$$
$$s_{ij}(\boldsymbol{v}) - \left(\rho_i^k - \rho_j^k\right) = 0, \quad \forall (i,j) \in \Omega^2, k \in K: x_{ij}^k > 0 \quad (5.3)$$
$$s_{ij}(\boldsymbol{v}) - \left(\rho_i^k - \rho_j^k\right) \geq 0, \quad \forall (i,j) \in \Omega^2, k \in K: x_{ij}^k = 0 \quad (5.4)$$
$$\left(\alpha_1^k \beta_1 + \alpha_2^k \beta_2\right) + \boldsymbol{E}_k^T \boldsymbol{\rho}^k = D_k^{-1}(\sigma_k + \eta_k), \quad \forall k \in K: \boldsymbol{x}^k \neq 0 \quad (5.5)$$
$$s_{ij}(\boldsymbol{v}) - \left(\gamma_i^k - \gamma_j^k\right) = 0, \quad \forall (i,j) \in \Omega^2, k \in K: y_{ij}^k > 0 \quad (5.6)$$
$$s_{ij}(\boldsymbol{v}) - \left(\gamma_i^k - \gamma_j^k\right) \geq 0, \quad \forall (i,j) \in \Omega^2, k \in K: y_{ij}^k = 0 \quad (5.7)$$
$$\boldsymbol{E}_k^T \boldsymbol{\gamma}^k = D_k^{-1}(\sigma_k + \eta_k), \quad \forall k \in K: \boldsymbol{y}^k \neq 0 \quad (5.8)$$
$$(\alpha_1^k, \alpha_2^k) \text{ solves } D2(\boldsymbol{v}), \quad \forall k \in K \quad (5.9)$$

Proof: For each $k \in K$, assume that there exist $\boldsymbol{\rho}^k, \boldsymbol{\gamma}^k, \alpha_1^k$, and α_2^k satisfying conditions (5.1) – (5.9). Below, we show that, for every OD pair $k \in K$, the generalized cost of all utilized routes, toll or toll-free, equal $D_k^{-1}(d_k)$ and the costs of those not utilized are at least as large.

Consider a toll-free route that is utilized with respect to any pair $(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v}))$ compatible with $(\boldsymbol{v}, \boldsymbol{d}) \in V^x$, i.e., $r \in NP_{++}^k(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v}))$. If $\delta_{ar} = 1$, then there must be flows on link a, i.e., $y_a^k(\boldsymbol{v}) > 0$. Summing together expression (5.6) for all a such that $\delta_{ar} = 1$ yields

$$0 = \sum_{(i,j) \in \Omega} \delta_{(i,j)r} \left(s_{ij}(\boldsymbol{v}) - \left(\gamma_i^k - \gamma_j^k\right)\right) = \sum_{a \in \Omega} \delta_{ar} s_a(\boldsymbol{v}) - \gamma_{o(k)}^k + \gamma_{d(k)}^k = \sum_{a \in \Omega} \delta_{ar} s_a(\boldsymbol{v}) - \boldsymbol{E}_k^T \boldsymbol{\gamma}^k \quad (5.10)$$

where $o(k)$ and $d(k)$ denote, respectively, the origin and destination of OD pair k. Thus, $\sum_{a \in \Omega} \delta_{ar} s_a(\boldsymbol{v}) = \boldsymbol{E}_k^T \boldsymbol{\gamma}^k$ and it follows from (5.8) that $\sum_{a \in \Omega} \delta_{ar} s_a(\boldsymbol{v}) = D_k^{-1}(\sigma_k + \eta_k) = D_k^{-1}(d_k)$. Thus, the cost of path r equals the value of the inverse demand function at the realized demand d_k. When a toll-free route r is not utilized, i.e., $r \in NP_0^k(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v}))$, some link on route r has no flow, i.e., $y_a^k(\boldsymbol{v}) = 0$ for some a such that $\delta_{ar} = 1$. For arcs satisfying the latter, (5.7) indicates that $s_{ij}(\boldsymbol{v}) - \left(\gamma_i^k - \gamma_j^k\right) \geq 0$ and the following holds:

$$0 \leq \textstyle\sum_{(i,j) \in \Omega} \delta_{(i,j)r} \left(s_{ij}(\boldsymbol{v}) - \left(\gamma_i^k - \gamma_j^k\right)\right) = \sum_{a \in \Omega} \delta_{ar} s_a(\boldsymbol{v}) - \gamma_{o(k)}^k + \gamma_{d(k)}^k = \sum_{a \in \Omega} \delta_{ar} s_a(\boldsymbol{v}) - \boldsymbol{E}_k^T \boldsymbol{\gamma}^k. \quad (5.11)$$

From above, $\sum_{a \in \Omega} \delta_{ar} s_a(\boldsymbol{v}) \geq \boldsymbol{E}_k^T \boldsymbol{\gamma}^k = D_k^{-1}(d_k)$, i.e., the cost of a non-utilized toll-free path cannot be smaller than the value of the inverse demand function. Thus, among the toll-free paths, the utilized ones have costs equal to $D_k^{-1}(d_k)$ and those not utilized cannot have a lower cost.

For a toll route $r \in TP_{++}^k(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v}))$, let $z_a^k = \delta_{ar}, \forall a \in \Omega$. As constructed, $\boldsymbol{z}^k$ is feasible to the modified Lagrangian subproblem associated with $\tilde{L}_v^k(\alpha_1^k, \alpha_2^{\boldsymbol{k}})$ at the end of Section 5.1. The dual of this subproblem can be written as follows:

$$\begin{Bmatrix} max & \boldsymbol{E}_k^t\boldsymbol{\rho^k} & \\ s.t. & \rho_i^k - \rho_j^k \leq (\alpha_1\mu_1 + \alpha_2\mu_2)l_{ij} + s_{ij}(\boldsymbol{v}), & \forall(i,j) \in \Omega^1 \\ & \rho_i^k - \rho_j^k \leq s_{ij}(\boldsymbol{v}), & \forall(i,j) \in \Omega^2 \\ & \boldsymbol{\rho^k} \text{ unresticted} & \end{Bmatrix}$$

