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Abstract

We consider distribution and location-planning models for supply chains that explicitly account for traffic congestion effects. The majority of facility location and transportation planning models in the operations research literature consider facility operations and transportation costs as separable (e.g., linear) by origin-destination pairs. Our goal is to understand how congestion costs and effects, which are not separable, influence supply chain location and distribution decisions. We study a competitive facility location and market-supply game with multiple firms competing in multiple markets in a congested distribution network. As a result of location and quantity decisions, firms are subject to location-specific transportation costs, convex traffic congestion costs and fixed facility location costs. The unit price in each market is a linear decreasing function of the total amount shipped to the market by all firms; that is, we consider an oligopolistic Cournot game and analyze the two-stage Nash Equilibrium. We discuss the results of extensive numerical studies that illustrate the effects of traffic congestion on a firm's equilibrium location and quantity decisions and demonstrate the efficiency of our solution approaches for finding equilibrium solutions.

Executive Summary

We consider distribution and location-planning models for supply chains that explicitly account for traffic congestion effects. The majority of facility location and transportation planning models in the operations research literature consider facility operations and transportation costs as separable (e.g., linear) by origin-destination pairs. Our goal is to understand how congestion costs and effects, which are not separable, influence supply chain location and distribution decisions.

We first study a competitive facility location and market-supply game with multiple identical firms competing in multiple markets in a congested distribution network. As a result of location and quantity decisions, firms are subject to location-specific transportation costs, convex traffic congestion costs and fixed facility location costs. First, we study the supply quantity decisions for any firm when the location choices of the firms are identical. An oligopolistic Cournot game is analyzed to determine a Pure Nash Equilibrium (PNE) for these quantity decisions, and we provide analytical results on the effects of traffic congestion costs on the equilibrium quantities flowing from supply facilities to markets. We then focus on the location decisions of the firms. As firms are identical, firms will choose identical facility locations, and we therefore study the optimal location decisions for any individual firm. We discuss the results of extensive numerical studies that illustrate the effects of traffic congestion on a firm's location and quantity decisions and demonstrate the efficiency of our solution approach.

We then study a set of heterogeneous competitive firms considering the location of uncapacitated facilities at a set of candidate locations in order to serve a set of markets. Each firm incurs firm-specific (linear) transportation costs, as well as convex congestion and fixed location costs as a result of location and distribution volume decisions. The unit price in each market is a linear decreasing function of the total amount shipped to the market by all firms; that is, we consider an oligopolistic Cournot game and analyze the two-stage Nash Equilibrium. This problem is referred to as the location-supply game, or competitive location game, and we first study the firms' market-supply decisions for given facility locations, i.e., the game's second stage. We formulate the problem of finding the equilibrium supply quantities as a variational inequality problem and provide a solution algorithm. Then we focus on the location decisions, i.e., the game's first stage. We provide rules to obtain a dominant location matrix, and use these rules in a heuristic solution approach to search for an equilibrium location matrix. Numerical results on the efficiency of the heuristic method are documented.

A Facility Location Problem with Supply Competition in a Congested Network

Dinçer Konur and Joseph Geunes

1. Background

Research on traffic network equilibrium problems, toll pricing (congestion pricing), and methods to mitigate traffic congestion have typically focused on the welfare of individual road users. However, recent studies identify the negative impacts traffic congestion has on supply chain operations. In particular, the performance of logistics operations is affected by traffic congestion, and these impacts are more drastic in Just-in-Time (JIT) production systems. Despite the fact that traffic congestion affects supply chain operations, most of the studies combining traffic congestion and supply chains are based on empirical data and lack theoretical results. Another problematic point is that traffic congestion effects are exogenous in past literature, and these effects are analyzed indirectly by assuming that increased congestion either implies increased travel times or decreased travel time reliability. More importantly, traffic congestion effects are only studied in the context of a distribution network of a single firm. In this study, we focus on the effects of traffic congestion on supply chain operations by modeling traffic congestion costs endogenously. We study two primary supply chain decisions: facility location decisions and supply quantity decisions. McKinnon et al. (2008) note that companies may restructure their distribution systems due to increased congestion. Moreover, Rao et al. (1991) mention that changes in facility locations are often a long-term reaction to increased traffic congestion. For example, Lee (2004) points out that when 7-11 Japan (SEJ), a convenience-store company, located stores in key locations, SEJ was subject to more dramatic effects of traffic congestion. Sankaran et al. (2005) also note that the effects of traffic congestion depend on the facility location choices of a company. Therefore, it is important to gain a better understanding of the effects of traffic congestion on facility location and distribution flow decisions. We study these factors in a competitive environment, i.e., when multiple firms compete in common markets.

McKinnon (1999) presents survey results on the negative effects of traffic congestion on the efficiency of logistics operations. In a similar study, McKinnon et al. (2008) note that, on average, traffic congestion accounts for 23 percent of the total delay times in shipments of the companies completing the survey. This rate can be higher (up to 34 percent) in some industries McKinnon et al. (2008). For instance, Fernie et al. (2000) point out that traffic congestion is one of the most important factors affecting cost and service in grocery retailing in the UK. Sankaran et al. (2005) also document the results of a survey and mention the effects of traffic congestion on supply chain operations. Weisbrod et al. (2001) provide a systematic review of the studies at the intersection of traffic congestion and supply chains and discuss how traffic congestion affects costs and productivity. Another stream of research studies traffic congestion in JIT systems. Rao et al. (1991) note that JIT systems require small lot sizes, which results in increased traffic congestion. Moreover, their survey results indicate that companies are aware of the associated congestion impacts and Rao et al. (1991) propose short-term and long-term methods to mitigate the effects of congestion. Moinzadeh et al. (1997) study the relationship between small lot sizes and traffic congestion for a company's distribution system, with multiple retailers using a common congested road. Rao and Grenoble (1991) also study the effects of JIT replenishment and the resulting traffic congestion on distribution costs. One other field of research that combines traffic congestion and supply chains focuses on freight distribution. Figliozzi (2009) studies the effects of traffic congestion on the costs associated with commercial vehicle tours, while Figliozzi (2006) and Figliozzi et al. (2007) analyze freight tours in congested urban areas. Golob and Regan (2001; 2003) also study the impacts of traffic congestion on freight operations.

The model we formulate considers facility locations and supply quantity decisions in a competitive environment on a congested distribution network. In particular, we study a competitive location game with multiple firms competing in multiple markets. The competitive location problem we study assumes the following settings. Competing firms are non-cooperative and they must simultaneously determine their facility locations (first stage decisions); then, the supply quantities flowing out of these facilities into each market (second stage decisions) must be determined. Note that a firm may locate more than one facility. Markets and possible facility locations are represented as vertices in a network. Firms are assumed to compete under a homogeneous cost structure; that is, they have identical cost parameters. For this reason, we assume that firms make the same facility location decisions when maximizing expected profits (when a unique Pure Nash Equilibrium does not exist). We assume an oligopolistic Cournot competition in the second stage, i.e., the total supply to a market determines the price in that market.

The first competitive location problem was introduced by Hotelling (1929). In this study, each of two competing firms tries to maximize its market share by locating a single facility on a line. Hotelling's problem is then extended by Teitz (1968) to the case in which firms may locate more than one facility. Studies exist in the literature that consider competitive location problems under different assumptions. The number of competing firms, the nature of strategic decisions of the competing firms and the competition assumptions are distinctive characteristics of different studies in the literature. Competitive facility location games focus mainly on the location decisions of competing firms and equilibrium conditions are analyzed to determine firms' location decisions along with other strategic decisions, such as pricing, production levels and capacity planning. Eiselt and Laporte (1996), Eiselt et al. (1993) and Plastria (2001) provide reviews of competitive facility location problems under different assumptions studied in the literature.

Most of the competitive location problems in the literature assume Cournot competition. Spatial competition of two firms with Cournot competition is studied by Labbé and Hakimi 1991). This study is extended to multiple firms by Sarkar et al. (1997). Both of these studies assume that firms locate a single facility. Pal and Sarkar (2002) consider spatial competition in a Cournot duopoly where the competing firms may locate more than one facility. The distinguishing assumption of these studies is that competing firms enter each market by supplying a positive quantity to each market. Rhim et al. (2003) and Sáiz and Hendrix (2008) relax this assumption and consider the case of free entry. The settings of the competitive location problems studied in Rhim et al. (2003) and Sáiz and Hendrix (2008) are similar to the settings of our problem. In both of these studies, the competition basis is that of Cournot and firms determine the location of their single facility and the quantities to be supplied from this facility to each market, if they choose to enter any market. While a homogeneous cost structure is assumed by Rhim et al. (2003) and Sáiz and Hendrix (2008) study a heterogeneous cost structure.

Our study extends the problems studied by Rhim et al. (2003) and Sáiz and Hendrix (2008) by allowing firms to locate more than one facility. Moreover, firms are subject to nonlinear traffic congestion costs. Konur and Geunes (2009) study a general competitive facility location game where firms are subject to nonlinear congestion costs and allowed to locate more than one facility. The problem we study is a special case of their problem in which we assume that supply firms are identical in terms of the costs they incur. Our goal is to analyze the effects of traffic congestion on the firms' equilibrium facility location and supply quantity decisions. Considering the special case involving identical supply firms enables us to explicitly analyze and characterize how congestion costs affect the structure of equilibrium decisions, and to use this analysis to provide insights into how equilibrium solutions change in response to changes in congestion levels and costs. We use a two-stage solution approach similar to those in Labbé and Hakimi (1991), Lederer and Thisse (1990), Pal and Sarkar (2002), Rhim et al. (2003), Sáiz and Hendrix (2008) and Sarkar et al. (1997). First, we study the second stage decisions for any firm when the location choices of the firms are identical. The Pure Nash Equilibrium (PNE) concept is used in the analysis of a Cournot oligopoly to determine the equilibrium supply quantities. We provide analytical results on the effects of traffic congestion costs on the equilibrium quantities flowing from supply points to markets in this stage. Then, we focus on the location decisions of the firms. We note that firms choose identical facility locations in the case of a unique PNE location decision. However, for other cases, since the equilibrium concept does not characterize what firms will actually do, we use the maximization of expected profits as an objective, assuming that any location decision is equally likely for each firm. We show that a mixed strategy Nash Equilibrium (MSNE) implies that it is equally likely for any firm to choose any given location decision. Thus, when firms are homogeneous, they will end up with identical facility locations, and therefore, we study the optimal location decision set for the individual firm. A heuristic solution method to determine a good location decision for a firm is also provided. We perform extensive numerical studies that illustrate the effects of traffic congestion on a firm's location and quantity decisions.

The rest of this paper is organized as follows. In Section 2, we discuss the detailed problem setting and solution approach. In Section 3, we study the properties of equilibrium supply quantities and propose a solution algorithm, given that firms make identical facility location decisions. Moreover, we analyze the effects of increased traffic congestion on equilibrium supply quantities. Section 4 discusses the rationale behind the assumption that firms choose identical facility locations, and a total enumeration scheme and heuristic method are provided to characterize facility location decisions. In Section 5, we provide the results of extensive numerical studies that characterize: (i) the effects of congestion on facility location and supply quantity decisions, (ii) the efficiency of the heuristic method and, (iii) the impacts of accounting for congestion in the decision making process. Concluding remarks, a summary of the contributions of our study, and future research directions are noted in Section 6.

2. Research Approach

We study a set of k competitive firms considering the location of facilities at m possible locations in order to serve customer markets at n locations. Each firm incurs transportation, traffic congestion and facility location costs as a result of their location and distribution volume decisions. More explicitly, firms are subject to linear transportation costs in the quantity shipped from a facility to a market and the traffic congestion cost incurred is convex in the total quantity shipped from a facility to a market. A fixed facility location cost exists for each location i. Moreover, we assume that any open facility is effectively uncapacitated and, hence, a firm will not open more than one facility at a given location. The notation we use throughout the text is summarized below. We will define additional notation as needed.

- r: index for firms, $r \in R = \{1, 2, \ldots, k\}$
- i: index for locations, $i \in I = \{1, 2, \ldots, m\}$
- j: index for markets, $j \in J = \{1, 2, \ldots, n\}$
- q_{ijr}: quantity shipped from the facility of firm r at location i to market j
- $q_{\bullet jr}$: total quantity shipped to market j by firm r, $q_{\bullet jr} = \sum_{i \in I} q_{ijr}$
- $q_{ij\bullet}$: total quantity shipped from location i to market j, $q_{ij\bullet} = \sum_{r \in R} q_{ijr}$
- $q_{\bullet j\bullet}$: total quantity shipped to market j, $q_{i\bullet r} = \sum_{r \in R} \sum_{i \in I} q_{ijr}$
- $\mathbf{Q}$: $k \times m \times n$ matrix of q_{ijr} values
- $\mathbf{x}_r$: m-vector representing location decisions of firm r
- $\mathbf{X}$: $m \times k$ matrix representing location decisions
- $p_j\left(q_{\bullet j\bullet}\right)$: price function for market j
- $g_{ij}(q_{ij\bullet})$: traffic congestion cost function for the link from location i to market j
- c_{ij}: transportation cost per unit of flow from location i to market j
- f_i: fixed cost of opening a facility at location i
- $f_r(\mathbf{x}_r)$: total facility location cost for firm r

We assume the unit price in each market is a linear and decreasing function of the total quantity of flow into the market. Explicitly, the unit price in market j, p_j, is defined by the function

$$p_j(q_{\bullet j\bullet}) = a_j - b_j q_{\bullet j\bullet}, \tag{1}$$

where the parameters $a_j \geq 0$ and $b_j > 0$ represent the level of maximum demand and the price sensitivity in market j. We assume that the transportation cost is linear in the quantity of flow on link (i, j) and $c_{ij} \geq 0$ represents a per unit transportation cost. It should be noted that c_{ij} can be assumed to include per-unit production costs as well. That is, a parameter $v_i > 0$ specific to location i can be included within c_{ij} to account for per-unit production cost at location i. The function g_{ij}, which is a function of the total quantity of flow on link (i, j), determines the traffic congestion cost coefficient on link (i, j). In particular, we assume that

$$g_{ij}(q_{ij\bullet}) = \alpha_{ij} q_{ij\bullet} \tag{2}$$

where $\alpha_{ij} > 0$ denotes the traffic congestion cost factor. Hence, the congestion cost incurred by a firm using link (i, j) increases with the total quantity of flow on the link as well as with the quantity sent by the firm on that link. In particular, the congestion cost for firm r is $\alpha_{ij} q_{ijr} q_{ij\bullet} = 0$ when $q_{ijr} = 0$. On the other hand, when $q_{ijr} > 0$, the congestion cost of firm r equals $\alpha_{ij} q_{ijr} q_{ij\bullet} > 0$ and is convex and increasing in q_{ijr} when the quantities sent by other firms on the link are fixed. Thus, the firm's congestion cost is a nondecreasing convex function of the quantity sent by the firm on the link. This choice of functional form reflects the nature of traffic congestion, as congestion costs increase in volume at an increasing rate. This is compatible with the note in Weisbrod et al. (2001), which emphasizes that companies with higher shipping levels are subject to a higher level of congestion related costs.

The profit function of firm r reads as

$$\Pi_r(\mathbf{Q},\mathbf{X}) \quad = \quad \sum_{j\in J} p_j\left(\sum_{i\in I}\sum_{r\in R} q_{ijr}\right)\sum_{i\in I} q_{ijr} - \sum_{j\in J}\sum_{i\in I} c_{ij} q_{ijr} - \sum_{j\in J}\sum_{i\in I} q_{ijr} g_{ij}\left(\sum_{r\in R} q_{ijr}\right) - f_r(\mathbf{x}_r), \quad (3)$$

where the first term is the revenue gained by serving markets, the second term is the total transportation cost, the third term is the total traffic congestion cost, and the last term is the total facility location costs. Konur and Geunes (2009) define the firms' profit function in a similar way, although they consider the case in which both c_{ij} and α_{ij} may be different for individual firms.

The supply firms first decide where to locate facilities and then determine how much to supply markets from each of their facilities. Clearly, under competition, a firm's resulting profit after making location and supply decisions depends on the location and supply decisions of all other firms. This implies a two-stage decision and associated solution approach. Stage-one decisions correspond to firm location decisions, while Stage-two decisions correspond to market supply decisions for each firm. Our solution approach first solves the Stage-two decisions for a fixed set of location decisions, assuming each firm chooses the same location decision vector. We will employ the Nash Equilibrium concept of Nash (1951) to determine the firms' supply quantity decisions and provide a method to find Pure Nash Equilibrium (PNE) quantities sent from any location to any market by each firm.

2.1. Stage-two Decisions: Market-Supply Game

In this section, we study the second-stage game, which determines the firms' supply quantity decisions for a given location decision. This restricted game to determine the equilibrium quantity decisions is referred as the *Market-Supply Game*. Note that, unlike the previous studies by Rhim et al. (2003) and Sáiz and Hendrix (2008), the firms not only compete based on price, but also as a result of the congestion cost functions on supply links. This *Market-Supply Game* is a non-cooperative game in which the supply firms are the players. Firms simultaneously determine how much to send from facilities to markets. To determine the firms' flows, we use the PNE concept, i.e., no firm will be better off by altering its supply quantity decisions under the given location decisions.

Now let us assume that the location decision for each firm, i.e., the vector $\mathbf{x}_r$ for each $r = 1, 2, \ldots, k$, is pre-determined. That is, $\mathbf{X}$ is fixed. Since $f_r(\mathbf{x}_r)$ is fixed for the given $\mathbf{X} = \mathbf{X}^0$, it can be omitted from Equation (3) for the analysis of the *Market-Supply Game*. Using the notation introduced in the previous section and Equations (1), (2), and (3), the profit function of firm r for the given $\mathbf{X} = \mathbf{X}^0$ reads as

$$\Pi_r(\mathbf{Q} \mid \mathbf{X} = \mathbf{X}^0) = \sum_{j \in J} \left[(a_j - b_j q_{\bullet j \bullet}) q_{\bullet jr} - \sum_{i \in I} c_{ij} q_{ijr} - \sum_{i \in I} \alpha_{ij} q_{ijr} q_{ij\bullet} \right]. \tag{4}$$

The function in Equation (4) is strictly concave in each $q_{ijr} \geq 0$, as $b_j > 0$ and $\alpha_{ij} > 0$. Note that $q_{ijr} = 0$ for all $j \in J, i \notin I_r^0$, where I_r^0 denotes the locations where firm *r* has facilities for the given $\mathbf{X} = \mathbf{X}^0$. The quantity decision for any firm will depend on the quantity decisions of the other firms; thus, we can apply the Nash Equilibrium concept in our solution approach. A Nash equilibrium solution for the *Market-Supply Game* will satisfy the first order conditions, $\partial \Pi_r(\mathbf{Q} \mid \mathbf{X} = \mathbf{X}^0) / \partial q_{ijr} = 0$, for a set of q_{ijr} values such that $q_{ijr} > 0$. Explicitly, for any Nash equilibrium solution the following equation must hold whenever $q_{ijr} > 0$:

$$a_j - b_j [q_{\bullet j \bullet} + q_{\bullet jr}] - c_{ij} - \alpha_{ij} [q_{ijr} + q_{ij\bullet}] = 0. \tag{5}$$

Note that Equation (5) depends on the total quantity supplied to market *j*, the total quantity supplied by firm *r* to market *j*, the total quantity supplied from location *i* to market *j* and the quantity supplied from location *i* to market *j* by firm *r*. On the other hand, Equation (5) does not depend on the other market parameters or variables and, hence, the quantity decision of any firm for market *j* can be made independently from the decisions related to the other markets. That is, each market can be analyzed separately. Let $\Pi_r^j(\mathbf{Q}_j \mid \mathbf{X} = \mathbf{X}^0)$ denote the profit function of firm *r* at market *j*, where $\mathbf{Q}_j$ denotes the vector of quantity decisions of the firms at market *j* for the given location decision $\mathbf{X} = \mathbf{X}^0$. Then, it follows from Equation (4) that

$$\Pi_r^j(\mathbf{Q}_j \mid \mathbf{X} = \mathbf{X}^0) = p_j(q_{\bullet j \bullet}) q_{\bullet jr} - \sum_{i \in I_r^0} c_{ij} q_{ijr} - \sum_{i \in I_r^0} \alpha_{ij} q_{ijr} q_{ij\bullet}. \tag{6}$$

In the rest of this section, we focus on the *Market-Supply Game* for market *j*, since the Stage-two decisions for each market can be analyzed separately. The results that hold for market *j* will also hold for the *Market-Supply Game* across all markets.

It follows from Equation (6) that Equation (5) gives the first order equilibrium condition for quantities such that $q_{ijr} > 0$. Let q_{ijr}^* denote the equilibrium quantities. Our goals are then (i) to determine the locations and the firms such that $q_{ijr}^* > 0$ and (ii) to simultaneously solve Equation (5) for each $q_{ijr}^* > 0$ for the given $\mathbf{X} = \mathbf{X}^0$.

At this point, we assume that $\mathbf{X}^0$ consists of identical columns, where the r^{th} column corresponds to the location decision of firm *r*. That is, $\mathbf{x}_r = \mathbf{x}^0 \;\; \forall r \in R$, where $\mathbf{x}^0$ denotes any column of $\mathbf{X}^0$. The rationale behind considering such location decisions will be explained in the following section when we study Stage-one decisions. This choice of $\mathbf{X}^0$ enables us to determine the quantity decisions using an iterative scheme and to analyze the effects of traffic congestion cost factors on the equilibrium quantity decisions. However, the equilibrium quantity decisions for any given $\mathbf{X}^0$ can also be solved using a variational inequality approach. See Konur and Geunes (2009) for an application of the variational inequality formulation to determine equilibrium quantity decisions for a competitive location-quantity game.

Now, suppose that the location decision matrix $\mathbf{X}^0$ consists of identical columns and, thus, the number of facilities at any candidate location is either k or 0 for some positive k. Note that we do not need to consider locations where no firm has located a facility. Therefore, we only study quantity decisions at supply locations with k facilities. In the next proposition, we show that the quantity supplied from location i to market j is the same for each firm.

