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# Bounded Tracking for Nonminimum Phase Nonlinear Systems with Fast Zero Dynamics

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**Abstract**

In this paper, we derive tracking control laws for nonminimum phase nonlinear systems with both fast and slow, possibly unstable, zero dynamics. The fast zero dynamics arise from a perturbation of a *nominal* system. These fast zeros can be problematic in that they may be in the right half plane and may cause large magnitude tracking control inputs. In this paper, we combine the ideas from recent work of Hunt, Meyer, and Su with that of Devasia, Paden, and Chen on an asymptotic tracking procedure for nonminimum phase nonlinear systems. We give (somewhat subtle) conditions under which the tracking control input is bounded as the magnitude of the perturbation of the nominal system becomes zero. Explicit bounds on the control inputs are calculated using some interesting non-standard singular perturbation techniques. The method is applied to the simplified planar dynamics of VTOL and CTOL aircraft.

**Keywords:** Nonlinear control, zero dynamics, exact and asymptotic tracking, nonminimum phase, singular perturbation.

## 1 Introduction

In this paper, we discuss tracking using bounded inputs for nonlinear nonminimum phase systems with "fast zero dynamics". While exact and asymptotic tracking for nonlinear minimum phase systems has now been well understood for some time (see [1] for a comprehensive discussion), tracking for nonminimum phase nonlinear systems has been a tougher nut to crack. Early progress was made by the nonlinear regulator approach of Isidori-Byrnes [2] which extended to the nonlinear case results of the Francis-Wonham regulator. A major advance in a general framework for tracking for nonminimum phase systems was made by Devasia, Paden and Chen [3], [4] in which they provide a *non-causal* exact tracking compensator for nonlinear (possibly multi-input multi-output) systems. These results generalize the earlier results of Lanari and Wen [5] for linear time invariant systems.

In parallel, we have been interested in the control of MIMO nonlinear systems where the decoupling matrix is close to being singular. We were heavily motivated in this regard by the flight control of a Vertical Take Off and Landing aircraft (VTOL) Harrier in [6]. For systems such as this, the presence of the small control terms not only meant large control effort, but was symptomatic of "fast zero dynamics" which were possibly nonminimum phase. We explained this phenomenon in [7] as a singular perturbation of the zero dynamics and discussed approximate methods for controlling the zero dynamics. Related methods are presented in [8] and [9]. In new work on the problem of stable tracking for MIMO systems with fast zero dynamics, we attempted in [10] to apply the Devasia-Paden-Chen techniques to a model of a Conventional Take Off and Landing (CTOL) aircraft and make comparisons with the other approximate techniques discussed above. The difficulty that we encountered was the presence of large magnitude control inputs in directly applying the Devasia-Paden-Chen scheme. More recently, Hunt, Meyer and Su in [11] and [12] proposed an interesting variant to the application of the Devasia-Paden-Chen scheme by applying the method not to the given system, but to an "error system" obtained by comparing the given system to a nominal version of the system, which does not have the fast zero dynamics.

Our paper attempts to close the loop on this entire circle of ideas and to provide a reasonably complete[1] description of conditions under which bounded tracking control laws for nonlinear control systems with fast zero dynamics exist (in the limit that the perturbation of the system dynamics goes to zero). The paper considers a general class of invertible (but not necessarily under static state feedback) nonlinear systems and as such is a generalization of the results in [11]. Unlike [11], we consider only systems which are affine in the inputs, yet this allows us to derive conditions under which bounded tracking may be proved and to work out the details of explicit bounds on the system inputs. What is striking about the current paper is the delicacy of the asymptotic calculations involving many interesting concepts from singular perturbations and differential equations.

The outline of this paper is as follows. In Section 2, we consider SISO systems: we review the characterization of the fast zero dynamics and the Devasia-Paden-Chen scheme and show how these can be combined to produce bounded tracking control laws for the SISO case. Section 3 applies the theory to two MIMO flight control examples, planar models of VTOL and CTOL aircraft.

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[1]We say *reasonably*, because we have some single time scale assumptions on the "fast" zero dynamics, which we would like to eventually remove.

Section 4 presents the conclusions. A longer version of this paper, containing a proof of Theorem 1, a detailed characterization of the zero dynamics for the MIMO case, and more examples, is available in [13].