The hypothesis that $\boldsymbol{\rho}^k, \boldsymbol{\gamma}^k, \alpha_1^k$, and α_2^k exist ensures that the above dual problem has a solution. Then, it follows from the strong duality theorem in linear programming (see, e.g., Bazaraa et al., 2010) that $\tilde{L}_v^k(\alpha_1^k, \alpha_2^k) = \boldsymbol{E}_k^T\boldsymbol{\rho}^k$ and both $\boldsymbol{z}^k$ and $\boldsymbol{\rho}^k$ are optimal to their respective problems. Because $D1(\boldsymbol{v})$ and $D2(\boldsymbol{v})$ are equivalent and $SUB1(\boldsymbol{v})$ has no duality gap, $\tilde{L}_v^k(\alpha_1^k, \alpha_2^k) = \boldsymbol{E}_k^T\boldsymbol{\rho}^k$ and the following holds:

$$\sum_{a\in\Omega} s_a(\boldsymbol{v})\boldsymbol{z}_a^k + w^k = L_v^k(\alpha_1^k, \alpha_2^k) = \tilde{L}_v^k(\alpha_1^k, \alpha_2^k) + (\alpha_1^k\beta_1 + \alpha_2^k\beta_2) = \boldsymbol{E}_k^T\boldsymbol{\rho}^k + (\alpha_1^k\beta_1 + \alpha_2^k\beta_2).$$

In the above, $w^k = \max\{\beta_1 + \mu_1 \sum_{a\in\Omega^1} l_a z_a^k,\ \beta_2 + \mu_2 \sum_{a\in\Omega^1} l_a z_a^k\} = T^{max}(\sum_{a\in\Omega^1} l_a z_a^k)$ because $(\boldsymbol{z}^k, w^k)$ is optimal to $SUB1(\boldsymbol{v})$. Replacing w^k with $T^{max}(\sum_{a\in\Omega^1} l_a z_a^k)$ in the preceding equation yields

$$\sum_{a\in\Omega} s_a(\boldsymbol{v})\boldsymbol{z}_a^k + T^{max}\left(\sum\nolimits_{a\in\Omega^1} l_a z_a^k,\right) = \boldsymbol{E}_k^T\boldsymbol{\rho}^k + (\alpha_1^k\beta_1 + \alpha_2^k\beta_2) = D_k^{-1}(d_k), \tag{5.12}$$

where the last equality follows from (5.5). Thus, the cost of a utilized toll path equals $D_k^{-1}(d_k)$.

When $r \in TP_0^k(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v}))$, letting $z_a^k = \delta_{ar}, \forall a \in \Omega$, may not yield an optimal solution to the modified Lagrangian subproblem. When the path is not utilized, $x_{ij}^k(\boldsymbol{v})$ may equal zero when $z_{ij}^k = 1$. For such link (i,j), (5.2) and (5.4) imply that $z_{ij}^k\left((\alpha_1^k\mu_1 + \alpha_2^k\mu_2)l_{ij} + s_{ij}(\boldsymbol{v}) - \rho_i^k + \rho_j^k\right) \geq 0$ and $z_{ij}^k(s_{ij}(\boldsymbol{v}) - \rho_i^k + \rho_j^k) \geq 0$, i.e., the complementary slackness condition may not hold and $\boldsymbol{z}^k$ may not solve the modified Lagrangian subproblem at the end of Section 5.1. However, because $\boldsymbol{z}^k$ is still feasible to the subproblem, the weak duality theorem applies and

$$\sum_{a\in\Omega^1} \left(s_a(\boldsymbol{v}) + (\alpha_1^k\mu_1 + \alpha_2^k\mu_2)l_a\right) z_a^k + \sum_{a\in\Omega^2} s_a(\boldsymbol{v}) z_a^k \geq \tilde{L}_v^k(\alpha_1^k, \alpha_2^k) = \boldsymbol{E}_k^T\boldsymbol{\rho}^k.$$

Adding $(\alpha_1^k\beta_1 + \alpha_2^k\beta_2)$ to both sides of the above and using (5.5) yields

$$\sum_{a\in\Omega} s_a(\boldsymbol{v})\, z_a^k + \sum_{m=1}^{2} \alpha_m^k \left(\beta_m + \mu_m \sum_{a\in\Omega_1} l_a z_a^k\right) \geq \boldsymbol{E}_k^T\boldsymbol{\rho}^k + (\alpha_1^k\beta_1 + \alpha_2^k\beta_2) = D_k^{-1}(d_k).$$

Since $\max\{\beta_1 + \mu_1 \sum_{a\in\Omega^1} l_a z_a^k, \beta_2 + \mu_2 \sum_{a\in\Omega^1} l_a z_a^k\} \ \geq \ \sum_{m=1}^{2} \alpha_m^k \left(\beta_m + \mu_m \sum_{a\in\Omega_1} l_a z_a^k\right)$ when $\alpha_1^k + \alpha_2^k = 1$ and $\alpha_1^k, \alpha_2^k \geq 0$, it follows from above that

$$\sum_{a\in\Omega} s_a(\boldsymbol{v})\boldsymbol{z}_a^k + T^{max}(\sum_{a\in\Omega^1} l_a z_{a}^k,) \geq \sum_{a\in\Omega} s_a(\boldsymbol{v})\, z_a^k + \sum_{m=1}^{2} \alpha_m^k\left(\beta_m + \mu_m \sum_{a\in\Omega_1} l_a z_a^k\right) \geq D_k^{-1}(d_k). \tag{5.13}$$

Thus, if a toll path is not utilized, its cost is no smaller than $D_k^{-1}(d_k)$. Finally, it follows from (5.10) – (5.13) that any pair $(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v}))$ compatible with $(\boldsymbol{v}, \boldsymbol{d})$ is in tolled UE.

For the converse, assume that the flows on toll and toll-free paths are in tolled UE. For $r \in TP_{++}^k\left(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v})\right)$, $z_a^k = \delta_{ar}, \forall a \in \Omega$, must solve $SUB1(\boldsymbol{v})$ because path r must be one with the least generalized cost by definition. The zero duality gap assumption and Lemma 5.1 imply that $\boldsymbol{z}^k$ also solves the modified Lagrangian subproblem at the end of Section 5.1. Then, it is easy to show that the optimal dual vector, $\boldsymbol{\rho}^k$, associated with the subproblem satisfies (5.1) – (5.5) with (α_1^k, α_2^k) as specified in (5.9). The similar also holds with $r \in NP_{++}^k\left(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v})\right)$, $\boldsymbol{\gamma}^k$, $SUB2a(\boldsymbol{v})$ and (5.6) – (5.8).□

There are also link-based equilibrium conditions without relying on the zero duality gap assumption. Typically, they are only sufficient. For example, the theorem below provides a set of such conditions. Unlike the previous theorem, there are two set of node potentials, $\boldsymbol{\rho}^k$ and $\boldsymbol{\psi}^k$, for the link flows x_{ij}^k.

