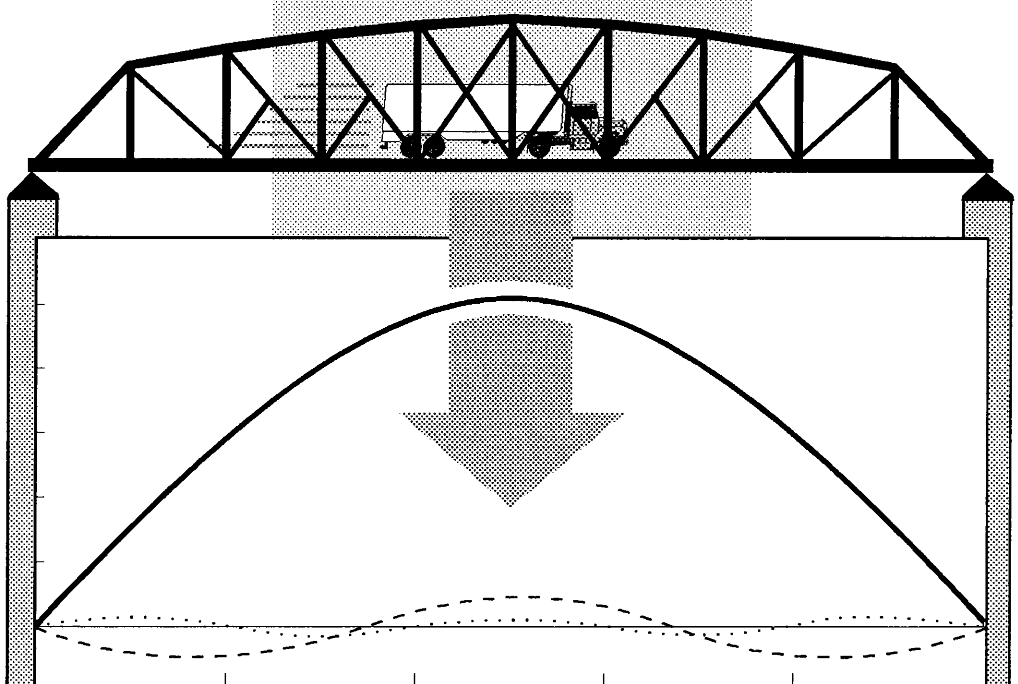
INTERIM REPORT

COMPARISON OF MODAL SUPERPOSITION METHODS FOR THE ANALYTICAL SOLUTION TO MOVING LOAD PROBLEMS



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Technical Report Documentation Page

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
FHWA/VA-95-R6		

4. Title and Subtitle

Comparison of Modal Superposition Methods for the Analytical Solution to Moving Load Problems

5. Report Date August 1994

6. Performing Organization Code

7. Author(s)

Baber, T. T., Ph.D., Massarelli, P.J.

8. Performing Organization Report No.

VTRC 95-R6

9. Performing Organization Name and Address

Virginia Transportation Research Council
530 Edgemont Road
Charlottesville, Virginia 22903

10. Work Unit No. (TRAIS)

11. Contract or Grant No.
3043

12. Sponsoring Agency Name and Address

Virginia Department of Transportation
1401 E. Broad Street
Richmond, Virginia 23219

13. Type of Report and Period Covered

Interim Report

14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

The response of bridge structures to moving loads is investigated using modal superposition methods. Two distinct modal superposition methods are available: the mode-displacement method and the mode-acceleration method. While the mode-displacement method is widely used in analysis of both continuous and discrete systems, the mode-acceleration method is typically given for discrete systems only. This report introduces general operator notation as a way to extend the mode acceleration method to the analysis of arbitrary continuous or discrete structural models and shows that the formation is available to arbitrary self-adjoint systems.

The problem of a damped and undamped beam subjected to a concentrated moving load is considered. The displacement, shear force, and bending moment solutions are formulated to accommodate any time-dependent moving force, including randomly varying vehicular loads. While both mode-displacement and mode-acceleration methods provide reasonable displacement responses, the mode-acceleration method provides superior moment and shear estimates in the vicinity of the moving load at a given level of approximation because of its ability to directly incorporate the pseudo-static response of the system into the solution. This insight, which is readily applicable to the study of transportation structures, suggests that mode-acceleration solution techniques can significantly reduce computational time and labor when modeling highway bridge behavior.

17. Key Words	18. Distribution Statement
Vibrations, Modal Analysis, Dynamic Loading, Bridge Modelling	No restrictions. This document is available to the public through NTIS, Springfield, VA 22161.

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	41	

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(The opinions, findings, and conclusions expressed in this report are those of the authors and not necessarily those of the sponsoring agencies.)

Virginia Transportation Research Council
(A Cooperative Organization Sponsored Jointly by the
Virginia Department of Transportation and
the University of Virginia)

In Cooperation with the U.S. Department of Transportation
Federal Highway Administration

Charlottesville, Virginia

July 1994
VTRC 95-R6

BRIDGE RESEARCH ADVISORY COMMITTEE

ABSTRACT

The response of bridge structures to moving loads is investigated using modal superposition methods. Two distinct modal superposition methods are available: the mode-displacement method and the mode-acceleration method. While the mode-displacement method is widely used in the analysis of both continuous and discrete systems, the mode-acceleration method is typically given for discrete systems only. This report introduces general operator notation as a way to extend the mode acceleration method to the analysis of arbitrary continuous or discrete structural models and shows that the formulation is available for arbitrary self-adjoint systems.

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INTRODUCTION

The dynamic response of bridge structures to transient vehicular loadings is a particularly important aspect of bridge design and evaluation. Although computational modelling of this response can bring valuable insight to the interpretation of field data, the development of analytical models for actual structures is quite complex. Analytical models of simpler systems can provide an understanding of the structural behavior characteristics caused by dynamically applied loads, and this understanding can then be applied to more complex systems. Moreover, studies of simple systems can sometimes provide insight into the suitability of particular methods for a certain class of problems.

One such simple system, fundamental to the study of modern transportation, is that of a load moving over a beam. This "moving load problem" offers an adequate analogy in many circumstances to a vehicle crossing a bridge. Moving loads are characterized as being variable in both time and space, and they impose this same characteristic upon the displacements and stresses which describe the behavior of the structure over which they travel. These loads induce vibrations which can substantially increase the dynamic stresses in the underlying structure, and this vibrational effect is largely dependent upon the velocity and magnitude of the moving force. If the force is moving slowly, the dynamic effects can be neglected and a simple static analysis performed. Up until the early nineteenth century the static load approach was reasonably applicable to virtually all transportation problems.

Historically, much of the research on the moving load problem has been geared toward rail transportation. With the advent of the locomotive and the building of the early railway bridges, a debate arose which provided the incentive for the initial study of the effects of moving loads. Some engineers at the time argued that these early trains traveled so quickly that a bridge didn't have time to deform, while others thought that the high train velocities caused an impact response in a bridge when crossing it.

Initial studies of the moving load problem assumed that the mass of the beam was small compared to the mass of the load. R. Willis is credited with finding the original approximate solution to this problem in 1851.[1] This problem is especially well-suited to rail transportation, because trains are relatively heavy compared to railroad bridges. Consequently, massless beams or beams whose mass is comparative to that of the load are of interest. Also, trains are generally longer than the bridges they cross, and so a moving uniform load, rather than a concentrated one, would perhaps be more appropriate to railway studies.

Fifty years later, the opposite problem was addressed by A. N. Krylov, S. P. Timoshenko and others who examined analytical solutions to the problem of a constant concentrated force crossing a simply-supported beam.[1] Although intended for other uses, this problem often serves as a good model for an automobile crossing a highway bridge, because highway bridges are generally longer and heavier than the vehicles which travel over them. This problem is the focus of this paper.

Since its conception, many variations of the moving load problem have been examined. A few notable studies include the analysis of vehicle/beam interaction caused by a series of moving loads, rather than a single load, crossing the beam, and the inclusion of corrections for shear deformation and rotary inertia effects in the beam response. Most recently, the moving load problem has been the subject of extensive finite element analyses. Y.-H. Lin and M. W. Trethewey have examined elastic beam response due to the movement of a spring-mass-damper system,[2] and G. N. Geannakakes and P.-C. Wang applied the increasingly popular B_3-spline method to studying the problem of a load moving across a plate.[3]

The analytical solution of the fundamental moving load problem is represented by an infinite series. Consequently, modal superposition techniques are most useful in examining this problem. Traditionally, the solution to this problem is formulated to yield the equivalent solution of the mode-displacement method. This mode-displacement solution converges rapidly when computing the deflection of an undamped beam caused by a moving constant force. However, often the stresses in the beam are of primary importance, and the mode-displacement solutions for the bending moment and shear force in the beam do not readily converge. Consequently, the number of solution terms needed for an adequate representation of beam stresses can be computationally expensive.

A more rapidly convergent solution is provided by the mode-acceleration method in which the total solution is broken down into two parts and represented by a pseudo-static response and a separate dynamic contribution. The pseudo-static response is not a series solution and thus convergence considerations for this component of the solution are unnecessary.

The superior convergence of the mode-acceleration method in finding the bending moment of an undamped beam subjected to a constant moving load was studied by G. B. Warburton.[4] However, Warburton's analysis does not consider beam damping or the application of the mode-acceleration method to calculating the shear force in the beam. It is through this application that the superior convergence of this method is best illustrated. Another solution was presented by L. Frýba who combined the general solution of the governing homogenous integro-differential equation with the particular solution of the non-homogenous integro-differential equation to arrive at the same result as Warburton.[1]

The mode-acceleration method is not often used to study continuous systems. However, it has been shown to provide rapid solution convergence and computational economy in the study of discrete vibrating systems as evidenced by Cornwell, Craig, and Johnson.[5] Léger and Wilson have also demonstrated the improved convergence achieved when pseudo-static corrections have been applied to finite element computations.[6]

PURPOSE AND SCOPE

This paper examines the convergence properties of two modal superposition methods: mode-displacement and mode-acceleration. Generalized solution formulations for both discrete and distributed dynamic systems based on each of the methods are derived. These solutions are then applied to the generalized problem of a time-dependent force moving across a uniform, simply supported beam.

The beam is assumed to be of the Bernoulli-Euler type and its behavior described by small deformation theory. It has a constant cross section and mass per unit length. The force is representative of the gravitational effect of a relatively light vehicle crossing a relatively heavy beam (i.e., the weight of the beam is considered to be much greater than the weight of the vehicle).

Three specific moving force problems are considered in this study. For the first problem, the beam is undamped and the force constant. The second problem considers the effect of beam damping on the response due to a moving constant force. The third problem is that of a moving harmonic force on an undamped beam.

The study of these problems provides insight into the modelling of bridge structures beneath moving vehicles. To estimate bridge system parameters from measured input and output values, it is first necessary to develop parametric models. These models, which are employed in system identification, provide *a priori* information about the system which facilitates parameter estimation.[7] Modal methods can be applied to develop these computational models. By examining the moving load problems, assumptions can be made as to which modal summation method will be most useful in the development of these models.

METHODOLOGY

In this report, the relative merits of the mode-displacement and the mode-acceleration methods for moving load problems are investigated. Customarily, the mode-acceleration method is developed in the context of discrete variable systems. However, as will be shown, the method is applicable to arbitrary self-adjoint systems, whether formulated in discrete, differential equation or integral equation form. Extension to certain classes of non-self-adjoint systems is also possible, but is not discussed here.

The following analysis generalizes upon that of Meirovitch (1967)[8] and Craig (1981).[9] Meirovitch presents the general response formulation of undamped continuous systems, and Craig generalizes the response of damped and undamped discrete systems. The following discussion permits a unified treatment of mode-displacement and mode-acceleration methods for discrete and distributed dynamic systems.