Proposition 1. Suppose that $\mathbf{X}^0$ consists of identical columns, i.e., $\mathbf{x}_r = \mathbf{x}^0 \quad \forall r \in R$. Then $q^*_{ijr} = Q^*_{ij} / k \quad \forall r \in R$, where Q^*_{ij} denotes the total quantity flow on link (i, j) at equilibrium.

Proof: All proofs can be found in the Appendix.

Proposition 1 implies that if we know the total equilibrium quantity supplied from location i to market j, denoted by Q^*_{ij}, we also know the quantity that each firm with a facility at location i supplies to market j. For the given $\mathbf{X}^0$, with $\mathbf{x}_r = \mathbf{x}^0 \quad \forall r \in R$, it follows from Proposition 1 that (i) $q^*_{ijr} = Q^*_{ij} / k$, (ii) $q^*_{ij\bullet} = Q^*_{ij}$, (iii) $q^*_{\bullet jr} = \sum_{i \in I^0} Q^*_{ij} / k$ and (iv) $q^*_{\bullet j \bullet} = \sum_{i \in I^0} Q^*_{ij}$, where I^0 denotes the set of locations with k facilities associated with $\mathbf{X}^0$.
Recall that Equation (5) gives the first order conditions for any $q^*_{ijr} > 0$; that is, it gives the first order condition when $Q^*_{ij} > 0$. Substituting (i)-(iv) into Equation (5), we get

$$\delta_{ij} - b_j \left(\sum_{i \in I^0} Q^*_{ij} + \sum_{i \in I^0} \frac{Q^*_{ij}}{k} \right) - \alpha_{ij} \left(Q^*_{ij} + \frac{Q^*_{ij}}{k} \right) = \delta_{ij} - \gamma b_j \sum_{i \in I^0} Q^*_{ij} - \gamma \alpha_{ij} Q^*_{ij} = 0, \qquad (7)$$

where $\delta_{ij} = a_j - c_{ij}$ and $\gamma = (k+1)/k$. Note that we may have at most $k \times m$ such first order conditions defined for market j. Nevertheless, the first order conditions associated with a location use the same equation for each firm, given by Equation (7). Therefore, we focus on simultaneously solving at most m first order conditions, one for each location, defined by Equation (7). The next proposition provides conditions that must be satisfied by the total equilibrium quantity on a link (i, j).

Proposition 2. The equilibrium quantities must satisfy the following conditions:

$$(a)\ Q^*_{ij} > 0 \text{ if and only if } \delta_{ij} > \gamma b_j \sum_{i \in I^0} Q^*_{ij},$$

$$(b)\ Q^*_{ij} = 0 \text{ if and only if } \delta_{ij} \le \gamma b_j \sum_{i \in I^0} Q^*_{ij}.$$

Proposition 2 implies that the δ_{ij} values are important in determining the active locations at market j. A location is referred to as *active* at a market whenever there is a positive supply from this location to the market. Similarly, a firm is referred as *active* at a market whenever there is a positive supply by this firm to the market. The next proposition is a direct result of Proposition 2 and states the *activeness* relations between two locations.

Proposition 3. Suppose that $\delta_{i_1 j} \ge \delta_{i_2 j}$ for locations $i_1, i_2 \in I^0$. Then in an equilibrium solution

$$(a) \quad \text{If } Q^*_{i_2 j} > 0, \text{ then } Q^*_{i_1 j} > 0,$$

$$(b) \quad \text{If } Q^*_{i_1 j} = 0, \text{ then } Q^*_{i_2 j} = 0.$$

Proposition 3 highlights the importance of sorting locations according to their δ_{ij} values which, for a given market j, is equivalent to sorting supply locations based on the c_{ij} values. In particular, if we know that ℓ locations are active at market j, these locations should be the ℓ locations with greatest δ_{ij} values (or, smallest c_{ij} values). Note that if there exist locations with identical δ_{ij} values, it follows from Proposition 3 that either all of these locations are active or none of them is active at market j. Moreover, for both of these cases, the equilibrium supply quantities at market j remain unchanged regardless of the sorting order of tied values of δ_{ij}, as the same first order conditions given in Equation (7) will be solved.

Now let us sort locations according to their δ_{ij} values, and without loss of generality, let us assume that $\delta_{ij} \geq \delta_{(i+1)j}$. Therefore; if ℓ locations are active at market j, these locations are $1,2,...,\ell$ with $\ell \leq |I^0|$, where $|I^0|$ denotes the cardinality of the set I^0. Then for any firm at any location i, $i \leq \ell$ (since $q^*_{ijr} > 0$ as $Q^*_{ij} > 0$) the following first order condition must be satisfied:

$$\delta_{ij} - \gamma b_j (Q^*_{1j} + Q^*_{2j} + \cdots + Q^*_{\ell j}) - \gamma \alpha_{ij} Q^*_{ij} = 0 \forall i \leq \ell.$$

In matrix notation, the first order conditions can be represented as

$$\begin{bmatrix} \delta_{1j} \\ \delta_{2j} \\ \vdots \\ \delta_{\ell j} \end{bmatrix} = \gamma \begin{bmatrix} \alpha_{1j} + b_j & b_j & \cdots & b_j \\ b_j & \alpha_{2j} + b_j & \cdots & \vdots \\ \vdots & \vdots & \ddots & b_j \\ b_j & \cdots & b_j & \alpha_{\ell j} + b_j \end{bmatrix} \begin{bmatrix} Q^*_{1j} \\ Q^*_{2j} \\ \vdots \\ Q^*_{\ell j} \end{bmatrix}.$$

It follows from the above representation that we can find the Q^*_{ij} values easily by inverting the $\ell \times \ell$ matrix for a given set of active locations. Note that inverting this matrix basically involves solving the first order conditions for locations $1,2,...,\ell$ together. However, our aim is to determine the set of active locations and then find the equilibrium quantities. In the next proposition, we provide an algorithm that determines the set of active locations at a market as well as the total equilibrium flows from these locations. The algorithm is based on Propositions 2 and 3.

Proposition 4. Suppose that $\mathbf{X}^0$ consists of identical columns, i.e., $\mathbf{x}_r = \mathbf{x}^0 \quad \forall r \in R$. Then Algorithm 1, stated below, determines the number of the active locations and the corresponding equilibrium flow quantities.

Algorithm 1.

Given $\mathbf{x}_r = \mathbf{x}^0 \quad \forall r \in R$, the number of firms, b_j, δ_{ij} and α_{ij} values for market j:

Step 0. Set $Q^*_{ij} = 0 \quad \forall i \notin I^0$. Sort the remaining locations such that location 1 has the greatest δ_{ij} value. If $\delta_{1j} > 0$, set $\ell = 2$ and go to Step 1; otherwise $Q^*_{ij} = 0$ $\forall i \in I^0$.

Step 1. For location ℓ, find $Q_{\ell j}^{(\ell)}$ by solving the following set of equations represented in matrix form

$$\begin{bmatrix} \delta_{1j} \\ \delta_{2j} \\ \vdots \\ \delta_{\ell j} \end{bmatrix} = \gamma \begin{bmatrix} \alpha_{1j}+b_j & b_j & \cdots & b_j \\ b_j & \alpha_{2j}+b_j & \cdots & \vdots \\ \vdots & \vdots & \ddots & b_j \\ b_j & \cdots & b_j & \alpha_{\ell j}+b_j \end{bmatrix} \begin{bmatrix} Q_{1j}^{(\ell)} \\ Q_{2j}^{(\ell)} \\ \vdots \\ Q_{\ell j}^{(\ell)} \end{bmatrix}. \tag{8}$$

Step 2. If $Q_{\ell j}^{(\ell)} > 0$ and $\ell < |I^0|$, set $\ell = \ell + 1$ and go to Step 1. If $Q_{\ell j}^{(\ell)} > 0$ and $\ell = |I^0|$, stop, locations $1, 2, ..., \ell$ are active and $Q_{ij}^* = Q_{ij}^{(\ell)} \quad \forall i \in I^0$. Else if, $Q_{\ell j}^{(\ell)} \leq 0$, stop; locations $1, 2, ..., \ell - 1$ are active at market j. $Q_{ij}^* = Q_{ij}^{(\ell-1)}$ for $i \leq \ell - 1$ and $Q_{ij}^* = 0$ for $i \geq \ell$.

We next analyze Algorithm 1, which will be helpful in characterizing the effects of the traffic congestion cost factor on the equilibrium flow quantities. Suppose that there are ℓ active locations at market j. Consider the w^{th} iteration of Step 2 in Algorithm 1. Let $Q_{ij}^{(w)}$ be the tentative quantities calculated at the w^{th} iteration using Equation (8). (Note that $Q_{ij}^{(\ell)} = Q_{ij}^*$.) In the w^{th} iteration, we have tentative equilibrium quantities for locations 1 to w, which are the solutions to

$$\delta_{ij} - \gamma\alpha_{ij}Q_{ij}^{(w)} = \gamma b_j (Q_{1j}^{(w)} + Q_{2j}^{(w)} + \cdots + Q_{wj}^{(w)}) \forall i \leq w. \tag{9}$$

It follows from Equation (9) that

$$\delta_{1j} - \gamma\alpha_{1j}Q_{1j}^{(w)} = \delta_{2j} - \gamma\alpha_{2j}Q_{2j}^{(w)} = \cdots = \delta_{wj} - \gamma\alpha_{\ell j}Q_{wj}^{(w)}. \tag{10}$$

Equations (9) and (10) imply that, for location s, $s \leq w$, in the w^{th} iteration

$$Q_{sj}^{(w)} = \frac{\delta_{\ell j} + b_j \sum_{i=1}^{w} \dfrac{(\delta_{sj} - \delta_{ij})}{\alpha_{ij}}}{\gamma \left(\alpha_{sj} + b_j \sum_{i=1}^{w} \dfrac{\alpha_{sj}}{\alpha_{ij}} \right)}. \tag{11}$$

Equation (11) gives the equilibrium quantity for location s when $w = \ell$. In the next proposition, we show that the quantity supplied from an active location decreases as the number of active locations increases at each iteration of Algorithm 1, whereas the total quantity supplied to the market increases.

Proposition 5. (a) $Q_{sj}^{(w)} > Q_{sj}^{(w+1)}$ for location s, $s \leq w$ and $w+1 \leq \ell$. (b) $\sum_{i=1}^{w} Q_{ij}^{(w)} < \sum_{i=1}^{w+1} Q_{ij}^{(w+1)}$, $w+1 \leq \ell$.

The following subsection characterizes the effects of the traffic congestion cost factor on the total equilibrium quantity flow from a location to market j, as well as on the total equilibrium quantity supplied to market j. Proposition 5 will be used in the analysis of these effects. Our goal is to understand how variation in α_{ij} influences the equilibrium solution.

2.2. Impact of Variation in α_{ij}

In this section, we analyze the changes in the equilibrium supply quantities when the congestion cost factor on one of the links connecting a supply location to a market increases. Note that we assume the facility location decisions of the firms are fixed and identical, i.e., firms have the same facility locations. Suppose that locations are sorted such that location 1 has the greatest δ_{ij} value. Hence, when there are ℓ locations active in a market, these locations will be the first ℓ locations under Proposition 3. We first note that when there are ℓ locations active initially, an increase in the traffic congestion cost factor for one of the active locations will not result in any of the initially active locations becoming inactive. The next proposition provides a formal proof of this.

Proposition 6. Consider α_{sj}^1 and α_{sj}^2 such that $\alpha_{sj}^1 < \alpha_{sj}^2$, and suppose that locations 1 to ℓ are active under the α_{sj}^1 value, $1 \le s \le k$. Then locations 1 to ℓ are also active under the α_{sj}^2 value.

Proposition 6 implies that when the traffic congestion cost factor for one of the initially active locations increases, it is possible that the total number of active locations may increase. Moreover, the initially active locations will continue to be active. Next, we study the cases (i) when the number of active locations remains the same and (ii) when the number of active locations increases.

(i) When the number of active locations remains the same, we know that all of the initially active locations will remain active. That is, the set of active locations remains the same. This case also captures the situation when all of the locations are initially active. In this case, the quantity supplied from the location for which the traffic congestion cost factor increased, will decrease. On the other hand, the quantity supplied from the other locations will increase. Moreover, the *total quantity* supplied to the market decreases. We formalize this discussion in the next proposition.

Proposition 7. Suppose that α_{sj}^1 and α_{sj}^2 are such that $\alpha_{sj}^1 < \alpha_{sj}^2$, and that locations 1 to ℓ are active under α_{sj}^1 and α_{sj}^2, $1 \le s \le \ell$; that is, the number of active locations and the set of active locations remain the same. Then (a) $Q_{ij}^{1*} > Q_{ij}^{2*}$ for $i = s$ and, $Q_{ij}^{1*} < Q_{ij}^{2*}$ $i \ne s$. Moreover, (b) $\sum_{i=1}^{\ell} Q_{ij}^{1*} > \sum_{i=1}^{\ell} Q_{ij}^{2*}$, where Q_{ij}^{1*} and Q_{ij}^{2*} denote the equilibrium quantities supplied from location i to market j under the α_{sj}^1 and α_{sj}^2 values, respectively.

Statement (a) of Proposition 7 implies that each firm will reduce the quantity that it supplies to market j on link (i, j) if the set of active locations does not change when the traffic congestion cost factor increases on the link. On the other hand, each firm will increase the quantity it supplies to market j on the other links in this case. Moreover, it follows from Statement (b) of

Proposition 7 that the total quantity sent to market j by any firm will decrease. This discussion highlights the fact that firms will reduce their supply to market j and, hence, increase the price in market j, while decreasing their transportation costs by supplying less, to balance the increase in their traffic congestion costs. Next we study the case when the number of active locations increases.

(ii) When the number of active locations increases, the total quantity supplied from the location for which the traffic congestion cost factor increases will decrease. On the other hand, the total quantity supplied from the other locations that were initially active may increase or decrease. However, if the total quantity supplied from one of the initially active locations (for which the traffic congestion cost factor remains the same) increases (decreases), the total quantity supplied from the other initially active locations (with unchanged traffic congestion cost factors) also increases (decreases). The next proposition formalizes this discussion.

Proposition 8. Suppose that α_{sj}^1 and α_{sj}^2 are such that $\alpha_{sj}^1 < \alpha_{sj}^2$, and suppose that locations 1 to ℓ are active under α_{sj}^1, $s \le \ell$, and locations 1 to $\ell + \wp$ are active under α_{sj}^2. Then (a) $Q_{ij}^{1*} > Q_{ij}^{2*}$ for $i = s$. Moreover, (b) if $Q_{ij}^{1*} < Q_{ij}^{2*}$ for a location i, $i \ne s$, then $Q_{ij}^{1*} < Q_{ij}^{2*}$ $\forall i \le \ell$, $i \ne s$ and $\sum_{i=1}^{\ell} Q_{ij}^{1*} > \sum_{i=1}^{\ell+\wp} Q_{ij}^{2*}$, where Q_{ij}^{1*} and Q_{ij}^{2*} denote the equilibrium quantities supplied from location i to market j under α_{sj}^1 and α_{sj}^2, respectively.

Proposition 8 implies that each firm will reduce the quantity it supplies to market j on link (s, j) if the number of active locations increases when the traffic congestion cost factor increases on link (s, j). On the other hand, each firm may increase or decrease the quantity it supplies to market j on the other links in this case. However, the reaction of the firms will be the same for the quantity decisions on the other links, i.e., if firms increase (decrease) the flow on link (i, j), $i \ne s$, they will increase (decrease) the flow on any link (i, j), $i \ne s$. Moreover, when firms increase (decrease) the flow on link (i, j), $i \ne s$, the total quantity supplied to market j and the total quantity supplied to market j by any firm decreases (increases). When the total quantity supplied to market j by a firm decreases, this implies that all of the firms decrease supply to market j, increasing the price in market j to balance the increase in the traffic congestion costs. Nevertheless, when the total quantity supplied to market j by a firm increases and the number of supply points increases, this illustrates how firms may choose to divert flow to market j using links that are not as close to market j but are less congested.

Our discussion of Propositions 7 and 8 implies that increased congestion hampers efficient planning of supply chain activities, because it pushes firms to supply a market using either more congested links or links that are not close to the market. In Section 5, we give the results of extensive numerical studies to characterize the effects of increased traffic congestion on the facility location decisions of the firms as well as the supply quantity decisions.

2.3. Stage-One Decisions: Facility Locations

In this section, we study the firms' supply facility location decisions. We first discuss the rationale behind our prior assumption that all firms make identical location decisions. Then, we seek the best location decision of a single firm, under the assumption that it will also be the best location decision of the other firms.

2.3.1. Identical location decisions

Suppose that we are able to determine the optimal supply quantity decisions for any given location decision matrix $\mathbf{X}^0$, which implies that we can determine the total profit, including the facility location costs, for any given $\mathbf{X}^0$ (see Konur and Geunes, 2009). In the next proposition, we show that if there exists a unique PNE location decision, then each firm chooses the same facility locations in equilibrium.

Proposition 9. Suppose that there exists a unique PNE location matrix, $\mathbf{X}^*$. Then, $\mathbf{x}_r = \mathbf{x}^*$ $\forall r \in R$, where $\mathbf{x}^*$ denotes the column vector decision for each firm in $\mathbf{X}^*$.

Proposition 9 also follows from the fact that location decisions of the firms form a multi-player symmetric (strongly symmetric; Brant et al., 2009) game with a finite number of strategies (Nash, 1951). For symmetric games, it is well known that a symmetric equilibrium exists, either under pure strategies or mixed strategies (Nash, 1951). Therefore, when there exists a unique PNE location matrix, it will be a symmetric PNE, i.e., each firm makes the same location decisions. Furthermore, Proposition 9 implies that when there exists a unique PNE location matrix, the search for an equilibrium location matrix can be restricted to location decisions such that each firm chooses the same facility locations. We can thus use the method described in the previous section to characterize the profit of each such location matrix and, hence, choose the best among all solutions with identical columns to determine the unique PNE.

On the other hand, it is possible that multiple PNE location decisions exist, or that a PNE location decision does not exist. While uniqueness of PNE location decisions implies existence of a symmetric PNE (which is the unique PNE location matrix itself as implied by Proposition 9), in the case of multiple PNE location decisions, it is possible that none of the equilibrium points under pure strategies is symmetric. Cheng et al. (2004) show that at least one PNE exists for multi-player symmetric games with two strategies. That is, if there exists a single location, the game corresponding to the location decisions of the firms has a PNE solution. We note that the single location case can be solved by considering 2^r solutions with each firm either locating or not locating a facility at the single location. It easily follows from the discussion in the previous section that for any such configuration, the quantity decisions of the firms with a facility will be identical. Moreover, Rhim et al. (2003) prove the existence of a PNE in a competitive facility location game in which firms are allowed to locate at most one facility, by noting that the game can be modeled as a congestion game under the assumption that each market will be supplied from a single location. It is a well-known result that congestion games have PNE points (Rosenthal, 1973). Nevertheless, the game we study cannot be modeled as a congestion game due to the fact that firms may locate more than one facility. It is noted by Cheng et al. (2004) that even for symmetric games with two strategies, the existence or uniqueness of a symmetric PNE (i.e., when each player chooses the same strategy) is not guaranteed. Amir et al. (2008) show that for supermodular, doubly symmetric games, there exists a Pareto dominant symmetric PNE. However, the location decisions for our problem do not constitute a doubly symmetric game.

When a symmetric PNE solution does not exist, this implies that either multiple PNE solutions exist or no PNE location exists. For both of these cases, as previously noted, the corresponding mixed strategy Nash equilibrium (MSNE) will be symmetric. Next, we study MSNE for such cases under the following assumptions:

Assumption 1. Given the location decisions of other firms, a firm will never locate an additional facility if locating this facility reduces profit.

Assumption 2. Given the location decisions of other firms, if locating an additional facility does not change the firm's total profit, the firm will add this facility.

Assumption 3. Given the location decisions of other firms, there do not exist multiple distinct location decisions containing an identical number of facilities that result in the same profit level for any firm.

Note that Assumptions 1-3 imply that, given the location decisions of other firms, a firm will have a unique choice of location vector. In the next proposition, we show that, under Assumptions 1-3, a MSNE exists such that the probability of a firm choosing any particular location vector $\mathbf{x}$ is either 0 or equal to some value ρ such that $1 \geq \rho > 0$.

Proposition 10. Suppose that Assumptions 1-3 hold and that no firm will choose a location decision that is weakly or strictly dominated. Then, there exists a mixed strategy Nash equilibrium with $\rho_r(\mathbf{x}) = \rho$ or $\rho_r(\mathbf{x}) = 0$ for any location vector $\mathbf{x}$, for all $r \in R$, where $\rho_r(\mathbf{x})$ denotes the probability that firm r will choose location vector $\mathbf{x}$ and $1 \geq \rho > 0$.

It follows from the proof of Proposition 10 that when there does not exist a unique symmetric PNE location decision, firms will assign the same probabilities to location vectors that are not dominated in a mixed strategy and dominated location vectors will be assigned probability 0. Moreover, due to the symmetry of the mixed strategy equilibrium, firms will assign the same probability to each particular location vector.

The problem with using the equilibrium concept as a decision mechanism for location decisions is that it fails to explain and characterize firms' actual decisions when multiple PNE solutions exist or when no PNE location decision exists. We already know from Proposition 9 that when the PNE is unique, all firms will choose the same locations and, hence, we can search over one firm's decisions to find an equilibrium solution, as the profits of the firms will be the same when the location decisions are the same. Nevertheless, when multiple PNE solutions exist or when no PNE solution exists, we cannot characterize the firms' actions using the PNE concept. Thus, if we assume that firms determine facility locations purely based on their expected profits (assuming that any location vector is equally likely for any firm), then since firms are homogeneous, they will make the same decisions. We can therefore determine firms' location decisions by choosing the best among all location matrices with identical columns. Moreover, as noted in Proposition 10, in the case of multiple or no PNE location solutions, when firms determine the probability of choosing a location vector, they will assign the same probabilities, and the probabilities associated with location vectors that are not weakly or strictly dominated are the same for each firm. Determining the best among all location matrices with identical columns will be equivalent to assuming that any location vector is equally likely as well, because a location matrix that consists of identical weakly or strictly dominated location vectors will not result in higher profits for any firm. Therefore, from this point on, we focus on determining the best location decision of a single firm, assuming that the other firms will choose the same locations. We note that the corresponding solution is a PNE when there exists a unique PNE location decision and it is the best symmetric PNE when there exist multiple symmetric PNE points. For both of these cases, the resulting solution will be a Subgame Perfect Nash equilibrium (Selten, 1975).