Theorem 5.3: A pair $(\boldsymbol{v}, \boldsymbol{d}) \in V^x$ is in tolled UE if there exist $\boldsymbol{\rho}^k, \boldsymbol{\psi}^k$, and $\boldsymbol{\gamma}^k$ such that following link-based conditions hold:

$$\mu_1 l_{ij} + s_{ij}(\boldsymbol{v}) - \left(\rho_i^k - \rho_j^k\right) = 0, \quad \forall (i,j) \in \Omega_1, k \in K: x_{ij}^k > 0 \tag{5.14}$$
$$\mu_1 l_{ij} + s_{ij}(\boldsymbol{v}) - \left(\rho_i^k - \rho_j^k\right) \geq 0, \quad \forall (i,j) \in \Omega_1, k \in K: x_{ij}^k = 0 \tag{5.15}$$
$$s_{ij}(\boldsymbol{v}) - \left(\rho_i^k - \rho_j^k\right) = 0, \quad \forall (i,j) \in \Omega_2, k \in K: x_{ij}^k > 0 \tag{5.16}$$
$$s_{ij}(\boldsymbol{v}) - \left(\rho_i^k - \rho_j^k\right) \geq 0, \quad \forall (i,j) \in \Omega_2, k \in K: x_{ij}^k = 0 \tag{5.17}$$
$$\mu_2 l_{ij} + s_{ij}(\boldsymbol{v}) - \left(\psi_i^k - \psi_j^k\right) = 0, \quad \forall (i,j) \in \Omega_1, k \in K: x_{ij}^k > 0 \tag{5.18}$$
$$\mu_2 l_{ij} + s_{ij}(\boldsymbol{v}) - \left(\psi_i^k - \psi_j^k\right) \geq 0, \quad \forall (i,j) \in \Omega_1, k \in K: x_{ij}^k = 0 \tag{5.19}$$
$$s_{ij}(\boldsymbol{v}) - \left(\psi_i^k - \psi_j^k\right) = 0, \quad \forall (i,j) \in \Omega_2, k \in K: x_{ij}^k > 0 \tag{5.20}$$
$$s_{ij}(\boldsymbol{v}) - \left(\psi_i^k - \psi_j^k\right) \geq 0, \quad \forall (i,j) \in \Omega_2, k \in K: x_{ij}^k = 0 \tag{5.21}$$
$$max\{\beta_1 + \boldsymbol{E}_k^T \boldsymbol{\rho}^k, \beta_2 + \boldsymbol{E}_k^T \boldsymbol{\psi}^k\} = D_k^{-1}(d_k), \quad \forall k \in K, \boldsymbol{x}^k \neq 0 \tag{5.22}$$
$$s_{ij}(\boldsymbol{v}) - \left(\gamma_i^k - \gamma_j^k\right) = 0, \quad \forall (i,j) \in \Omega_2, k \in K: y_{ij}^k > 0 \tag{5.23}$$
$$s_{ij}(\boldsymbol{v}) - \left(\gamma_i^k - \gamma_j^k\right) \geq 0, \quad \forall (i,j) \in \Omega_2, k \in K: y_{ij}^k = 0 \tag{5.24}$$
$$\boldsymbol{E}_k^T \boldsymbol{\gamma}^k = D_k^{-1}(d_k), \quad \forall k \in K, \boldsymbol{y}^k \neq 0 \tag{5.25}$$

Proof: For any $\left(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v})\right)$ compatible with $(\boldsymbol{v}, \boldsymbol{d})$ and $r \in TP_{++}^{k}\left(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v})\right)$, it follows from arguments similar to those in Theorem 5.2 that (5.14), (5.16), (5.18), and (5.20) lead to the following:

$$\mu_1 \sum_{a \in \Omega^1} \delta_{ar} l_a + \sum_{a \in \Omega} \delta_{ar} s_a(\boldsymbol{v}) = \boldsymbol{E}_k^T \boldsymbol{\rho}^k$$
$$\mu_2 \sum_{a \in \Omega^1} \delta_{ar} l_a + \sum_{a \in \Omega} \delta_{ar} s_a(\boldsymbol{v}) = \boldsymbol{E}_k^T \boldsymbol{\psi}^k$$

Substituting the above expressions for $\boldsymbol{E}_k^T \boldsymbol{\rho}^k$ and $\boldsymbol{E}_k^T \boldsymbol{\psi}^k$ into (5.22) yields

$$max \left\{ \beta_1 + \mu_1 \sum_{a \in \Omega^1} \delta_{ar} l_a + \sum_{a \in \Omega} \delta_{ar} s_a(\boldsymbol{v}), \beta_2 + \mu_2 \sum_{a \in \Omega^1} \delta_{ar} l_a + \sum_{a \in \Omega} \delta_{ar} s_a(\boldsymbol{v}) \right\} = D_k^{-1}(d_k)$$

$$max \left\{ \beta_1 + \mu_1 \sum_{a \in \Omega^1} \delta_{ar} l_a, \beta_2 + \mu_2 \sum_{a \in \Omega^1} \delta_{ar} l_a \right\} + \sum_{a \in \Omega} \delta_{ar} s_a(\boldsymbol{v}) = D_k^{-1}(d_k)$$

$$T^{max} \left(\sum_{a \in \Omega^1} \delta_{ar} l_a \right) + \sum_{a \in \Omega} \delta_{ar} s_a(\boldsymbol{v}) = D_k^{-1}(d_k)$$

Similarly, the following must hold

$$T^{max} \left(\sum_{a \in \Omega^1} \delta_{ar} l_a \right) + \sum_{a \in \Omega} \delta_{ar} s_a(\boldsymbol{v}) \geq D_k^{-1}(d_k), \forall r \in TP_0^k$$
$$\sum_{a \in \Omega^2} \delta_{ar} s_a(\boldsymbol{v}) = D_k^{-1}(d_k), \forall r \in NP_{++}^k$$
$$\sum_{a \in \Omega^2} \delta_{ar} s_a(\boldsymbol{v}) \geq D_k^{-1}(d_k), \forall r \in NP_0^k$$

Then, the last four equations imply that the costs for all utilized paths, toll-free or otherwise, equal $D_k^{-1}(d_k)$ and the costs of those not utilized cannot be lower, i.e., the tolled equilibrium conditions hold for any $(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v}))$ compatible with $(\boldsymbol{v}, \boldsymbol{d})$. □

5.3. Link-based Equilibrium Conditions: Two-part Pricing

This section considers two special cases in nonlinear pricing: area-based and two-part pricing. Mathematically, the latter corresponds to setting β_2 and μ_2 in the tolling function to zero. Doing so yields $T^{max}(\ell) = \beta_1 + \mu_1 \ell$ (see case (a) in Figure 2.2). Additionally, if $\mu_1 = 0$, then $T^{max}(\ell) = \beta_1$ and two-part pricing reduces to area-based pricing. The results below demonstrate that $SUB1(v)$ has no duality gap and provide link-based UE conditions for two-part pricing.