Operator Equations for Dynamic Systems

The general problem of forced vibration of a linear system may be written in the form

$$\frac{\partial^2}{\partial t^2}\boldsymbol{M}[v(x,t)] + \frac{\partial}{\partial t}C[v(x,t)] + \boldsymbol{K}[v(x,t] = f(x,t) \tag{1}$$

where $\boldsymbol{M}[\cdot]$, $\boldsymbol{C}[\cdot]$ and $\boldsymbol{K}[\cdot]$ are spatial operator matrices and $f(x,t)$ represents the load matrix. For example, the equations for a discrete system take the form

$$\boldsymbol{M}\ddot{v} + \boldsymbol{C}\dot{v} + \boldsymbol{K}v = \frac{d^2}{dt^2}\tilde{M}\tilde{v} + \frac{d}{dt}\tilde{C}\tilde{v} + \tilde{K}\tilde{v} = F(t) \tag{2}$$

where M, C, and K are constant matrices, while for a vibrating beam the scalar partial differential equation is

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 v}{\partial x^2}\right] + \rho A\frac{\partial^2 v}{\partial t^2} = f(x,t) \tag{3}$$

In this equation the operator expression

$$K(\cdot) = \frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2(\cdot)}{\partial x^2}\right]$$

clearly represents the system stiffness, while the expression

$$M(\cdot) = \rho A(\cdot)$$

contains the system mass. In this system, no damping has been included although a generalized damping of the form

$$\frac{\partial}{\partial t}\left\{\frac{\partial^2}{\partial x^2}\left[CI\frac{\partial^2(\cdot)}{\partial x^2}\right] + cA(\cdot)\right\}$$

could be added to incorporate both strain rate and velocity related damping terms. In general, a typical term of the $\boldsymbol{K}[\cdot]$ operator for two independent spatial variables, x and y, takes the form

$$\begin{aligned} k_{ij}(\cdot) = {} & k_{ij}^{(0)}(\cdot) + k_{ij}^{(1)}\frac{\partial}{\partial x}(\cdot) + k_{ij}^{(2)}\frac{\partial}{\partial y}(\cdot) + k_{ij}^{(3)}\frac{\partial^2}{\partial x^2}(\cdot) + k_{ij}^{(4)}\frac{\partial^2}{\partial x \partial y}(\cdot) \\ & + k_{ij}^{(5)}\frac{\partial^2}{\partial y^2}(\cdot) + ---+ k_{ij}^{(m)}\frac{\partial^{2p}}{\partial x^{2p}}(\cdot) + k_{ij}^{(m+1)}\frac{\partial^{2p}}{\partial x^{2p-1}\partial y}(\cdot) \\ & + --- + k_{ij}^{(n-1)}\frac{\partial^{2p}}{\partial x \partial y^{2p-1}}(\cdot) + k_{ij}^{(n)}\frac{\partial^{2p}}{\partial y^{2p}}(\cdot) \end{aligned}$$

while a term of $\boldsymbol{M}[\cdot]$ has the general form

$$\begin{aligned} m_{ij} = {} & \int_D m_{ij}^{(-1)}(\boldsymbol{x},\boldsymbol{\xi})(\cdot)d + m_{ij}^{(0)}(\cdot) + m_{ij}^{(1)}\frac{\partial}{\partial x}(\cdot) + m_{ij}^{(2)}\frac{\partial}{\partial y}(\cdot) \\ & + --- + m_{ij}^{(r)}\frac{\partial^{2q}}{\partial x^{2q}}(\cdot) + --- + m_{ij}^{(s)}\frac{\partial^{2q}}{\partial y^{2q}}(\cdot) \end{aligned}$$

and $q < p$. This formulation includes as special cases the discrete (matrix) formulation, partial differential equation formulation and integral equation formulation of forced vibration. Moreover, it assumes a vector of response functions, $\mathbf{v}^T(\mathbf{x},t) = \{v_1(\mathbf{x},t) - - - v_\ell(\mathbf{x},t)\}$, of the spatial variables $\mathbf{x}^T = \{x \vdots y\}$. Generalization to three spatial dimensions is straightforward. While this formulation is more general than any encountered in practice, it contains all of the usual models for bars, beams, discrete systems, and finite element models as special cases, and thus permits a unified and efficient development.

Symbolically the $\boldsymbol{M}[\cdot]$, $\boldsymbol{C}[\cdot]$ and $\boldsymbol{K}[\cdot]$ operators represent the spatial distributions of mass, damping and stiffness, respectively, within the system. As formulated, equation (1) admits a variety of structural system models that may be subjected to conservative or non-conservative loadings.[10, 11] For present purposes, it may be assumed that $\boldsymbol{K}[\cdot]$ and $\boldsymbol{M}[\cdot]$ are self-adjoint operators. Thus, classical normal modes, $\phi_j(\mathbf{x})$, exist that satisfy the equation

$$\boldsymbol{K}[\boldsymbol{\phi}_j] = \omega_j^2 \boldsymbol{M}[\boldsymbol{\phi}_j] \tag{4}$$

where ω_j is the j^{th} natural frequency of the system.[7] Moreover, the $\boldsymbol{\phi}_j(\mathbf{x})$ form a basis for the solution space of the vibration problem and satisfy the orthogonality relationships

$$\begin{aligned} \langle \boldsymbol{\phi}_k, \boldsymbol{K}[\boldsymbol{\phi}_j] \rangle &= \delta_{jk}\,\omega_k^2 \\ \langle \boldsymbol{\phi}_k, \boldsymbol{M}[\boldsymbol{\phi}_j] \rangle &= \delta_{jk} \end{aligned} \tag{5}$$

where δ_{jk} is the Kronecker delta. The expressions $\langle \phi_k, \mathbf{K}[\phi_j] \rangle$ and $\langle \phi_k, \mathbf{M}[\phi_j] \rangle$ represent bilinear forms, special cases of which are well known in vibration analysis. For example, the orthogonality of the eigenvectors with regard to the mass and stiffness matrices is well known in discrete vibration analysis. For any operator matrix $\boldsymbol{G}[\cdot]$, the bilinear form takes the form

$$\langle \boldsymbol{u}, \boldsymbol{G}[\boldsymbol{w}] \rangle = \boldsymbol{u}^T \boldsymbol{G}\ \boldsymbol{w} \tag{6}$$

for discrete problems and

$$\langle \boldsymbol{u}, \boldsymbol{G}[\boldsymbol{w}] \rangle = \int_D \boldsymbol{u}^T(\boldsymbol{x}) \boldsymbol{G}[\boldsymbol{w}(\boldsymbol{x})]\, d\boldsymbol{x} \tag{7}$$

for continuous (integral or differential equation) problems. Moreover, it is assumed that $\boldsymbol{C}[\cdot]$ is a damping operator that is decoupled by the eigenvector transformation. One such operator is the generalized Rayleigh damping operator given by

$$\boldsymbol{C}[\cdot] = \gamma_k \boldsymbol{K}[\cdot] + \gamma_m \boldsymbol{M}[\cdot] \tag{8}$$

From equation (5) the orthogonality relationship

$$\langle \boldsymbol{\phi}_k, \boldsymbol{C}[\boldsymbol{\phi}_j] \rangle = 2\delta_{jk}\zeta_k\omega_k \tag{9}$$

is obtained for the operator of equation (8) where

$$2\zeta_k\omega_k = \gamma_k\omega_k^2 + \gamma_m \tag{10}$$

from equation (5) and (8). Clearly, from equation (8) and (10), γ_k and γ_m are coefficients which must establish the correct dimensionality for the damping operator matrix. The suggested damping for the beam vibration equations given earlier is one example of such a damping operator.

The form of damping admitted by equation (8) requires that ζ_k increases linearly with ω_k in direct proportion to the $\boldsymbol{K}[\cdot]$ operator contribution and varies with $1/\omega_n$ in proportion to the $\boldsymbol{M}[\cdot]$ operator contribution. Damping operators of the form of equation (8) represent a sufficient, but not a necessary, condition for decoupling of the $\boldsymbol{C}[\cdot]$ operator. Somewhat more general $\boldsymbol{C}[\cdot]$ operators that permit ζ_k to vary independently from mode to mode can also be used. However, systematic methods for obtaining explicit forms for these operators appear to be difficult to implement. The usual approach in practice is to perform modal decomposition on the undamped system, and to simply to specify the values of ζ_k in an *ad hoc* fashion, and this approach is followed in the remainder of this report.

Solution of General Forced Vibration Problem by Mode-Displacement Method

A set of eigenfunctions, $\{\phi_j(\mathbf{x})\}^N_{j=1}$, that satisfy equation (4) and possess the orthogonality properties in equation (5) and (9) provide a basis for the N dimensional solution space of the operator equation (1). N is finite if the formulation is discrete and it is infinite if the formulation is for a continuous system. Thus, a solution of equation (1) can be represented as a series of eigenfunctions as follows

$$v(\boldsymbol{x},t) = \sum_{j=1}^{N} \alpha_j(t)\boldsymbol{\phi}_j(\boldsymbol{x}) \tag{11}$$

Substituting into equation (1), and using the linearity of the operators $\boldsymbol{M}[\cdot]$, $\boldsymbol{C}[\cdot]$ and $\boldsymbol{K}[\cdot]$ leads to

$$\sum_{j=1}^{N}\left[\ddot{\alpha}_j\boldsymbol{M}(\boldsymbol{\phi}_j) + \dot{\alpha}_j\boldsymbol{C}(\phi_j) + \alpha_j\boldsymbol{K}(\boldsymbol{\phi}_j)\right] = \boldsymbol{f}(\boldsymbol{x},t) \tag{12}$$

Taking the inner product of equation (12) with $\phi_k(\mathbf{x})$ and using equations (5) and (9) yields

$$\ddot{\alpha}_k + 2\zeta_k\omega_k\dot{\alpha}_k + \omega_k^2\alpha_k = f_k(t) \qquad k = 1,\ldots,N \tag{13}$$

Thus, permitting the solution to be obtained as the sum of N single degree of freedom responses.

Solution of General Forced Vibration Problem by Mode-Acceleration Method

An alternative solution for the operator equation (1) often displays superior convergence to the mode-displacement solution. Rewrite equation (1) as

$$\boldsymbol{K}[\boldsymbol{v}] = \boldsymbol{f}(\boldsymbol{x},t) - \frac{\partial}{\partial t}\boldsymbol{C}[\boldsymbol{v}] - \frac{\partial^2}{\partial t^2}\boldsymbol{M}[\boldsymbol{v}] \tag{14}$$

and replace **v** on the right-hand side of equation (14) with the mode-displacement solution of equation (11) to obtain

$$\boldsymbol{K}[\boldsymbol{v}] = \boldsymbol{f}(\boldsymbol{x},t) - \sum_{j=1}^{N} \dot{\alpha}_j \boldsymbol{C}[\boldsymbol{\phi}_j] - \sum_{j=1}^{N} \ddot{\alpha}_j \boldsymbol{M}[\boldsymbol{\phi}_j] \tag{15}$$

Now, separate the solution $\boldsymbol{v}(\boldsymbol{x},\mathrm{t})$ on the left-hand side into two parts such that

$$\boldsymbol{v}(\boldsymbol{x},t) = \boldsymbol{v}_{ps}(\boldsymbol{x},t) + \boldsymbol{v}_D(\boldsymbol{x},t) \tag{16}$$

where $\boldsymbol{v}_{\mathrm{D}}(\boldsymbol{x},\mathrm{t})$ is the dynamic component of the solution and $\boldsymbol{v}_{\mathrm{ps}}(\boldsymbol{x},\mathrm{t})$ is the pseudo-static solution of the operator equation which satisfies

$$\boldsymbol{K}[\boldsymbol{v}_{ps}(\boldsymbol{x},t)] = \boldsymbol{f}(\boldsymbol{x},t) \tag{17}$$

Physically $\boldsymbol{v}_{\mathrm{ps}}$ may be interpreted as the response of an undamped massless system to $\boldsymbol{f}(\boldsymbol{x},\mathrm{t})$. Since $\boldsymbol{K}[\cdot]$ is a spatial operator and is the same as that for the static response of the system under consideration, the pseudo-static portion of the solution may be formally constructed using the static influence function of the system.

Substituting equations (16) and (17) into equation (15) yields

$$\boldsymbol{K}[\boldsymbol{v}_D] = -\sum_{j=1}^{N} \dot{\alpha}_j \boldsymbol{C}[\boldsymbol{\phi}_j] - \sum_{j=1}^{N} \ddot{\alpha}_j \boldsymbol{M}[\boldsymbol{\phi}_j] \tag{18}$$

Since the eigenfunctions are a basis, $\boldsymbol{v}_{\mathrm{D}}(\boldsymbol{x},\mathrm{t})$ may be written as

$$\boldsymbol{v}_D(\boldsymbol{x},t) = \sum_{j=1}^{N} \beta_j(t)\boldsymbol{\phi}_j(\boldsymbol{x}) \tag{19}$$

Substituting equation (19) into equation (18) yields

$$\sum_{j=1}^{N} \beta_j(t)\boldsymbol{K}[\boldsymbol{\phi}_j(\boldsymbol{x})] = -\sum_{j=1}^{N} \dot{\alpha}_j \boldsymbol{C}[\boldsymbol{\phi}_j] - \sum_{j=1}^{N} \ddot{\alpha}_j \boldsymbol{M}[\boldsymbol{\phi}_j] \tag{20}$$

Then, taking the inner product of equation (20) with $\boldsymbol{\phi}_{\mathrm{k}}(\boldsymbol{x})$ and utilizing the orthogonality relationships of equations (5) and (9) yields

$$\omega_j^2 \beta_j(t) = -2\zeta_j \omega_j \dot{\alpha}_j(t) - \ddot{\alpha}_j(t) \tag{21}$$

Hence,

$$\beta_j(t) = -\frac{2\zeta_j}{\omega_j}\dot{\alpha}_j(t) - \frac{1}{\omega_j^2}\ddot{\alpha}_j(t) \tag{22}$$

Thus, the response, $v_D(x,t)$, may be written in the form

$$v_D(x,t) = -\sum_{j=1}^{N}\left(\frac{2\zeta_j}{\omega_j}\dot{\alpha}_j(t) + \frac{1}{\omega_j^2}\ddot{\alpha}_j(t)\right)\phi_j(x) \tag{23}$$

and substituting equation (23) into equation (17) leads to the total displacement response

$$v(x,t) = v_{ps}(x,t) - \sum_{j=1}^{N}\left(\frac{2\zeta_j}{\omega_j}\dot{\alpha}_j(t) + \frac{1}{\omega_j^2}\ddot{\alpha}_j(t)\right)\phi_j(x) \tag{24}$$

Noting that for typical vibrating systems $\omega_j >> 1$ as j increases and that typically $\zeta_j < 1$, it is apparent that the terms in the summation of equation (24) die out quite rapidly. Thus, the summation in equation (25) may often be truncated at some $n << N$, yielding the approximate solution

$$v(x,t) = v_{ps}(x,t) - \sum_{j=1}^{n}\left(\frac{2\zeta_j}{\omega_j}\dot{\alpha}_j(t) + \frac{1}{\omega_j^2}\ddot{\alpha}_j(t)\right)\phi_j(x) \tag{25}$$

which is the generalized mode-acceleration solution.