## 2 Bounded Tracking for SISO systems

In this section, we will be concerned by a family of systems depending on a parameter $\epsilon$, described by equations of the form

$$\begin{aligned} \dot{x} &= f(x,\epsilon) + g(x,\epsilon)u \\ y &= h(x,\epsilon) \end{aligned} \tag{1}$$

where $f(x,\epsilon)$ and the columns of $g(x,\epsilon)$ are smooth vector fields and $h(x,\epsilon)$ is a smooth function, defined in a neighborhood of $(x_0, 0)$ in $\mathbb{R}^n \times \mathbb{R}_+$. We will refer to the system of (1) with $\epsilon = 0$ as the *nominal* system and with $\epsilon \neq 0$ as the *perturbed* system. We will assume that $x = x_0$ is an equilibrium point for the nominal system, that is $f(x_0, 0) = 0$, and without loss of generality we will assume that $h(x_0, 0) = 0$.

### 2.1 Singularly Perturbed Driven Dynamics

In [7] it was shown that if the system (1) has relative degree $r(\epsilon) = r$ for $\epsilon \neq 0$, and relative degree $r(\epsilon) = r + d$ for $\epsilon = 0$, then there are *fast time scale zero dynamics* for the perturbed nonlinear system. This is in itself a rather surprising conclusion: we review one such result from these papers. As a consequence of the definition of relative degree we have that $r(\epsilon) = r$ and $r(0) = r + d$ implies that $\forall \epsilon \neq 0$

$$\begin{gathered} L_g h(x,\epsilon) = L_g L_f h(x,\epsilon) = \cdots = L_g L_f^{r-2} h(x,\epsilon) = 0 \\ \forall x \text{ near } x_0, \text{ and } L_g L_f^{r-1} h(x_0,\epsilon) \neq 0 \end{gathered} \tag{2}$$

and for $\epsilon = 0$,

$$\begin{gathered} L_g h(x,0) = L_g L_f h(x,0) = \cdots = L_g L_f^{r+d-2} h(x,0) = 0 \\ \forall x \text{ near } x_0, \text{ and } L_g L_f^{r+d-1} h(x_0,0) \neq 0 \end{gathered} \tag{3}$$

To keep the singularly perturbed zero dynamics from demonstrating *multiple time scale* behavior[2] we assume that for $0 \leq k \leq d$

$$L_g L_f^{r-1+k} h(x,\epsilon) = \epsilon^{d-k} \alpha_k(x,\epsilon) \tag{4}$$

where each $\alpha_k(x,\epsilon)$ is a smooth function of $(x,\epsilon)$ in a neighborhood of $(x_0, 0)$. The choice of $L_g L_f^{r-1} h(x,\epsilon) = O(\epsilon^d)$ rather than $O(\epsilon)$ is made to keep from having to use fractional powers of $\epsilon$. What is critical about the assumption (4) is the decreasing powers of $\epsilon$ dependence as $k$ increases from 0 to $d$.

As is standard in the literature, we will denote by $\xi \in \mathbb{R}^{r+d}$ the vector corresponding to the first $r + d$ derivatives of the output of the system in (1), given by

$$\xi = \begin{pmatrix} h(x,\epsilon) \\ L_f h(x,\epsilon) \\ \vdots \\ L_f^{r-1} h(x,\epsilon) \\ \vdots \\ L_f^{r+d-1} h(x,\epsilon) \end{pmatrix} \tag{5}$$

where the first $r$ coordinates correspond to the first $r$ derivatives of the output, and the full set of $r + d$ coordinates, at $\epsilon = 0$, are the first $r + d$ derivatives of the output of the nominal system. It was shown in [7] that for small $\epsilon$ we have the following "normal form" (in the sense of [1]):

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_r &= \xi_{r+1} + \epsilon^d \alpha_0(\xi,\eta,\epsilon)u \\ \dot{\xi}_{r+1} &= \xi_{r+2} + \epsilon^{d-1} \alpha_1(\xi,\eta,\epsilon)u \\ \dot{\xi}_{r+2} &= \xi_{r+3} + \epsilon^{d-2} \alpha_2(\xi,\eta,\epsilon)u \\ &\vdots \\ \dot{\xi}_{r+d} &= b(\xi,\eta,\epsilon) + a(\xi,\eta,\epsilon)u \\ \dot{\eta} &= q(\xi,\eta,\epsilon) \end{aligned} \tag{6}$$

Here, we have introduced the smooth functions $a$, $b$, and $q$; the details of how $a$ and $b$ depend on $f$, $g$, and $h$ are discussed in [7].