Lemma 5.4: If $T^{max}(\ell) = \beta_1 + \mu_1 \ell$, where β_1 and μ_1 are both nonnegative, then $SUB1(\boldsymbol{v})$ has no duality gap.

Proof: For $T^{max}(\ell)$ as given, $SUB1(\boldsymbol{v})$ reduces to

$$\begin{aligned} SUB1(\boldsymbol{v}): \quad & min \quad \sum_{a\in\Omega} s_a(\boldsymbol{v}) z_a^k + w^k \\ & s.t. \quad A\boldsymbol{z}^k = \boldsymbol{E}_k \\ & \qquad \beta_1 + \mu_1 \sum_{a\in\Omega^1} l_a z_a^k \le w^k \\ & \qquad z_a^k \in \{0,1\}, \forall a \in \Omega. \end{aligned}$$

The Lagrangian dual problem (or $D1(\boldsymbol{v})$) of the above can be written as follows:

$$\max\{L_v^k(\alpha_1^k): 0 \le \alpha_1^k \le 1\}$$

where $L_v^k(\alpha_1^k) = \alpha_1^k \beta_1 + \min\{\sum_{a\in\Omega} s_a(\boldsymbol{v}) z_a^k + \alpha_1^k \mu_1 \sum_{a\in\Omega^1} l_a z_a^k : A\boldsymbol{z}^k = \boldsymbol{E}_k, z_a^k \ge 0\}$. As before, we can replace the binary restriction with $z_a^k \ge 0$ because A is totally unimodular. Observe that the second constraint in $SUB1(\boldsymbol{v})$ must be hold at equality, i.e., $\beta_1 + \mu_1 \sum_{a\in\Omega^1} l_a z_a^k = w^k$ in order to minimize the objective function. Thus, w^k in the objective of $SUB1(\boldsymbol{v})$ can be replaced by $\beta_1 + \mu_1 \sum_{a\in\Omega^1} l_a z_a^k$ and the problem can be written as

$$\begin{aligned} SUB1(\boldsymbol{v}): \quad & min \quad \sum_{a\in\Omega} s_a(\boldsymbol{v}) z_a^k + \beta_1 + \mu_1 \sum_{a\in\Omega^1} l_a z_a^k \\ & s.t. \quad A\boldsymbol{z}^k = \boldsymbol{E}_k \\ & \qquad z_a^k \ge 0, \forall a \in \Omega. \end{aligned}$$

Comparing the two equivalent forms of $SUB1(\boldsymbol{v})$ yields that

$$\min\left\{\sum_{a\in\Omega} s_a(\boldsymbol{v}) z_a^k + w^k\right\} = \min\left\{\sum_{a\in\Omega} s_a(\boldsymbol{v}) z_a^k + \mu_1 \sum_{a\in\Omega^1} l_a z_a^k\right\} + \beta_1 = L_v^k(1)$$

Thus, $\alpha_1^k = 1$ is optimal to the Lagrangian dual problem and the objective values of $SUB1(\boldsymbol{v})$ and its Lagrangian dual problem are the same, i.e., there is no duality gap. □

Theorem 5.5: Let $T^{max}(\ell) = \beta_1 + \mu_1 \ell$. Then, a pair $(\boldsymbol{v}, \boldsymbol{d}) \in V^x$ is in tolled UE if and only if there exist $\boldsymbol{\rho}^k$ and $\boldsymbol{\eta}^k$ such the following link-based conditions hold:

$$\mu_1 l_{ij} + s_{ij}(\boldsymbol{v}) - \left(\rho_i^k - \rho_j^k\right) = 0, \quad \forall (i,j) \in \Omega_1, k \in K: x_{ij}^k > 0 \quad (5.26)$$
$$\mu_1 l_{ij} + s_{ij}(\boldsymbol{v}) - \left(\rho_i^k - \rho_j^k\right) \geq 0, \quad \forall (i,j) \in \Omega_1, k \in K: x_{ij}^k = 0 \quad (5.27)$$
$$s_{ij}(\boldsymbol{v}) - \left(\rho_i^k - \rho_j^k\right) = 0, \quad \forall (i,j) \in \Omega_2, k \in K: x_{ij}^k > 0 \quad (5.28)$$
$$s_{ij}(\boldsymbol{v}) - \left(\rho_i^k - \rho_j^k\right) \geq 0, \quad \forall (i,j) \in \Omega_2, k \in K: x_{ij}^k = 0 \quad (5.29)$$
$$\beta_1 + \boldsymbol{E}_k^T \boldsymbol{\rho}^k = D_k^{-1}(d_k), \quad \forall k \in K, \boldsymbol{x}^k \neq 0 \quad (5.30)$$
$$s_{ij}(\boldsymbol{v}) - \left(\gamma_i^k - \gamma_j^k\right) = 0, \quad \forall (i,j) \in \Omega_2, k \in K: y_{ij}^k > 0 \quad (5.31)$$
$$s_{ij}(\boldsymbol{v}) - \left(\gamma_i^k - \gamma_j^k\right) \geq 0, \quad \forall (i,j) \in \Omega_2, k \in K: y_{ij}^k = 0 \quad (5.32)$$
$$\boldsymbol{E}_k^T \boldsymbol{\gamma}^k = D_k^{-1}(d_k), \quad \forall k \in K, \boldsymbol{y}^k \neq 0 \quad (5.33)$$

Proof: The result follows directly from Theorem 5.2 and Lemma 5.5. When applying Theorem 5.2 to the case where $T^{max}(\ell) = \beta_1 + \mu_1 \ell$, observe that there is no α_2^k. In addition, the argument in Lemma 5.5 shows that $\alpha_1^k = 1$ solves $D2(\boldsymbol{v})$ in condition (5.9) of Theorem 5.2. □

Observe that (5.26) – (5.33) are the KKT conditions of the following optimization problem or the tolled UE problem under two-part pricing (TUE2):

$$TUE2: \quad min \quad \sum_{a \in \Omega^1} \int_0^{\sum_k x_a^k} s_a(z)dz + \sum_{a \in \Omega^2} \int_0^{\sum_k x_a^k + y_a^k} s_a(z)dz - \sum_{k \in K} \int_0^{\sigma_k + \eta_k} D_k^{-1}(z)dz$$
$$+\beta_1 \sum_{k \in K} \sigma_k + \mu_1 \sum_{a \in \Omega^1} \sum_{k \in K} l_a x_a^k$$
$$\begin{aligned} s.t. \quad & [A_1 : A_2]x^k - E_k \sigma_k = 0, \forall k \in K \\ & [0 : A_2]y^k - E_k \eta_k = 0,, \forall k \in K \\ & x^k, y^k, \sigma_k, \eta_k \geq 0,, \forall k \in K \end{aligned}$$

In the objective, the first three terms are convex functions and represent the objective function of a problem for finding a (toll-free) UE flow-demand pair when demands are elastic. The last two terms determine the total toll collected in two parts, the access and VMT fee. The first two constraints are flow-balance constraints for users who pay, σ_k, and do not pay toll, η_k. By letting $\boldsymbol{\rho}^k$ and $\boldsymbol{\gamma}^k$ be the multiplier vectors associated with the first two constraints, it is straightforward to show that the KKT conditions of the above problem reduce to conditions (5.26) – (5.33). Thus, the solution to the above problem yields a UE flow-demand pair $(\boldsymbol{v}, \boldsymbol{d})$ under two-part pricing, where $\boldsymbol{v} = \sum_{k \in K} \boldsymbol{x}^k + \boldsymbol{y}^k$ and $d_k = \sigma_k + \eta_k$.