Now suppose that either $\mathbf{x}^{1*}$ or $\mathbf{x}^{2*}$ is the best location decision for firm r. To determine which of these is better for firm r, we need to compare the profits of firm r given $\mathbf{X}^0 = \mathbf{X}^1$ and $\mathbf{X}^0 = \mathbf{X}^2$, where each column of $\mathbf{X}^1$ equals $\mathbf{x}^{1*}$ and each column of $\mathbf{X}^2$ equals $\mathbf{x}^{2*}$. Note that we can find the total profit for firm r associated with $\mathbf{X}^1$ and $\mathbf{X}^2$ by determining the profit from supplying markets using the method described in the previous section, and then subtracting the facility location costs associated with $\mathbf{x}^{1*}$ and $\mathbf{x}^{2*}$. A total enumeration scheme would determine the profit for each $\mathbf{X}^0$ such that $\mathbf{X}^0$ has identical columns, and pick the one with maximum profits. In case of alternative optimal solutions, Assumptions 1-2 can be used as a selection tool.

The resulting matrix $\mathbf{X}^*$ will give the best location decision for firm r as well as for all other firms. However, total enumeration requires evaluating exponentially many location decisions for a firm. In particular, a firm must determine the profit for 2^m location decisions, and choose the one with the maximum profit. As total enumeration is computationally burdensome, we next provide a heuristic method intended to be representative of how individual firms may approach simultaneous location decisions in practice. Our heuristic method first chooses the number of facilities to be located based on a ranking of locations derived from the problem parameters and then, chooses the best locations for these facilities. The comparison of the heuristic method with total enumeration that we later provide in Section 5 will characterize conditions under which the method of analyzing location decisions in two steps leads to optimal or near-optimal performance.

2.3.2. Heuristic method for identifying location matrix

Because we consider a simultaneous game in which a player may not possess all relevant information associated with the other players, it is impossible to provide a general characterization of how an individual firm will approach the decision problem (and to, therefore, characterize the solution that will result). In an attempt to emulate a reasonable approach that might be taken by an individual firm under such conditions, we have constructed a ranking-based heuristic approach in which potential locations are ranked in a preference order based on problem data. The heuristic method we provide is thus based on assigning weights to locations. In particular, the weight of location i is determined by the expression

$$\omega_i = \frac{1}{n}\sum_{j \in J}(c_{ij} + \alpha_{ij}^2) + f_i. \tag{12}$$

The weight of location i, ω_i, is the sum of average transportation cost coefficients and the square of traffic congestion cost factors from location i to all of the markets, plus the facility location cost at location i. As a result, the weight factor for a location will be lower for locations with low facility location costs and low average transportation and congestion costs. Therefore, a location with lower weight is more favorable.

The heuristic method has two phases. In the first phase, a firm decides on the number of facilities to locate as follows. Suppose that a firm is planning to locate ℓ facilities, $\ell \le m$. We assume that the locations of these ℓ facilities will be the ℓ locations with the lowest weights, and we compute the profit associated with such a location decision. We repeat this process for each $0 \le \ell \le m$, and assume that the firm chooses the number of facilities that provides the maximum profit. In the second phase of the heuristic method, a firm determines the best locations for the number of facilities determined in the first phase. Below, we provide a step-by-step description of the algorithm.

Algorithm 2. 2-Phase Heuristic method:

Phase I: Determining the number of facilities to be located

Step 0. Calculate the location weights using Equation (12). Sort locations in non-increasing order of weight. Set $\ell = 0$, $\ell^* = 0$ $\pi^* = 0$ and go to Step 1.

Step 1. Construct $\mathbf{x}^\ell$ by locating facilities at locations 1 to ℓ and determine the profit of any firm, π^ℓ, using Algorithm 1 with $\mathbf{X}^0 = \mathbf{X}^\ell$, where each column of $\mathbf{X}^\ell$ is $\mathbf{x}^\ell$. Go to Step 2.

Step 2. If $\pi^\ell \ge \pi^*$, $\pi^* = \pi^\ell$ and $\ell^* = \ell$. If $\ell \le m-1$ set $\ell = \ell + 1$ and Go to Step 1. If $\ell = m$, go to Step 3.

Step 3. If $\pi^* \ge 0$, $\pi^* = \pi^\ell$ and $\ell^* = \ell^*$ and, go to Step 4. Else, set $\pi^* = 0$ and $\ell^* = 0$, and stop.

Phase II: Finding the best location decision with ℓ^* facilities

Step 4: Find the best location decision with ℓ^* facilities by enumerating the location decisions containing ℓ^* ones (locations). Return the best solution.

Algorithm 2 assumes that a firm determines facility locations in two phases; first, the number of facilities to be located is decided and then the locations for these facilities are determined. We note that Algorithm 2 provides the best location decision of a firm when the firm believes that all other firms will utilize the same weight ranking based approach in deciding the number of facilities to be located.

In the next section, we present the results of a numerical study to analyze the effects of traffic congestion costs on the quantity and facility location decisions of the firms. Moreover, we provide numerical results on the efficiency of the heuristic method described in Algorithm 2.

3. Findings and Applications

Our numerical studies focus on three kinds of analysis. We first consider the effects of traffic congestion cost factors on the firms' best decisions. Following this, we characterize the efficiency of the heuristic method provided in the previous section. We then compare the firms' best decisions (i) when firms consider traffic congestion costs in decision making and (ii) when firms disregard traffic congestion costs in decision making.

3.1. Analysis 1: Effects of Traffic Congestion

Our first analysis documents the effects of traffic congestion cost on the best decisions of the firms. We generate data for our computational tests in the following way. We consider four problem classes, where each problem class differs in transportation costs, c_{ij}, and facility location costs, f_i. For each of the classes, we use all combinations of $k \in \{3,5\}$, $n \in \{3,5,7\}$ and $m \in \{3,5,7,10\}$, resulting in 24 combinations of the values of k, n, and m. For each of these combinations, we generate 10 problem instances and each problem instance is solved for 16 different intervals of traffic congestion cost factor, α_{ij}, starting from 0 and increasing to 8 in increments of 0.5. This way we can analyze the effects of increasing congestion cost on the facility location and supply quantity decisions of the firms. For every problem, we let $a_j \sim U[50,150]$ and $b_j \sim U[1,2]$, where $U[l,u]$ denotes the uniform distribution on $[l,u]$. Table 1 gives the distribution interval of c_{ij} and f_i values in each problem class.

Table 1. Data Intervals for Problem Classes 1-4

	c_{ij}	f_i
Class 1	(0,50]	[75,125]
Class 2	(0,50]	[100,150]
Class 3	[25,75]	[75,125]
Class 4	[25,75]	[100,150]

In each problem class, we solve 240 problem instances, and each problem instance is solved 16 times, once for each interval of α_{ij} values. For each problem instance, we determine the best location decision (using total enumeration) and the corresponding equilibrium quantity decisions for a single firm. We document the following average statistics over 960 problem instances (240 in each in each Problem Class) for each interval of α_{ij} values in Table 2: a given firm's number of facilities (# of fac.), total quantity supplied to markets (Supply Quant.), total transportation costs (Trans. Cost), total traffic congestion costs (Cong. Cost), total facility location costs (Loc. Cost) and total profit. A graph of each statistic in Table 2 is given in Figure 1. The following conclusions can be drawn by analysis of the statistics shown in Table 2.

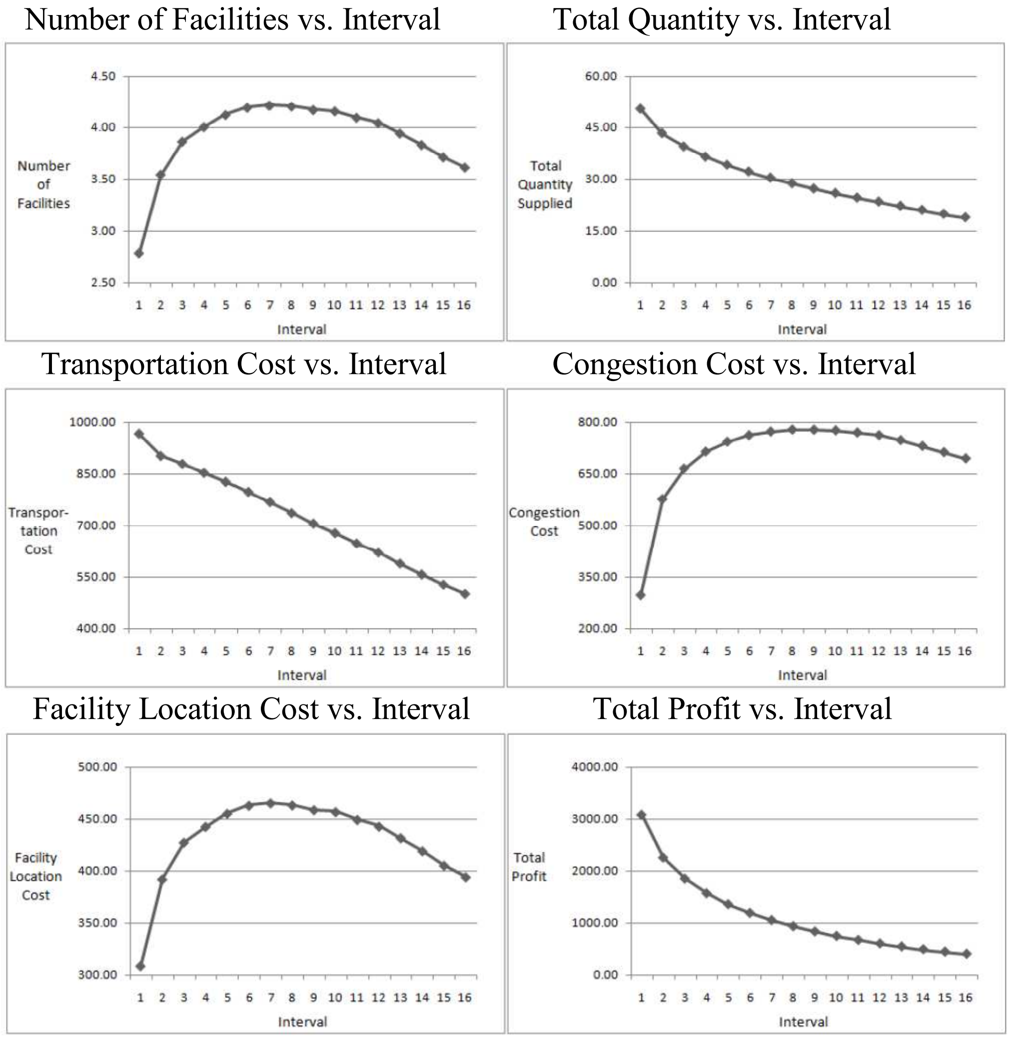
1. The number of facilities located increases with the congestion cost parameters up to Interval 8. After Interval 8, it decreases. That is, firms will locate more facilities when congestion cost parameters increase up to a point. However, after a point, firms will locate fewer facilities with the increase in congestion cost parameters. Note that the facility location cost follows the same pattern. See Figures 1a and 1e.
2. The total quantity supplied by a firm decreases as the congestion cost parameters increase. This result is parallel with Propositions 7 and 8. Total transportation cost also follows the same pattern. See Figures 1b and 1c.
3. The total traffic congestion cost increases with the congestion cost parameters up to Interval 8. After Interval 8, it decreases. See Figure 1d.
4. The total profit decreases as the congestion cost parameters increase. See Figure 1f.

We note that the patterns observed in Table 2 were also observed within each problem class individually. Considering the points noted above, a firm's reaction to an increase in congestion cost can be explained as follows. Up to a point, a firm will locate more facilities and supply less to markets, in order to maximize profit by increasing the market price and decreasing transportation costs to compensate for the increase in congestion costs. However, when the congestion costs becomes significantly high, the firm will send less supply to markets from fewer supply points to avoid congestion costs in order to retain profitability. Note that if congestion cost were to continue increasing, the firm would tend to locate fewer and fewer facilities, and ultimately discontinue supplying markets.

Table 2. Average Statistics over Problem Classes 1-4

	α_{ij}	# of fac.	Supply Quant.	Trans. Cost	Cong. Cost	Loc. Cost	Total Profit
Interval 1	$(0, 0.5]$	2.78	50.26	967.00	297.91	308.61	3089.29
Interval 2	$[0.5, 1]$	3.54	43.22	902.27	576.97	391.98	2261.97
Interval 3	$[1, 1.5]$	3.86	39.37	879.07	665.10	427.38	1855.16
Interval 4	$[1.5, 2]$	4.00	36.42	853.04	714.36	442.61	1574.12
Interval 5	$[2, 2.5]$	4.12	34.09	827.08	743.01	455.17	1361.77
Interval 6	$[2.5, 3]$	4.19	32.12	797.90	761.66	462.93	1194.12
Interval 7	$[3, 3.5]$	4.21	30.34	768.56	772.38	465.25	1054.87
Interval 8	$[3.5, 4]$	4.20	28.70	736.29	777.83	463.28	936.21
Interval 9	$[4, 4.5]$	4.17	27.19	705.62	777.58	458.74	835.39
Interval 10	$[4.5, 5]$	4.16	25.90	678.09	775.77	456.82	748.39
Interval 11	$[5, 5.5]$	4.09	24.59	647.55	769.48	449.28	671.05
Interval 12	$[5.5, 6]$	4.04	23.45	620.06	761.90	442.83	603.85
Interval 13	$[6, 6.5]$	3.94	22.23	587.91	747.70	431.71	544.33
Interval 14	$[6.5, 7]$	3.83	21.06	556.68	730.90	419.15	491.24
Interval 15	$[7, 7.5]$	3.71	19.94	526.62	712.79	405.20	444.18
Interval 16	$[7.5, 8]$	3.61	18.93	499.51	694.84	394.05	402.12

Figure 1. Patterns of Each Column in Table 2



3.2. Analysis 2: Efficiency of the Heuristic Method

We next focus on characterizing the efficiency of the heuristic method provided in the previous section. We generate data for our computational tests in the following way. We consider eight problem classes, where each problem class differs in congestion cost factors, α_{ij}, transportation costs, c_{ij}, and facility location costs, f_i. By considering different problem classes, we aim at providing a more conclusive analysis (rather than solving a specific class of problem for which the heuristic method is quite efficient). For each of the classes, we use all combinations of $k \in \{3,5\}$, $n \in \{3,5,7\}$ and $m \in \{3,5,7,10,15\}$, resulting in 30 combinations of the values of *k*, *n*, and *m*. For each of these combinations, we generate 10 problem instances. For every problem, we let $a_j \sim U[50,150]$ and $b_j \sim U[1,2]$. Table 3 gives the distribution range for α_{ij}, c_{ij} and f_i values in each problem class.

Table 3. Data Ranges for Problem Classes 1-8

	α_{ij}	c_{ij}	f_i
Class 1	(0,4]	(0,50]	[75,125]
Class 2	(0,4]	(0,50]	[100,150]
Class 3	(0,4]	[25,75]	[75,125]
Class 4	(0,4]	[25,75]	[100,150]
Class 5	[4,8]	(0,50]	[75,125]
Class 6	[4,8]	(0,50]	[100,150]
Class 7	[4,8]	[25,75]	[75,125]
Class 8	[4,8]	[25,75]	[100,150]

In each problem class, we solve 300 problem instances and each problem instance is solved using total enumeration and the heuristic method stated in Algorithm 2. Table 4 compares total enumeration with Algorithm 2. As can be seen from Table 4, the 2-Phase heuristic method is of course faster than total enumeration, and the average solution obtained by the 2-Phase heuristic method has an average optimality gap of 2.95%. Moreover, the 2-Phase heuristic solution results in more facility locations, whereas the total quantities supplied to markets are very close to those when using total enumeration for each problem class. In Table 5, we compare the total enumeration and 2-Phase heuristic method solutions for problem instances with the same number of potential facility locations, i.e., for problems with $m=3$, $m=5$, $m=7$, $m=10$ and $m=15$. We note that as the number of potential locations increases, the computation time advantage of the 2-Phase heuristic method increases as well. On the other hand, the optimality gap does not show a clear increasing or decreasing trend in the number of locations increases. Therefore, we believe that 2-Phase heuristic method is robust. For instance, while the optimality gap for problem instances with $m=7$ potential facilities is smaller than the optimality gap for problem instances with $m=10$, the optimality gap for problem instances with $m=15$ is also smaller than the optimality gap for problem instances with $m=10$. Thus, we can say that the solution quality of Algorithm 2 is not clearly decreasing as the problem size increases, although Algorithm 2 becomes substantially more efficient computationally.

Table 4. Comparison of Total Enumeration and 2-Phase Heuristic Method

	Total Enumeration				2-Phase Heuristic				
	# of fac.	Supply Quant.	Total Profit	CPU time	# of fac.	Supply Quant.	Total Profit	CPU time	Optimality gap
Class 1	4.28	49.73	2907.08	51.65	5.06	50.12	2871.09	5.01	1.54%
Class 2	3.86	49.24	2757.59	47.97	4.62	49.76	2717.95	5.05	1.75%
Class 3	3.25	34.23	1453.33	49.55	3.97	34.70	1416.88	4.62	3.48%
Class 4	2.84	32.86	1325.44	49.86	3.47	33.38	1280.68	3.65	4.41%
Class 5	5.67	32.33	1092.79	54.37	5.84	32.45	1086.52	4.96	1.42%
Class 6	5.19	31.17	955.78	51.23	5.36	31.29	947.79	5.68	1.88%
Class 7	3.68	18.82	373.40	49.69	3.71	18.74	367.20	4.36	3.71%
Class 8	2.91	16.69	292.51	53.80	2.90	16.46	284.44	3.28	5.45%
average	3.96	33.13	1394.74	51.01	4.37	33.36	1371.57	4.58	2.95%

Table 5. Comparison of Total Enumeration and 2-Phase Heuristic Method for Each m

	Total Enumeration				2-Phase Heuristic				
m	# of fac.	Supply Quant.	Total Profit	CPU time	# of fac.	Supply Quant.	Total Profit	CPU time	Optimality gap
3	2.39	26.37	1018.68	0.02	2.46	26.48	1015.27	0.01	1.65%
5	3.52	31.45	1269.39	0.09	3.75	31.67	1257.00	0.03	2.16%
7	4.06	33.42	1393.05	0.46	4.45	33.70	1372.79	0.09	2.87%
10	4.64	35.95	1561.23	5.04	5.20	36.19	1525.46	0.63	4.20%
15	5.19	38.48	1731.36	249.46	5.97	38.78	1687.32	22.12	3.88%
average	3.96	33.13	1394.74	51.01	4.37	33.36	1371.57	4.58	2.95%

From the analysis of Tables 4 and 5, we conclude that when a firm determines its facility locations using a two-phase approach (such that in the first phase, the number of facilities is determined by sorting potential facility locations with respect to weights; Equation (12) in our case), the resulting solution approach is computationally efficient, and the relative performance as measured by the optimality gap is relatively strong. Furthermore, the number of potential locations does not heavily influence the optimality gap. This suggests that the strategy of deciding locations in two phases makes sense. This also suggests a future research direction beyond the scope of this paper, in which the game of the firms corresponds to a three-stage game. In the first stage, the number of facilities to be located is determined; then, in the second stage facility locations are chosen and, finally, in the third stage, the supply quantities are determined.

3.3. Analysis 3: Accounting for Congestion in Decision Making

This section compares the decisions of the firms (i) when all of the firms explicitly consider traffic congestion costs and (ii) when all firms disregard traffic congestion costs in their location and supply quantity decisions. In particular, we compare two cases: (i) when all of the firms are aware of congestion in the network and account for congestion costs in their decisions and (ii) when all of the firms are not aware of congestion in the network and exclude congestion costs in their decisions, but still face congestion costs after they implement their decisions. Firms in Case (i) will determine their quantity decisions using Algorithm 1, and determine facility location decisions using total enumeration. Firms in Case (ii) do not consider traffic congestion costs in their decisions and, hence, we cannot use Algorithm 1 directly to determine equilibrium quantity decisions. On the other hand, using the next proposition, we show that when firms are not aware of congestion, they will supply a market from the closest facility to the market, and each firm will supply the same quantity.

Proposition 11. Suppose that $\alpha_{ij} = 0 \;\; \forall i \in I, j \in J$. Given $\mathbf{X}^0$ such that $\mathbf{X}^0$ consists of identical columns, $q^*_{ijr} = b_j \delta_{ij} / (k+1)$ for $i = i^*$ and $q^*_{ijr} = 0$ for $i \neq i^* \;\; \forall r \in R$, where $i^* = \text{argmax}_{i \in I^0} \{\delta_{ij}\}$.

Proposition 11 provides a solution method to find the equilibrium quantities for given location decisions $\mathbf{X}^0$ such that $\mathbf{X}^0$ consists of identical columns for Case (ii). Regarding the discussion in the previous section, total enumeration can still be used for Case (ii) to determine the location decisions.

We generate data for our computational tests in the following way. We consider two problem classes, where each problem class has eight parameter distribution settings, as shown in Table 6. That is, for each problem class, and for each of the three parameters of interest (c_{ij}, f_i, and α_{ij}), we have two uniform distributions from which parameter values are drawn (resulting in eight combinations of distribution settings). For each of these eight combinations within a class, we use all combinations of $k \in \{3,5\}$, $n \in \{3,5,7\}$ and $m \in \{3,5,7,10\}$, resulting in 24 combinations of the values of k, n, and m. For each of these combinations, we generate 25 problem instances. For every problem, we let $a_j \sim U[50,150]$ and $b_j \sim U[1,2]$. Table 6 gives the distribution range for the α_{ij}, c_{ij} and f_i values in each data category, where B_i denotes data category i.