Using the change of coordinates for the perturbed system given by

$$z_1 = \xi_{r+1}, \quad z_2 = \epsilon \xi_{r+2}, \quad \cdots \quad z_d = \epsilon^{d-1} \xi_{r+d} \tag{7}$$

it may be verified that the zero dynamics (corresponding to the output of the perturbed system being held identically to zero) have the form

$$\begin{aligned} \epsilon \dot{z}_1 &= -\tfrac{\alpha_1}{\alpha_0} z_1 + z_2 \\ \epsilon \dot{z}_2 &= -\tfrac{\alpha_2}{\alpha_0} z_1 + z_3 \\ &\vdots \\ \epsilon \dot{z}_d &= -\tfrac{a}{\alpha_0} z_1 + \epsilon^d b \\ \dot{\eta} &= q(z,\eta,\epsilon) \end{aligned} \tag{8}$$

Note that $\eta \in \mathbb{R}^{n-r-d}, z \in \mathbb{R}^d$. Also, we have abused notation for $q$ from equation (6). Thus, the zero dynamics appear in singularly perturbed form, ie.

$$\begin{aligned} \epsilon \dot{z} &= r(z,\eta,\epsilon) \\ \dot{\eta} &= q(z,\eta,\epsilon) \end{aligned} \tag{9}$$

with $n - r - d$ slow states ($\eta$) and $d$ fast states ($z$). This is now consistent with the zero dynamics for the system at $\epsilon = 0$ given by

$$\dot{\eta} = q(0,\eta,0) \tag{10}$$

Thus, the presence of small terms in $L_g L_f^{r-1+k} h(x,\epsilon)$ for $0 \leq k \leq d$, causes the presence of singularly perturbed zero dynamics. The Jacobian matrix evaluated

[2] This is an interesting case and though it is no different conceptually, the notation and the details of the assumptions needed are more involved.

at $z = 0, \epsilon = 0$ of the fast zero subsystem is obtained to be

$$\begin{bmatrix} a_1(0,\eta,0) & 1 & 0 & \cdots & 0 \\ a_2(0,\eta,0) & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{d-1}(0,\eta,0) & 0 & 0 & \cdots & 1 \\ a_d(0,\eta,0) & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{11}$$

Here $a_i = -\frac{\alpha_i}{\alpha_0}(\xi,\eta,\epsilon)$ for $1 \leq i < d$, and $a_d = -\frac{a}{\alpha_0}(\xi,\eta,\epsilon)$. It is clear that the perturbed system may be *nonminimum phase* either for positive $\epsilon$, negative $\epsilon$, or both positive and negative $\epsilon$ (according to whether the matrix in (11) has eigenvalues in $\mathbb{C}_-$, $\mathbb{C}_+$ or has indefinite inertia, respectively). If (10) has a stable equilibrium point at the origin (corresponding to the nominal system being minimum phase), but the origin of the system (8) is unstable, (corresponding to the perturbed system being nonminimum phase), we refer to these systems as *slightly nonminimum phase*.

We will need to generalize the preceding discussion of *zero dynamics* to the *driven dynamics* corresponding to the problem of tracking a desired output trajectory $y_D(t)$. If the output $y(t) \equiv y_D(t)$, it follows that, for the perturbed system,

$$\xi_D(t) = \begin{pmatrix} y_D(t) \\ \dot{y}_D(t) \\ \vdots \\ y_D^{(r-1)}(t) \end{pmatrix} \tag{12}$$