Statement of Generalized Moving Load Problem

To demonstrate the applicability of the mode-superposition methods to structural vibration problems, the moving load system represented in Figure 1 will be examined. The beam is oriented along the x-axis with its origin at $x = 0$ and is simply-supported at both ends. It has a length, L, a constant stiffness, EI, and a constant mass per unit length, ρA. The force, F(t), is moving at a uniform velocity, c, along the beam. Thus, at any instant, t, the position of the force is given by

$$x_f(t) = ct \tag{26}$$

Furthermore, the transverse displacement of the beam is v(x,t) and designated as positive downward.

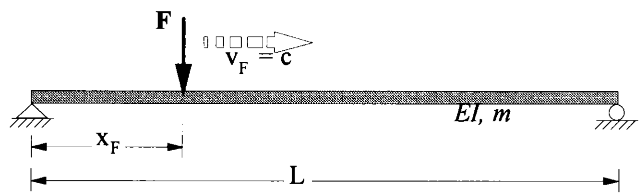


Figure 1 Force moving across a uniform, simply-supported beam at a constant velocity, c.

Hamilton's principle or another means can be utilized to obtain the governing differential equation for the moving load system so that it conforms to equation (1). For this specific case, the problem is formulated such that

$$\boldsymbol{M}[v(x,t)] = \rho A v(x,t) \tag{27}$$

and

$$\boldsymbol{K}[v(x,t)] = EI\frac{\partial^4 v(x,t)}{\partial x^4} \tag{28}$$

The $\boldsymbol{C}[v(x,t)]$ operator is not explicitly given, but it is assumed to lead to constant ζ for each mode.

The force is described by

$$f(x,t) = \begin{cases} F(t)\,\delta(x-ct) & \text{for} \quad 0 \le ct \le L \\ 0 & \text{for} \quad ct > L \end{cases} \tag{29}$$

where $F(t)$ is the time-dependent forcing function.

Solution of Generalized Moving Load Problem by Mode-Displacement Method

From equation (11), the solution is assumed to have the form

$$v(x,t) = \sum_{j=1}^{N} \alpha_j(t)\phi_j(x) \tag{30}$$

where N is infinite. The normalized eigenfunctions of a uniform beam, simply-supported at both ends are

$$\phi_j(x) = \sqrt{\frac{2}{\rho AL}}\sin\left(\frac{j\pi x}{L}\right) \qquad j = 1,2,\ldots \tag{31}$$

with the corresponding eigenvalues

$$\lambda_j = \frac{j\pi}{L} \qquad j = 1,2,\ldots \tag{32}$$

and natural frequencies

$$\omega_j = \sqrt{\frac{EI}{\rho A}}\left[\frac{j\pi}{L}\right]^2 \qquad j = 1,2,\ldots \tag{33}$$

Taking the inner product of equation (29) with the eigenfunction yields

$$f_j(t) = \sqrt{\frac{2}{\rho AL}}F(t)\sin(\acute{\omega}_j t) \tag{34}$$

where the loading frequency, $\acute{\omega}_j$, is given as follows:

$$\acute{\omega}_j = \frac{j\pi c}{L}. \tag{35}$$

The governing equation of the time-variant component of the solution is then obtained by plugging equation (34) into equation (13) to yield

$$\ddot{\alpha}_j + 2\zeta\omega_j\dot{\alpha}_j + \omega_j^2\alpha_j = \sqrt{\frac{2}{\rho AL}}F(t)\sin(\dot{\omega}_j t) \qquad j = 1,\ldots,N \tag{36}$$

Because the system is linear and assumed to be critically underdamped (i.e., $\zeta_j < 1$), the solution to equation (36) can be expressed by the Duhamel integral as follows

$$\begin{aligned}\alpha_j(t) &= \left(\frac{\sqrt{2}}{\sqrt{\rho AL}\omega_{d_j}}\right)\int_0^t F(\tau)\sin(\dot{\omega}_j\tau)e^{-\zeta\omega_j(t-\tau)}\sin\left[\omega_{d_j}(t-\tau)\right]d\tau \\ &+ \alpha_j(0)e^{-\zeta\omega_j t}\cos(\omega_{d_j}t) \\ &+ \left(\frac{1}{\omega_{d_j}}\right)\left[\dot{\alpha}_j(o) + \zeta\omega_j\alpha_j(0)\right]e^{-\zeta\omega_j t}\sin(\omega_{d_j}t)\end{aligned} \tag{37}$$

where the damped frequency is defined as

$$\omega_{d_j} = \omega_j\sqrt{1-\zeta^2} \tag{38}$$

By substituting equations (31) and (37) into equation (30) and assuming zero initial conditions, the following mode-displacement solution of the generalized moving load problem is obtained

$$v(x,t) = \frac{2}{\rho AL}\sum_{j=1}^{N}\frac{1}{\omega_{d_j}}\int_0^t F(\tau)\sin(\dot{\omega}_j\tau)e^{-\zeta\omega_j(t-\tau)}\sin\left[\omega_{d_j}(t-\tau)\right]d\tau\,\sin\left(\frac{j\pi x}{L}\right) \tag{39}$$

Solution of Generalized Moving Load Problem by Mode-Acceleration Method

The mode-acceleration solution of the moving-load problem is obtained by separating the total solution into its pseudo-static and dynamic components. The dynamic contribution can be readily obtained by substituting the first and second time derivatives of equation (37) into equation (23). By using the Leibnitz' Formula, the first time derivative of $\alpha_j(t)$ is found to be

$$\dot{\alpha}_j(t) = \sqrt{\frac{2}{\rho AL}}\int_0^t F(\tau)\sin(\dot{\omega}_j\tau)e^{-\zeta\omega_j(t-\tau)}\left\{\cos\left[\omega_{d_j}(t-\tau)\right] - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left[\omega_{d_j}(t-\tau)\right]\right\}d\tau \tag{40}$$

and taking the second-time derivative yields

$$
\begin{aligned}
\ddot{\alpha}_j(t) = {} & \frac{\sqrt{2}\omega_j(2\zeta^2-1)}{\sqrt{\rho AL(1-\zeta^2)}}\int_0^t F(\tau)\sin(\dot{\omega}_j\tau)e^{-\zeta\omega_j(t-\tau)}\sin\left[\omega_d(t-\tau)\right]d\tau \\
& - \frac{2\sqrt{2}\zeta\omega_j}{\sqrt{\rho AL}}\int_0^t F(\tau)\sin(\dot{\omega}_j\tau)e^{-\zeta\omega_j(t-\tau)}\cos\left[\omega_d(t-\tau)\right]d\tau \\
& + \sqrt{\frac{2}{\rho AL}}F(t)\sin(\dot{\omega}_j t)
\end{aligned}
\tag{41}
$$

Substituting equations (40) and (41) into equation (23) gives the following expression for the generalized dynamic component of the moving load problem

$$
v_d(x,t) = \left(\frac{2}{\rho AL}\right)\sum_{j=1}^{N}\left\{\begin{matrix} \dfrac{1}{\omega_{d_j}}\displaystyle\int_0^t F(\tau)\sin(\dot{\omega}_j\tau)e^{-\zeta\omega_j(t-\tau)}\sin\left[\omega_{d_j}(t-\tau)\right]d\tau \\ -\dfrac{1}{\omega_J^2}F(t)\sin(\dot{\omega}_j t) \end{matrix}\right\}\sin\left(\frac{j\pi x}{L}\right)
\tag{42}
$$

The governing equation of the pseudo-static contribution is derived by substituting equations (28) and (29) into equation (17) which produces

$$
EI\frac{\partial^4 v_{ps}(x,t)}{\partial x^4} = F(t)\,\delta(x-ct) \qquad for\ 0\le ct\le L
\tag{43}
$$

Equation (43) can be directly integrated to solve for $v_{ps}(x,t)$. The result is almost identical to the deflection equation of a simply-supported beam subjected to a static force at some position x_F. This well-known solution can be written as

$$
v_{static}(x) = \frac{F}{6EIL}\begin{cases} x(L-x_F)(2x_FL-x_F^2-x^2) & x \le x_F \\ x_F(L-x)(2xL-x^2-x_F^2) & x \ge x_F. \end{cases}
\tag{44}
$$

However, the pseudo-static displacement solution differs from equation (44) in that the position of the force is a function of time as indicated by equation (26). Also, the magnitude of the force does not have to be constant but may also be time-dependent. Thus, the equation for the pseudo-static component of the moving-load problem solution is

$$v_{ps}(x,t) = \frac{F(t)}{6EIL}\begin{cases} x(L-ct)[2ctL-(ct)^2-x^2] & x \le x_F \\ ct(L-x)[2xL-x^2-(ct)^2] & x \ge x_F. \end{cases} \tag{45}$$

Alternatively, the above expression can be represented by the following equivalent infinite series solution:

$$v_{ps}(x,t) = \frac{2F(t)L^3}{EI\pi^4}\sum_{j=1}^{\infty}\frac{1}{j^4}\sin\left(\frac{j\pi ct}{L}\right)\sin\left(\frac{j\pi x}{L}\right). \tag{46}$$

This pseudo-static solution formulation of equation (46) is sometimes useful in deriving the dynamic component of the total solution for particular moving load problems.

RESULTS AND DISCUSSION

This study considers three specific moving force problems. In the first, the damping of the beam is neglected and the moving force is assumed to be constant. In the second, a non-zero beam damping factor is introduced into the constant moving force problem. In the third, the undamped beam is re-examined, but it is subjected to a harmonic force instead of a constant one.

Comparison of Modal-Superposition Methods for an Undamped Beam Subjected to a Constant Moving Force

For this sub-problem, it is assumed that the time-dependent force is replaced by a constant one, $F \neq F(t)$, and that $\zeta = 0$. The mode-displacement solution is found by substituting these parameters into equation (39) and performing the integration to obtain

$$v_{MD}(x,t) = \frac{2FL^3}{EI\pi^4}\sum_{j=1}^{\infty}\frac{1}{j^2(j^2-\alpha^2)}\left[\sin\left(\frac{j\pi ct}{L}\right)-\frac{\alpha}{j}\sin\left(\frac{j^2\pi ct}{\alpha L}\right)\right]\sin\left(\frac{j\pi x}{L}\right) \qquad \textit{for } \alpha \neq j \tag{47}$$

and

$$\begin{aligned} v_{MD}(x,t) = {} & \frac{FL^3}{EI\pi^4\alpha^4}\left[\sin\left(\frac{\alpha^2\pi ct}{L}\right)-\frac{\alpha^2\pi ct}{\alpha L}\cos\left(\frac{\alpha^2\pi ct}{\alpha L}\right)\right]\sin\left(\frac{\alpha\pi x}{L}\right) \\ & + \frac{2FL^3}{EI\pi^4}\sum_{\substack{j=1 \\ j\neq\alpha}}^{\infty}\frac{1}{j^2(j^2-\alpha^2)}\left[\sin\left(\frac{j\pi ct}{L}\right)-\frac{\alpha}{j}\sin\left(\frac{j^2\pi ct}{\alpha L}\right)\right]\sin\left(\frac{j\pi x}{L}\right) \end{aligned} \qquad \textit{for } \alpha = j \tag{48}$$

where the dimensionless quantity, α, is defined as follows

$$\alpha = j\left(\frac{\dot{\omega}_j}{\omega_j}\right) = \frac{cL}{\pi}\sqrt{\frac{\rho A}{EI}}. \tag{49}$$

Note that α is a function of the structural properties of the beam, and typically $\alpha = 1.0$ corresponds to vehicle speeds greater than 400 km/h.[12] Thus, for most applications and the scope of this study equation (47) rather than equation (48) will be of primary practical importance.