We solve each problem instance for firms in Cases (i) and (ii). If the total profit of any single firm in Case (ii) is negative, we exclude this instance from our analysis since we assume that firms will stop their actions when they have negative profits. In particular, this results in more than 15 problem instances in each of 24 sets for each of the eight categories for Problem Classes 1 and 2.

Table 6. Data Categories for Problem Classes 1 and 2

	Class 1			Class 2		
	α_{ij}	c_{ij}	f_i	α_{ij}	c_{ij}	f_i
B1	(0,0.25]	[25,75]	[50,100]	[0.5,1]	[25,75]	[50,100]
B2	(0,0.25]	[25,75]	[75,125]	[0.5,1]	[25,75]	[75,125]
B3	(0,0.25]	[50,100]	[50,100]	[0.5,1]	[50,100]	[50,100]
B4	(0,0.25]	[50,100]	[75,125]	[0.5,1]	[50,100]	[75,125]
B5	[0.25,0.5]	[25,75]	[50,100]	[0.75,1.25]	[25,75]	[50,100]
B6	[0.25,0.5]	[25,75]	[75,125]	[0.75,1.25]	[25,75]	[75,125]
B7	[0.25,0.5]	[50,100]	[50,100]	[0.75,1.25]	[50,100]	[50,100]
B8	[0.25,0.5]	[50,100]	[75,125]	[0.75,1.25]	[50,100]	[75,125]

Table 7. Statistics of Cases (i) and (ii) for Problem Classes 1 and 2

Class 1

	# of fac.	Supply Quant.	Trans. Cost	Cong. Cost	Loc. Cost	Total Profit
Case (i)	2.47	34.38	1467.66	201.54	210.56	1628.94
Case (ii)	2.24	39.08	1642.61	415.57	191.85	1661.44

Class 2

	# of fac.	Supply Quant.	Trans. Cost	Cong. Cost	Loc. Cost	Total Profit
Case (i)	3.05	28.63	1275.48	365.74	257.76	1107.56
Case (ii)	2.26	39.30	1656.18	1427.80	193.18	662.54

Intuitively, we would expect that firms in Case (i) have higher profits since they consider traffic congestion in their decisions, whereas, firms in Case (ii) disregard the traffic congestion in their decisions but pay for congestion after their decisions are implemented. However, our numerical results imply that the opposite is also possible.

Table 7 compares Cases (i) and (ii) for each Problem Class. For Problem Class 1, we see that the average total profit for a single firm in Case (ii) is higher than the average total profit of a single firm in Case (i), whereas, we have the opposite for Problem Class 2. This result for Problem Class 1 implies that firms may actually increase their profits if they do not consider traffic congestion in their decisions. This phenomenon can be explained as follows. For our problem, firms are competing on two dimensions: the price in a market and the congestion on links connecting supply locations and markets. For Case (ii), since the congestion cost is disregarded in the decision making process, firms compete only on market price. So when the impact of congestion cost is relatively small and when firms compete only on market price, they may actually end up with higher profit. Next, we provide a simple example to illustrate the phenomenon in which Case (ii) results in higher profit.

Example 1. Consider two firms competing in a single market, market 1. There are two potential locations, 1 and 2 , at which the firms may locate facilities. Suppose that facility location costs are 0 at both locations, i.e., $f_1 = f_2 = 0$. Let $c_{11} = 80$, $c_{21} = 90$, and $\alpha_{11} = 0.25$, $\alpha_{21} = 0.5$. The market parameters are $a_1 = 100$ and $b_1 = 1$. Table 8 gives the total quantity supplied to the market and the corresponding total profit for a single firm for Cases (i) and (ii), when firms have facilities at both locations.

Table 8. Solution for Example 1

	Quantity Supplied	Total Profit
Case (i)	5.33	64.00
Case (ii)	6.67	66.67

In both of the cases, only the facilities at location 1 supply market 1. As can be seen in Table 8, a firm is more profitable under Case (ii). Moreover, we note that when both firms locate facilities at both locations, this corresponds to a PNE location decision, since facility location costs are 0.

As is clear from Example 1, disregarding congestion costs in the decision making process may result, in some cases, in higher profits even under a PNE solution for both the quantity and facility location decisions.

4. Conclusions, Recommendations, and Suggested Research

This paper studied facility location and supply quantity decisions for multiple firms in a competitive environment on a congested network. Our contributions are primarily twofold: (i) we determine facility location and supply quantity decisions for firms under a homogeneous cost structure, where the firms may locate more than one facility and are subject to nonlinear cost terms, and (ii) we analyze the effects of traffic congestion on facility location and quantity decisions in a competitive environment.

We provided a solution method to determine the PNE quantity decisions of the firms in Section 3. Our solution method is based on determining the equilibrium total quantities sent from any location to any market given that the facility location decisions are the same for each firm. Section 4 discussed the facility location decisions of the firms. We proposed a 2-Phase heuristic method, together with a total enumeration scheme. As implied by our numerical studies, the heuristic method is an efficient method that ranks locations based on certain problem parameters in the first phase. The analysis of the heuristic method suggests a future research direction: firms' decisions can be modeled as a three-stage game. In this game, firms first determine the number of facilities (first stage), then the locations of these facilities (second stage), and, finally, the supply quantities (third stage). The competitive location problem we studied was a non-cooperative simultaneous entry game. Future additional research might allow for cooperation between competing firms. Also, studying this problem as a sequential entry game, i.e., when there exists a sequential order of decision making among firms, is an interesting future research direction.

We modeled traffic congestion costs endogenously and provided analytical results on how traffic congestion cost affects equilibrium supply quantity decisions. Increased traffic congestion hinders efficient use of the distribution network as firms may choose to supply a market from multiple distant decentralized facilities. Moreover, our numerical studies characterize the effects of congestion on facility location decisions as well. In our numerical studies, we illustrate how a continuous increase in traffic congestion can drive firms out of markets and out of business. Furthermore, we highlighted a counter-intuitive result in our numerical studies. We showed that firms may increase profits when they ignore congestion-based competition in some cases. We note that this point is an important future research area. When competitors compete over more than one resource, e.g., market price and congestion in our case, analyzing which of these should be considered in competition to produce higher profit is an interesting problem.

Our results document the negative effects of traffic congestion on firms. As a result, it is possible that firms may be willing to cooperate with government agencies to reduce the traffic congestion. It is even possible that firms may cooperate among each other to mitigate traffic congestion, and, thereby reduce the negative effects of traffic congestion, as noted by Hensher and Puckett (2005). Studying such traffic congestion mitigation policies, with mathematical bases, remains as a future research area.

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Appendix A

A.1. Proof of Proposition 1

First note that for the locations where there is no facility, $q^*_{ijr}=0 \quad \forall r\in R$. Hence, we only focus on the locations where there are k facilities. We prove the statement using the KKT conditions defined for the optimal q_{ijr} values. Together with $q_{ijr}\ge 0$, the KKT conditions read

$$\begin{aligned} \delta_{ij}-b_j q_{\bullet j \bullet}-b_j q_{\bullet jr}-\alpha_{ij}q_{ij\bullet}-\alpha_{ij}q_{ijr}+u_{ijr} &= 0,\\ u_{ijr}q_{ijr} &= 0,\\ u_{ijr} &\ge 0, \end{aligned}$$

where $\delta_{ij}=a_j-c_{ij}$. Now consider any two firms r_1 and r_2. We show that $q^*_{ijr_1}=q^*_{ijr_2} \quad \forall i\in I^0$, where I^0 denotes the locations with k facilities for the given $\mathbf{X}^0$, by considering the following two cases.

Case I: Suppose that $q^*_{ijr_1}>0$ and $q^*_{ijr_2}>0 \quad \forall i\in I^0$. Then $q^*_{ijr_1}$ and $q^*_{ijr_2}$ must satisfy the first order conditions, i.e., $u_{ijr_1}=u_{ijr_2}=0 \quad \forall i\in I^0$. Without loss of generality, we assume that $I^0=\{1,2,3,...,s\}$ such that $s\le m$. The first order conditions, then, read

$$\begin{aligned} &(\mathbf{a}1) \quad \delta_{ij}-b_j q^*_{\bullet j\bullet}-b_j q^*_{\bullet jr_1}-\alpha_{ij}q^*_{ij\bullet}-\alpha_{ij}q^*_{ijr_1} = 0 \quad \forall i\in I^0,\\ &(\mathbf{a}2) \quad \delta_{ij}-b_j q^*_{\bullet j\bullet}-b_j q^*_{\bullet jr_2}-\alpha_{ij}q^*_{ij\bullet}-\alpha_{ij}q^*_{ijr_2} = 0 \quad \forall i\in I^0. \end{aligned}$$

It follows from $(\mathbf{a}1)$ and $(\mathbf{a}2)$ that

$$\begin{aligned} &(\mathbf{b}1) \quad b_j q^*_{\bullet jr_1}+\alpha_{ij}q^*_{ijr_1} = b_j q^*_{\bullet jr_2}+\alpha_{ij}q^*_{ijr_2} \quad \forall i\in I^0,\\ &(\mathbf{b}2) \quad \alpha_{zj}(q^*_{zjr_1}-q^*_{zjr_2}) = -b_j(q^*_{\bullet jr_2}-q^*_{\bullet jr_2}) = -b_j\sum\nolimits_{i\in I^0}(q^*_{ijr_1}-q^*_{ijr_2}) \quad \forall z\in I^0. \end{aligned}$$

Subtracting $(\mathbf{b}1)$ for location 2 from $(\mathbf{b}1)$ for location 1, we get

$$(\mathbf{c}) \quad \alpha_{1j}q^*_{1jr_1}-\alpha_{2j}q^*_{2jr_1} = \alpha_{1j}q^*_{1jr_2}-\alpha_{2j}q^*_{2jr_2}.$$

It follows from $(\mathbf{c})$ that $\alpha_{1j}(q^*_{1jr_1} - q^*_{1jr_2}) = \alpha_{2j}(q^*_{2jr_1} - q^*_{2jr_2})$. Following similar argument, subtracting $(\mathbf{b}1)$ for location $i+1$ from expression $(\mathbf{b}1)$ for location i, $i \le s-1$, we get

$$(\mathbf{d}) \quad \alpha_{1j}(q^*_{1jr_1} - q^*_{1jr_2}) = \alpha_{2j}(q^*_{2jr_1} - q^*_{2jr_2}) = \alpha_{3j}(q^*_{3jr_1} - q^*_{3jr_2}) = \ldots = \alpha_{sj}(q^*_{sjr_1} - q^*_{sjr_2}).$$

Considering $(\mathbf{d})$, $(\mathbf{b}2)$ can be written as

$$(\mathbf{e}) \quad \alpha^2_{zj}(q^*_{zjr_1} - q^*_{zjr_2}) = -b_j \sum\nolimits_{i \in I^0} \alpha_{ij}(q^*_{zjr_1} - q^*_{zjr_2}).$$

Since $\alpha_{ij} > 0$, expression $(\mathbf{e})$ is only satisfied when $q^*_{zjr_1} = q^*_{zjr_2}$ for any location $z \in I^0$. Thus, it follows that $q^*_{ijr_1} = q^*_{jr_2} \quad \forall i \in I^0$.

Case II: Suppose that $q^*_{ijr_1} = 0$ for locations $i \in I^0_{r_1} \subset I^0$ and $q^*_{ijr_2} = 0$ for locations $i \in I^0_{r_2} \subset I^0$. We consider the following three subcases of Case II.

Subcase I: $I^0_{r_1} = I^0_{r_2} = I^0_r \subset I^0$, i.e., $q^*_{ijr_1} = q^*_{ijr_2} = 0$ for locations $i \in I^0_r \subset I^0$. For locations $i \notin I^0_r$, that is, for locations $i \in I^0 \setminus I^0_r$, we have $q^*_{ijr_1} > 0$ and $q^*_{ijr_2} > 0$. Thus, Subcase I reduces to Case I with $I^0 \setminus I^0_r$ instead of I^0, which means we have $q^*_{ijr_1} = q^*_{ijr_2}$ for locations $i \in I^0 \setminus I^0_r$. Thus, for Subcase I, we have $q^*_{ijr_1} = q^*_{ijr_2} \quad \forall i \in I^0$.

Subcase II: $I^0_{r_1} \ne I^0_{r_2}$ and either $I^0_{r_1} = \varnothing$ or $I^0_{r_2} = \varnothing$. Without loss of generality, suppose that $I^0_{r_2} = \varnothing$.

We first consider Situation (i): $I^0_{r_1} = I^0$. Situation (i) implies that $q^*_{ijr_1} = 0 \quad \forall i \in I^0$, thus, $q^*_{\bullet jr_1} = 0$. Now consider any location $z \in I^0$ and suppose that $q^*_{zjr_2} > 0$. It follows from the KKT conditions that $u_{zjr_2} = 0$. Moreover, from the KKT conditions for $q^*_{zjr_1}$ and $q^*_{zjr_2}$, we have

$$(\mathbf{f}1) \qquad \delta_{zj} - b_j q^*_{\bullet j \bullet} - a_{zj} q^*_{zj\bullet} + u_{zjr_1} \qquad = 0$$

$$(\mathbf{f}2) \quad \delta_{zj} - b_j q^*_{\bullet j \bullet} - b_j q^*_{\bullet jr_2} - a_{zj} q^*_{zj\bullet} - a_{zj} q^*_{ijr_2} \quad = 0$$

It follows from $(\mathbf{f}1)$ and $(\mathbf{f}2)$ that $-u_{zjr_1} = b_j q^*_{\bullet jr_2} + a_{zj} q^*_{zj\bullet} + a_{zj} q^*_{ijr_2}$, which implies

$$(\mathbf{g}) \quad b_j q^*_{\bullet jr_2} + \alpha_{zj} q^*_{zj\bullet} + \alpha_{zj} q^*_{zjr_2} = 0.$$

since $u_{zjr_1} \ge 0$, $\alpha_{zj} > 0$ and $b_j > 0$. Moreover, since $q^*_{ijr_2} \ge 0$, $(\mathbf{g})$ is only satisfied when $q^*_{\bullet jr_2} = q^*_{zj\bullet} = q^*_{zjr_2} = 0$. Therefore, we have a contradiction with $q^*_{zjr_2} > 0$, thus, $q^*_{zjr_1} = q^*_{zjr_2} = 0$ for any location $z \in I^0$ for Situation (i), i.e., $q^*_{ijr_1} = q^*_{jr_2} \quad \forall i \in I^0$.

Now we consider Situation (ii): $I^0_{r_1} \subset I^0$. Situation (ii) implies that there is at least one location, say location t, $t \in I^0 \setminus I^0_{r_1}$ such that $q^*_{tjr_1} > 0$ and $q^*_{tjr_2} > 0$. We show by contradiction that $q^*_{ijr_2} = 0$ $\forall i \in I^0_{r_1}$. Suppose that $q^*_{zjr_2} > 0$ for any location $z \in I^0_{r_1}$. It follows from the KKT conditions that $u_{zjr_2} = 0$. Moreover, from the KKT conditions for $q^*_{zjr_1}$ and $q^*_{zjr_2}$, we have

$$(\mathbf{h}1) \quad \delta_{zj} - b_j q^*_{\bullet j\bullet} - b_j q^*_{\bullet jr_1} - \alpha_{zj} q^*_{zj\bullet} + u_{zjr_1} = 0,$$

$$(\mathbf{h}2) \quad \delta_{zj} - b_j q^*_{\bullet j\bullet} - b_j q^*_{\bullet jr_2} - \alpha_{zj} q^*_{zj\bullet} - \alpha_{zj} q^*_{ijr_2} = 0.$$

It follows from $(\mathbf{h}1)$ and $(\mathbf{h}2)$ that $b_j q^*_{\bullet jr_1} - u_{zjr_1} = b_j q^*_{\bullet jr_2} + \alpha_{zj} q^*_{ijr_2}$. Since $u_{zjr_1} \geq 0$, $\alpha_{zj} > 0$ and $q^*_{zjr_2} > 0$, it implies that

$$(\mathbf{k}) \quad b_j q^*_{\bullet jr_1} > b_j q^*_{\bullet jr_2}.$$

Now consider any location $t \in I^0 \setminus I^0_{r_1}$ such that $q^*_{tjr_1} > 0$ and $q^*_{tjr_2} > 0$. Then it follows from KKT conditions that $u_{tjr_1} = u_{tjr_2} = 0$ and $b_j q^*_{\bullet jr_1} + \alpha_{tj} q^*_{tjr_1} = b_j q^*_{\bullet jr_2} + \alpha_{tj} q^*_{tjr_2}$. Since $b_j q^*_{\bullet jr_1} > b_j q^*_{\bullet jr_2}$, we have $q^*_{tjr_1} < q^*_{tjr_2}$ for any location $t \in I^0 \setminus I^0_{r_1}$. Thus, it follows that

$$(\mathbf{l}) \quad \sum_{t \in I^0 \setminus I^0_{r_1}} q^*_{tjr_1} < \sum_{t \in I^0 \setminus I^0_{r_1}} q^*_{tjr_2}.$$

Moreover, since $q^*_{zjr_1} = 0$ and $q^*_{zjr_2} > 0$ for any location $z \in I^0_{r_1}$, we have

$$(\mathbf{m}) \quad \sum_{t \in I^0_{r_1}} q^*_{tjr_1} < \sum_{t \in I^0_{r_1}} q^*_{tjr_2}.$$

Inequalities $(\mathbf{l})$ and $(\mathbf{m})$ together imply that $b_j q^*_{\bullet jr_1} < b_j q^*_{\bullet jr_2}$, which is contradiction with inequality $(\mathbf{k})$. Thus, we should have $q^*_{zjr_1} = q^*_{zjr_2} = 0$ for any location $z \in I^0_{r_1}$. For other locations, i.e., any location $i \in I^0 \setminus I^0_{r_1}$, we have $q^*_{ijr_1} > 0$ and $q^*_{ijr_2} > 0$. Since, we know $q^*_{zjr_1} = q^*_{zjr_2} = 0$ for any location $z \in I^0_{r_1}$, we can ignore such locations. Then, Situation (ii) reduces to Case I with $I^0 \setminus I^0_r$ instead of I^0, which means we have $q^*_{ijr_1} = q^*_{ijr_2}$ for locations $i \in I^0 \setminus I^0_r$. Thus, $q^*_{ijr_1} = q^*_{ijr_2} \ \forall i \in I^0$ for Situation (ii).

Situations (i) and (ii) together imply that $q^*_{ijr_1} = q^*_{ijr_2} \ \forall i \in I^0$ for Subcase II.

Subcase III: $I^0_{r_1} \neq \varnothing$, $I^0_{r_2} \neq \varnothing$ and $I^0_{r_1} \neq I^0_{r_2}$. First note that for any location $i \in I^0_{r_1} \cap I^0_{r_2}$, we have $q^*_{ijr_1} = q^*_{ijr_2} = 0$, thus, we can disregard such locations and only study the situation when $I^0_{r_1} \neq \varnothing$, $I^0_{r_2} \neq \varnothing$ and $I^0_{r_1} \cap I^0_{r_2} = \varnothing$. This situation implies the following conditions:

$$(\mathbf{n}1) \quad q^*_{ijr_1} = 0 \text{ and } q^*_{ijr_2} > 0 \forall i \in I^0_{r_1},$$

$$(\mathbf{n}2) \quad q^*_{ijr_1} > 0 \text{ and } q^*_{ijr_2} = 0 \forall i \in I^0_{r_2}.$$

We now show by contradiction that conditions $(\mathbf{n}1)$ and $(\mathbf{n}1)$ cannot be satisfied at the same time. Consider any location $z \in I^0_{r_1}$, that is, $q^*_{zjr_1} = 0$ and $q^*_{zjr_2} > 0$. It follows from the KKT conditions that $u_{zjr_2} = 0$ and (i) $b_j q^*_{\bullet jr_1} - u_{zjr_1} = b_j q^*_{\bullet jr_2} + \alpha_{zj} q^*_{zjr_2}$, which means that $b_j q^*_{\bullet jr_1} > b_j q^*_{\bullet jr_2}$ as $u_{zjr_2} \geq 0$, $\alpha_{zj} > 0$ and $q^*_{zjr_2} > 0$. Now consider any location $t \in I^0_{r_2}$, that is, $q^*_{tjr_1} > 0$ and $q^*_{tjr_2} = 0$. It follows from the KKT conditions that $u_{tjr_2} = 0$ and (ii) $b_j q^*_{\bullet jr_1} + \alpha_{tj} q^*_{tjr_1} = b_j q^*_{\bullet jr_2} - u_{zjr_2}$, which means that $b_j q^*_{\bullet jr_1} < b_j q^*_{\bullet jr_2}$ as $u_{tjr_1} \geq 0$, $\alpha_{tj} > 0$ and $q^*_{tjr_1} > 0$. (i) and (ii) establishes a contradiction. That is, we cannot satisfy the conditions $(\mathbf{n}1)$ and $(\mathbf{n}1)$ at the same time. Without loss of generality, suppose that we do not have condition $(\mathbf{n}1)$, i.e., $q^*_{ijr_1} = 0$ and $q^*_{ijr_2} = 0 \ \forall i \in I^0_{r_1}$. Hence, we can disregard any location $i \in I_{r_1}^0$. Then, Subcase III reduces to Subcase II. Thus, $q_{ijr_1}^* = q_{ijr_2}^*$ $\forall i \in I^0$ for Subcase III. Subcases I-III captures all of the possibilities of Case II. Therefore, we have $q_{ijr_1}^* = q_{ijr_2}^*$ $\forall i \in I^0$. Cases I and II imply that $q_{ijr_1}^* = q_{ijr_2}^*$ $\forall i \in I^0$ for any two firms r_1 and r_2. Thus, we have shown that q_{ijr}^* is the same for all of the firms. Now, letting Q_{ij}^* denote the total equilibrium quantity flow on the link (i, j), since there exist k firms at any location $i \in I^0$, it follows that $q_{ijr}^* = Q_{ij}^* / k$.