Then, the driven dynamics of the system are given by (6) with the choice of error coordinates

$$\begin{aligned} v_1 = \xi_{r+1} - y_D^{(r)}(t),\ v_2 = \epsilon(\xi_{r+1} - y_D^{(r+2)}(t)), \ldots \\ v_d = \epsilon^{d-1}(\xi_{r+d} - y_D^{(r+d-1)}(t)) \end{aligned} \tag{13}$$

and the input

$$u = \frac{-v_1}{\epsilon^d \alpha_0}$$

to be

$$\begin{aligned} \epsilon\dot{v}_1 &= a_1 v_1 + v_2 \\ \epsilon\dot{v}_2 &= a_2 v_1 + v_3 \\ &\vdots \\ \epsilon\dot{v}_d &= a_d v_1 + \epsilon^d(b - y_D^{(r+d)}(t)) \\ \dot{\eta} &= q(\xi,\eta,\epsilon) \end{aligned} \tag{14}$$

### 2.2 Two Step Procedure for Bounded Tracking

For systems of the form (1), difficulties with bounded tracking, that is the problem of finding bounded control laws for making $y(t)$ track a prescribed bounded trajectory $y_D(t)$ (with its first $r+d$ derivatives also bounded) may arise for two reasons:

1. **The nominal system may be nonminimum phase**. This means that the zero dynamics (10) of the nominal system are unstable.
2. **The presence of terms of $O(\epsilon^d)$ for $L_g L_f^{r-1} h(x,\epsilon)$ for the perturbed system.** This, in turn, may cause two different kinds of problems:
   (a) *The (exact) tracking control law given by (15) may become unbounded as $\epsilon \to 0$.*
   $$u = \frac{1}{L_g L_f^{r-1} h(x,\epsilon)}(y_D^{(r)} - L_f^r h(x,\epsilon)) \tag{15}$$
   (b) *The fast time scale zero dynamics of the perturbed system are likely to be nonminimum phase* as noted in the discussion following (11).

In this subsection, we combine some interesting new results of Devasia, Paden and Chen [3], [4] on output tracking using bounded inputs for nonlinear systems with hyperbolic[3](but not necessarily minimum phase) zero dynamics, with a two step procedure suggested by Hunt, Meyer and Su in [11] and [12] which we use to derive conditions for boundedness of the tracking control law (15). The algorithm proceeds in two steps:

**Step 1:** One finds a bounded input to cause the *nominal* system to track $y_D(t)$. If the nominal system is nonminimum phase, the algorithm of Devasia-Paden-Chen is applied, as follows. The nominal system with relative degree $r+d$ has driven dynamics given by

$$\dot{\eta} = q(\xi_D, \eta, 0) \tag{16}$$

with $\eta \in \mathbb{R}^{n-r-d}$ and $\xi_D$ given by (12). The Devasia-Paden-Chen scheme (time invariant version) consists of defining a linear approximant to the smooth function $q$, usually

$$Q := \frac{\partial q}{\partial \eta}(\xi_{D_0}, \eta_0, 0)$$

and then under the hypothesis that $Q$ is hyperbolic (i.e it has no eigenvalues on the $j\omega$ axis) and that the residual error defined by $r(\xi_D,\eta,0) := q(\xi_D,\eta,0) - Q\eta$ is Lipschitz continuous in both of its arguments, a condition referred to as *locally approximately linear*:

$$|r(\xi_1,\eta_1,0) - r(|\xi_2,\eta_2,0)| < K_1|\xi_1 - \xi_2| + K_2|\eta_1 - \eta_2|$$

with Lipschitz constants $K_1, K_2$ small enough, there exists for given bounded $\xi_D$ a bounded solution $\eta(t)$ satisfying $\lim_{t\to\pm\infty} \eta(t) = 0$, which is obtained as the fixed point of the following integral equation:

$$\eta(t) = \int_{-\infty}^{\infty} \Phi(t-\tau) r(\xi_D, \eta, 0) d\tau \tag{17}$$

Here $\Phi(t)$ is the Caratheodory solution of the matrix differential equation

$$\dot{X} = QX \quad X(\pm\infty) = 0 \quad X(0+) - X(0-) = I$$

Furthermore, one can find $K_3$ such that

$$|\eta(t)| \leq K_3 \sup_t |\xi_D(t)|$$

The strategy for solving the fixed point equation (17) is to use a Picard Lindelöf iteration scheme with any initial guess $\eta^0(t) : -\infty < t < \infty$,

$$\eta^{m+1}(t) = \int_{-\infty}^{\infty} \Phi(t-\tau) r(\xi_D, \eta^m, 0) d\tau$$