Similarly, the mode-acceleration solution to this problem is found by replacing the time dependent force, F(t), in equations (42) and (45) by F and setting ζ equal to zero. After integrating the dynamic component, it is then added to the pseudo-static contribution to obtain

$$\begin{aligned} v_{MA}(x,t) &= \frac{2FL^3}{EI\pi^4}\left(\frac{\pi^4}{12L^4}\right)\begin{cases} x(L-x_F)(2x_{FL}-x_F^2-x^2) & for\ x\le x_F \\ x_F(L-x)(2xL-x^2-x_F^2) & for\ x>x_F \end{cases} \\ &+ \frac{2FL^3}{EI\pi^4}\sum_{k=1}^{N}\frac{\alpha^2}{k^4(k^2-\alpha^2)}\left[\sin\left(\frac{k\pi ct}{L}\right)-\frac{k}{\alpha}\sin\left(\frac{k^2\pi ct}{\alpha L}\right)\right]\sin\left(\frac{k\pi x}{L}\right) \end{aligned} \tag{50}$$

Figure 2 shows the superimposed plots of the normalized mid-span displacements for the 1-term and 25-term mode-displacement solutions and the 1-term mode-acceleration solution when $\alpha = 0.5$. The mode-displacement and mode-acceleration solutions were normalized by dividing them by the mid-span deflection due to a static force, F, at x=L/2. These normalized solutions are called the *displacement dynamic magnification factors*. Note that when the moving load is at mid-span (ct/L = 0.5) the displacement is 20% greater than the deflection due to the static load. This magnification factor is highly dependent upon α. As α approaches zero, the magnification factor at ct/L = 0.5 goes to one, as would be expected, but in general the magnification factor is a non-linear function of α. Olsson provides a detailed examination of this behavior in reference 12.

Figure 2 shows that convergence of the displacement solutions occurs rapidly. Both 1-term solutions are good approximations of the 25-term solution; however, the 1-term mode-acceleration solution is slightly better. Because both displacement solutions converge so rapidly, there is little apparent advantage to using one over the other; however, the superior convergence properties of the mode-acceleration become evident when calculating the bending moment and shear force. This will be demonstrated in the next two sections.

Bending-moment solutions

The bending moment, $M_N(x,t)$, of the beam is approximated from the relation

$$M_N(x,t) = -EI\frac{\partial^2 v_N(x,t)}{\partial x^2}. \quad \textbf{(51)}$$

Substituting equation (47) into (51) gives the mode-displacement moment equation

$$M_{MD}(x,t) = \frac{2FL}{\pi^2}\sum_{j=1}^{N}\frac{1}{(j^2-\alpha^2)}\left[\sin\left(\frac{j\pi ct}{L}\right)-\frac{\alpha}{j}\sin\left(\frac{j^2\pi ct}{\alpha L}\right)\right]\sin\left(\frac{j\pi x}{L}\right) \quad \textbf{(52)}$$

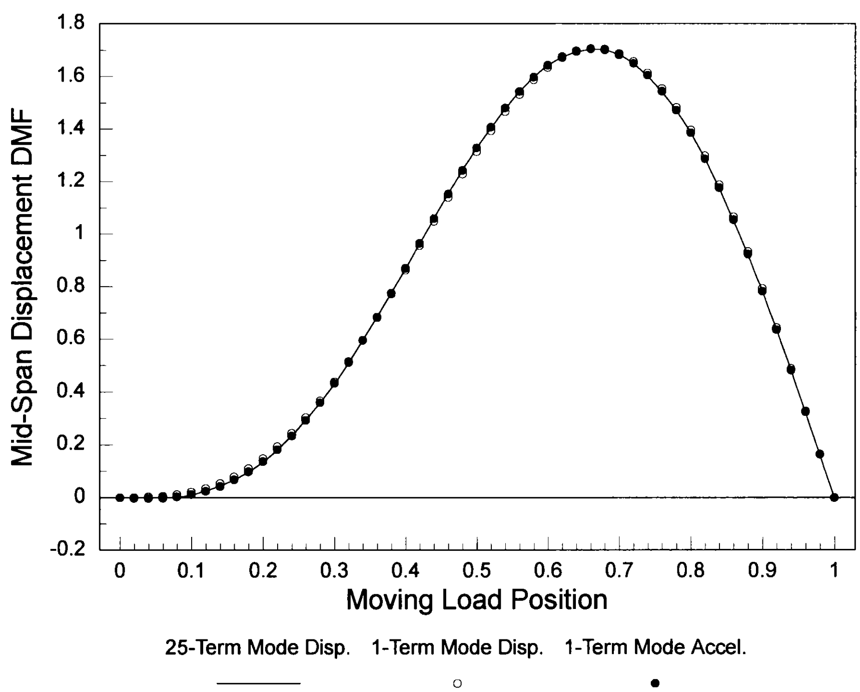


Figure 2 Dynamic magnification factors for mid-span displacement based on 1-term and 25-term mode-displacement solutions and a 1-term mode-acceleration solution.

This solution does not converge as quickly as the solution for the deflection does because the series coefficient for the moment equation is $1/(j^2-\alpha^2)$ whereas the displacement solution series coefficient has an additional j^2 multiplier term in the denominator. Although, each additional term will contribute less and less to the total moment solution, the convergence is not as dramatic as it was for the displacement solution.

Substituting equation (50) into (51) gives the mode-acceleration moment equation

$$M_{MA}(x,t) = \frac{2FL}{\pi^2}\left(\frac{\pi^2}{2L^2}\right)\begin{cases} x(L-x_F) & for\ x \le x_F \\ x_F(L-x) & for\ x > x_F \end{cases} + \frac{2FL}{\pi^2}\sum_{k=1}^{N}\frac{\alpha^2}{k^2(n^2-\alpha^2)}\left[\sin\left(\frac{k\pi ct}{L}\right) - \frac{k}{\alpha}\sin\left(\frac{k^2\pi ct}{\alpha L}\right)\right]\sin\left(\frac{k\pi x}{L}\right). \tag{53}$$

Figure 3 shows the superimposed plots of the normalized mid-span bending moments for the 1-term, 3-term and 25-term mode-displacement solutions and the 1-term mode-acceleration solution as the load moves along the beam ($\alpha = 0.5$). The bending moments were normalized with respect to the mid-span bending moment of the beam subjected to a centrally-applied static load, F. This normalized parameter is called the *moment dynamic magnification factor.*

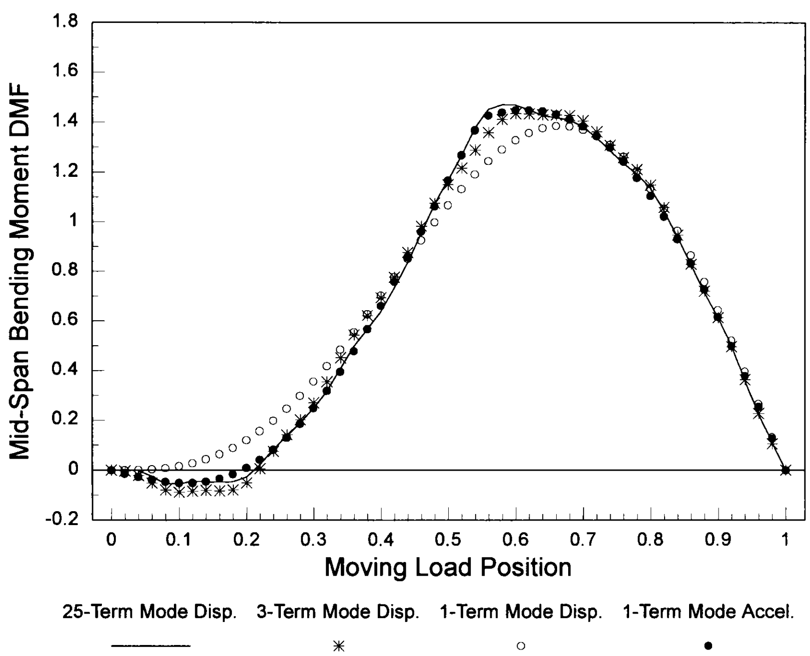


Figure 3 Dynamic magnification factors for mid-span bending moment, 1, 3 and 25-term mode displacement & 1-term mode acceleration solutions.

As can be seen from the plot, convergence of the mode-displacement moment solution is slower than it was for mode-displacement displacement solution, but the 1-term mode-acceleration moment solution remains fairly accurate. In fact, the 1-term mode-acceleration solution is a better approximation than even the 3-term mode-displacement solution. It can also be shown that as the load velocity increases (i.e., as α exceeds 0.5) more than three terms are required for a reasonable mode-displacement approximation, but the 1-term mode-acceleration approximation remains very accurate.

Shear-force solutions

The shear force, $T_N(x,t)$, of the beam is approximated from the relation

$$T_N(x,t) = -EI\frac{\partial^3 v_N(x,t)}{\partial x^3}. \tag{54}$$

Substituting equation (47) into (54) gives the mode-displacement shear equation

$$T_{MD}(x,t) = \frac{2F}{\pi}\sum_{j=1}^{N}\frac{j}{(j^2-\alpha^2)}\left[\sin\left(\frac{j\pi ct}{L}\right)-\frac{\alpha}{j}\sin\left(\frac{j^2\pi ct}{\alpha L}\right)\right]\cos\left(\frac{j\pi x}{L}\right), \tag{55}$$

Note that the series coefficient for the shear force solution is simply the bending moment coefficient multiplied by j. As would be expected, this multiplier further impedes rapid convergence and, consequently, more series terms are needed for the mode-displacement shear force solution than were needed for the mode-displacement bending moment solution.

Substituting equation (50) into (54) gives the following mode-acceleration shear equation

$$\begin{aligned} T_N(x,t) = {} & \frac{2F}{\pi}\left(\frac{\pi}{2L}\right)\begin{cases}(L-x_F) & for\ x\le x_F\\ -x_F & for\ x>x_F\end{cases}\\ & + \frac{2F}{\pi}\sum_{n=1}^{N}\frac{\alpha^2}{n(n^2-\alpha^2)}\left[\sin\left(\frac{n\pi ct}{L}\right)-\frac{n}{\alpha}\sin\left(\frac{n^2\pi ct}{\alpha L}\right)\right]\cos\left(\frac{n\pi x}{L}\right). \end{aligned} \tag{56}$$

The shear equation converges more slowly than the moment and displacement expressions, but a 2-term mode-acceleration approximation is capable of providing a fairly accurate representation of the exact solution. Note that two terms are needed because the odd-numbered terms do not contribute to the total shear solution at mid-span, just as the even-numbered terms did not contribute to the total mid-span displacement and moment solutions.

Figure 4 shows the superimposed plots of the normalized mid-span shear forces for a 2-term, a 10-term, and a 100-term mode-displacement solution and a 1-term mode-acceleration solution when $\alpha = 0.5$. The shear forces were normalized with respect to the quarter-span shear of a beam under a centrally-applied static load to give the *shear dynamic magnification factor.*

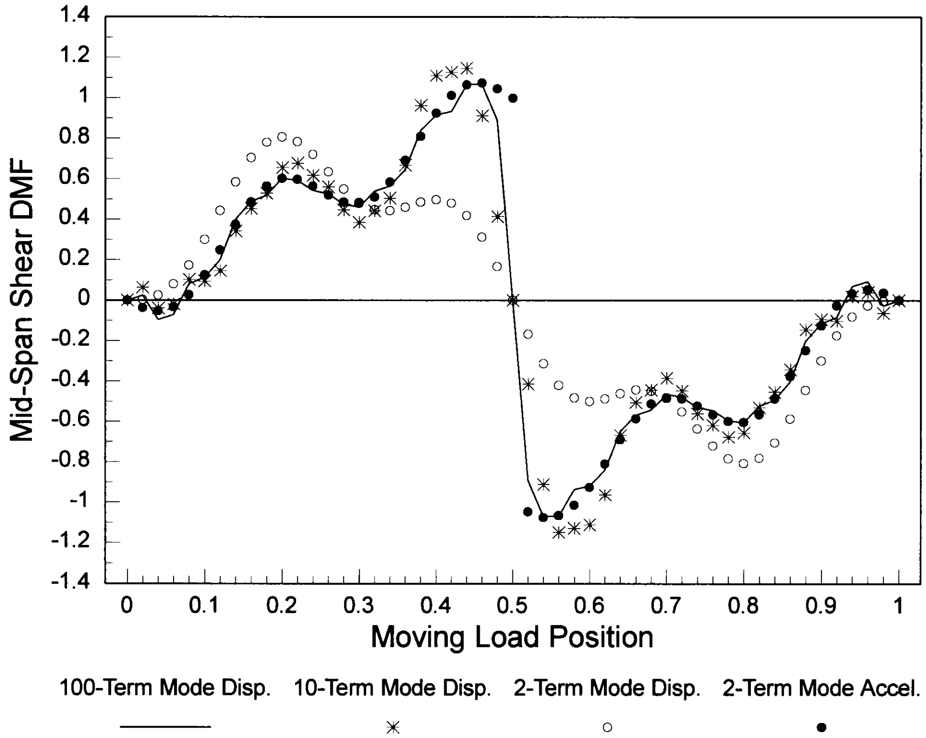


Figure 4 Comparison of dynamic magnification factors for mid-span shear force based on 2-term, 10-term and 100-term mode displacement solutions and a 2-term mode acceleration solution.