A.2. Proof of Proposition 2

Considering Proposition 1 and Equation(7), the KKT conditions for any firm at location i, $i \in I^0$, can be written as follows:

$$\begin{aligned} \delta_{ij} - \gamma b_j \textstyle\sum_{i \in I^0} Q_{ij}^* - \alpha_{ij} Q_{ij}^* + u_i &= 0, \\ u_i Q_{ij}^* &= 0, \\ u_i &\geq 0. \end{aligned}$$

We first prove Statement (a). Suppose $Q_{ij}^* > 0$, then it implies that $u_i = 0$, hence, we have $\delta_{ij} = \gamma b_j \sum_{i \in I^0} Q_{ij}^* + \alpha_{ij} Q_{ij}^*$. Since $\alpha_{ij} Q_{ij}^* > 0$ as $\alpha_{ij} > 0$ and $Q_{ij}^* > 0$, it follows that $\delta_{ij} > \gamma b_j \sum_{i \in I^0} Q_{ij}^*$. Now suppose that $\delta_{ij} > \gamma b_j \sum_{i \in I^0} Q_{ij}^*$ and $Q_{ij}^* = 0$, then it implies that $u_i = \gamma b_j \sum_{i \in I^0} Q_{ij}^* - \delta_{ij} < 0$ which is a contradiction since $u_i \geq 0$. Hence, $Q_{ij}^* > 0$. We now prove Statement (b). Suppose $Q_{ij}^* = 0$ and $\delta_{ij} > \gamma b_j \sum_{i \in I^0} Q_{ij}^*$, then it implies that $u_i = \gamma b_j \sum_{i \in I^0} Q_{ij}^* - \delta_{ij} < 0$ which is a contradiction since $u_i \geq 0$. Hence, $\delta_{ij} \leq \gamma b_j \sum_{i \in I^0} Q_{ij}^*$. Now suppose that $\delta_{ij} \leq \gamma b_j \sum_{i \in I^0} Q_{ij}^*$ and $Q_{ij}^* > 0$, then it implies that $u_i = 0$, hence, we have $\delta_{ij} - \gamma b_j \sum_{i \in I^0} Q_{ij}^* = \alpha_{ij} Q_{ij}^* > 0$, which is a contradiction since $\alpha_{ij} > 0$.

A.3. Proof of Proposition 3

Suppose that $\delta_{i_1 j} \geq \delta_{i_2 j}$ for locations $i_1, i_2 \in I^0$. Now suppose $Q_{i_2 j}^* > 0$. Then it follows from Proposition 2 that $\delta_{i_2 j} > \gamma b_j \sum_{i \in I^0} Q_{ij}^*$. This implies that $\delta_{i_1 j} > \gamma b_j \sum_{i \in I^0} Q_{ij}^*$. Thus, it follows from Proposition 2 that $Q_{i_1 j}^* > 0$, which proves Statement (a). Now suppose $Q_{i_1 j}^* = 0$. Then it follows from Proposition 2 that $\delta_{i_1 j} \leq \gamma b_j \sum_{i \in I^0} Q_{ij}^*$. This implies that $\delta_{i_2 j} \leq \gamma b_j \sum_{i \in I^0} Q_{ij}^*$. Thus, it follows from Proposition 2 that $Q_{i_2 j}^* = 0$, which proves Statement (b).

A.4. Proof of Proposition 4

When Algorithm 1 stops at $(\ell - 1)^{th}$ iteration (an iteration refers to an execution of Step 2), it means that $Q_{\ell j}^{(\ell)} < 0$. We now show that if we continue the algorithm one step further, that is, if we assume that first $\ell + 1$ locations are active, we should have $Q_{\ell+1}^{(\ell+1)} < 0$. Hence, this means that

$\ell+1$ locations cannot be active. Moreover, since $Q_{\ell j}^{(\ell)} < 0$, ℓ locations cannot be active. Now suppose that $Q_{\ell j}^{(\ell)} < 0$. Note that $Q_{\ell j}^{(\ell)}$ is determined by the solution of the Equation (8), of which solution should satisfy (i) $\delta_{ij} - \gamma\alpha_{ij}Q_{ij}^{(\ell)} = \gamma b_j (Q_{1j}^{(\ell)} + Q_{2j}^{(\ell)} + \cdots + Q_{\ell j}^{(\ell)}) \forall i \le \ell$ and (ii) $\delta_{1j} - \gamma\alpha_{1j}Q_{1j}^{(\ell)} = \delta_{2j} - \gamma\alpha_{2j}Q_{2j}^{(\ell)} = \cdots = \delta_{\ell j} - \gamma\alpha_{\ell j}Q_{\ell j}^{(\ell)}$. It follows from (i) and (ii) that

$$Q_{\ell j}^{(\ell)} = \frac{\delta_{\ell j} + b_j \sum_{i=1}^{\ell} \frac{(\delta_{\ell j} - \delta_{ij})}{\alpha_{ij}}}{\gamma\left(\alpha_{\ell j} + b_j \sum_{i=1}^{\ell} \frac{\alpha_{\ell j}}{\alpha_{ij}}\right)},$$

which means, if $Q_{\ell j}^{(\ell)} < 0$, then $\delta_{\ell j} + b_j \sum_{i=1}^{\ell} \frac{(\delta_{\ell j} - \delta_{ij})}{\alpha_{ij}} < 0$. Similarly,

$$Q_{(\ell+1)j}^{(\ell+1)} = \frac{\delta_{(\ell+1)j} + b_j \sum_{i=1}^{(\ell+1)} \frac{(\delta_{(\ell+1)j} - \delta_{ij})}{\alpha_{ij}}}{\gamma\left(\alpha_{(\ell+1)j} + b_j \sum_{i=1}^{\ell+1} \frac{\alpha_{(\ell+1)j}}{\alpha_{ij}}\right)}.$$

Now suppose that $Q_{(\ell+1)j}^{(\ell+1)} > 0$, then $\delta_{(\ell+1)j} + b_j \sum_{i=1}^{(\ell+1)} \frac{(\delta_{(\ell+1)j} - \delta_{ij})}{\alpha_{ij}} = \delta_{(\ell+1)j} + b_j \sum_{i=1}^{\ell} \frac{(\delta_{(\ell+1)j} - \delta_{ij})}{\alpha_{ij}} > 0$. On the other hand, $\delta_{(\ell+1)j} + b_j \sum_{i=1}^{\ell} \frac{(\delta_{(\ell+1)j} - \delta_{ij})}{\alpha_{ij}} < \delta_{\ell j} + b_j \sum_{i=1}^{\ell} \frac{(\delta_{\ell j} - \delta_{ij})}{\alpha_{ij}}$ as $\delta_{(\ell+1)} < \delta_{\ell}$, which implies $\delta_{(\ell+1)j} + b_j \sum_{i=1}^{(\ell+1)} \frac{(\delta_{(\ell+1)j} - \delta_{ij})}{\alpha_{ij}} < 0$. This is a contradiction, thus, $Q_{(\ell+1)j}^{(\ell+1)} < 0$.

A.5. Proof of Proposition 5

Suppose that $Q_{sj}^{(w)} \le Q_{sj}^{(w+1)}$ for any $s \le w < \ell$. By Equation (11) this means

$$\frac{\delta_{sj} + b_j \sum_{i=1}^{w} \frac{(\delta_{sj} - \delta_{ij})}{\alpha_{ij}}}{\alpha_{sj} + b_j \sum_{i=1}^{w} \frac{\alpha_{sj}}{\alpha_{ij}}} \le \frac{\delta_{sj} + b_j \sum_{i=1}^{w} \frac{(\delta_{sj} - \delta_{ij})}{\alpha_{ij}} + b_j \frac{(\delta_{sj} - \delta_{(w+1)j})}{\alpha_{(m+1)j}}}{\alpha_{sj} + b_j \sum_{i=1}^{w} \frac{\alpha_{sj}}{\alpha_{ij}} + b_j \frac{\alpha_{sj}}{\alpha_{(w+1)}}}$$

It follows from the above inequality that

$$\delta_{(w+1)j} \le \frac{-\alpha_{sj}A + \delta_{sj}B}{B + \alpha_{sj}} = \delta_{sj} - \alpha_{sj}\frac{\delta_{sj} + A}{\alpha_{sj} + B}$$

where $A = b_j \sum_{i=1}^{w} \frac{(\delta_{sj} - \delta_{ij})}{\alpha_{ij}}$ and $B = b_j \sum_{i=1}^{w} \frac{\alpha_{sj}}{\alpha_{ij}}$. Thus, considering Equation (11), the above inequality reads as

$$\delta_{(w+1)j} \le \delta_{sj} - \gamma\alpha_{sj}Q_{sj}^{(w)}.$$

Equation (9) implies that the above inequality can be written as $\delta_{(w+1)j} \le \gamma b_j (Q_{1j}^{(w)} + Q_{2j}^{(w)} + \cdots + Q_{wj}^{(w)})$. Furthermore, it follows from Equation (9), when $Q_{sj}^{(w)} \le Q_{sj}^{(w+1)}$, we have $\gamma b_j \sum_{i=1}^{w} Q_{ij}^{(w)} \ge \gamma b_j \sum_{i=1}^{w} Q_{ij}^{(w+1)}$. This implies that

$\delta_{(w+1)j} \le \gamma b_j (Q_{1j}^{(w+1)} + Q_{2j}^{(w+1)} + \cdots + Q_{wj}^{(w+1)})$. We can write the last inequality as $\delta_{(w+1)j} \le \gamma b_j (Q_{1j}^{(w+1)} + Q_{2j}^{(w+1)} + \cdots + Q_{wj}^{(w+1)} + Q_{(w+1)j}^{(w+1)}) - \gamma b_j Q_{(w+1)j}^{(w+1)}$. Moreover, from Equation (9), we have that $\gamma b_j (Q_{1j}^{(w+1)} + Q_{2j}^{(w+1)} + \cdots + Q_{wj}^{(w+1)} + Q_{(w+1)j}^{(w+1)}) = \delta_{(w+1)j} - \gamma \alpha_{(w+1)j} Q_{(w+1)j}^{(w+1)}$. Thus, we have $\delta_{(w+1)j} \le \delta_{(w+1)j} - \gamma \alpha_{(w+1)j} Q_{(w+1)j}^{(w+1)} - \gamma b_j Q_{(w+1)j}^{(w+1)}$, which means $(\gamma \alpha_{(w+1)j} + \gamma b_j) Q_{(w+1)j}^{(w+1)} \le 0$, which is a contradiction since at the $(w+1)^{th}$ iteration of the algorithm we check that $Q_{(w+1)j}^{(w+1)} > 0$ and then define $Q_{sj}^{(w+1)}$ values. This contradiction proves Statement (a). Statement (b) is a direct result of Statement (a) and Equation (9).

A.6. Proof of Proposition 6

We first show that $Q_{zj}^{1(\ell)} < Q_{zj}^{2(\ell)}$ for location $z \le \ell$, $z \ne s$, where $Q_{zj}^{1(\ell)}$ and $Q_{zj}^{2(\ell)}$ represents the quantities at the ℓ^{th} iteration of Algorithm 1 under α_{sj}^1 and α_{sj}^2 values, respectively. Suppose that $Q_{zj}^{1(\ell)} \ge Q_{zj}^{2(\ell)}$. By Equation (11) this means

$$\frac{\delta_{zj} + b_j \sum_{i=1, i\ne s}^{\ell} \frac{(\delta_{zj} - \delta_{ij})}{\alpha_{ij}} + b_j \frac{(\delta_{zj} - \delta_{sj})}{\alpha_{sj}^1}}{\alpha_{zj} + b_j \sum_{i=1, i\ne s}^{\ell} \frac{\alpha_{zj}}{\alpha_{ij}} + b_j \frac{\alpha_{zj}}{\alpha_{sj}^1}} \ge \frac{\delta_{zj} + b_j \sum_{i=1, i\ne s}^{\ell} \frac{(\delta_{zj} - \delta_{ij})}{\alpha_{ij}} + b_j \frac{(\delta_{zj} - \delta_{sj})}{\alpha_{sj}^2}}{\alpha_{zj} + b_j \sum_{i=1, i\ne s}^{\ell} \frac{\alpha_{zj}}{\alpha_{ij}} + b_j \frac{\alpha_{zj}}{\alpha_{sj}^2}}.$$

After simplifications, the above inequality implies that $\delta_{sj} \le 0$, which is a contradiction since location s is assumed to be active initially. This contradiction establishes that $Q_{zj}^{1(\ell)} < Q_{zj}^{2(\ell)}$. Hence, as $Q_{zj}^{1(\ell)} > 0$, we have $Q_{zj}^{2(\ell)} > 0$, i.e., location z is still active. Moreover, considering Equation (9), $Q_{zj}^{1(\ell)} < Q_{zj}^{2(\ell)}$ implies that $\sum_{i=1}^{\ell} Q_{ij}^{1(\ell)} > \sum_{i=1}^{\ell} Q_{ij}^{2(\ell)}$. Since, $\delta_{sj} > \gamma b_j \sum_{i=1}^{\ell} Q_{ij}^{1(\ell)}$ we have $\delta_{sj} > \gamma b_j \sum_{i=1}^{\ell} Q_{ij}^{2(\ell)}$, i.e., we have $Q_{sj}^{2(\ell)} > 0$ and location s is still active.

A.7. Proof of Proposition 7

Since the number of active locations and the set of active locations remain the same, $Q_{ij}^{1*} = Q_{ij}^{1(\ell)}$ and $Q_{ij}^{2*} = Q_{ij}^{2(\ell)}$. Now it directly follows from Equation (11) that $Q_{ij}^{1*} > Q_{ij}^{2*}$ for $i = s$ and, it follows from the proof of Proposition 6 that $Q_{ij}^{1*} < Q_{ij}^{2*}$ $i \ne s$. This completes the proof of Statement (a). Statement (b) is a direct result of Equation (9) and Statement (a).

A.8. Proof of Proposition 8

Note that, $Q_{ij}^{1*} = Q_{ij}^{1(\ell)}$ and $Q_{ij}^{2*} = Q_{ij}^{2(\ell+\wp)}$. We know from Proposition 7, that $Q_{sj}^{1(\ell)} > Q_{sj}^{2(\ell)}$. Moreover, we know from Proposition 5 that $Q_{sj}^{2(\ell)} > Q_{sj}^{2(\ell+1)}$. Thus it follows that $Q_{sj}^{1(\ell)} > Q_{sj}^{2(\ell+\wp)}$, which proves Statement (a). Statement (b) directly follows from Equation (9). In particular, suppose $Q_{tj}^{1*} < Q_{tj}^{2*}$ for location t, $t \le k$, $t \ne s$. Then it follows from Equation (9) that

$\sum_{i=1}^{\ell} Q_{ij}^{1*} > \sum_{i=1}^{\ell+\wp} Q_{ij}^{2*}$. Then it again follows from Equation (9) that $Q_{ij}^{1*} < Q_{ij}^{2*}$ for any location i, $i \le k$, $i \ne s$. This completes the proof of Statement (b).

A.9. Proof of Proposition 9

Suppose that $\mathbf{X}^*$ is the unique PNE location decision such that $\mathbf{x}_{r_1} \ne \mathbf{x}_{r_2}$ for any two firms r_1 and r_2. Then there exists at least one location, say location i, such that firm r_1 does not have a facility while firm r_2 has a facility at location i, that is, $x_{ir_1}^* = 1$ and $x_{ir_2}^* = 0$. Now, if we make $x_{ir_1}^* = 0$ and $x_{ir_2}^* = 1$ in $\mathbf{X}^*$ and construct $\mathbf{X}^{**}$, then $\mathbf{X}^{**}$ is also a PNE location decision since firms are homogeneous with respect to transportation, traffic congestion and facility location costs. This contradicts that $\mathbf{X}^*$ is the unique PNE location decision.

A.10. Proof of Proposition 10

We first note that, the location decisions of the firms corresponds to a symmetric game. It is well known that for symmetric games, there exist a symmetric MSNE in cases of multiple equilibria or there does not exist a PNE. Symmetry of MSNE means that the probability of choosing a specific location decision is the same for each firm, hence, if we know the probability assigned to location vector $\mathbf{x}$ by a firm, we know the probabilities assigned by each firm at equilibrium. Now, let us focus on a single firm and consider any location vector $\mathbf{x}$. Suppose Assumptions 1-3 hold. To capture the preferences in Assumptions 1 and 2, we formulate utility function of a firm and use this function as the firm's objective. We characterize the utility function of firm r, given the location decisions of all other firms as a function of $\mathbf{x}_r$ as follows

$$\mu_r(\mathbf{x}_r) = \begin{cases} -M & \text{if } \exists \mathbf{x} \text{ such that } \Pi_r(\mathbf{Q}^*(\mathbf{x}), \mathbf{x}) > \Pi_r(\mathbf{Q}^*(\mathbf{x}_r), \mathbf{x}_r), \\ -M & \text{if } \exists \mathbf{x} \text{ such that } \Pi_r(\mathbf{Q}^*(\mathbf{x}), \mathbf{x}) = \Pi_r(\mathbf{Q}^*(\mathbf{x}_r), \mathbf{x}_r) \text{ and } |\mathbf{x}| > |\mathbf{x}_r|, \\ \Pi_r(\mathbf{Q}^*(\mathbf{x}_r), \mathbf{x}_r) & \text{otherwise,} \end{cases} \quad (13)$$

where $M \to \infty$, $\Pi_r(\mathbf{Q}^*(\mathbf{x}_r), \mathbf{x}_r)$ denotes the total profit, including facility location costs, of firm r when $\mathbf{x}_r$ is the location vector, and $|\mathbf{x}|$ denotes the number of facilities located under location vector $\mathbf{x}$. Note that, the purpose of formulating a utility function as in Equation 13 and letting $M \to \infty$ is just to reflect Assumptions 1 and 2 mathematically. Now given that any firm uses Equation (13) as an objective, we focus on determining the probability assigned to location vector $\mathbf{x}$ by any firm, say firm r_1, using the utilities of any other firm, say firm r_2, i.e., we compare two firms. Suppose there are T possible location vectors and firm r_1 assigns probability $\rho_{r_1 t}$ to location vector $t \le T$. Now let us focus on utility matrix of firm r_2, say $\mathbf{A}$. Due to Equation (13), each row of $\mathbf{A}$ consists of 1 nonnegative and $t-1$ of $-M$ values. We consider two cases:

Case I: Each column on A has 1 nonnegative value. In this case, no strategy is weakly or strictly dominated, hence, $\rho_{r_1 t} > 0$ $\forall 0 \le t \le T$. Then, we should have $a\rho_{r_1 t} - M(1-\rho_{r_1 t}) = b\rho_{r_1 z} - M(1-\rho_{r_1 z})$, where $a > 0$ and $b > 0$ for any t and z, $1 \le t \le T$ and

$1 \le z \le T$. Then it follows that $\frac{\rho_{r_1 t}}{\rho_{r_1 z}} = \frac{a+M}{b+M}$. Then $\lim_{M\to\infty} \frac{\rho_{r_1 t}}{\rho_{r_1 z}} = 1$, i.e., $\rho_{r_1 t} = \rho_{r_1 z}$ for any $1 \le t \le T$ and $1 \le z \le T$. Moreover, since there are finite number of strategies for any firm, $\rho_{r_1 t} = \rho_{r_1 z} > 0$. Letting ρ denote this probability, we have $\rho_{r_1}(\mathbf{x}) = \rho$ for any location vector $\mathbf{x}$, as $M \to \infty$. Then it easily follows from the symmetry of the MSNE, $\rho_r(\mathbf{x}) = \rho$ for any firm $r \in R$.

Case II: There are columns with no nonnegative values. In this case, the location vectors corresponding to the columns with nonnegative values weakly or strictly dominates the location vectors corresponding to the columns without nonnegative values. Hence, we can assign probability 0 to the columns without nonnegative values. For the remaining columns, then, Case II reduces to Case I. We note that a weakly or strictly dominated strategy, i.e., a location vector, when utility function is used as an objective, is also weakly or strictly dominated when the profit function is used as an objective by the firms. It follows from Cases I and II that any firm will assign probability 0 to weakly or strictly dominated location vectors and any firm will assign probability ρ to any other location vector as $M \to \infty$.

A.11. Proof of Proposition 11

We first show that any firm r will supply market j from a single location. Considering Equation (5) when $\alpha_{ij} = 0 \;\; \forall i \in I, j \in J$, we have (a) $\delta_{ij} - b_j q_{\bullet j \bullet} - b_j q_{\bullet jr} = 0$. Now, if we assume that firm r supplies market j from two locations, we see that expression (a) for these two locations will imply that $q_{\bullet jr} \ne q_{\bullet jr}$, which is a contradiction. This further implies that (b) $q_{\bullet jr} = q_{i^* jr}$ for some i^* $i^* \in I^0$. Next, we show that $i^* = argmax_{i\in I^0}\{\delta_{ij}\}$. The profit of firm r is (c) $(p_j^* - c_{ij})q_{ijr}$, where the p_j^* is the equilibrium market price. (c) is maximized when $i^* = argmax_{i\in I^0}\{\delta_{ij}\}$. Up to now, we have shown that, any firm will supply market j from a single location and this location is the closest one to market j. Then, it follows from expression (a) that $q^*_{i^* jr_1} = q^*_{i^* jr_2}$ for any two firms r_1 and r_2. Moreover, it follows from (a) and (b) that $q^*_{ijr} = b_j \delta_{ij} / (k+1)$.

APPENDIX B: Heterogeneous Cost Case

Abstract

We study a set of competitive firms considering the location of uncapacitated facilities at a set of candidate locations in order to serve a set of markets. Each firm incurs firm-specific (linear) transportation costs, as well as convex congestion and fixed location costs as a result of location and distribution volume decisions. The unit price in each market is a linear decreasing function of the total amount shipped to the market by all firms; that is, we consider an oligopolistic Cournot game and analyze the two-stage Nash Equilibrium. This problem is referred to as the location-supply game, or competitive location game, and we first study the firms' market-supply decisions for given facility locations, i.e., the game's second stage. We formulate the problem of finding the equilibrium supply quantities as a variational inequality problem and provide a solution algorithm. Then we focus on the location decisions, i.e., the game's first stage. We provide rules to obtain a dominant location matrix, and use these rules in a heuristic solution approach to search for an equilibrium location matrix. Numerical results on the efficiency of the heuristic method are documented.