[3]More precisely, in the slowly time varying case, kinematically equivalent to uniformly hyperbolic

The resulting controller is synthesized by using the bounded $\eta(t)$ to obtain

$$u(t) = \frac{y_D^{(r+d)}(t) - b(\xi_D(t), \eta(t), 0)}{a(\xi_D(t), \eta(t), 0)} \tag{18}$$

A drawback of this control law is that it is *non-causal.* One way this is remedied is to use a preview of a certain duration. Also, while the algorithm as stated is used for exact tracking, asymptotic tracking is achieved by stabilizing the linearization of the system (1). Thus, for small enough $\xi_D$ and $(x(0) - x_0)$, bounded tracking is achieved using a non-causal input. A time varying version of the algorithm [4] may be used to linearize equation (16) about $\xi_D(t)$ to produce a time varying matrix

$$Q(t) := \frac{\partial q}{\partial \eta}(\xi_D(t), \eta_0, 0)$$

If $Q(t)$ is slowly time varying, and is kinematically equivalent to a two-block diagonal matrix with an exponentially stable and an exponentially unstable state transition matrix (*uniformly hyperbolic*) the Picard iteration can be applied as before.

**Step 2:** Denote by $u^0(t)$ the input (18) required to produce exact tracking with bounded inputs for the nominal system and let the resultant state trajectory be given by $\xi^0(t), \eta^0(t)$. That is,

$$\begin{array}{rcl} \dot{y}_D = & \dot{\xi}_1^0 = & \xi_2^0 \\ \ddot{y}_D = & \dot{\xi}_2^0 = & \xi_3^0 \\ \vdots & \vdots & \\ y_D^{(r)} = & \dot{\xi}_r^0 = & \xi_{r+1}^0 \\ y_D^{(r+1)} = & \dot{\xi}_{r+1}^0 = & \xi_{r+2}^0 \\ y_D^{(r+2)} = & \dot{\xi}_{r+2}^0 = & \xi_{r+3}^0 \\ \vdots & \vdots & \\ y_D^{(r+d)} = & \dot{\xi}_{r+d}^0 = & b(\xi^0, \eta^0, 0) + a(\xi^0, \eta^0, 0)u^0 \\ & \dot{\eta}^0 = & q(\xi^0, \eta^0, 0) \end{array} \tag{19}$$

Now, define the input $u(t)$ for the perturbed system for exact tracking. The system equations are given by the equations (6). Note that the first $r$ coordinates of the perturbed system match those of the nominal system. To obtain the control $u(t)$ for the perturbed system, the expression for $\dot{\xi}_r$ from (6) is equated with that of $\dot{\xi}_r^0$ from (19). Also, define as before

$$\begin{array}{rll} v_i = & \epsilon^{i-1}(\xi_{r+i} - \xi_{r+i}^0) & i = 1, \ldots, d \\ v_{i+d} = & \eta_i - \eta_i^0 & i = 1, \ldots, n-r-d \end{array}$$

Subtracting equations (19) from equations (6) yields an algebraic equation for the control, namely:

$$v_1 = -\epsilon^d \alpha_0(\xi, \eta, \epsilon)u \tag{20}$$

and an error system

$$\begin{array}{rl} \epsilon\dot{v}_1 = & v_2 + a_1(\xi, \eta, \epsilon)v_1 \\ \epsilon\dot{v}_2 = & v_3 + a_2(\xi, \eta, \epsilon)v_1 \\ \vdots & \\ \epsilon\dot{v}_d = & a_d(\xi, \eta, \epsilon)v_1 + \epsilon^d(b(\xi, \eta, \epsilon) \\ & -b(\xi^0, \eta^0, 0) - a(\xi^0, \eta^0, 0)u^0) \\ \dot{v}_{d+1} = & q_1(\xi, \eta, \epsilon) - q_1(\xi^0, \eta^0, 0) \\ \vdots & \\ \dot{v}_{n-r} = & q_{n-r-d}(\xi, \eta, \epsilon) - q_{n-r-d}(\xi^0, \eta^0, 0) \end{array} \tag{21}$$