As stated above, $TUE2$ involves no path flow (or f_r^k) and is a linearly constrained convex program, a problem that can be solved by commercial software such as CONOPT (see, e.g., Drud et al., 2002). To illustrate that standard algorithms in the literature with some modifications are applicable to $TUE2$, we state the Frank-Wolfe algorithm as it applies to $TUE2$ in the Appendix.

CHAPTER 6: FINDING OPTIMAL NONLINEAR TOLLING SCHEMES

For the tolling function based on $T^{max}(\cdot)$, the problem of finding an optimal nonlinear tolling scheme can be formulated as follows:

$$
\begin{aligned}
NLT: \quad \max \quad & \sum_{k \in K} \int_0^{d_k} D_k^{-1}(\chi) d\chi - s(\boldsymbol{v})^T \boldsymbol{v} \\
\text{s.t.} \quad & \text{restrictions on } \beta_1, \beta_2, \mu_1, \text{and } \mu_2 \\
& (\boldsymbol{v}, \boldsymbol{d}) \in V^f \\
& (\boldsymbol{v}, \boldsymbol{d}) \text{ satisfies } (3.1) - (3.4)
\end{aligned}
$$

The objective of the above is to maximize the social benefit. In the constraints, restrictions on the four pricing parameters depend on the pricing function of interest. For example, setting β_2, μ_1, and μ_2 to zero and allowing β_1 to be in the interval $[0, \beta_1^{max}]$ yield an area-based pricing scheme. On the other hand, setting β_2 and μ_2 to zero and allowing β_1 and μ_1 to be in the intervals $[0, \beta_1^{max}]$ and $[0,\ \mu_1^{max}]$, respectively, would generate a two-part pricing scheme instead. The remaining constraints ensure that the flow-demand pair is feasible and satisfies the tolled UE conditions. In words, NLT finds a set of pricing parameters such that the associated UE flow-demand pair yields the maximum social benefit.

As stated, NLT is a mathematical program with equilibrium constraints (see, e.g., Luo et al., 1996), a class of optimization problems generally difficult to solve. However, NLT contains at most four main decision variables–the pricing parameters. The other variables $(\boldsymbol{v}, \boldsymbol{d})$ react to or are induced by the pricing parameters via the last two set of constraints. As such, NLT can be solved approximately using a coordinate search technique (see, e.g., Bazaraa et al., 2006), one that sequentially searches for an optimal solution one decision variable (or coordinate) at a time. Because the feasible region of NLT is not convex, search and other algorithms in nonlinear programming typically produce locally optimal solutions. For techniques that yield globally optimal solutions, see, e.g., Rinnooy Kan et al. (1989).

In the coordinate search algorithm below, $TUE(\beta_1, \beta_2, \mu_1, \mu_2)$ denotes the TUE problem in Chapter 4 with the pricing function based on $T^{max}(\ell) = \max\{\beta_1 + \mu_1 \ell, \beta_2 + \mu_2 \ell\}$. The algorithm assumes that $\beta_1 \in [0, \beta_1^{max}], \mu_1 \in [0, \mu_1^{max}], \beta_2 \in [0, \beta_2^{max}]$ and $\mu_2 \in [0, \mu_2^{max}]$.

Coordinate Search Algorithm

<u>Step 1</u>: Set $(\beta_1^1, \beta_2^1, \mu_1^1, \mu_2^1) = (0,0,0,0)$ and $m = 1$.
<u>Step 2</u>: Let β_1^{m+1} solves the following problem:

$$
\max \left\{ \sum_{k \in K} \int_0^{d_k} D_k^{-1}(\chi) d\chi - s(\boldsymbol{v})^T \boldsymbol{v} : 0 \le \beta_1 \le \beta_1^{max}, (\boldsymbol{v}, \boldsymbol{d}) \in V^f, (\boldsymbol{v}, \boldsymbol{d}) \text{ solves } TUE(\beta_1, \beta_2^m, \mu_1^m, \mu_2^m) \right\}
$$

Step 3: Let μ_1^{m+1} solves the following problem:

$$\max\left\{\sum_{k\in K}\int_0^{d_k} D_k^{-1}(\chi)d\chi - s(\boldsymbol{v})^T\boldsymbol{v}: 0 \le \mu_1 \le \mu_1^{max}, (\boldsymbol{v},\boldsymbol{d}) \in \boldsymbol{V^f}, (\boldsymbol{v},\boldsymbol{d}) \text{ solves } TUE(\beta_1^{m+1},\beta_2^m,\mu_1,\mu_2^m)\right\}$$

Step 4: Let β_2^{m+1} solves the following problem:

$$\max\left\{\sum_{k\in K}\int_0^{d_k} D_k^{-1}(\chi)d\chi - s(\boldsymbol{v})^T\boldsymbol{v}: 0 \le \beta_2 \le \beta_2^{max}, (\boldsymbol{v},\boldsymbol{d}) \in \boldsymbol{V^f}, (\boldsymbol{v},\boldsymbol{d}) \text{ solves } TUE(\beta_1^{m+1},\beta_2,\mu_1^{m+1},\mu_2^m)\right\}$$

Step 5: Let μ_2^{m+1} solves the following problem:

$$\max\left\{\sum_{k\in K}\int_0^{d_k} D_k^{-1}(\chi)d\chi - s(\boldsymbol{v})^T\boldsymbol{v}: 0 \le \mu_2 \le \mu_2^{max}, (\boldsymbol{v},\boldsymbol{d}) \in \boldsymbol{V^f}, (\boldsymbol{v},\boldsymbol{d}) \text{ solves } TUE(\beta_1^{m+1},\beta_2^{m+1},\mu_1^{m+1},\mu_2)\right\}$$

Step 6: If $\|(\beta_1^{m+1},\beta_2^{m+1},\mu_1^{m+1},\mu_2^{m+1}) - (\beta_1^m,\beta_2^m,\mu_1^m,\mu_2^m)\| \le \epsilon$, stop and $(\beta_1^{m+1},\beta_2^{m+1},\mu_1^{m+1},\mu_2^{m+1})$ solves NLT approximately. Otherwise, set $m = m + 1$ and return to Step 2.