B.1. Introduction and Literature Review

Facility location problems have been extensively studied in the literature. Most of the past operations research studies on facility location theory focus on formulating a single decision maker's problem in the absence of competitive factors. This stream of research is discussed in the facility location books by Drezner [4] and Drezner and Hamacher [5], and the review papers by Hale and Moberg [10], Owen and Daskin [29], and Tansel et al. [41], as well as the references contained therein. As noted by Plastria [31], an assumption of no competition is often impractical. Rhim et al. [34] also observe that location competition is an important factor in competitive supply chains. As a result, another stream of research focuses on facility location problems under competition. The problems studied within this research stream comprise the fundamentals of competitive location theory. In this problem class, firms' location decisions (along with other strategic decisions, such as pricing decisions, supply quantity decisions, or capacity decisions) are studied by applying competitive equilibrium tools and concepts.

The classical study of Hotelling [17] introduces the first competitive location problem. In this study, two firms compete in a market and each wishes to maximize its market share under a demand inelasticity assumption. Smithies [40] considers the same problem with demand elasticity. Teitz [42] extends Hotelling's problem by allowing firms to locate more than one facility. Following these basic studies, competitive location problems have been studied under different settings in the literature. These settings differ in their assumptions on the number of competing firms (two firms versus more general multiple firm problems), the number of strategic decisions (facility locations, product pricing, supply quantities and facility capacities), and the nature of the competition and strategic game (sequential facility location decisions, simultaneous location decisions, and decisions when facilities already exist at some locations). The reader may refer to Eiselt and Laporte [6], Eiselt et al. [7] and Plastria [31] for reviews of competitive facility location problems under different assumptions.

The problem we study in this paper assumes competition between multiple firms supplying a single product to multiple markets. Each firm must determine its supply facility locations and the quantities to be supplied from each facility to every market. Firms are non-cooperative and must make simultaneous decisions. We assume that potential facility locations and markets are located on finite number of vertices of a network. The competition base is that of Cournot, i.e., the price in a market is determined by the total quantity supplied to the market. Labbé and Hakimi [22] study spatial competition in a Cournot duopoly setting with multiple markets. Sarkar et al. [38] extend the problem to a Cournot oligopoly. Pal and Sarkar [30] extend spatial competition in a Cournot duopoly setting by allowing competing firms to locate more than one facility. In these studies, the underlying assumptions imply that each firm is active in every market, i.e., there is a positive supply from each firm to every market. This assumption is then relaxed by Rhim et al. [34] and Sáiz and Hendrix [36] to capture the concept of free market entry. In both of these studies, Cournot competition exists and firms choose the location of their single facility and the quantity they will supply from this facility to each market, where markets and potential facility locations are located on the vertices of a network. We note that defining potential facility locations as vertices of a network is more practical and parallels the results of Labbé and Hakimi [22], Lederer and Thisse [23] and Sarkar et al. [38], which state that equilibrium facility locations tend to be on the vertices of an underlying network under spatial competition. In these studies, the price in a market is a linear and decreasing function of the total quantity supplied to the market, and firms are subject to linear transportation and production costs as well as fixed facility location costs. While Rhim et al. [34] assume a homogeneous cost structure, i.e, transportation, production and facility location costs are location specific, Sáiz and Hendrix [36] consider a heterogeneous cost structure, i.e., transportation, production and facility location costs are both location and firm specific. Moreover, Rhim et al. [34] consider the capacity of facilities as a strategic decision of any firm as well.

The problem we consider in this paper applies similar assumptions as those of Rhim et al. [34] and Sáiz and Hendrix [36]. However, we allow each firm to locate more than one facility, and firms are subject to firm-specific nonlinear traffic congestion costs, along with firm-specific linear transportation and fixed facility location costs. We use a two-stage solution approach as in [22], [23], [30], [34], [36] and [38]: first, Pure Nash Equilibrium (PNE) supply quantities are determined for given facility locations (the stage-two game) and these are then used to determine equilibrium facility locations (the stage-one game). That is, backward induction is used and a Subgame Perfect Nash equilibrium set of locations is found, if one exists. However, due to the complexity of each firm's profit function, it is not possible to use a simple method that solves the first order equilibrium conditions to determine the PNE supply quantities, as in [34] and [36]. Hence, we use a variational inequality approach. Gabay and Moulin [9] suggest that variational inequalities can be used to determine equilibrium solutions in non-cooperative games. One may refer to [8], [13] and [18] for an introduction to variational inequalities, solution approaches and the problems studied in variational inequality theory. Applications of variational inequalities on equilibrium problems can be seen in [19] and [26]. Dong et al. [3] and Nagurney et al. [27] provide representative examples of variational inequality formulations of equilibrium problems in competitive supply chains. In the literature, different solution approaches have been proposed for different types of variational inequality problems (VIP). Han and Lo [12], He [14], He and Liao [15] and Wang et al. [43] consider nonlinear VIPs, whereas Andreani et al. [1], He and

Zhou [16] and Liao and Wang [24] focus on linear VIPs. The stage-two game of the problem we study corresponds to an asymmetric linear VIP, for which the method proposed by Han [11] has been shown to be efficient. We then focus on solving the stage-one game.

Finding a solution to the stage-one game is important for understanding and characterizing the structural properties of equilibrium facility locations. Government agencies, land-use planners, and suppliers to competing firms may benefit from understanding the actions private decision-makers will take in equilibrium. On the other hand, finding equilibrium facility locations is challenging. A total enumeration approach for finding a solution to the stage-one game is computationally burdensome (because the number of potential solutions increases exponentially in the number of competing firms and potential locations). Therefore, we provide a heuristic search algorithm for the stage-one game that can be used to understand the the structural properties of equilibrium solutions. A genetic algorithm is proposed by Rhim [33] to determine equilibrium locations for the problem studied by Rhim et al. [34], while Sáiz and Hendrix [36] provide a multi-start search algorithm. Our method is a search algorithm that evaluates the conditions that must be satisfied by any equilibrium-location decision.

Our work extends Rhim et al. [34] and Sáiz and Hendrix [36] by including nonlinear traffic congestion cost terms and by allowing firms to locate more than one supply facility. The effects of traffic congestion on supply chain activities are discussed in [25], [32], [37] and [44], which combine supply chain and traffic congestion analysis. Konur and Geunes [20] study the effects of traffic congestion on supply chains by modeling traffic congestion costs endogenously, unlike the studies mentioned above. We model traffic congestion costs in a similar manner to Konur and Geunes [20], although they study a competitive location problem involving identical suppliers, whereas we permit heterogeneous suppliers. The solution methods we will discuss can enable efficient planning of supply chain activities (facility locations and market-supply quantity decisions) within a congested distribution network in the presence of competition.

The rest of this paper is organized as follows. In Section 2, we formulate our problem and discuss the details of the problem setting and the solution approach. In Section 3, the variational inequality formulation of the stage-two game is stated and a solution method is provided. In Section 4, conditions that an equilibrium location decision must satisfy are analyzed and the heuristic search method is explained. In Section 5, numerical studies on the efficiency of the heuristic method are documented. Finally, we provide concluding remarks and future research directions in Section 6.

B.2. Problem Formulation and Solution Approach

Consider a set of k firms who wish to supply a set of n customer markets. The firms compete with each other in the markets for the sales of a single product. The firms may locate supply facilities at m potential locations in order to supply the markets. The costs incurred by supply firms include transportation, traffic congestion and facility location costs. These costs depend on the firms' location choices and the quantities they send from these locations to the markets. A market's price for the good is a linear, decreasing function of the total quantity supplied to the market from all firms, and each firm wishes to maximize its own profit. We assume that variable transportation costs are a linear function of the quantity shipped from facilities to markets, and that these costs are firm dependent. Furthermore, firms incur traffic congestion costs, which are convex and non-decreasing in the quantity supplied from a facility to a market, and these costs are also firm-specific. If a firm locates a facility at a potential location, it incurs a fixed location cost that depends on the location and the firm. Moreover, we assume that a firm will not open more than one facility at a given location, implying that the firm will create sufficient capacity at the location to accommodate the quantity supplied by the facility to all markets in equilibrium. The following list defines the notation we use in defining our model.

- r: index for firms, $r \in R = \{1, 2, \cdots, k\}$
- i: index for locations, $i \in I = \{1, 2, \cdots, m\}$
- j: index for markets, $j \in J = \{1, 2, \cdots, n\}$
- q_{ijr}: quantity shipped from the facility of firm r at location i to market j
- $q_{\bullet jr}$: total quantity shipped to market j by firm $r, q_{\bullet jr} = \sum_{i \in I} q_{ijr}$
- $q_{i \bullet r}$: total quantity shipped from location i by firm $r, q_{i \bullet r} = \sum_{j \in J} q_{ijr}$
- $q_{ij \bullet}$: total quantity shipped from location i to market $j, q_{ij \bullet} = \sum_{r \in R} q_{ijr}$
- $q_{\bullet j \bullet}$: total quantity shipped to market $j, q_{i \bullet r} = \sum_{r \in R} \sum_{i \in I} q_{ijr}$
- $\mathbf{Q}$: $k{\times}m{\times}n$ matrix of q_{ijr} values
- $\mathbf{x}_r$: m-vector representing the location decisions of firm r
- $\mathbf{X}$: $m{\times}k$ matrix representing all location decisions
- $p_j\left(q_{\bullet j \bullet}\right)$: price function for market j
- $g_{ijr}\left(q_{ij \bullet}\right)$: traffic congestion cost coefficient from location i to market j for firm r
- c_{ijr}: transportation cost coefficient for sending units from location i to market j for firm r
- f_{ir}: fixed cost of opening a facility at location i for firm r
- $f_r\left(\mathbf{x}_r\right)$: total facility location cost for firm r

Then, we can formulate the profit function for each firm r as follows:

$$\Pi_r(\mathbf{Q}, \mathbf{X}) \quad = \quad \sum_{j \in J} p_j \left(\sum_{i \in I} \sum_{r \in R} q_{ijr} \right) \sum_{i \in I} q_{ijr} - \sum_{j \in J} \sum_{i \in I} c_{ijr} q_{ijr} - \sum_{j \in J} \sum_{i \in I} q_{ijr} g_{ijr} \left(\sum_{r \in R} q_{ijr} \right) - f_r(\mathbf{x}_r). \quad \text{(B.1)}$$

The above profit function consists of the supply firm's total revenue, less variable costs, traffic congestion costs, and facility location costs.

We assume that p_j, the price in market j, is determined by the function

$$p_j(q_{\bullet j \bullet}) = a_j - b_j q_{\bullet j \bullet}, \quad \text{(B.2)}$$

where $a_j \geq 0$ and $b_j > 0$ denote the price at zero demand and the price sensitivity for market j. Note that Equation (B.2) is the inverse demand function associated with Cournot competition. As illustrated by the profit function, the transportation cost is linear in the quantity sent from facility i to market j with marginal cost $c_{ij} \geq 0$. Note that c_{ij} can be easily adjusted to account for any per-unit production costs without loss of generality. That is, a location-specific parameter $v_i \geq 0$

denoting the per-unit production cost at location i can be added to c_{ij}. The traffic congestion cost coefficient for link (i, j) is defined as g_{ij} (which is a function of the total quantity of flow on the link) below.

$$g_{ijr}(q_{ij\bullet}) = \alpha_{ijr} q_{ij\bullet} \tag{B.3}$$

The parameter $\alpha_{ijr} > 0$ is a traffic congestion cost multiplier for flow on link (i, j) for firm r. Hence, the congestion cost incurred by a firm using link (i, j) increases in the total flow on the link. It follows from Equations (1) and (3) that firm r's congestion cost equals $\alpha_{ijr} q_{ijr} q_{ij\bullet} = 0$ when $q_{ijr} = 0$. On the other hand, when $q_{ijr} > 0$, the firm's congestion cost equals $\alpha_{ijr} q_{ijr} q_{ij\bullet} > 0$, and increases quadratically with q_{ijr} for fixed values of the quantities sent by the other firms on the link. This implies that the congestion cost incurred by firm r on link (i, j) is nondecreasing and convex in q_{ijr}. A convex and nondecreasing function is consistent with typical traffic congestion costs, which increase in volume at an increasing rate. Moreover, permitting α_{ijr} to be firm-specific reflects the fact that firms may price congestion differently. We note that Konur and Geunes [20] model traffic congestion costs in a similar manner. However, they assume that all firms cost parameters are identical, i.e., $\alpha_{ijr} = \alpha_{ij}$ and $c_{ijr} = c_{ij}$ $\forall r \in R$ (they also assume identical facility location costs, which we do not).

As in typical practice, a firm will determine the locations of facilities prior to deciding on the quantities to supply from facilities to markets. Moreover, the individual firm's profit is a function of the decisions of other firms. Because of this, we use a two-stage solution approach. In the first stage, all firms choose locations. In the second stage, they then determine supply quantities from facilities to markets. We first solve the Stage-two problem for given location decisions. We use game-theoretic techniques (in particular, Nash Equilibria) to characterize equilibrium supply quantities and we provide an approximation scheme to find Pure Nash Equilibrium (PNE) solutions. For the Stage-one decisions, since total enumeration of all possible location decisions requires analyzing exponentially many location matrices (and each location matrix requires determining the profits for exponentially many other location matrices to check whether the location matrix is an equilibrium point), we provide a heuristic method for determining PNE locations. Therefore, we use a backward induction approach to find the PNE location decisions, and the resulting solution is a Subgame Perfect Nash Equilibrium [39]. We provide numerical results on the efficiency of the heuristic method by comparing it with a random search method.

B.3. Stage-two Game: Quantity Decisions

In this section, we formulate the problem corresponding to the game of determining equilibrium supply quantities for given location decisions. We refer to this restricted game as the *Stage-two Game*. The first-order conditions for the *Stage-two Game* will imply that the supply quantities for each market can be analyzed separately; thus, we study the *Stage-two Game* for a specific market. We first present the variational inequality formulation of the *Stage-two Game* for a specific market and characterize important properties of the variational inequality formulation, which enable us to use an efficient algorithm for the solution of the *Stage-two Game* for a specific market.

We first consider the Stage-two game for a given first-stage solution, i.e., when the vectors $\mathbf{x}_r$ for each $r = 1, 2, ..., k$, have been pre-determined, which implies that the matrix $\mathbf{X}$ is fixed. This implies that $f_r(\mathbf{x}_r)$ is also fixed at $\mathbf{X} = \mathbf{X}^0$, and can be ignored when analyzing the *Stage-two Game*. Based on the profit function (B.1) and our definitions of the price (B.2) and congestion (B.3) functions, firm r's profit function at $\mathbf{X} = \mathbf{X}^0$ can then be written as

$$\Pi_r(\mathbf{Q}|_{\mathbf{X}=\mathbf{X}^0}) = \sum_{j \in J} \left[(a_j - b_j q_{\bullet j \bullet}) q_{\bullet jr} - \sum_{i \in I} c_{ijr} q_{ijr} - \sum_{i \in I} \alpha_{ijr} q_{ijr} q_{ij\bullet} \right]. \tag{B.4}$$

It is straightforward to show that Equation (B.4) is a strictly concave function in each variable $q_{ijr} \geq 0$, because $b_j > 0$ and $\alpha_{ijr} > 0$. Letting I_r^0 denote the set of locations at which firm r has opened a facility (in the corresponding Stage-one solution at $\mathbf{X} = \mathbf{X}^0$), we have $q_{ijr} = 0$ for all $j \in J, i \notin I_r^0$. The first-order conditions ($\partial \Pi_r(\mathbf{Q}|_{\mathbf{X}=\mathbf{X}^0}) / \partial q_{ijr} = 0$, for q_{ijr} values such that $q_{ijr} > 0$) must be satisfied at a Nash equilibrium solution for the *Stage-two Game*. In particular, if $q_{ijr} > 0$ then a Nash equilibrium solution must satisfy the condition

$$a_j - b_j[q_{\bullet j \bullet} + q_{\bullet jr}] - c_{ijr} - \alpha_{ijr}[q_{ijr} + q_{ij\bullet}] = 0. \tag{B.5}$$

Observe that Equation (B.5) is independent of markets other than j, which implies that we can analyze the equilibrium conditions for market j independently of other markets (similar results are given in [34] and [36]).

We define $\mathbf{Q}_j|_{\mathbf{X}=\mathbf{X}^0}$ as the vector of supply quantities for market j given the location matrix $\mathbf{X}^0$. Then, the profit function for firm r in market j under these supply quantities, denoted by $\Pi_r^j(fQ_j|_{\mathbf{X}=\mathbf{X}^0})$, can be written as

$$\Pi_r^j(\mathbf{Q}_j|_{\mathbf{X}=\mathbf{X}^0}) = p_j(q_{\bullet j \bullet}) q_{\bullet jr} - \sum_{i \in I_r^0} c_{ijr} q_{ijr} - \sum_{i \in I_r^0} \alpha_{ijr} q_{ijr} q_{ij\bullet}. \tag{B.6}$$

Because the equilibrium conditions for market j are independent of the other markets for the given location matrix $\mathbf{X}^0$ we focus on the *Stage-two Game* for an arbitrary market j. Next, we provide a variational inequality formulation for the *Stage-two Game* for market j. Note that we cannot use simple methods similar to those in [34] and [36] to find equilibrium quantities due to the complexity of the first order Nash equilibrium conditions stated in Equation (B.5).

Define $|I_r^0|$ as the cardinality of the set I_r^0 for the given location decisions $\mathbf{X} = \mathbf{X}^0$. The supply quantities flowing to a market can be represented by the λ-vector $\mathbf{Q}_j$, where $\lambda = \sum_{r \in R} |I_r^0|$. Since $q_{ijr} = 0 \ \ \forall i \notin I_r^0$ for any firm r, $r \in R$, these flows are not accounted for in the vector $\mathbf{Q}_j$. Thus, $\mathbf{Q}_j \in R_+^{\lambda}$. The Nash equilibrium quantities must be optimal for each firm, given the optimal decisions of all other firms. As the profit function of each firm is strictly concave in every q_{ijr}, the optimality conditions for each firm can be written in variational inequality form. It follows from [9], [26] and [27] that $\mathbf{Q}_j^*$ is a Nash equilibrium if it satisfies the following variational inequality:

$$-\sum_{r\in R}\sum_{i\in I_r^0}\frac{\partial\Pi_r^j(\mathbf{Q}_j\,|_{\mathbf{X}=\mathbf{X}^0})}{\partial q_{ijr}}\times(q_{ijr}-q_{ijr}^*)\geq 0,\forall\mathbf{Q}_j\in R_+^{\lambda}. \quad \text{(B.7)}$$

The variational inequality in Equation (B.7) for market j then takes the following explicit form

$$\sum_{r\in R}\sum_{i\in I_r^0}\left[c_{ijr}-a_j+b_j(q_{\bullet j\bullet}+q_{\bullet jr})+\alpha_{ijr}(q_{ijr}+q_{ij\bullet})\right]\times(q_{ijr}-q_{ijr}^*)\geq 0,\forall\mathbf{Q}_j\in R_+^{\lambda}. \quad \text{(B.8)}$$

Next, we present the problem of determining equilibrium quantities at market j, assuming given location decisions, using a classical variational inequality representation. We then study qualitative properties of this variational inequality problem formulation.

Let $\mathbf{Q}_j^r\in S_r$ denote the vector of supplies from firm r facilities to market j; that is, $\mathbf{Q}_j^r=(q_{1jr},\ldots,q_{|I_r^0|jr})^T\quad\forall r\in R$, where S_r denotes the strategy set of firm r. Then $\mathbf{Q}_j^*=((\mathbf{Q}_j^{1*})^T,\ldots,(\mathbf{Q}_j^{k*})^T)^T\in S$, where $S=S_1\times\ldots\times S_k$, is a Nash equilibrium solution if it satisfies

$$\Pi_r^j(\mathbf{Q}_j^{r*},\mathbf{Q}_j^{-r*}\,|_{\mathbf{X}=\mathbf{X}^0})\geq\Pi_r^j(\mathbf{Q}_j^r,\mathbf{Q}_j^{-r*}\,|_{\mathbf{X}=\mathbf{X}^0})\forall\mathbf{Q}_j^r\in S_r,\forall r\in R, \quad \text{(B.9)}$$

where $\mathbf{Q}_j^{-r*}=\left((\mathbf{Q}_j^{1*})^T,\ldots,(\mathbf{Q}_j^{(r-1)*})^T,(\mathbf{Q}_j^{(r+1)*})^T,\ldots,(\mathbf{Q}_j^{k*})^T\right)^T$. The next theorem characterizes the *Stage-two Game* solution for market j under a variational inequality approach. The proof of the theorem is based on the Nash equilibrium conditions stated in Equation (B.9) and the first-order optimality conditions of the strictly concave profit function for any firm $r\in R$.

Theorem B.1. $\mathbf{Q}_j^*\in S$ solves the Stage-two Game at market j if it solves the following variational inequality problem for given location decision $\mathbf{X}=\mathbf{X}^0$:

$$\left\langle F(\mathbf{Q}_j^*),\mathbf{Q}_j-\mathbf{Q}_j^*\right\rangle\geq 0,\forall\mathbf{Q}_j\in S, \quad \text{(B.10)}$$

where $F(\mathbf{Q}_j)=(-\nabla_{\mathbf{Q}_j^1}\Pi_1^j(\mathbf{Q}_j\,|_{\mathbf{X}=\mathbf{X}^0}),\ldots,-\nabla_{\mathbf{Q}_j^k}\Pi_k^j(\mathbf{Q}_j\,|_{\mathbf{X}=\mathbf{X}^0}))$ is a λ-row vector function and we have $\nabla_{\mathbf{Q}_j^r}\Pi_r^j(\mathbf{Q}_j\,|_{\mathbf{X}=\mathbf{X}^0})=(\partial\Pi_r^j(\mathbf{Q}_j\,|_{\mathbf{X}=\mathbf{X}^0})/\partial q_{1jr},\ldots,\partial\Pi_r^j(\mathbf{Q}_j\,|_{\mathbf{X}=tbfX^0})/\partial q_{|I_r^0|jr})\ \ \forall r\in R$.