One now applies the Devasia-Paden algorithm [4] to the system of (21) to find the bounded control $u(t)$ for exact tracking. For the purpose of applying this algorithm, it is necessary to consider the linear approximant to the right hand side of (21). This is conveniently chosen to be

$$\begin{bmatrix} \begin{bmatrix} a_1(\xi^0, \eta^0, 0) & 1 & \cdots & 0 \\ a_2(\xi^0, \eta^0, 0) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ a_{d-1}(\xi^0, \eta^0, 0) & 0 & \cdots & 1 \\ a_d(\xi^0, \eta^0, 0) & 0 & \cdots & 0 \end{bmatrix} & 0 \\ \frac{\partial q(\xi^0, \eta^0, 0)}{\partial \xi} & \frac{\partial q(\xi^0, \eta^0, 0)}{\partial \eta} \end{bmatrix} \tag{22}$$

The matrix in (22) is a time varying one: the time dependence of $\xi^0, \eta^0$ has been dropped for brevity. To apply the results of Devasia-Paden to this system we need to assume that it is *slowly varying* in time and *kinematically equivalent to a uniformly hyperbolic matrix.* One convenient way to do this is to *assume that the nominal trajectory $y_D$ and its first $r+d$ derivatives are small enough and that the functions $a_i, 1 \leq i \leq d$ are Lipschitz continuous in their arguments.*

Thus for fixed $\epsilon > 0$, the control law $u$ calculated in (20) is bounded. What is less clear is the magnitude of the control law as $\epsilon \to 0$. The following theorem gives conditions under which the control law remains bounded as $\epsilon \to 0$. The proof is available in [13].

**Theorem 1 (Bounded Tracking as $\epsilon \to 0$)** *Assuming that: the driven dynamics of the nominal system (16) is hyperbolic and slowly time varying; the error system (21) is hyperbolic, and each function $a_i(\xi, \eta, \epsilon)$, $\frac{\partial q(\xi, \eta, \epsilon)}{\partial \xi}$, $\frac{\partial q(\xi, \eta, \epsilon)}{\partial \eta}$ in the Jacobian (22) is smooth and slowly time varying; and, in addition, the functions $a_i(\xi, \eta, \epsilon)$ in the Jacobian (22) satisfy the following Lipschitz condition:*

$$\begin{array}{l} |a_i(\xi^1, \eta^1, \epsilon) - a_i(\xi^2, \eta^2, 0)| \leq \\ L_{i,1}(\epsilon)|\xi_{r+1}^1 - \xi_{r+1}^2| + \ldots + L_{i,d}(\epsilon)|\xi_{r+d}^1 - \xi_{r+d}^2| + \\ L_{i,d+1}(\epsilon)|\eta_1^1 - \eta_1^2| + \ldots + L_{i,n-r}(\epsilon)|\eta_{n-r-d}^1 - \eta_{n-r-d}^2| \end{array} \tag{23}$$

*where*

$$\begin{array}{rll} L_{i,j}(\epsilon) = & o(\epsilon^{j+1}) & i = 1, \ldots, d, j = 1, \ldots, d \\ L_{i,j}(\epsilon) = & k\epsilon^{1+\alpha} & i = 1, \ldots, d, j = d+1, \ldots, n-r \end{array} \tag{24}$$

*then, under these assumptions, the input $u(t)$ required for exactly tracking a desired output signal $y_D(t)$ with bounded derivatives (all sufficiently small), is bounded as $\epsilon \to 0$.* ■

The derivation of the singularly perturbed zero and driven dynamics for the MIMO counterpart of (1) is presented in [13].

## 3 Flight Control Examples

Our examples are motivated by our study of flight control for vertical take off and landing (VTOL) and conventional take off and landing (CTOL) aircraft, in [6] and [10]. These are two-input two-output systems in which the nominal systems have no zero dynamics.

### 3.1 PVTOL Aircraft

We consider a model of a planar vertical takeoff and landing (PVTOL) aircraft, an example being the YAV-8B Harrier of the McDonnell Douglas Corporation. The simplified PVTOL equations, corresponding to the aircraft in the hover mode, derived in [6], are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_5 u_1 + \epsilon^2 \cos x_5 u_2 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \cos x_5 u_1 + \epsilon^2 \sin x_5 u_2 - 1 \\ \dot{x}_5 &= x_6 \\ \dot{x}_6 &= u_2 \end{aligned} \tag{25}$$

where $x_1 = x$, $x_3 = y$, $x_5 = \theta$. Note that we have used $\epsilon^2$ in the equations instead of the standard $\epsilon$. We choose the standard outputs $y_1 = x_1$, $y_2 = x_3$.