In Step 1, it is also possible to use other values for $(\beta_1^1,\beta_2^1,\mu_1^1,\mu_2^1)$. The problems in Steps 2 – 5 essentially have only one decision variable, i.e., they can be viewed as line search problems and there are many line search algorithms in the literature (see, e.g., Bazaraa et al., 2006), all of which guarantee a globally optimal solution under some assumptions. In our implementation below, we solve, e.g., the $TUE(\beta_1,\beta_2^m,\mu_1^m,\mu_2^m)$ problem in Step 2 by SD to obtain UE flow-demand pairs at 20 equally spaced β_1-values in the interval $[0,\beta_1^{max}]$ and choose one whose UE flow-demand pair $(\boldsymbol{v},\boldsymbol{d})$ yields the best social benefit, i.e., $\sum_{k\in K}\int_0^{d_k} D_k^{-1}(\chi)d\chi - s(\boldsymbol{v})^T\boldsymbol{v}$**,** as the solution to the problem in Step 2. The procedures for Steps 3 – 5 are similar. The order in which to optimize the pricing parameters in Steps 2 – 5 is heuristic. Other orderings are possible and may lead to a faster convergence. In Step 6, the algorithm terminates when the change between two consecutive solutions is small.

CHAPTER 7: NUMERICAL EXAMPLES

Using GAMS (Brooks et al., 1992), we implemented the SD algorithm in Chapter 4 to find tolled UE flow-demand pairs for some nonlinear pricing functions and used the coordinate search in Chapter 6 to find the pricing parameters whose associated UE flow-demand pair maximizes the social benefit. The CPU times reported below are from a 2 GHz Dell Computer with 2037 MB of RAM. The network used for all results below is displayed in Figure 7.1 and it has 36 OD pairs and a (disconnected) tolling area as shown.

The travel time function for each link is of the form $s_a(v_a) = T_a(1 + 0.15(v_a/c_a)^4)$, where the values of T_a and c_a are randomly selected from the intervals (5, 20) and (50, 100), respectively. The demand function for every OD pair is linear, i.e., $D_k(t) = a_k + b_k t$, where a_k and b_k are randomly chosen. For each $k \in K$, we first choose a demand, d_k, randomly from the interval (10, 30) and let τ_k^1 and τ_k^2 denote, respectively, the free-flow and user-equilibrium travel time. The latter assumes that the demand is fixed and equals d_k. Then, a_k and b_k are the intercept and slope of the line that passes through two points, $(\tau_k^1, \mu d_k)$ and (τ_k^2, d_k), where μ is a random number between 2 and 3.

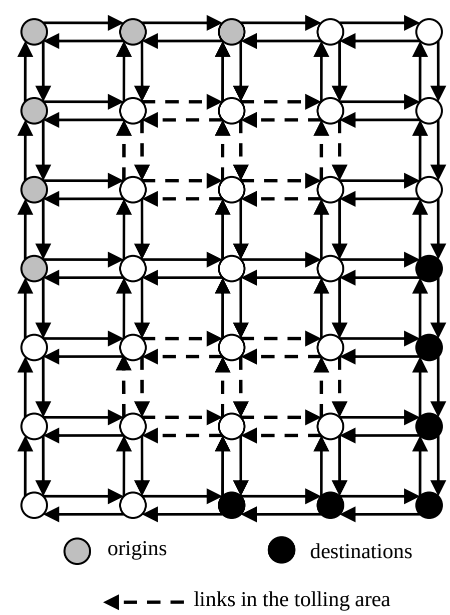

Figure 7.1: Network for area-based pricing

Table 7.1 displays the information about each iteration of simplicial decomposition for the TUE problem with $T^{max}(\ell) = max\{0.35\ell, 4.0 + 0.5\ell\}$. As explained in Chapter 3, we solved $SUB1(\boldsymbol{v})$ and $SUB2a(\boldsymbol{v})$ instead of the subproblem in Step 2 of the SD algorithm. On average, doing so reduces the CPU time by approximately 30% to 50%. In addition to the CPU times for solving the master and subproblem, the table provides at the end of each iteration the objective value of TUE, the average number of paths generated for each OD pair, and the maximum relative gap among all OD pairs. Using the notation from the algorithm in Chapter 4, the relative gap for OD pair k at the end of iteration n is $\left(c^k - D_k^{-1}(d_k^n)\right)/D_k^{-1}(d_k^n)$. For the network in Figure 7.1, SD requires only six iterations to find a solution to the TUE problem with a small relative gap. This is similar to the results in Hearn et al. (1987).

Table 7.1: Simplicial Decomposition for the TUE problem

			Max.	CPU Times (sec.)	
Iter.	Obj. Val.	Ave. Path	Rel. Gap	Master	Subproblem
1	-29842.85	1.00	1.0475	0.01	0.24
2	-42081.20	1.97	0.5243	0.01	0.28
3	-45625.04	2.89	0.1819	0.02	0.51
4	-46135.80	3.72	0.1334	0.03	0.99
5	-46317.35	4.06	0.0016	0.03	0.96

We also used the coordinate search algorithm to find the best parameters for two-part pricing, i.e., $T^{max}(\ell) = \beta_1 + \mu_1\ell$, where $\beta_1 \in [0, 100]$ and $\mu_1 \in [0, 5.0]$. (See case (a) in Figure 2.2.) We solved the TUE problem using SD and each iteration of the coordinate search for two-part pricing does not require Step 4 and 5 because β_2 and μ_2 are zero.

As displayed in Table 7.2, the coordinate search algorithm requires only four iterations to terminate with approximately 27544 in social benefit. We surmise that the efficiency of the coordinate search is due in part to the form of the social benefit as a function of β_1 and μ_1 shown in Figure 7.2. Although non-convex, the function in this figure is unimodal and well suited for the coordinate search.

Table 7.2: Coordinate search of two-part pricing

Iter.	β_1	μ_1	Social Benefit	CPU (sec.)
1	38	0.1	27379.62	8.54
2	34	0.2	27512.64	3.27
3	32	0.2	27544.09	2.42
4	32	0.2	27544.09	2.45

Observe that the first search iteration in Table 7.2 requires significantly more CPU time than the rest and the social benefit of the last (or 4th) iteration is the same as one in the third. During the first search iteration, evaluating the social benefit at a particular set of pricing parameters requires solving a TUE problem with SD, an algorithm that must generate paths for every OD pair if suitable ones are not available. These paths are saved and used in later search iterations, during which saved paths are often sufficient for finding a UE flow-demand pair.