The 2-term mode-acceleration solution proves to be a far better approximation of the exact shear behavior than either the 2-term or the 10-term mode-displacement solutions. In fact, when the load is traveling over the mid-span, the 2-term mode-acceleration solution provides a better representation of the discontinuity than even the 100-term mode-displacement solution.

Comparison of Modal-Superposition Methods for a Damped Beam Subjected to a Constant Moving Force

Next, the constant moving force problem will be re-examined, but damping of the beam will be taken into account. The mode-displacement solution is found by integrating equation (39) where the force parameter is once again kept constant, but the damping factor, ζ, is retained to give

$$v_{MD}(x,t) = \frac{2FL^3}{EI\pi^4}\sum_{j=1}^{N}\left(\frac{1}{j^2[(j^2-\alpha^2)^2+(2\zeta j\alpha)^2]}\right)\left\{\begin{array}{c} 2\zeta j\alpha\left[e^{-\zeta\omega_j t}\cos(\omega_{d_j}t)-\cos(\dot{\omega}_j t)\right]+(j^2-\alpha^2)\sin(\dot{\omega}_j t) \\ +\left[\dfrac{\alpha}{j\sqrt{1-\zeta^2}}\right][2j^2\zeta^2-(j^2-\alpha^2)e^{-\zeta\omega_j t}]\sin(\omega_{d_j}t) \end{array}\right\}\sin\left(\frac{j\pi x}{L}\right) \tag{57}$$

The mode-acceleration solution is obtained by taking the same considerations into account for equations (42) and (45) and then adding the two contributions to yield

$$\begin{aligned} v_{MA}(x,t) = {} & \frac{2FL^3}{EI\pi^4}\left(\frac{\pi^4}{12L^4}\right)\begin{cases} x(L-x_F)(2x_{FL}-x_F^2-x^2) & for\ x \le x_F \\ x_F(L-x)(2xL-x^2-x_F^2) & for\ x > x_F \end{cases} \\ & + \frac{2FL^3}{EI\pi^4}\sum_{k=1}^{N}\left(\frac{1}{k^4[(k^2-\alpha^2)^2+(2\zeta k\alpha)^2]}\right)\left\{\begin{array}{c} 2\zeta k^3\alpha\left[e^{-\zeta\omega_k t}\cos(\omega_{d_k}t)-\cos(\dot{\omega}_k t)\right] \\ +\alpha^2[k^2(1-4\zeta^2)-\alpha^2]\sin(\dot{\omega}_k t) \\ +\left[k\dfrac{\alpha}{\sqrt{1-\zeta^2}}\right][2k^2\zeta^2-(k^2-\alpha^2)e^{-\zeta\omega_k t}]\sin(\omega_{d_k}t) \end{array}\right\}\sin\left(\frac{k\pi x}{L}\right) \end{aligned} \tag{58}$$

Figure 5 shows the displacement dynamic magnification factors at mid-span for the 1-term and 25-term mode-displacement solutions and the 1-term mode-acceleration solution of a damped beam subjected to a moving load with $\alpha = 0.5$ and a constant modal damping ratio $\zeta = 0.1$. Note that for $\alpha = 0.5$ the damping attenuates the mid-span displacement, and, consequently, the dynamic magnification factors for the damped beam are smaller than those for the undamped beam shown in Figure 2. Once again, the mode-acceleration solution converges quickly and is only slightly better than the 1-term mode-displacement solution.

Bending-moment solutions

The mode-displacement bending moment solution is obtained by substituting equation (57) into equation (51) to achieve

$$M_{MD}(x,t) = \frac{2FL}{\pi^2}\sum_{j=1}^{N}\left(\frac{1}{(j^2-\alpha^2)+(2\zeta j\alpha)^2}\right)\left\{\begin{matrix} 2\zeta j\alpha\left[e^{-\zeta\omega_j t}\cos(\omega_{d_j}t)-\cos(\dot{\omega}_j t)\right]+(j^2-\alpha^2)\sin(\dot{\omega}_j t) \\ +\left[\dfrac{\alpha}{j\sqrt{1-\zeta^2}}\right][2j^2\zeta^2-(j^2-\alpha^2)e^{-\zeta\omega_j t}]\sin(\omega_{d_j}t) \end{matrix}\right\}\sin\left(\frac{j\pi x}{L}\right). \tag{59}$$

Equation (58) is substituted into equation (51) to obtain the following mode-acceleration bending moment solution

$$\begin{aligned} M_{MA}(x,t) = {} & \frac{2FL}{\pi^2}\left(\frac{\pi^2}{2L^2}\right)\begin{cases} x(L-x_F) & for\ x\le x_F \\ x_F(L-x) & for\ x>x_F \end{cases} \\ & + \frac{2FL}{\pi^2}\sum_{k=1}^{N}\left(\frac{1}{k^2[(k^2-\alpha^2)^2+(2\zeta k\alpha)^2]}\right)\left\{\begin{matrix} 2\zeta k^3\alpha\left[e^{-\zeta\omega_k t}\cos(\omega_{d_k}t)-\cos(\dot{\omega}_k t)\right] \\ +\alpha^2[k^2(1-4\zeta^2)-\alpha^2)\sin(\dot{\omega}_k t) \\ +\left[k\dfrac{\alpha}{\sqrt{1-\zeta^2}}\right][2k^2\zeta^2-(k^2-\alpha^2)e^{-\zeta\omega_k t}]\sin(\omega_{d_k}t) \end{matrix}\right\}\sin\left(\frac{k\pi x}{L}\right) \end{aligned} \tag{60}$$

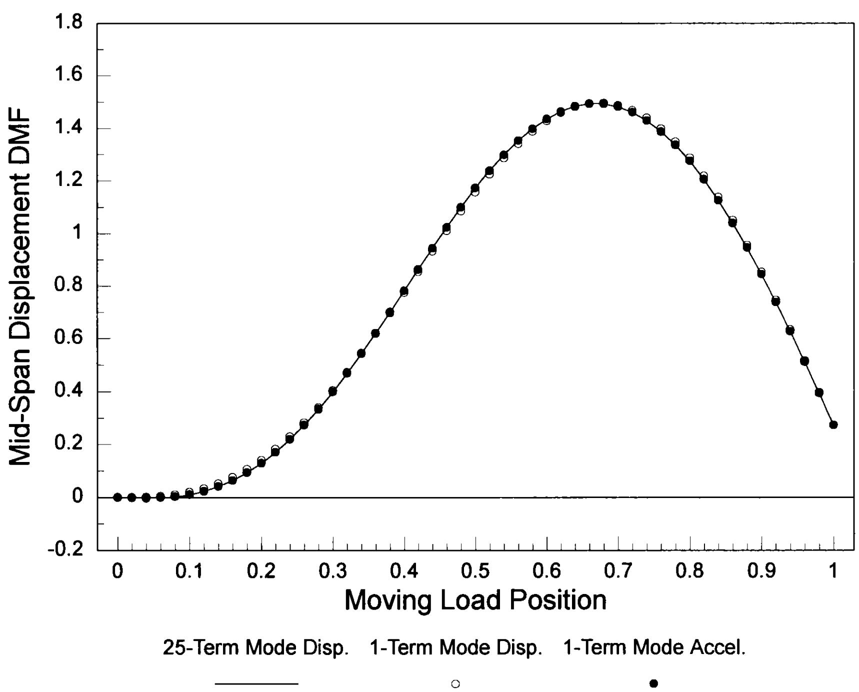


Figure 5 Comparison of dynamic magnification factors for mid-span displacement based on 1-term and 25-term mode-displacement solutions and 1-term mode-acceleration solution.

Figure 6 shows the superimposed plots of the bending moment dynamic magnification factors at mid-span for the 1-term, 3-term, and 25-term mode-displacement solutions and the 1-term mode-acceleration solution as the load moves along the damped beam ($\alpha = 0.5$) for a constant modal damping ratio $\zeta = 0.1$. As can be seen from the plot, convergence of the mode-displacement moment solution is slower than it was for the displacement solution. The 1-term mode-acceleration moment solution is still very accurate and better approximates the exact solution than even the 3-term mode-displacement solution. Note that the introduction of damping does not affect the accuracy of the mode-acceleration bending moment solution, but it does appear to impede the convergence of the mode-displacement solution.

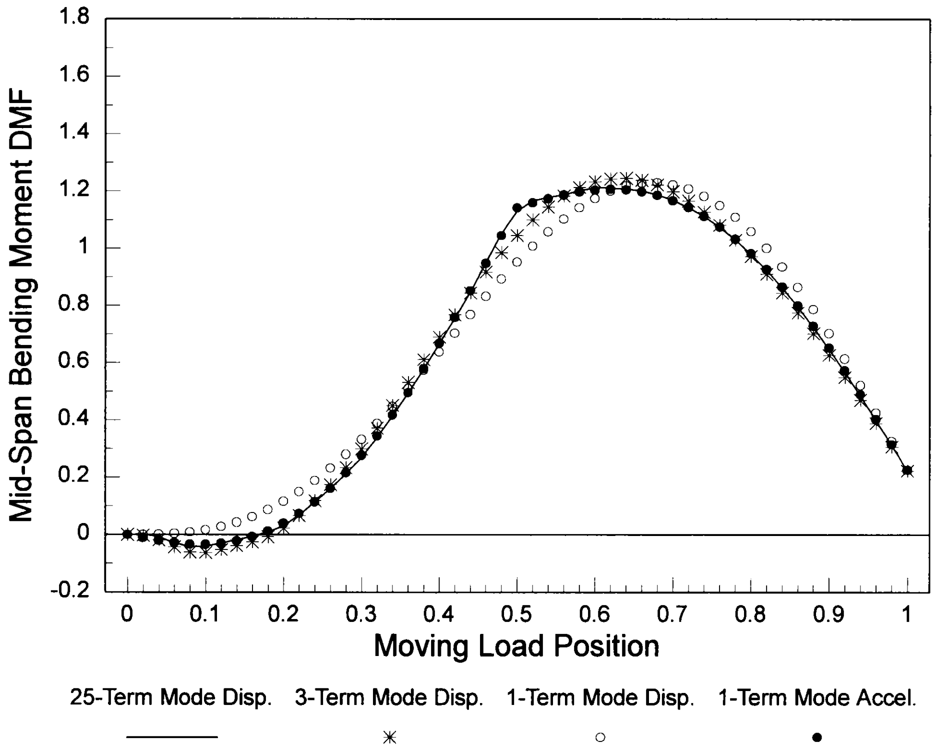


Figure 6 Comparison of dynamic magnification factors for mid-span bending moment based on 1-term, 3-term, and 25-term mode-displacement and 1-term mode acceleration solutions.

Shear-force solutions

The mode-displacement shear force solution is obtained by substituting equation (57) into equation (54) to achieve

$$T_{MD}(x,t) = \frac{2F}{\pi}\sum_{j=1}^{N}\left(\frac{j}{(j^2-\alpha^2)+(2\zeta j\alpha)^2}\right)\left\{\begin{array}{l} 2\zeta j\alpha\left[e^{-\zeta\omega_j t}\cos(\omega_{d_j}t)-\cos(\dot{\omega}_j t)\right]+(j^2-\alpha^2)\sin(\dot{\omega}_j t) \\ +\left[\frac{\alpha}{j\sqrt{1-\zeta^2}}\right][2j^2\zeta^2-(j^2-\alpha^2)e^{-\zeta\omega_j t}]\sin(\omega_{d_j}t) \end{array}\right\}\cos\left(\frac{j\pi x}{L}\right) \tag{61}$$

Equation (57) is substituted into equation (54) to obtain the following mode-acceleration shear force solution

$$T_{MA}(x,t) = \frac{2F}{\pi}\left(\frac{\pi}{2L}\right)\begin{cases}(L-x_F) & for\ x \le x_F \\ -x_F & for\ x > x_F\end{cases}$$
$$+ \frac{2F}{\pi}\sum_{k=1}^{N}\left(\frac{1}{k[(k^2-\alpha^2)^2+(2\zeta k\alpha)^2]}\right)\left\{\begin{gathered}2\zeta k^3\alpha\left[e^{-\zeta\omega_k t}\cos(\omega_d t)-\cos(\dot{\omega}_k t)\right] \\ +\alpha^2[k^2(1-4\zeta^2)-\alpha^2)\sin(\dot{\omega}_k t) \\ +\left[k\frac{\alpha}{\sqrt{1-\zeta^2}}\right][2k^2\zeta^2-(k^2-\alpha^2)e^{-\zeta\omega_k t}]\sin(\omega_d t)\end{gathered}\right\}\cos\left(\frac{k\pi x}{L}\right) \tag{62}$$

The superimposed plots of the normalized mid-span shear forces for a 2-term, 16-term, and a 150-term mode-displacement solution and a 2-term mode-acceleration solution ($\alpha = 0.5$) for a constant modal damping ratio $\zeta = 0.1$ are shown in Figure 7. Once again, the 2-term mode-acceleration solution proves to be a far better approximation of the exact shear behavior than either the 2-term or the 16-term mode-displacement solutions and it provides a better representation of the discontinuity than even the 150-term mode-displacement solution.