Proof: See [26].

Note that $F(\mathbf{Q}_j)$ is a linear, continuous and differentiable function. Now let us consider the set S. We already know that supply quantities are bounded from below, since $q_{ijr}\geq 0$. If we assume that the price in any market will not be negative, then there is an upper limit on the total quantity supplied to that market. This implies that for any market j, $(a_j-b_j\sum_{r\in R}\sum_{i\in I_r^0}q_{ijr})\geq 0$. Thus, we have that $q_{ijr}\leq a_j/b_j$, i.e., a_j/b_j is a natural upper bound for q_{ijr}. This implies that S is a polytope, i.e., it is a compact and convex set. The next property demonstrates the existence of a solution to the variational inequality problem stated in Equation (B.10).

Property B.1. Suppose that $(a_j-b_j\sum_{r\in R}\sum_{i\in I_r^0}q_{ijr})\geq 0\quad\forall\mathbf{Q}_j\in S$. Then the variational inequality problem $<F(\mathbf{Q}_j^*),\mathbf{Q}_j-\mathbf{Q}_j^*>\geq 0,\forall\mathbf{Q}_j\in S$ admits at least one solution $\mathbf{Q}_j^*$.

Proof: All proofs can be found in the Appendix.

Property B.1 also follows from [35], as the profit function $\Pi_r^j(\mathbf{Q}_j|_{\mathbf{X}=\mathbf{X}^0})$ is strictly concave and the set of strategies, *S*, is compact and convex for each player, i.e., for each firm. The following property characterizes the strict monotonicity of the variational inequality function *F*.

Property B.2. $F(\mathbf{Q}_j)$ is strictly monotone on S.

It can be seen from the proof of Property B.2 that $F(\mathbf{Q}_j)$ is strictly monotone on the entire space R_+^λ. As a direct result of Properties B.1 and B.2, the uniqueness of the solution to the variational inequality problem is stated next.

Property B.3. Suppose that $(a_j - b_j \sum_{r\in R} \sum_{i\in I_r^0} q_{ijr}) \geq 0 \quad \forall \mathbf{Q}_j \in S$. Then the variational inequality problem $\left\langle F(\mathbf{Q}_j^*)^t, \mathbf{Q}_j - \mathbf{Q}_j^* \right\rangle \geq 0, \forall \mathbf{Q}_j \in S$ has a unique solution.

The uniqueness of the solution to the variational inequality problem also follows from the fact that the profit function of each firm is strictly concave. Moreover, when the condition of Property B.3 is satisfied, the strategy set for each firm is compact and convex, as noted previously. Hence, it follows that the Nash equilibrium point is unique. In Properties B.1 and B.3, S is assumed to be a compact set. While compactness is required for the uniqueness of the solution, we do not require S to be compact for the existence result. The following property gives an existence condition when S is the nonnegative orthant.

Property B.4. The variational inequality problem, $\left\langle F(\mathbf{Q}_j^*), \mathbf{Q}_j - \mathbf{Q}_j^* \right\rangle \geq 0, \forall \mathbf{Q}_j \in S$ admits at least one solution $\mathbf{Q}_j^*$ given that $S = R_+^\lambda$.

Before presenting a solution method for the variational inequality problem formulation, we note that F is Lipschitz continuous. That is, there exists an $L > 0$ such that $\| F(\mathbf{Q}_j^a) - F(\mathbf{Q}_j^b) \| \leq L \| \mathbf{Q}_j^a - \mathbf{Q}_j^b \|, \ \forall \mathbf{Q}_j^a, \mathbf{Q}_j^b \in S$. The Lipschitz continuity of F follows from the fact that F is a continuously differentiable and linear function.

We are now ready to present an algorithm that solves the variational inequality problem stated in Equation (B.10). Because we have a linear variational inequality problem, an algorithm for linear variational inequality problems will be stated. However, it should be noted that *the modified projection algorithm* of Korpelevich [21] can also be used to solve our problem, and that this algorithm will converge to a solution. The convergence of *the modified projection algorithm* follows from Properties B.1, B.2, and the Lipschitz continuity of F. See [27] for a discussion of the algorithm's application to a supply chain network equilibrium problem. The algorithm we use is the *self-adaptive projection method* proposed by Han [11] for solving linear variational inequalities in the following form

$$(Mx^* + z)^T (x - x^*) \geq 0, \forall x \in K,$$

where K is a nonempty, convex and closed subset of R^n, $M \in R^{n\times n}$ is a given matrix, and

$z \in R^n$ is a given vector. In our case, the resulting linear variational inequality problem is asymmetric, for which M is defined by the partial derivatives as stated in Theorem B.1, and the vector z consists of the $c_{ijr} - a_j$ values. Moreover, $K = S = R_+^\lambda$. The algorithm can be formalized as follows.

Algorithm B.1.

Self-adaptive Projection method for the variational inequality formulation of the *Stage-two Game* at market j.

Step 0. Start with a $\mathbf{Q}_j^0 \in R^\lambda$. Let ℓ denote an iteration counter. Set $0 < \gamma < 2$, $\beta_0 > 0$, $\varepsilon \geq 0$, and a sequence $\{\tau_\ell\} \subseteq [0, \infty)$ with $\sum_{\ell=0}^{\infty} \tau_\ell < \infty$. Set $\ell := 0$.

Step 1. If $\| e(\mathbf{Q}_j^\ell, \beta_\ell) \|_\infty \leq \varepsilon$, then stop; else go to Step 2. Here, $e(x, \beta) = x - P_S[x - \beta(Mx + z)]$ where $P_S[.]$ denotes the orthogonal projection from R^λ onto S.

Step 2. Compute the next iterate using $\mathbf{Q}_j^{\ell+1} = \mathbf{Q}_j^\ell - \gamma(I + \beta_\ell M)^{-1} e(\mathbf{Q}_j^\ell, \beta_\ell)$.

Step 3. Choose the next parameter $\beta_{\ell+1}$ from the interval $\frac{1}{1+\tau_\ell}\beta_\ell \leq \beta_{\ell+1} \leq (1+\tau_\ell)\beta_\ell$. Set $\ell := \ell + 1$ and go to Step 1.

The algorithm requires calculating the inverse of a matrix and taking the projection of a point onto the set S. In our problem, S is the nonnegative orthant and, hence, projection is easily carried out. In particular, as noted by Han [11] as well, projection onto S using the Euclidean-norm is defined component-wise for each element of the vector to be projected. Explicitly, $P_S[x]_j = x_j$ if $x_j \geq 0$, and, $P_S[x]_j = 0$ if $x_j < 0$. The next theorem establishes the convergence of Algorithm B.1.

Theorem B.2. The Self-adaptive Projection method, stated in Algorithm B.1, converges to a solution of the variational inequality formulation in Equation (B.10).

In the next Section, we study the *Stage-one Game*, which seeks an equilibrium location decision matrix if one exists, or concludes that an equilibrium location decision does not exist.

B.4. Stage-one Game: Location Decisions

The goal of this section is to determine an equilibrium location decision matrix. We first focus on defining dominant strategies, which are candidates for an equilibrium location matrix. Later, for a given location matrix in a dominant strategy set, we check to see whether it is an equilibrium location matrix. Note that a location matrix $\mathbf{X}$ consists of the location vectors for each firm, where the vector for firm r is denoted by the column vector $\mathbf{x}_r$; that is, the r^{th}

column of $\mathbf{X}$ represents the location decisions for firm r. Moreover, the i^{th} row of $\mathbf{X}$ determines the set of firms with a facility at location i. Let x_{ir} denote the entry in the i^{th} row and j^{th} column of $\mathbf{X}$. Then $x_{ir} = 1$ if firm r has a facility at location i and $x_{ir} = 0$ otherwise. Recall from Section 1 that f_{ir} denotes the fixed facility location cost associated with location i and firm r.

Now let $\mathbf{Q}^*(\mathbf{X})$ denote the equilibrium quantities for a given location matrix $\mathbf{X}$. Observe that we know how to find $\mathbf{Q}^*(\mathbf{X})$ for any given $\mathbf{X}$ from the previous section. In particular, we can determine the equilibrium quantities for each market using the variational inequality formulation and Algorithm B.1. Recall that Equation (B.9) gives the Nash equilibrium conditions for the equilibrium quantities. Similarly, the condition required for a location matrix $\mathbf{X}$ to correspond to an equilibrium decision reads

$$\Pi_r\left(\mathbf{x}_r^*, \mathbf{X}^{-r*}\,|_{\mathbf{Q}^*(\mathbf{X}^*)}\right) \geq \Pi_r\left(\mathbf{x}_r, \mathbf{X}^{-r*}\,|_{\mathbf{Q}^*(\tilde{\mathbf{X}})}\right) \forall \mathbf{x}_r, \forall r \in R, \qquad \text{(B.11)}$$

where $\mathbf{x}_r^*$ denotes the equilibrium location decision of firm r, $\mathbf{X}^{-r*}$ denotes the equilibrium decisions of all other firms, i.e., $\mathbf{X}^{-r*} = [\mathbf{x}_1^*, ..., \mathbf{x}_{r-1}^*, \mathbf{x}_{r+1}^*, ..., \mathbf{x}_k^*]$, $\tilde{\mathbf{X}} = [\mathbf{x}_1^*, ..., \mathbf{x}_{r-1}^*, \mathbf{x}_r, \mathbf{x}_{r+1}^*, ..., \mathbf{x}_k^*]$, and $\mathbf{X}^*$ denotes the equilibrium location matrix. Note that the profit function given in Equation (B.11) includes facility location costs $f_r(\mathbf{X}) = f_r(\mathbf{x}_r) = \sum_{i \in I} f_{ir} x_{ir}$. In the next proposition, we state a simple condition that must be satisfied by any equilibrium location decision.

Proposition B.1. Let $\mathbf{X}^*$ be an equilibrium location decision. Then, if $f_{ir} > 0$, we must have $x_{ir}^* = 0$ when $q_{i\bullet r}^*(\mathbf{X}^*) = 0$.

Proposition B.1 implies that there may exist multiple equilibria when $f_{ir} = 0$ for some $i \in I$ and $r \in R$. In particular, if $q_{i\bullet r}^* = 0$ and $f_{ir} = 0$, the profits of any firm will be the same when $x_{ir} = 0$ or $x_{ir} = 1$. It also follows from Proposition B.1 that while checking whether $\mathbf{X}$ is an equilibrium location decision, we should first set $x_{ir} = 0$ if $q_{i\bullet r}^*(\mathbf{X}) = 0$, and then check the conditions in Equation (B.11). Hence, we define the following rule that we will use in our heuristic approach.

Rule B.0: The following procedure generates a dominating location matrix:

Step 0. Let $\mathbf{X}$ be a given location matrix with entries x_{ir} and let L be a given list of location matrices.

Step 1. If $\mathbf{X} \in L$, set $continue = 0$, stop and return $continue = 0$. Else, set $continue = 1$, $L = L \cup \{\mathbf{X}\}$, solve for the equilibrium quantities corresponding to $\mathbf{X}$ and go to Step 2.

Step 2. Construct $\mathbf{X}^0$ with entries x_{ir}^0 by defining $x_{ir}^0 = 0$ if $q_{i\bullet r}^*(\mathbf{X}) = 0$, and $x_{ir}^0 = 1$ otherwise. Return $\mathbf{X}^0$.

In Step 0 of Rule B.0, we are given a list of matrices denoted by L. We use L to keep track of the location matrices that we have analyzed in our heuristic method and we will explain its use when we state our heuristic method. The list L consists of the location matrices which are not equilibrium and, hence, if the given location matrix is in L, we do not need to continue and check whether it satisfies the equilibrium conditions. However, when $\mathrm{X} \notin L$, the output of Rule B.0 is the location matrix $\mathbf{X}^0$, which dominates the original location matrix $\mathbf{X}$ for at least one firm (unless $\mathbf{X}^0 = \mathbf{X}$), which is why we use the *dominating location matrix* phrase. Now, let T^* denote the set of equilibrium location matrices and let T^0 denote the set of location matrices generated using Rule B.0. Then, it follows from Proposition B.1 that

$$T^* \subseteq T^0.$$

We next focus on dominance relations within the columns of $\mathbf{X}^0$, i.e., the location decisions of a specific firm. The following proposition characterizes such a dominance relation.

Proposition B.2. Let $\mathbf{X}^0$ be generated from a location matrix $\mathbf{X}$ by using Rule B.0. If $\Pi_r(\mathbf{Q}^*(\mathbf{X}^0), \mathbf{X}^0) < 0$ for some firm $r \in R$; then $\mathbf{X}^0$ cannot be an equilibrium location matrix.

Proposition B.2 implies that $\mathbf{X}^0$ can be an equilibrium decision if and only if each one of its non-zero columns produces positive profit for the corresponding firm. This condition is referred to as a *viability* condition by Dobson and Karmarkar [2] and Rhim et al. [33]. At this point, it is important to mention the *Stable Set* concept introduced in [2] and used by [34] and [36] to study location decisions in a similar competitive facility location setting. In [34], each firm may open at most one facility, and a set of facility locations is defined to be *stable* as long as those firms with a facility make a positive profit (viability condition) and the firms without a facility cannot make a positive profit by opening a facility (survival condition). A firm is referred to as an entrant whenever the firm has a facility. However, in our study, a firm may open more than one facility. Thus, a firm is an entrant if the firm opens at least one facility. Proposition B.2 implies that an entrant firm should make positive profit as a result of its location and corresponding quantity decisions, which can be referred to as the viability condition for our case. On the other hand, defining a survival condition can be ambiguous in our case. We note that the survival condition defined in [34] does not imply that an entrant firm must choose each location that is individually profitable. This follows from the fact that the facility location decisions of an entrant firm are not independent. In particular, suppose that an entrant firm may make positive profit by locating a facility at a location where the firm has no facility. It is possible that locating a facility at that location may decrease the overall profit of the entrant firm. However, a non-entrant firm, by definition, can not make a positive profit by locating a set of facilities at any subset of the locations. We note that this discussion is already implied by Proposition B.2. In the next subsection, we study a heuristic method to move to a viable location decision from a randomly given location decision $\mathbf{X}$.

B.4.1. Generating a Viable Location Decision

Now suppose that we are given a random location matrix $\mathbf{X}$ with entries x_{ir}. Let $\mathbf{X}^0$ be generated from $\mathbf{X}$ by using Rule B.0. Next we define a rule to move from matrix $\mathbf{X}^0$, which is not viable, to a viable location decision. The intuition behind this rule is as follows. If $\mathbf{X}^0$ is not viable, then there exists at least one firm with negative total profit. This further implies that there exists at least one facility of that firm with negative profit, i.e., for which the facility location cost exceeds the total profit of the firm gained by supplying markets from the facility. This discussion does not imply that each facility must be profitable in a viable location matrix. Instead, it implies that there must be a facility with negative profit in a location matrix that is not viable. For such matrices that are not viable, we first set $x_{ir} = 0$ when $\pi_{ir}(\mathbf{X}^0) < 0$ for a firm r such that $\Pi_r(\mathbf{X}^0) < 0$, where $\Pi_r(\mathbf{X}^0)$ and $\pi_{ir}(\mathbf{X}^0)$ denote the total profit of firm r under $\mathbf{X}^0$ and the firm's profit at location i, respectively. Then, we determine the equilibrium quantities for the modified matrix $\mathbf{X}$ and generate the corresponding modified matrix $\mathbf{X}^0$. We repeat this process until we find a viable location matrix. We only make changes in columns that cause inviability. Specifically, this rule is defined as follows.

Rule B.1: The following procedure generates a viable location matrix:

Step 0. Let $\mathbf{X}$ be a given location matrix with entries x_{ir} and let L be a given list of location matrices.

Step 1. Apply Rule B.0 with $\mathbf{X}$ and L. If $continue = 0$, stop and return $continue = 0$. Else, generate $\mathbf{X}^0$. Define R^- as the set of firms with negative total profit at $\mathbf{X}^0$, i.e., $\Pi_r(\mathbf{X}^0) < 0$, $\forall r \in R^-$. Go to Step 2.

Step 2. If $R^- = \varnothing$, stop, $\mathbf{X}^0$ is viable and return $\mathbf{X}^1 = \mathbf{X}^0$. Else, let $(\hat{\imath}, \hat{r}) = \text{argmin}\{\pi_{ir}(\mathbf{X}^0) : i \in I, r \in R^-\}$ and set $x_{\hat{\imath}\hat{r}} = 0$ in $\mathbf{X}$. Go to Step 0.

Note that the quantity decisions for $\mathbf{X}$ and $\mathbf{X}^0$ are the same, i.e., $q_{i\bullet r}(\mathbf{X}) = q_{i\bullet r}(\mathbf{X}^0)$. Hence, the $\pi_{ir}(\mathbf{X}^0)$ values can easily be calculated from the $\pi_{ir}(\mathbf{X})$ values by simply letting $\pi_{ir}(\mathbf{X}^0) = \pi_{ir}(\mathbf{X})$ when $x^0_{ir} = 1$ or $x_{ir} = 0$ and, $\pi_{ir}(\mathbf{X}^0) = 0$ when $x^0_{ir} = 0$ and $x_{ir} = 1$. In Step 2 of Rule B.1, we modify the location matrix $\mathbf{X}$ given in Step 0; that is, we do not close the facilities with negative profits under $\mathbf{X}^0$ and try to get a viable location matrix that has fewer facilities located than $\mathbf{X}^0$. The reason we use such a modification is that we try to capture the possibility that the facilities closed due to zero supply with Rule B.0 can have positive supply after we close the facilities with positive supply but with negative profits. Thus, we generate a viable location matrix with more open facilities. At this point, it should be noted that the original matrix $\mathbf{X}$ and the modified matrix differ in only one entry.

Now suppose that $\mathbf{X}_{(i)}$ and $\mathbf{X}_{(i+1)}$ are two consecutive location matrices entering Step 0 of Rule B.1 and let $\mathbf{X}^0_{(i)}$ and $\mathbf{X}^0_{(i+1)}$ be the viable matrices generated by Rule B.0 in Step 1 of Rule B.1 corresponding to $\mathbf{X}_{(i)}$ and $\mathbf{X}_{(i+1)}$, respectively. As noted previously, $\mathbf{X}_{(i)}$ and $\mathbf{X}_{(i+1)}$ differ in only one entry. However, $\mathbf{X}^0_{(i)}$ and $\mathbf{X}^0_{(i+1)}$ may differ in more than one entry. We also close the facility with the most negative profit, as such a facility is less likely to be open in an equilibrium solution. It should be noted that in a viable location matrix, there may be some facilities with negative profit for some firms, although no firm will have negative total profit. Rule B.1 will always find a viable location matrix after starting with a random location matrix. This follows because we may repeat Step 2 of Rule B.1 at most $m \times k$ times, and after at most $m \times k$ repetitions, we would end up with $\mathbf{X} = 0$, which is viable, in the worst case. We denote a viable location matrix generated at the end of Rule B.1 corresponding to the given location matrix $\mathbf{X}$ by $\mathbf{X}^1$. At this point we have the following relations:

$$T^* \subseteq T^1 \subseteq T^0,$$

where T^1 denotes the set of location matrices generated using Rule B.1. In the next subsection, we focus on checking whether the location decisions for any firm represent the best response in a viable location matrix.

B.4.2. Equilibrium Check

Now suppose that $\mathbf{X}^1$ is a viable location matrix generated from $\mathbf{X}$ by using Rule B.1. To determine whether $\mathbf{X}^1$ is an equilibrium location matrix, we need to check if $\mathbf{x}^1_{.r}$, the r^{th} column of $\mathbf{X}^1$, is the best response of firm r, $\forall r \in R$. To do so, we need to check all possible location vectors for firm r while the location decisions of the other firms are fixed. Note that there are 2^m different location decisions for each firm and, hence, we need to check the profit of firm r for $2^m - 1$ location vectors, while keeping the other firms' location decisions unchanged. When we find that $\mathbf{x}^1_{.r}$ is not the best response of firm r for some $r \in R$, then we know that $\mathbf{X}^1$ is not an equilibrium location matrix. As a result, we will need to consider another viable location matrix as a potential equilibrium location matrix. Note that it is sufficient to show that there exists a better location decision for at least one firm in $\mathbf{X}^1$ to conclude that $\mathbf{X}^1$ is not an equilibrium location matrix. To this end, we define two additional rules, referred to as Rule B.2 and Rule B.3, to check whether a better location decision exists for firm r under $\mathbf{X}^1$. The intuition behind Rule B.2 is as follows. If there exists a firm r facility with negative profit, we check whether closing this facility will increase firm r's total profit. We noted in the previous section that there may exist facilities with negative profits for a viable location matrix. If the total profit increases by closing this facility, then we know that the current location vector for the firm under $\mathbf{X}^1$ is not the best response of the firm and, hence, $\mathbf{X}^1$ is not an equilibrium location matrix. We should therefore consider another viable location matrix as a candidate equilibrium matrix. This rule proceeds in the same way as Rule B.1, but is applied only to one column of $\mathbf{X}^1$ each time.

Rule B.2: The following procedure searches for an improving location matrix for a firm:

Step 0. Let $\mathbf{X}$ be a given location matrix with entries x_{ir} and let L be a given list of location matrices.

Step 1. Apply Rule B.1 with $\mathbf{X}$ and L. If $continue = 0$, stop and return $continue = 0$. Else, generate $\mathbf{X}^1$ with entries x^1_{ir} and determine the profit at equilibrium associated with $\mathbf{X}^1$. Set $r = 1$ and go to Step 2.

Step 2. If $r > k$ go to Step 5. Otherwise, define I_r^- as the set of facilities with negative profit under $\mathbf{X}^1$, i.e., $\pi_{ir}(\mathbf{X}^1) < 0$, $\forall i \in I_r^-$, and go to Step 3.

Step 3. If $I_r^- = \varnothing$, set $r = r+1$ and go to Step 2. Otherwise, let $(\hat{\imath}, r) = \arg\min\{\pi_{ir}(X^1) : i \in I_r^-\}$ and construct $\mathbf{X}^2$ by letting $x_{ir}^2 = x_{ir}^1$ and $x_{\hat{\imath}r}^2 = 0$. Go to Step 4.