Thus, later search iterations are typically less intensive computationally. In iteration 4, the algorithm essentially verifies that the stopping criterion in Step 6 is satisfied.

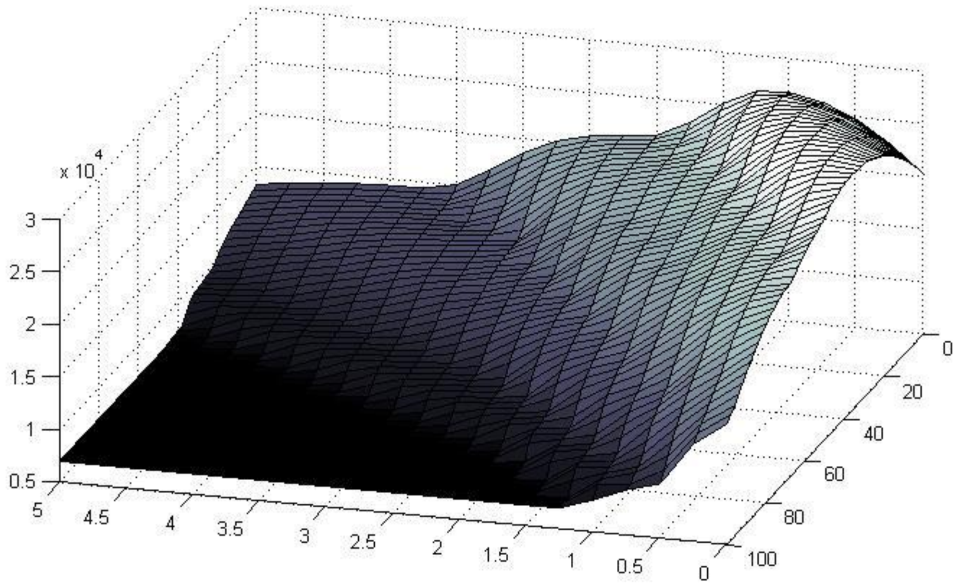

Figure 7.2: Social benefit from two-part pricing as a function of $\boldsymbol{\beta_1}$ and $\boldsymbol{\mu_1}$

For comparison, Table 7.3 displays the results from using the coordinate search algorithm to find the best parameters for three-part pricing, i.e., $T^{max}(\ell) = \max\{\mu_1\ell, \beta_2\}$. (See case (b) in Figure 2.2.) Not counting the last iteration whose purpose is to verify the stopping criterion, it took the search algorithm only one iteration to produce a solution. On average, the CPU times in Table 7.3 are significantly larger than those in Table 7.2 because $SUB1(\boldsymbol{v}^n)$ for three-part pricing contains binary variables. On the other hand, three-part pricing achieves a slightly higher social benefit, approximately 27802, than two-part pricing.

We also used the search algorithm to find the best parameters when $T^{max}(\ell) = \max\{\mu_1\ell, \beta_2 + \mu_2\ell\}$. The search took more CPU times and did not yield a social benefit better than the one from three-part pricing. This suggests that a more complex pricing structure may not be beneficial for the network in Figure 7.1.

Table 7.3: Coordinate search of three-part pricing

Iter.	μ_1	β_2	Social Benefit	CPU (sec.)
1	1.2	38	27801.66	22.06
2	1.2	38	27801.66	36.63

CHAPTER 8: CONCLUSIONS

This report considers the case where the amounts of toll that users pay varies nonlinearly with the distance they travel inside a tolling area. Instead of allowing any nonlinear function to represent the toll amount, this report assumes that the tolling function is piecewise linear. According to Wilson (1997), piecewise linear functions can realize most of the advantages of general nonlinear functions. Moreover, when the number of linear pieces is small, piecewise linear functions are more practical because they are easier to understand. Technically, piecewise linear functions lead to a UE problem that can be formulated as a convex program and solved using simplicial decomposition. When the zero duality gap assumption holds (e.g., as in the case of two-part pricing), link-based UE conditions exist and the UE problem can be stated using link flows. This eliminates the need to maintain information about individual paths and lessens the computational resources required to, e.g., solve the UE problem. To illustrate, we implemented a coordinate search algorithm and used it to find pricing functions that maximize the social benefit. A small road network with randomly generated data is used to empirically show how the algorithms behave.

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APPENDIX

This appendix presents a modification of the Frank-Wolfe algorithm for solving the tolled UE problem with two-part pricing or $TUE2$. For linearly constrained convex programs, the Frank-Wolfe algorithm begins with an initial feasible solution, finds an improving feasible direction by solving a linear program that approximates the original problem, and performs a line search along the direction found to obtain an improved solution. In theory, the algorithm repeats these steps until it finds a feasible solution for which no improving feasible direction exists. When applied to the toll-free UE problems in the literature, finding an improving feasible direction reduces to solving a shortest path problem for each OD pair. The similar is true when applied to $TUE2$. Instead of one, the algorithm below solves two shortest path problems for each OD pair, one to obtain a path using the tolling area and the other to find one that bypasses it instead.

The algorithm below applies the Frank-Wolfe algorithm to $TUE2$ with the assumption that $D_k^{-1}(0) = M_k < \infty$, i.e., M_k is the maximum demand for OD pair k.

Frank-Wolfe Algorithm for TUE2

Step 1: Let $(\boldsymbol{x}^1, \boldsymbol{y}^1, \boldsymbol{\sigma}^1, \boldsymbol{\eta}^1) \in V^x$ and set $m = 1$.
Step 2: Let $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\eta}})$ solve the following (sub)problem:

$$
\begin{aligned}
SUB3^m: \quad \min \quad & \sum_{a \in \Omega_1} (s_a(v_a^m) + \mu_1 l_a) x_a^k + \sum_{a \in \Omega_2} s_a(v_a^m)(x_a^k + y_a^k) + \sum_{k \in K} \beta \sigma_k - \sum_{k \in K} D_k^{-1}(d_k^m)(\sigma_k + \eta_k) \\
\text{s.t.} \quad & A_1 \boldsymbol{x}_1^k + A_2 \boldsymbol{x}_2^k = \sigma_k \boldsymbol{E}_k, \forall k \in K \\
& A_2 \boldsymbol{y}_2^k = \eta_k \boldsymbol{E}_k, \forall k \in K \\
& \sigma_k + \eta_k \le M_k, \forall k \in K \\
& x_a^k, y_a^k, \sigma_k, \eta_k \ge 0, \forall k \in K, a \in \Omega.
\end{aligned}
$$

Step 3: If the following holds, stop and $(\boldsymbol{x}^m, \boldsymbol{y}^m, \boldsymbol{\sigma}^m, \boldsymbol{\eta}^m)$ is optimal. Otherwise, go to Step 4.

$$
\sum_{a \in \Omega^1} (s_a(\boldsymbol{v}^m) + \mu_1 l_a)(\hat{v}_a - v_a^m) + \sum_{a \in \Omega^2} s_a(\boldsymbol{v}^m)(\hat{v}_a - v_a^m) + \sum_{k \in K} \beta(\hat{\sigma}_k - \sigma_k^m) - \boldsymbol{D}^{-1}(\boldsymbol{d}^m)^T(\hat{\boldsymbol{d}} - \boldsymbol{d}^m) \ge 0
$$

where $\boldsymbol{d}^m = \boldsymbol{\sigma}^m + \boldsymbol{\eta}^m$ and $\hat{\boldsymbol{d}} = \hat{\boldsymbol{\sigma}} + \hat{\boldsymbol{\eta}}$.