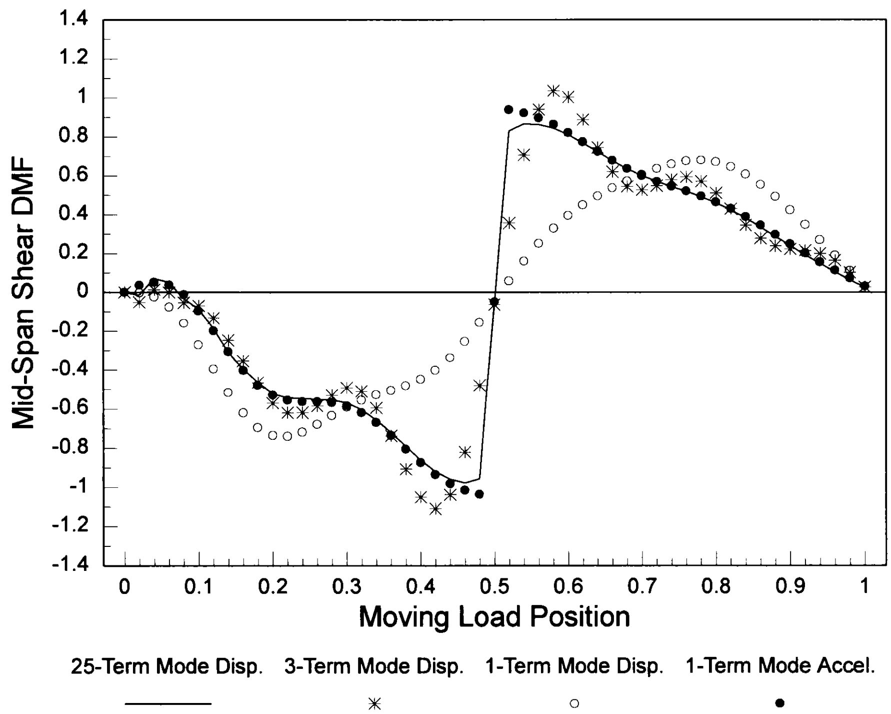


Figure 7 Comparison of dynamic magnification factors for mid-span shear force based on 2-term, 10-term and 100-term mode-displacement and 2-term mode-acceleratiom solutions.

Comparison of Modal-Superposition Methods for an Undamped Beam Subjected to a Harmonic Moving Force

As a final illustration of the superior convergence of modal acceleration solutions, consider a harmonic force traveling across the undamped beam. For this sub-problem, it is assumed that $\zeta = 0$ and the time-dependent force is represented by

$$F(t) = F\sin(\Omega t) \tag{63}$$

where Ω is the forcing frequency of the harmonic load. The mode-displacement solution is found by substituting equation (63) into equation (39) and performing the integration to obtain

$$v_{MD}(x,t) = \frac{FL^3}{EI\pi^4}\sum_{j=1}^{N}\left[\frac{\omega_j t\sin(\omega_j t)}{2j^4} - \frac{\cos[(\Omega+\acute{\omega}_j)t]-\cos(\omega_j t)}{j^4-(\gamma+j\alpha)^2}\right]\sin\left(\frac{j\pi x}{L}\right) \qquad \textit{when } \gamma = j(j+\alpha) \tag{64}$$

and

$$v_{MD}(x,t) = \frac{FL^3}{EI\pi^4}\sum_{j=1}^{N}\left[\frac{\cos[(\Omega-\acute{\omega}_j)t]-\cos(\omega_j t)}{j^4-(\gamma-n\alpha)^2} - \frac{\omega_j t\sin(\omega_j t)}{2j^4}\right]\sin\left(\frac{j\pi x}{L}\right) \qquad \textit{when } \gamma = j(j-\alpha) \tag{65}$$

and

$$v_{MD}(x,t) = \frac{FL^3}{EI\pi^4}\sum_{j=1}^{N}\left[\frac{\cos[(\Omega-\acute{\omega}_j)t]-\cos(\omega_j t)}{j^4-(\gamma-n\alpha)^2} - \frac{\cos[(\Omega+\acute{\omega}_j)t]-\cos(\omega_j t)}{j^4-(\gamma+j\alpha)^2}\right]\sin\left(\frac{j\pi x}{L}\right) \qquad \textit{when } \gamma \neq j(j\pm\alpha) \tag{66}$$

where the new dimensionless quantity γ, defined as

$$\gamma = n^2\frac{\Omega}{\omega_n}, \tag{67}$$

represents the ratio of the forcing frequency to the fundamental natural frequency of the beam.

The pseudo-static and dynamic displacement components of the beam under the moving harmonic load can be obtained by substituting equation (63) into equations (42) and (45) and setting ζ equal to zero. This yields the following expression for the pseudo-static contribution

$$v_{ps}(x,t) = \frac{F\sin(\Omega t)}{6EIL}\begin{cases} x(L-ct)[2ctL-x(ct)^2-x^2] & for\ x \le ct \\ ct(L-x)[2xL-x^2-(ct)^2] & for\ x \ge ct \end{cases} \tag{68}$$

and the dynamic contribution is obtained as

$$v_d(x,t) = \frac{FL^3}{EI\pi^4}\sum_{k=1}^{N}\frac{1}{k^4}\left[\begin{array}{c} \dfrac{\omega_k t\sin(\omega_k t)}{2} - \cos[(\Omega-\omega_k)t] \\ -\dfrac{(\gamma+k\alpha)^2\cos[(\Omega+\omega_k)t]-k^4\cos(\omega_k t)}{k^4-(\gamma+k\alpha)^2} \end{array}\right]\sin\left(\frac{k\pi x}{L}\right) \quad when\ \gamma = k(k+\alpha) \tag{69}$$

and

$$v_d(x,t) = \frac{FL^3}{EI\pi^4}\sum_{k=1}^{N}\frac{1}{k^4}\left[\begin{array}{c} \cos[(\Omega+\omega_k)t] - \dfrac{\omega_k t\sin(\omega_k t)}{2} \\ +\dfrac{(\gamma-k\alpha)^2\cos[(\Omega-\omega_k)t]-k^4\cos(\omega_k t)}{k^4-(\gamma-k\alpha)^2} \end{array}\right]\sin\left(\frac{k\pi x}{L}\right) \quad when\ \gamma = k(k+\alpha) \tag{70}$$

and

$$v_d(x,t) = \frac{FL^3}{EI\pi^4}\sum_{k=1}^{N}\frac{1}{k^4}\left[\begin{array}{c} \dfrac{(\gamma-k\alpha)^2\cos[(\Omega-\omega_k)t]-k^4\cos(\omega_k t)}{k^4-(\gamma-k\alpha)^2} \\ -\dfrac{(\gamma+k\alpha)^2\cos[(\Omega+\omega_k)t]-k^4\cos(\omega_k t)}{k^4-(\gamma+k\alpha)^2} \end{array}\right]\sin\left(\frac{k\pi x}{L}\right) \quad when\ \gamma \ne k(k\pm\alpha) \tag{71}$$

Thus, the mode-acceleration solution for the displacement of the beam under the moving harmonic load is

$$v_{MA}(x,t) = v_d(x,t) + \frac{F\sin(\Omega t)}{6EIL}\begin{cases} x(L-ct)[2ctL-x(ct)^2-x^2] & for\ x \le ct \\ ct(L-x)[2xL-x^2-(ct)^2] & for\ x \ge ct \end{cases} \tag{72}$$

where v_d(x,t) is given by equations (69) to (71).

The number of terms necessary for an adequate solution depends upon the forcing frequency, Ω, of the harmonic load. If the load is oscillating at or near one of the system's natural frequencies, then it is usually necessary to include all of the terms up to and including that which corresponds to the mode of excitation. The parameter which indicates how many terms are needed is γ which represents the number of sine waves the force will oscillate through as it traverses the beam. A reasonable truncated solution should include all terms less than $\gamma^{1/2} + 1$.

In order to examine the effect of γ on solution convergence, we will consider a beam subjected to a load traveling at $\alpha = 0.5$ and oscillating at three different forcing frequencies: $\gamma = 0.5$, $\gamma = 2.5$, and $\gamma = 6.5$. Figure 8 shows the displacement dynamic magnification factors at mid span for the 1-term and 25-term mode-displacement solutions and the 1-term mode-acceleration solution of a beam subjected to a moving harmonic load with $\gamma = 0.5$. Figures 9 and 10 show the same plots for $\gamma = 2.5$ and $\gamma = 6.5$, respectively. Note that these plots were normalized with respect to a stationary *harmonic* load at mid-span.

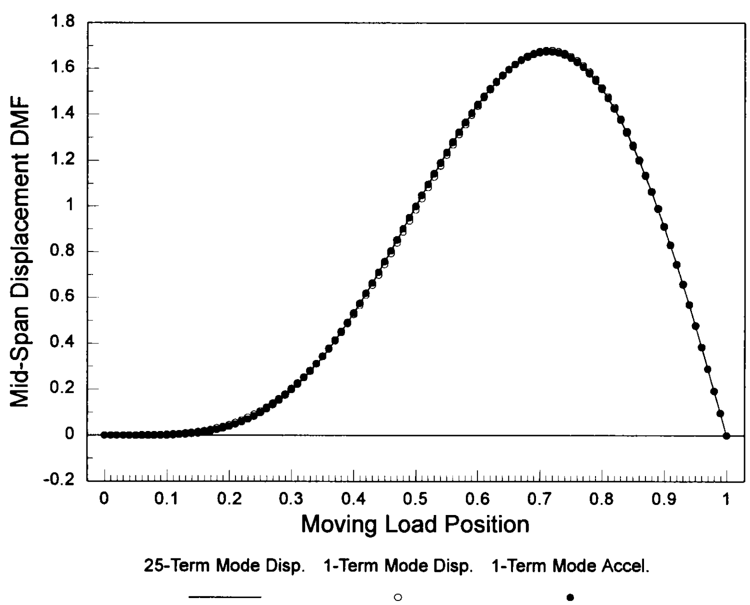


Figure 8 Dynamic magnification factors for mid-span displacement based on 1- and 25-term mode-displacement and 1-term mode acceleration solutions (γ=0.5).

As was the case for the non-harmonic moving load, the displacement solution should converge relatively quickly. Figure 8 shows that both 1-term solutions are good approximations of the exact solution when $\gamma = 0.5$. As γ increases to 2.5, the 1-term mode-displacement solution loses accuracy, but the mode-acceleration solution still provides excellent convergence. However, at $\gamma = 6.5$, at least three terms of either solution are needed because the harmonic force is oscillating at a frequency between the second and third natural frequencies of the beam. In general, the mode-acceleration solution provides better convergence than the mode-displacement solution, but the number of terms needed for convergence is dependent upon the forcing frequency.

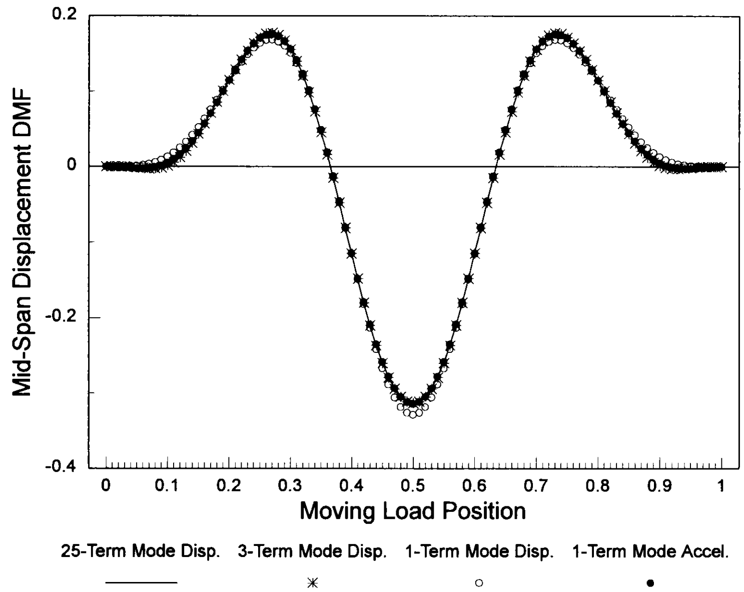


Figure 9 Dynamic magnification factors for mid-span displacement based on 1-, 3- and 25-term mode-displacement and 1-term mode-acceleration solutions (γ=2.5).