The vector relative degree is well defined for the perturbed system:

$$\begin{aligned} r_1(\epsilon) &\equiv s = 2 \\ r_2(\epsilon) &\equiv r = 2 \end{aligned}$$

At $\epsilon = 0$, the system (25) does not have vector relative degree. The zero dynamics manifold of the unperturbed system is trivial, and the two time scales assumption is satisfied.

The singularly perturbed zero dynamics are given by

$$\begin{aligned} \epsilon\dot{z}_1 &= z_2 \\ \epsilon\dot{z}_2 &= \sin z_1 \end{aligned} \tag{26}$$

where $z_1 = x_5$ and $z_2 = \epsilon x_6$.

The bounded inputs required for the nominal system to track $y_{D1}$ and $y_{D2}$ are calculated as:

$$\ddot{u}_1^0 = x_6^2 u_1^0 - \sin x_5^0 y_{D1}^{(4)} + \cos x_5^0 y_{D2}^{(4)} \tag{27}$$

$$u_2^0 = -2x_6^0 \dot{u}_1^0 / u_1^0 - y_{D1}^{(4)} \cos x_5^0 / u_1^0 - y_{D2}^{(4)} \sin x_5^0 / u_1^0 \tag{28}$$

The control law $u = [u_1\ u_2]$ for the perturbed system is calculated by solving the differential equations

$$\begin{aligned} \epsilon\dot{v}_1 &= v_2 \\ \epsilon\dot{v}_2 &= u_1^0 \sin v_1 - \epsilon^2 u_2^0 \end{aligned} \tag{29}$$

where

$$\begin{aligned} \sin v_1 &= \epsilon^2 u_2 / u_1^0 \\ v_2 &= \epsilon(x_6 - x_6^0) \end{aligned} \tag{30}$$

and by solving the algebraic equation

$$u_1 \sin(v_1 + x_5^0) = u_1^0 \sin x_5^0 + \epsilon^2 \cos(v_1 + x_5^0) u_2 \tag{31}$$

The control

$$u_2 = u_1^0 \sin v_1 / \epsilon^2 \tag{32}$$

$$u_1 = \frac{u_1^0 \sin x_5^0 + \epsilon^2 \cos(v_1 + x_5^0) u_2}{\sin(v_1 + x_5^0)} \tag{33}$$

is bounded as $\epsilon \to 0$. The proof does not follow directly from Theorem 1, because the form of (29) is not in the standard "quasi-linear" form of (21). We maintain the appealing simplicity of (29) and use the following proposition, the proof of which is available from the authors.

**Proposition 2** *Consider the nonlinear differential equation (34)*

$$\dot{x} = Ax + \psi(x) + \epsilon^2 v \tag{34}$$

*in which $A \in \mathbb{R}^{n\times n}$ is hyperbolic and $\psi$ belongs to a class of functions called Lip(r) in [14]:*

$$\sup_{|x|\le r} |\psi(x)| \le L(r)|x|$$

*and $L$ is a continuous, nondecreasing, nonnegative function on $[0, \infty)$ with $L(0) = 0$. Then if $\epsilon$ is small enough the unique bounded solution of (34) is of $O(\epsilon^2)$.*

### 3.2 CTOL Aircraft

The second aircraft model we consider is the planar conventional take off and landing (PCTOL) aircraft introduced in [10]. The simplified PCTOL equations are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (-D + u_1)\cos x_5 - (L - \epsilon^2 u_2)\sin x_5 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= (-D + u_1)\sin x_5 + (L - \epsilon^2 u_2)\cos x_5 - 1 \\ \dot{x}_5 &= x_6 \\ \dot{x}_6 &= u_2 \end{aligned} \tag{35}$$

where $x_1 = x$, $x_3 = y$, $x_5 = \theta$. Unlike the example of the hovering VTOL, we now have aerodynamic forces: $L$ and $D$, the aerodynamic lift and drag forces given by

$$L = a_L(x_2^2 + x_4^2)(1 + c\alpha) \tag{36}$$

$$D = a_D(x_2^2 + x_4^2)(1 + b(1 + c\alpha)^2) \tag{37}$$

and $\alpha$ is the angle of attack

$$\alpha = x_5 - \tan^{-1}(x_4/x_2) \tag{38}$$

The coordinates are illustrated in Figure 1. The angle of attack $\alpha$ is assumed to be zero for these calculations. The outputs are $y_1 = x_1$, $y_2 = x_3$.