Step 4: Let $\lambda^m \in [0,1]$ solve the following one-dimensional problem:

$$\min_{\lambda\in[0,1]} \sum_{a\in\Omega^1} \int_0^{\lambda\hat{v}_a+(1-\lambda)v_a^m} (s_a(\chi)+\mu_1 l_a)d\chi + \sum_{a\in\Omega^2} \int_0^{\lambda\hat{v}_a+(1-\lambda)v_a^m} s_a(\chi)d\chi$$
$$+\beta \sum_{k\in K} \lambda\hat{\sigma}_k + (1-\lambda)\sigma_k^m - \sum_{k\in K} \int_0^{\lambda\hat{d}_k+(1-\lambda)d_k^m} D_k^{-1}(\chi)d\chi$$

Set $(\boldsymbol{x}^{m+1}, \boldsymbol{y}^{m+1}, \boldsymbol{\sigma}^{m+1}, \boldsymbol{\eta}^{m+1}) = \lambda^m(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\eta}}) + (1-\lambda^m)(\boldsymbol{x}^m, \boldsymbol{y}^m, \boldsymbol{\sigma}^m, \boldsymbol{\eta}^m)$ and $m = m+1$. Return to Step 2.

It is possible to let $(\boldsymbol{x}^1, \boldsymbol{y}^1, \boldsymbol{\sigma}^1, \boldsymbol{\eta}^1) = (0,0,0,0)$ in Step 1. Problem $SUB3^m$ in Step 2 is a linear program that approximates $TUE2$ around the current solution $(\boldsymbol{x}^m, \boldsymbol{y}^m, \boldsymbol{\sigma}^m, \boldsymbol{\eta}^m)$. The first two constraints in $SUB3^m$ balance the flows at each node for paths that use and do not use the tolling area, respectively. The third set of constraints ensures that the demands for toll and toll-free routes do not exceed the maximum for each OD pair. Equivalently, $SUB3^m$ can be written as follows:

$$\begin{aligned} SUB3a^m: \quad min \quad & \sum_{k\in K} \pi_k^1(\sigma_k) - D_k^{-1}(d_k^m)\sigma_k + \sum_{k\in K} \pi_k^2(\eta_k) - D_k^{-1}(d_k^m)\eta_k \\ s.t. \quad & \sigma_k + \eta_k \le M_k, \forall k \in K \\ & \sigma_k, \eta_k \ge 0, \forall k \in K \end{aligned}$$

where, for each $k \in K$,

$$\pi_k^1(\sigma_k) = \beta\sigma_k + \min\left\{\sum_{a\in\Omega} (s_a(v_a^m) + \mu_1 l_a)x_a^k : A\boldsymbol{x}^k = \sigma_k \boldsymbol{E}_k, x^k \ge 0\right\},$$

$$\pi_k^2(\eta_k) = \min\left\{\sum_{a\in\Omega^2} s_a(v_a^m)y_a^k : A_2\boldsymbol{y}^k = \eta_k \boldsymbol{E}_k, z^k \ge 0\right\}.$$

The minimization problems in the definition of $\pi_k^1(\sigma_k)$ and $\pi_k^2(\eta_k)$ are minimum cost flow problems (see, e.g., Ahuja et al., 1993). Because there is no capacity constraint on any link, these minimizations correspond to sending σ_k and η_k along the least-cost path using and not using the tolling area, respectively.

To solve $SUB3a^m$, evaluate $\pi_k^1(M_k)$ and $\pi_k^2(M_k)$, i.e., solve two shortest path problems, one using the full network and the other bypassing the tolling area, and send M_k units of flows along each route. Then, the solution to $SUB3a^m$ is, for each k,

$$(\hat{\sigma}_k, \hat{\eta}_k) = \begin{cases} (0,0), & \text{if } \min\{\pi_k^1(M_k), \pi_k^2(M_k)\} - D_k^{-1}(d_k^m)M_k \geq 0 \\ (M_k, 0), & \text{if } \pi_k^1(M_k) \leq \pi_k^2(M_k) \ \& \ \min\{\pi_k^1(M_k), \pi_k^2(M_k)\} - D_k^{-1}(d_k^m)M_k < 0 \\ (0, M_k), & \text{if } \pi_k^2(M_k) < \pi_k^1(M_k) \ \& \ \min\{\pi_k^1(M_k), \pi_k^2(M_k)\} - D_k^{-1}(d_k^m)M_k < 0 \end{cases}$$

Then, the corresponding the optimal solution to $SUB3^m$ is

$$\left(\hat{x}^k, \hat{y}^k, \hat{\sigma}_k, \hat{\eta}_k\right) = \begin{cases} (0,0,0,0), & \text{if } \min\left\{\pi_k^1(M_k), \pi_k^2(M_k)\right\} - D_k^{-1}(d_k^m)M_k \geq 0 \\ \left(x^k, 0, M_k, 0\right), & \text{if } \pi_k^1(M_k) \leq \pi_k^2(M_k) \ \& \ \min\left\{\pi_k^1(M_k), \pi_k^2(M_k)\right\} - D_k^{-1}(d_k^m)M_k < 0 \\ \left(0, y^k, 0, M_k\right), & \text{if } \pi_k^2(M_k) < \pi_k^1(M_k) \ \& \ \min\left\{\pi_k^1(M_k), \pi_k^2(M_k)\right\} - D_k^{-1}(d_k^m)M_k < 0 \end{cases}$$

where x^k and y^k are optimal solutions to the minimization problems in $\pi_k^1(M_k)$ and $\pi_k^2(M_k)$, respectively.

To obtain a more efficient algorithm, it is also possible to modify or extend the above algorithm via simplicial decomposition (see, e.g., Lawphongpanich and Hearn, 1984, Hearn et al., 1987, and Patriksson, 1994).