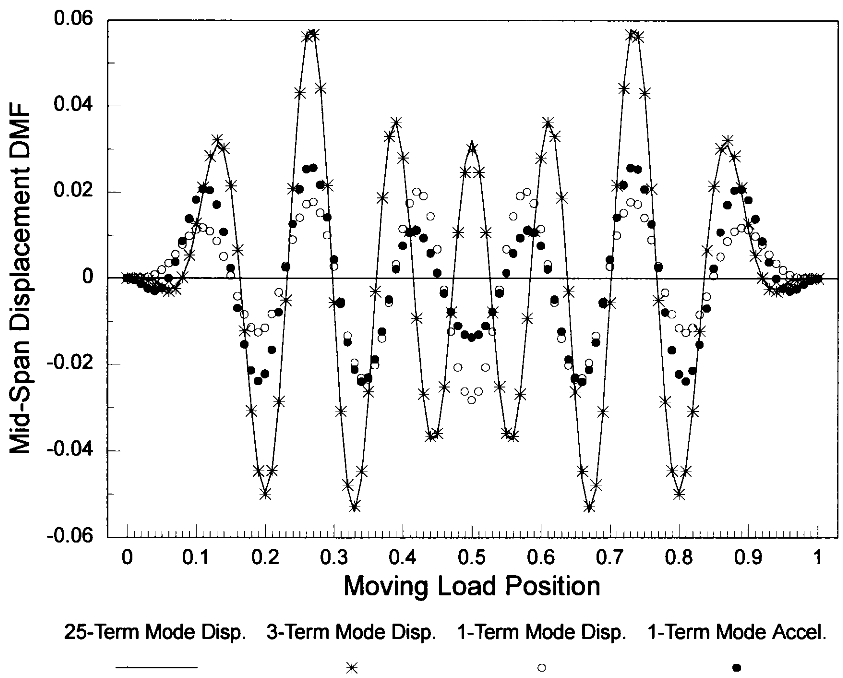


Figure 10 Dynamic magnification factors for mid-span displacement based on 1-, 3- and 25-term mode-displacement and 1-term mode-acceleration solutions (γ=6.5).

Bending-moment solutions

The mode-displacement bending moment solution is obtained by substituting equations (64) to (66) into equation (51) which gives

$$M_{MD}(x,t) = \frac{FL}{\pi^2}\sum_{j=1}^{N}\left[\frac{\omega_j t\sin(\omega_j t)}{2j^2} - j^2\left(\frac{\cos[(\Omega+\omega_j)t]-\cos(\omega_j t)}{j^4-(\gamma+j\alpha)^2}\right)\right]\sin\left(\frac{j\pi x}{L}\right) \qquad when\ \gamma = j(j+\alpha) \tag{73}$$

and

$$M_{MD}(x,t) = \frac{FL}{\pi^2}\sum_{j=1}^{N}\left[j^2\left(\frac{\cos[(\Omega-\omega_j)t]-\cos(\omega_j t)}{j^4-(\gamma-j\alpha)^2}\right) - \frac{\omega_j t\sin(\omega_j t)}{2j^2}\right]\sin\left(\frac{j\pi x}{L}\right) \qquad when\ \gamma = j(j-\alpha) \tag{74}$$

and

$$M_{MD}(x,t) = \frac{FL}{\pi^2}\sum_{j=1}^{N} j^2\left[\frac{\cos[(\Omega-\omega_j)t]-\cos(\omega_j t)}{j^4-(\gamma-j\alpha)^2} - \frac{\cos[(\Omega+\omega_j)t]-\cos(\omega_j t)}{j^4-(\gamma+j\alpha)^2}\right]\sin\left(\frac{j\pi x}{L}\right) \qquad when\ \gamma \neq j(j\pm\alpha) \tag{75}$$

The mode-acceleration bending moment solution is derived by substituting the mode-acceleration displacement solution for the harmonic force problem into equation (51), so that

$$M_{MA}(x,t) = M_d(x,t) + \frac{FL\sin(\Omega t)}{\pi^2}\left(\frac{\pi^2}{L^2}\right)\begin{cases} x(L-ct) & for\ x \le x_F \\ ct(L-x) & for\ x \ge x_F. \end{cases} \tag{76}$$

where

$$M_d(x,t) = \frac{FL}{\pi^2}\sum_{k=1}^{N}\frac{1}{k^2}\left[\begin{array}{c}\dfrac{\omega_k t\sin(\omega_k t)}{2} - \cos[(\Omega-\omega_k)t] \\ -\dfrac{(\gamma+k\alpha)^2\cos[(\Omega+\omega_k)t]-k^4\cos(\omega_k t)}{k^4-(\gamma+k\alpha)^2}\end{array}\right]\sin\left(\frac{k\pi x}{L}\right) \qquad when\ \gamma = k(k+\alpha) \tag{77}$$

and

$$M_d(x,t) = \frac{FL}{\pi^2}\sum_{k=1}^{N}\frac{1}{k^2}\left[\begin{array}{c}\cos[(\Omega+\omega_k)t] - \dfrac{\omega_k t\sin(\omega_k t)}{2} \\ +\dfrac{(\gamma-k\alpha)^2\cos[(\Omega-\omega_k)t]-k^4\cos(\omega_k t)}{k^4-(\gamma-k\alpha)^2}\end{array}\right]\sin\left(\frac{k\pi x}{L}\right) \qquad when\ \gamma = k(k-\alpha) \tag{78}$$

and

$$M_d(x,t) = \frac{FL}{\pi^2}\sum_{k=1}^{N}\frac{1}{k^2}\left[\begin{array}{c}\dfrac{(\gamma-k\alpha)^2\cos[(\Omega-\omega_k)t]-k^4\cos(\omega_k t)}{k^4-(\gamma-k\alpha)^2} \\ -\dfrac{(\gamma+k\alpha)^2\cos[(\Omega+\omega_k)t]-k^4\cos(\omega_k t)}{k^4-(\gamma+k\alpha)^2}\end{array}\right]\sin\left(\frac{k\pi x}{L}\right) \qquad when\ \gamma \neq k(k\pm\alpha) \tag{79}$$

Figures 11, 12, and 13 show the bending moment dynamic magnification factors at mid-span for the three harmonic loads described above. The superior convergence properties of the mode-acceleration solution is illustrated in both Figures 11 and 12 where $\gamma = 0.5$ and 2.5, respectively. In each of these cases, the 1-term mode-acceleration solution proves to be more representative of the exact solution than either the 1-term or 3-term mode-displacement solutions. However, Figure 13 again shows that as γ increases to 6.5 the 1-term solutions are inadequate. The 3-term mode-displacement solution is approaching convergence, but a 5-term solution would be better. Although it is not included on the graph, the 3-term mode-acceleration solution for $\gamma = 6.5$ provides excellent results.

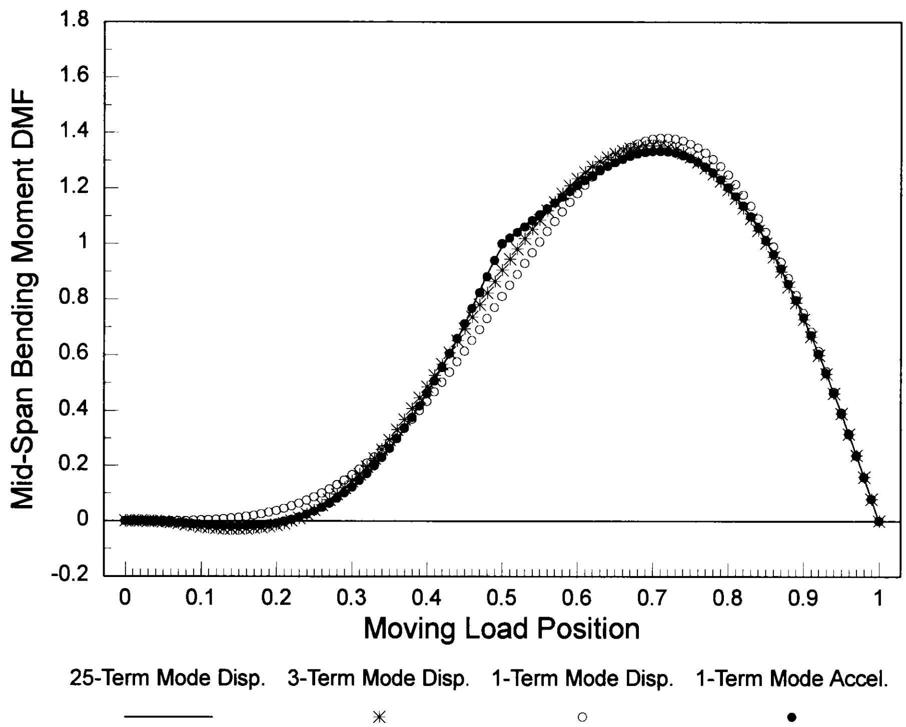


Figure 11 Dynamic magnification factors for mid-span bending moment based on 1-, 3- and 25-term mode-displacement and 1-term mode-acceleration solutions (γ=0.5).

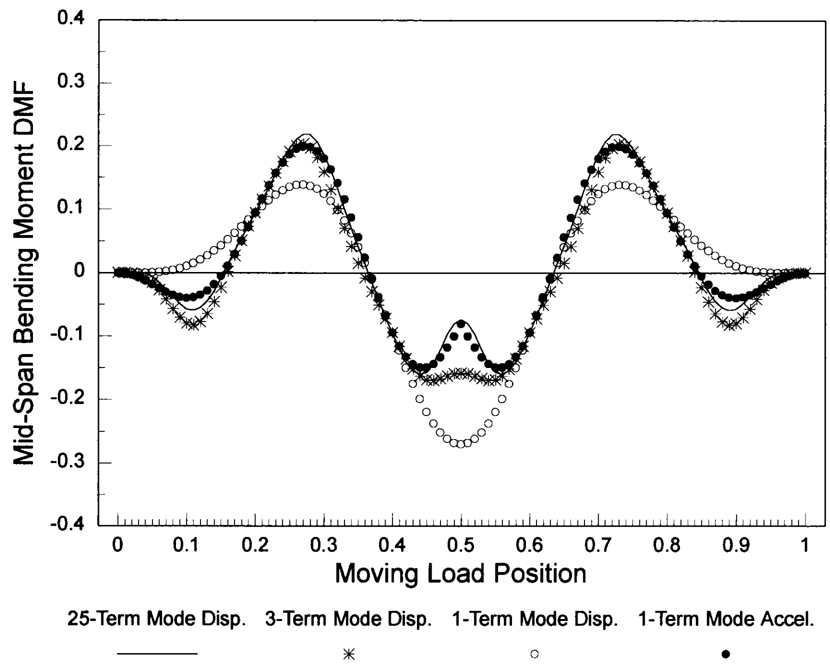


Figure 12 Dynamic magnification factors for mid-span bending moment based on 1-, 3- and 25-term mode-displacement and 1-term mode-acceleration solutions (γ=2.5).

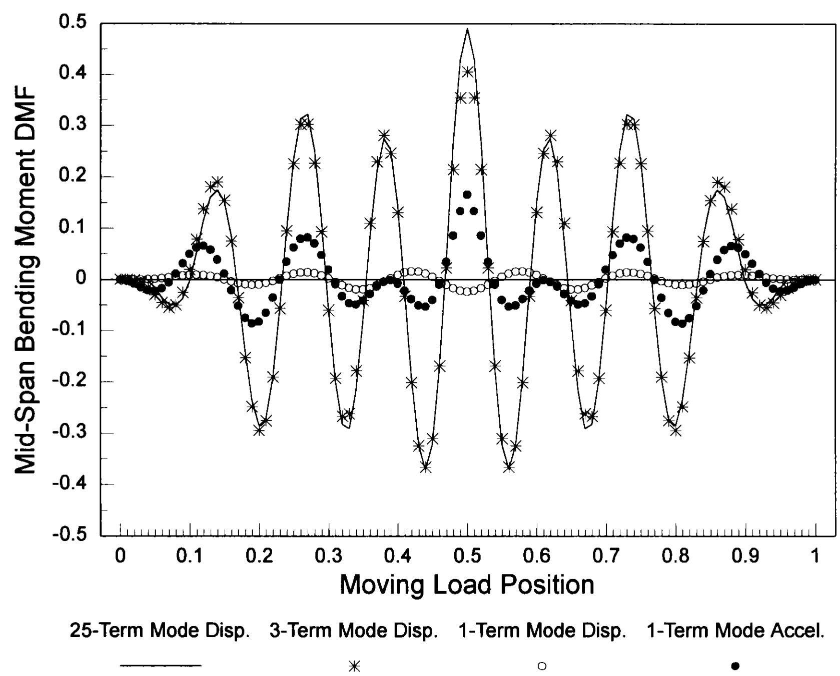


Figure 13 Dynamic magnification factors for mid-span bending moment based on 1-, 3- and 25-term mode-displacement and 1-term mode-acceleration solutions (γ=6.5).