The vector relative degree is well defined for the perturbed system:

$$\begin{aligned} r_1(\epsilon) &\equiv s = 2 \\ r_2(\epsilon) &\equiv r = 2 \end{aligned}$$

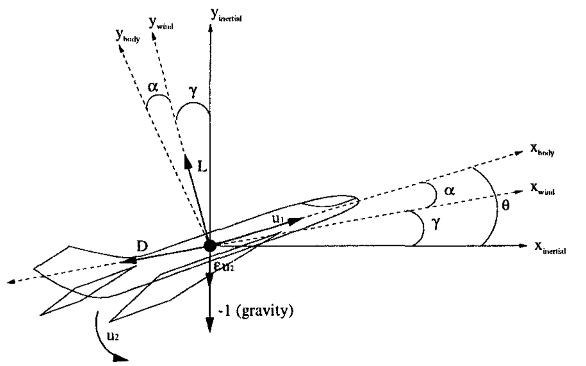


**Figure 1:** PCTOL aircraft

At $\epsilon = 0$, the system (35) does not have vector relative degree. The zero dynamics manifold of the unperturbed system is trivial, and the two time scales assumption is satisfied. The singularly perturbed zero dynamics are given by

$$\epsilon \dot{z}_1 = z_2 \quad (39)$$
$$\epsilon \dot{z}_2 = -\cos z_1 \quad (40)$$

where $z_1 = x_5$ and $z_2 = \epsilon x_6$. Note that we have again used a non-standard form for simplicity. The bounded inputs required for the nominal system to track $y_{D1}$ and $y_{D2}$ are calculated as in the PVTOL case by dynamic extension. The calculations are more involved because of the presence of the lift and drag and are available from the authors. As before, they result in bounded values for $\ddot{u}_1^0$ and $u_2^0$. The control law $u = [u_1 \ u_2]$ for the perturbed system is calculated by solving the differential equations

$$\epsilon \dot{v}_1 = v_2 \quad (41)$$
$$\epsilon \dot{v}_2 = u_1^0 \sin v_1 - D^0 \sin v_1 - L^0 \cos v_1 + L - \epsilon^2 u_2^0 \quad (42)$$

where

$$\begin{aligned} \sin v_1 &= \tfrac{1}{u_1^0}(\epsilon^2 u_2 - L + D^0 \sin v_1 + L^0 \cos v_1) \\ v_2 &= \epsilon(x_6 - x_6^0) \end{aligned} \quad (43)$$

and $L^0$, $D^0$ correspond to the aerodynamic lift and drag at $\epsilon = 0$.

The control

$$u_2 = \frac{1}{\epsilon^2}(u_1^0 \sin v_1 + L - D^0 \sin v_1 - L^0 \cos v_1) \quad (44)$$
$$\begin{aligned} u_1 = {} & \cos^{-1}(v_1 + x_5^0)((u_1^0 - D^0)\cos x_5^0 - L^0 \sin x_5^0 \\ & + D\cos(v_1 + x_5^0) + (L - \epsilon^2 u_2)\sin(v_1 + x_5^0)) \end{aligned} \quad (45)$$

is bounded as $\epsilon \to 0$. The proof follows from Proposition 2.

## 4 Conclusions

This work presents a method for tracking systems with singularly perturbed zero dynamics. We combined recent results in exact tracking by Devasia, Paden, and Chen, and Hunt, Meyer, and Su, with a general framework for describing nonlinear nonminimum phase systems with singularly perturbed zero dynamics. Using this framework, we prove boundedness of the control inputs required for exact tracking. We showed, using planar dynamic models of VTOL and CTOL aircraft, that this method may be successfully applied to the slightly nonminimum phase systems characteristic of flight control.

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