Step 4. Determine the profit at equilibrium for $\mathbf{X}^2$. If $\pi_r(\mathbf{X}^1) < \pi_r(\mathbf{X}^2)$, set $x_{\hat{\imath}r} = 0$ in $\mathbf{X}$ and go to Step 0. Otherwise, set $I_r^- = I_r^- \setminus \{\hat{\imath}\}$ and go to Step 3.

Step 5. Return $\mathbf{X}^2$.

The purpose of Rule B.2 is to determine whether the viable location matrix $\mathbf{X}^1$ generated from $\mathbf{X}$ is not in equilibrium without finding the best response of any firm. Rule B.2 checks whether closing one of the facilities with negative profit will improve the total profit of a firm. If there is an improvement, $\mathbf{X}$ is updated and a new viable location matrix is generated. Now suppose that $\mathbf{X}_{(i)}$ and $\mathbf{X}_{(i+1)}$ are two consecutive location matrices entering Step 0 of Rule B.2 and let $\mathbf{X}_{(i)}^1$ and $\mathbf{X}_{(i+1)}^1$ be the viable matrices generated by Rule B.1 in Step 1 of Rule B.2 corresponding to $\mathbf{X}_{(i)}$ and $\mathbf{X}_{(i+1)}$, respectively. We note that $\mathbf{X}_{(i)}^1 \neq \mathbf{X}_{(i+1)}^{\mathbf{1}}$, because we change an entry of $\mathbf{X}_{(i)}$ to 0 whose value is 1 in $\mathbf{X}_{(i)}^1$; hence, $\mathbf{X}_{(i+1)}^1$ cannot have 1 in this entry. Note that Rule B.2 terminates when either (i) there is no facility with negative profit or (ii) closing any facility with negative profit does not increase the profit of the corresponding firm. The output of Rule B.2 is either $\mathit{continue} = 0$ or $\mathbf{X}^2$,which is a viable location matrix and satisfies (i) or (ii). When $\mathit{continue} = 0$, this means that if we continue, we will end up with a matrix $\mathbf{X}^2$ that has already been analyzed and, hence, we should start applying Rule B.2 to another location matrix $\mathbf{X}$. We have the following relations:

$$T^* \subseteq T^2 \subseteq T^1 \subseteq T^0,$$

where T^2 denotes the set of location matrices generated using Rule B.2. Now suppose that we have applied Rule B.2 and generated $\mathbf{X}^2$. We still do not know whether $\mathbf{X}^2$ is an equilibrium location matrix. The next step is to determine whether $\mathbf{X}^2$ contains the best responses for each firm. For this purpose, we perform a full neighborhood search for each firm as explained in Rule B.3.

Rule B.3: The following procedure determines whether a location matrix is in equilibrium:

Step 0. Let $\mathbf{X}$ be a given location matrix with columns $\mathbf{x}_r$. Set $r = 1$ and go to Step 1.

Step 1. If $r > k$, stop; $\mathbf{X}$ is an equilibrium location matrix, and return $\mathbf{X}^* = \mathbf{X}$ and $\mathit{equilibrium} = 1$. Otherwise, enumerate all possible location decisions of firm r, i.e., generate all possible $\mathbf{x}_r$ vectors. Determine the best response of firm r when the other firms' location decisions are fixed at $\mathbf{x}_i$ $i \neq r$, by comparing the total profit of firm r for each matrix $\overline{\mathbf{X}}$ (which differs from $\mathbf{X}$ only in the r^{th} column). Let $x_r^*(\mathbf{X})$ denote the best response of firm r and go to Step 2.

Step 2. If $x_r^*(\mathbf{X}) = \mathbf{x}_r$, set $r = r+1$ and go to Step 1. Else, stop; $\mathbf{X}$ is not an equilibrium location matrix and return $equilibrium = 0$.

The purpose of Rules B.2 and B.3 is to determine if a given viable location matrix is not in equilibrium as quickly as possible. If Rule B.2 cannot guarantee that the viable location matrix is not an equilibrium location matrix, then Rule B.3 completes the check by considering all other options for each firm. Hence, at the conclusion of Rule B.3, we will either have an improved location decision for a firm, which implies that the location matrix is not in equilibrium, or conclude that the location matrix is in equilibrium. Next, we define our heuristic method to search for an equilibrium location matrix.

B.4.3. Heuristic Algorithm for Finding an Equilibrium Location Decision

In this section we provide a heuristic algorithm to search for an equilibrium location matrix. The algorithm starts with a random location matrix and first moves to a viable location matrix. Then, the algorithm checks whether the equilibrium conditions are satisfied by this viable matrix. During the move from a random location matrix to a viable location matrix, Rule B.1 is utilized. Rules B.2 and B.3 are used to check for equilibrium conditions. Rules B.2 and B.3 are mainly aimed at simplifying the process of checking equilibrium conditions by easily showing whether the equilibrium conditions are not satisfied, when the current viable matrix is not an equilibrium location matrix. However, a full search is needed to determine an equilibrium location matrix. It should be emphasized that the algorithm does not perform a full search for each non-equilibrium location matrix, which eases the computational burden, as a complete search is burdensome. In particular, a total enumeration scheme to find all of the equilibrium location decisions, or to find out that there does not exist any equilibrium location decision, requires checking the equilibrium conditions for $2^{m \times k}$ locations. Moreover, checking the equilibrium conditions for any given location decision requires analyzing $k(2^m - 1)$ other options. Then it follows that a total enumeration scheme would require solving for equilibrium quantities $k2^{m \times k}(2^m - 1)$ times, which is exponential in both m and k. Hence, we next propose a heuristic method that utilizes the rules defined in the previous sections.

Algorithm B.2. The following algorithm is a heuristic method to find an equilibrium location matrix, if one exists.

Step 0. Let $L = \varnothing$. Go to Step 1.

Step 1. If $|L| = 2^{m \times k}$, stop; there does not exist an equilibrium location matrix. Else, generate a random location matrix, $\mathbf{X}$, such that $\mathbf{X} \notin L$. Go to Step 2.

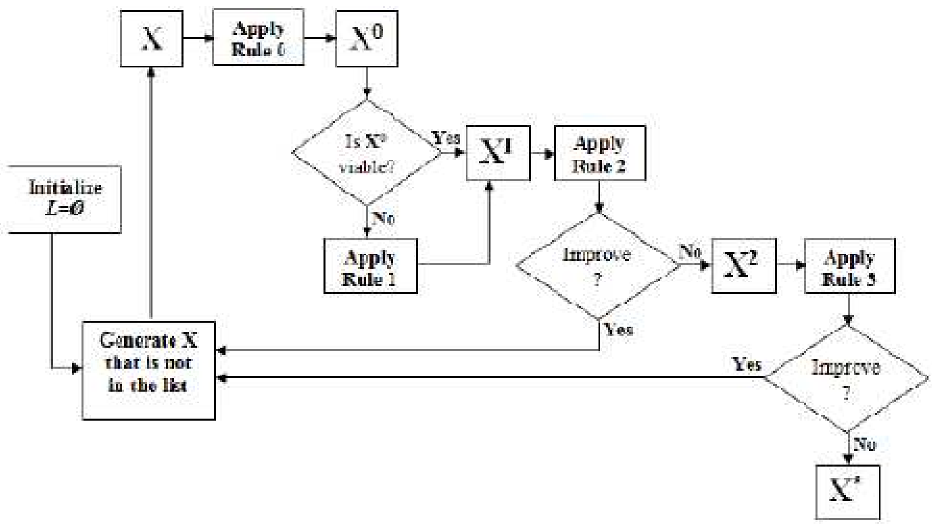
Step 2. Apply Rule B.2 with $\mathbf{X}$ and L. If $continue = 0$, go to Step 1. Else, generate $\mathbf{X}^2$

and go to Step 3.

Step 3. Apply Rule B.3 to $\mathbf{X}^2$. If $equilibrium = 0$, go to Step 1. Else, equilibrium is found, stop and return $\mathbf{X}^*$.

Figure B.1 illustrates how the heuristic method proceeds. During the algorithm, we define the list L to keep track of the location matrices that have been processed. Note that if the algorithm does not stop at Step 3, then we know that the locations in L are not equilibrium location decisions, and we cannot generate an equilibrium location decision from these location matrices using Rules B.0, B.1, B.2 and B.3. Hence, we generate a new location matrix that is not in L. Moreover, since there are $2^{m \times k}$ possible location decisions, we conclude that there does not exist an equilibrium location decision when $|L| = 2^{m \times k}$.

Figure B.1. Illustration of Algorithm B.2



The efficiency of the heuristic method follows from the fact that it reduces the number of full equilibrium checks. In Algorithm B.2, we do not perform a full equilibrium check for any non-viable location matrix, and even for some of the viable location matrices (those that are non-equilibrium). In the next section, we compare Algorithm B.2 with a random search algorithm and present numerical results that demonstrate that Algorithm B.2 is more efficient in finding an equilibrium location matrix.

B.5. Numerical Studies

In this section, we compare the heuristic method proposed in the previous section with a random search method. We note that to solve for the equilibrium flow quantities for a given location matrix, we use Algorithm B.1 with the parameter settings provided in [11] (the efficiency of Algorithm B.1 is discussed in [11]). Our aim in this section is to demonstrate the efficiency and benefits of Algorithm B.2. However, because our model is new to the literature, no benchmark algorithm exists for comparison. Thus, we demonstrate the potential benefits of our proposed heuristic algorithm when compared to a naïve or brute force random search algorithm that might be applied in practice in the absence of an alternative approach.

As a result, we compare Algorithm B.2 with the following random search algorithm.

Algorithm B.3. The following algorithm is a random search method to find an equilibrium location matrix, if one exists.

Step 0. Let $L = \varnothing$. Go to Step 1.

Step 1. If $|L| = 2^{m \times k}$, stop; there does not exist an equilibrium location matrix. Else, generate a random location matrix, $\mathbf{X}$, such that $\mathbf{X} \notin L$. Go to Step 2.

Step 2. Apply Rule B.3 to $\mathbf{X}$. If $equilibrium = 0$, go to Step 1. Else, equilibrium is found, stop and return $\mathbf{X}^*$.

Note that the only difference between Algorithm B.3 and Algorithm B.2 is that Rule B.2 is not used in Algorithm B.3. That also means that Rule B.1 (embedded in Rule B.2) and, hence, Rule B.0 (embedded in Rule B.1) are not used in Algorithm B.3 as well. Algorithm B.3 applies a full equilibrium check to the given random location matrix and repeats Steps 1 and 2 until either an equilibrium location matrix is found or all of the location matrices are determined to be non-equilibrium. It is worth pointing out that the list of matrices in Algorithm B.3 increases by 1 at each occurrence of Step 1. On the other hand, the list of matrices in Algorithm B.2 may increase by more than 1 in each occurrence of Step 1. Furthermore, Algorithm B.3 applies Rule B.3 to each element of the list, whereas Algorithm B.2 applies Rule B.3 only to the location matrices generated by Rule B.2.

For comparison purposes, we use the same sequence of random location matrices within Algorithm B.3 and Algorithm B.2 for each problem instance. We solve 10 randomly generated problem instances for each of the 12 different combinations of $k = \{2,3\}$, $m = \{3,4,5\}$ and $n = \{2,3\}$. We repeat this process for 3 classes of problems, and note that the parameters for each problem instance are uniformly distributed. For each class of problems, $a_j \sim U[50,100]$ and $b_j \sim U[1,2]$, where $U[l,u]$ denotes the uniform distribution on $[l,u]$. Table B.1 gives the distribution ranges for c_{ijr}, α_{ijr} and f_{ir} values for each problem class.

Table B.1. Data Categories for Problem Classes 1, 2 and 3

	Class 1	Class 2	Class 3
c_{ijr}	(0,25]	[25,75]	[75,125]
α_{ijr}	(0,0.5]	[0.5,1]	[1,1.5]
f_{ir}	(0,50]	[50,150]	[150,200]

An equilibrium location decision is determined for every problem instance. Each row in Table B.2 summarizes the average of 30 problem instances (10 from each problem class), for each combination of k, m and n (resulting in 360 total instances) for the following data: length of the list at termination (list length), number of full equilibrium checks (# of checks) and CPU time in seconds.

Table B.2. Comparison of Heuristic Method with Random Search Method

			Algorithm 2			Algorithm 3		
k	m	n	list length	# of checks	CPU time	list length	# of checks	CPU time
2	3	2	15.83	13.17	0.64	34.90	34.90	1.15
3	3	2	24.00	17.07	1.17	247.10	247.10	10.41
2	4	2	39.47	30.67	2.91	117.10	117.10	9.26
3	4	2	124.50	102.67	16.02	2093.80	2093.80	216.43
2	5	2	48.60	31.80	6.68	422.03	422.03	68.57
3	5	2	256.70	173.30	51.52	16078.43	16078.43	4093.96
2	3	3	19.03	15.70	1.08	28.83	28.83	1.42
3	3	3	55.77	40.00	4.13	228.37	228.37	13.71
2	4	3	56.53	48.43	7.52	131.17	131.17	15.12
3	4	3	415.77	347.40	75.69	2169.17	2169.17	325.07
2	5	3	138.37	106.30	37.58	488.43	488.43	131.50
3	5	3	718.30	473.67	202.56	15314.13	15314.13	5665.67
average			159.41	116.68	33.96	3112.79	3112.79	879.36

As can be seen from Table B.2, the heuristic method is much faster than the random search method. This is due to the following two points: (i) the heuristic method does not perform a full equilibrium check (which is computationally burdensome) for each element within the list and, (ii) it moves to a viable location matrix from the given random location matrix and may determine that the viable matrix is not an equilibrium location matrix without a full equilibrium check. We can see by comparison of the list length at termination that Algorithm B.2 analyzes fewer matrices and performs full equilibrium checks for fewer than 75 percent of these matrices when compared to Algorithm B.3, which performs full equilibrium checks for all of the location matrices it analyzes. However, these values differ by problem class. Table B.3 compares the average values of 120 problem instances within each problem class.

Table B.3. Comparison of Heuristic Method with Random Search Method for Each Problem Class

	Algorithm 2			Algorithm 3		
	list length	# of checks	CPU time	list length	# of checks	CPU time
Class 1	289.87	261.53	77.73	3017.80	3017.80	1172.99
Class 2	174.13	85.05	23.49	3103.92	3103.92	775.12
Class 3	14.22	3.47	0.65	3216.65	3216.65	689.96
average	159.41	116.68	33.96	3112.79	3112.79	879.36

It can be noted from Table B.3 that Algorithm B.2 performs full equilibrium checks approximately for 90 percent, less than 50 percent and less than 25 percent of the location matrices in the list, for problem classes 1, 2 and 3, respectively. This difference is due to the fact that for Class 1 (compared to Classes 2 and 3) and for Class 2 (compared to Class 3) problems, the cost parameters are relatively low and, hence, this reduces the likelihood that a solution associated with a given location matrix will contain facilities with negative profits. However, Algorithm B.2 outperforms Algorithm B.3 in average computational time, as well as in average list size and the average number of full equilibrium checks, for each problem class.

B.6. Conclusion and Future Research

In this study, we formulated a general competitive location game with non-linear costs for multiple firms in a multiple-market setting under Cournot competition. Each firm incurs firm-specific linear transportation costs, convex traffic congestion costs and fixed facility location costs. That is, we study a heterogeneous cost structure. Unlike the studies in [34] and [36], we allow each firm to locate more than one facility, which increases the problem's complexity. A two-stage solution approach is used to find the equilibrium supply quantities and facility locations. First, we solve for the equilibrium supply quantities for given facility locations via formulating the Stage-two Game as a variational inequality problem. In particular, the resulting formulation is an asymmetric linear variational inequality problem defined over the nonnegative orthant. We noted that projection methods can be used as a solution method and we discussed a self-adaptive projection method proposed in [11], which has been shown to be an efficient solution tool for asymmetric linear variational inequality problems. Second, we focused on determining the equilibrium facility locations, i.e., the Stage-one Game. Rules B.0, B.1, and B.2 were defined to ease the computational burden of the search for an equilibrium location decision. Utilizing these rules along with Rule B.3, which performs a full equilibrium check, we propose a heuristic search method that finds an equilibrium location decision, if one exists. Numerical studies implied that the proposed heuristic method is quite efficient when compared to a random search method.

We study the problem at hand assuming that the firms are non-cooperative and they take simultaneous actions. On the other hand, studying competitive location games with non-linear cost terms with cooperation allowed among the firms or under sequential actions remain as future research directions. One other future research direction would include studying competitive facility location games in multi-echelon supply chains (an equilibrium problem for a two-echelon supply chain is studied in [27]; however, this study assumes that facility locations are predetermined). We noted that understanding equilibrium location decisions of competing firms is important for external suppliers as well as government agencies and land-use planners. Therefore, our study establishes a basis for analysis of strategic decisions of these parties as well.

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7. Appendix

7.1. Proof of Property B.1

When $\left(a_j - b_j \sum_{i \in I_X} \sum_{r \in R_X} q_{ijr}\right) \geq 0$, S is compact and convex. Moreover, $F(\mathbf{Q}_j)$ is continuous on S. Then it follows from [26] that the variational inequality problem admits at least one solution.

7.2. Proof of Property B.2

We know that $F(\mathbf{Q}_j) = (-\nabla_{\mathbf{Q}_j^1} \Pi_1^j(\mathbf{Q}_j \mid \mathbf{X} = \mathbf{X}^0), ..., -\nabla_{\mathbf{Q}_j^k} \Pi_k^j(\mathbf{Q}_j \mid \mathbf{X} = \mathbf{X}^0))$ where $\nabla_{\mathbf{Q}_j^r} \Pi_r^j(\mathbf{Q}_j \mid \mathbf{X} = \mathbf{X}^0) = (\partial \Pi_r^j(\mathbf{Q}_j \mid \mathbf{X} = \mathbf{X}^0) / \partial q_{1jr}, ..., \partial \Pi_r^j(\mathbf{Q}_j \mid \mathbf{X} = \mathbf{X}^0) / \partial q_{|I_r^0|jr}) \ \forall r \in R.$ Moreover, $\nabla_{\mathbf{Q}_j^r} \Pi_r^j(\mathbf{Q}_j \mid \mathbf{X} = \mathbf{X}^0) =$

$([c_{1jr} - a_j + b_j(q_{\bullet j \bullet} + q_{\bullet jr}) + \alpha_{1jr}(q_{1jr} + q_{1j\bullet})], ..., [c_{|I_r^0|jr} - a_j + b_j(q_{\bullet j \bullet} + q_{\bullet jr}) + \alpha_{|I_r^0|jr}(q_{|I_r^0|jr} + q_{|I_r^0|j\bullet})]).$

Then the Jacobian matrix of $F(\mathbf{Q}_j)$, $\nabla F(\mathbf{Q}_j)$, consists of the following values: $2b_j + \alpha_{ijr}, 2b_j, b_j + \alpha_{ijr}$ or b_j. Noting that $b_j > 0$ and $\alpha_{ijr} > 0$, it follows that each component of the Jacobian matrix is positive. Thus, for any $\mathbf{Q}_j \neq 0$, we have $\mathbf{Q}_j^T \nabla F(\mathbf{Q}_j) \mathbf{Q}_j > 0$. The result then follows from the mid-value theorem as noted in [26].

7.3. Proof of Property B.3

We know from Property B.1 that there exists at least one solution when $(a_j - b_j \sum_{r \in R} \sum_{i \in I_r^0} q_{ijr}) \geq 0 \quad \forall \mathbf{Q}_j \in S$. Moreover, Property B.2 states that $F(\mathbf{Q}_j)$ is strictly monotone. Then it follows from [26] that the solution of the variational inequality problem is unique.

7.4. Proof of Property B.4

It follows from [18] that a necessary and sufficient condition for existence of a solution to the variational inequality problem, $\langle F(\mathbf{Q}_j^*), \mathbf{Q}_j - \mathbf{Q}_j^* \rangle \geq 0, \forall \mathbf{Q}_j \in S$, where S is closed and convex and F is continuous, is that there exists an $R > 0$ such that a solution $q_R \in S_R$ to $\langle F(q_R), q - q_R \rangle \geq 0, \forall q \in S_R$ satisfies $|q_R| < R$. When we assume that $(a_j - b_j \sum_{i \in I_{\mathbf{X}}} \sum_{r \in R_{\mathbf{X}}} q_{ijr}) \geq 0,$ we know from Property B.1 that a solution exists. Hence, it follows that a solution exists when S is closed and convex.

7.5. Proof of Theorem B.2

It follows from the proof of Property B.2 that M is positive definite for the variational inequality formulation in Equation (B.10). Moreover, Property B.4 states that there exists a solution for Equation (B.10) when S is the nonnegative orthant. Then, it follows from [11] that the algorithm converges to a solution of Equation (B.10).

7.6. Proof of Proposition B.1

Let $\mathbf{X}^*$ be an equilibrium location decision. Suppose that $x_{ir}^* = 1$ when $f_{ir} > 0$ and $q_{i\bullet r}^*(\mathbf{X}^*) = 0$ for some $i \in I$ and $r \in R$. Now letting $x_{ir}^* = 0$ will not change the equilibrium quantities and, hence, when firm r closes her/his facility at location i, she/he improves her/his profit by f_{ir}. This implies that $\mathbf{X}^*$ cannot be an equilibrium matrix.

7.7. Proof of Proposition B.2

Suppose that $\Pi_r(\mathbf{Q}^*(\mathbf{X}^0), \mathbf{X}^0) < 0$ for some firm r, $r \in R$. Then, this implies that $\mathbf{x}_r^0$ has at least one nonzero component. Letting all nonzero components in $\mathbf{x}_r^0$ be 0, we have $\Pi_r = 0$, which implies that firm r is better off by not locating any facility. Hence, it follows from Equation (B.11) that $\mathbf{x}_r^0$ cannot be in an equilibrium location matrix, i.e., $\mathbf{X}^0$ is not an equilibrium location decision.