Shear Force solutions

By substituting the mode-displacement displacement solution for the harmonic force problem into equation (54), the following mode-displacement shear force solution is obtained:

$$T_{MD}(x,t) = \frac{F}{\pi}\sum_{j=1}^{N}\left[\frac{\omega_j t\sin(\omega_j t)}{2j} - j^3\left(\frac{\cos[(\Omega+\dot{\omega}_j)t]-\cos(\omega_j t)}{j^4-(\gamma+j\alpha)^2}\right)\right]\cos\left(\frac{j\pi x}{L}\right) \quad when\ \gamma=j(j+\alpha) \tag{80}$$

and

$$T_{MD}(x,t) = \frac{F}{\pi}\sum_{j=1}^{N}\left[j^3\left(\frac{\cos[(\Omega-\dot{\omega}_j)t]-\cos(\omega_j t)}{j^4-(\gamma-j\alpha)^2}\right)-\frac{\omega_j t\sin(\omega_j t)}{2j}\right]\cos\left(\frac{j\pi x}{L}\right) \quad when\ \gamma=j(j-\alpha) \tag{81}$$

and

$$T_{MD}(x,t) = \frac{F}{\pi}\sum_{j=1}^{N}j^3\left[\frac{\cos[(\Omega-\dot{\omega}_j)t]-\cos(\omega_j t)}{j^4-(\gamma-j\alpha)^2}-\frac{\cos[(\Omega+\dot{\omega}_j)t]-\cos(\omega_j t)}{j^4-(\gamma+j\alpha)^2}\right]\cos\left(\frac{j\pi x}{L}\right) \quad when\ \gamma\neq j(j\pm\alpha) \tag{82}$$

The mode-acceleration shear force solution is derived by substituting the mode-acceleration displacement solution for the harmonic force problem into equation (54), so that

$$T_{MA}(x,t) = T_d(x,t) + \frac{F\sin(\Omega t)}{\pi}\left(\frac{\pi}{L}\right)\begin{cases}(L-ct) & for\ x\leq x_F\\ -ct & for\ x\geq x_F\end{cases} \tag{83}$$

where

$$T_d(x,t) = \frac{F}{\pi}\sum_{k=1}^{N}\frac{1}{k}\left[\begin{array}{c}\dfrac{\omega_k t\sin(\omega_k t)}{2} - \cos[(\Omega-\dot{\omega}_k)t]\\ -\dfrac{(\gamma+k\alpha)^2\cos[(\Omega+\dot{\omega}_k)t]-k^4\cos(\omega_k t)}{k^4-(\gamma+k\alpha)^2}\end{array}\right]\cos\left(\frac{k\pi x}{L}\right) \quad when\ \gamma=k(k+\alpha) \tag{84}$$

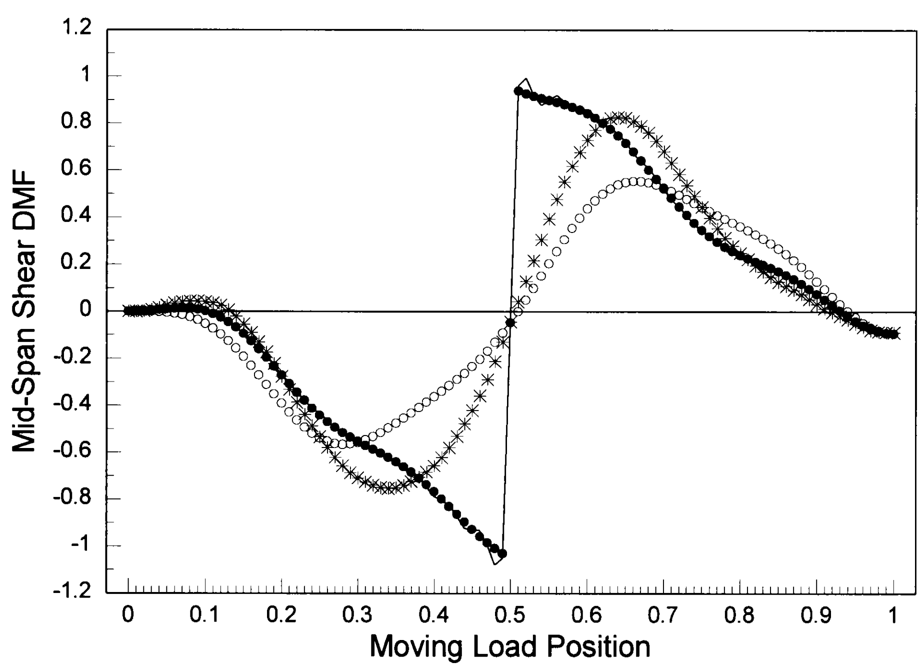
and

$$T_d(x,t) = \frac{F}{\pi}\sum_{k=1}^{N}\frac{1}{k}\left[\begin{array}{c}\cos[(\Omega+\dot{\omega}_k)t] - \dfrac{\omega_k t\sin(\omega_k t)}{2}\\ +\dfrac{(\gamma-k\alpha)^2\cos[(\Omega-\dot{\omega}_k)t]-k^4\cos(\omega_k t)}{k^4-(\gamma-k\alpha)^2}\end{array}\right]\cos\left(\frac{n\pi x}{L}\right) \quad when\ \gamma=k(k-\alpha) \tag{85}$$

and

$$T_d(x,t) = \frac{F}{\pi}\sum_{k=1}^{N}\frac{1}{k}\left[\begin{array}{c}\dfrac{(\gamma-k\alpha)^2\cos[(\Omega-\acute{\omega}_k)t-k^4\cos(\omega_k t)}{k^4-(\gamma-k\alpha)^2} \\ -\dfrac{(\gamma+k\alpha)^2\cos[(\Omega+\acute{\omega}_k)t]-k^4\cos(\omega_k t)}{k^4-(\gamma+k\alpha)^2}\end{array}\right]\cos\left(\frac{k\pi x}{L}\right) \quad when\ \gamma \neq k(k\pm\alpha) \tag{86}$$

Again, the mode-acceleration shear force solution for the harmonic force problem converges far more rapidly than the mode-displacement solution. Figure 14 shows the shear dynamic magnification factors at mid-span for the 2-term, 4-term, and 150-term mode-displacement solutions and the 2-term mode-acceleration solution when $\gamma = 0.5$. The 2-term mode-acceleration solution is far better than the 4-term mode-displacement solution and almost identical to the 150-term mode-displacement solution. The discontinuity at ct/L = 0.5 is fully represented by the 2-term mode-acceleration solution and still only approximated by the 150-term mode-displacement solution. Figures 15 and 16 show that for $\gamma = 2.5$ and $\gamma = 6.5$, the 2-term mode-acceleration solution remains superior to the 4-term mode-displacement solution, especially in the vicinity of the discontinuity at ct/L = 0.5.



150-Term Mode Disp. 4-Term Mode Disp. 2-Term Mode Disp. 2-Term Mode Accel.

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Figure 14 Dynamic magnification factors for mid-span shear force based on 2-, 4- and 150-term mode-displacement and 2-term mode-acceleration solutions (γ=0.5).

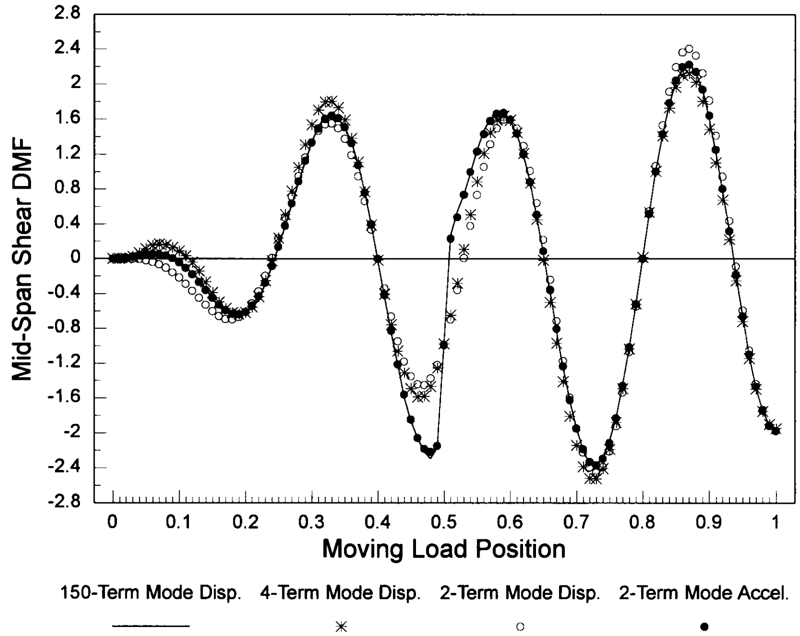

Figure 15 Dynamic magnification factors for mid-span shear force based on 2-, 4- and 150-term mode-displacement and 2-term mode-acceleration solutions (γ=2.5).

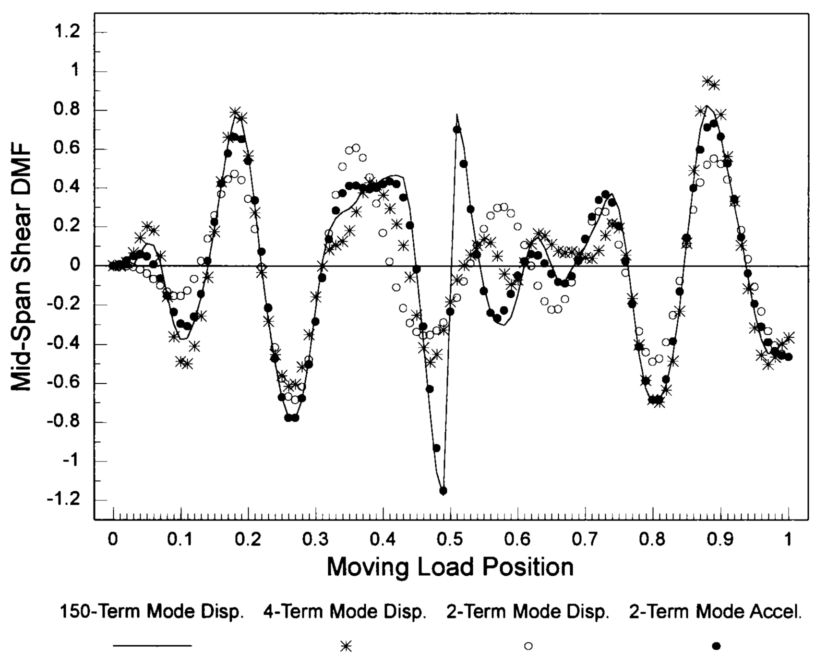

Figure 16 Dynamic magnification factors for mid-span shear force based on 2-, 4- and 150-term mode-displacement and 2-term mode-acceleration solutions (γ=6.5).

CONCLUSIONS

The analytical solution to the fundamental moving load problem is an infinite series solution. The solution for the beam deflection converges rapidly and a very accurate approximate solution can be obtained by using either the mode-displacement or mode-acceleration method and truncating the infinite series. For a constant load moving across a damped or undamped beam, one term of the solution is usually all that is needed for a reasonable approximation. When the moving load is harmonic, the forcing frequency determines how many modes must be considered, and, in general, it is not necessary to include terms in the displacement solution which correspond to natural frequencies higher than the forcing frequency.

However, the convergence property of the mode-displacement solution deteriorates when it is used to calculate the bending moment and shear force in the beam. The mode-acceleration solution obtained by separating the pseudo-static response from the dynamic component of the solution greatly improves solution convergence for all of the moving load cases presented in this paper.

Another conclusion relates to the fact that the pseudo-static accelerations are much smaller than the dynamic accelerations, even though the pseudo-static term dominates the displacement and strains. Therefore, the practical use of accelerometer data is likely to be limited for such a structure.

RECOMMENDATIONS

The insight gained from this solution technique is directly applicable to the study of transportation structures, because, in general, the pseudo-static response is the dominant component of the total response of most highway bridges. This is attributed to the fact that vehicle weight is frequently less than one percent of the total weight of the bridges which the vehicles must cross. When modeling highway bridge behavior, the mode-acceleration solution techniques can significantly reduce computational time and labor. Furthermore, the generalized mode-acceleration solution to the moving load problem presented in this paper can be readily applied to any time-dependent moving force, including randomly varying vehicular loads.

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