ON-ORBIT COLLISION HAZARD ANALYSIS IN LOW EARTH ORBIT USING THE POISSON PROBABILITY DISTRIBUTION

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INTRODUCTION

Any object placed in orbit around the Earth has a risk of colliding with other orbiting objects. For mission planning and safety reasons, it is useful to understand and assess the level of collision risk by determining the probability that a given object will collide with currently orbiting objects. Several techniques can be used to calculate the collision hazard for orbiting spacecraft. This primer introduces one method for analyzing collision risks that utilizes the Poisson distribution and principles of the kinetic theory of gases. Also, this primer addresses its application to low Earth orbit (LEO) only.

THE PROBABILITY OF COLLISION

The Poisson probability distribution function is given by the equation:

(1)

$$P_k = \frac{R^k e^{-R}}{k!}$$

where: P_k = Probability of k events
k = Number of events, and
R = Rate of occurrence parameter.

The probability of no events occurring (k = 0) can be found:

(2)

$$P_0 = \frac{(R^0 e^{-R})}{0!} = e^{-R}$$

In general, a Poisson distribution is used in those situations when the probability of two or more events occurring is very small. If an assumption is made that the probability of more than one event is negligible, the probability of one collision can be represented by:

(3)

$$P_1 = 1 - P_0 = 1 - e^{-R}$$

The rate of occurrence parameter, R, is obtained from the kinetic theory of gasses. It is defined as the number of collisions during a given length of time between one molecule of gas with others in a confined volume, and is given by the product of the molecule's size,[1] density of the gas, relative speed, and the length of time considered.

By applying the same kinetic gas assumptions to a confined volume in space, the collision rate between a specified object and all other objects in a given volume can be found for a specified length of time. For an object in orbit, the rate of occurrence parameter would then be the product of that object's cross-sectional area (AC), average spatial density of all other objects (SPD), average relative velocity (VR) between the object under consideration and all other objects, and the time (T), which is the length of time the object is exposed to the given population density, SPD. Substituting the analytic form of the rate of occurrence parameter (AC*SPD*VR*T) for orbital collisions into Equation (3) yields Equation (4); the probability that there will be a collision between a given object and any other object in a given volume.

(4)

$$PC = 1 - \exp(-AC*SPD*VR*T)$$

where: PC = Probability of collision for the duration of time, T
AC = Cross-sectional area, km^2
SPD = Spatial density, objects/km^3
VR = Relative velocity, km/s, and
T = Time at risk, seconds.

CALCULATING THE PROBABILITY OF COLLISION

Equation (4) shows there are four parameters to be considered when calculating the probability of a collision. Each of these terms are discussed below.

Spatial Density (SPD): Spatial density is the number of objects that reside in a given volume of space. Figure 1 shows the object spatial density up to 2000 km altitude for 50 km altitude "shells." For example, on the average[2], 40 objects reside in the 1700 - 1750 km region. The volume of the "shell," with an inner radius of 8078 135 km (1700 km altitude) and an outer radius of 8128.135 km (1750 km altitude), is 3.69E+10 km^3. The spatial density is then the number of objects divided by the volume or 1.06E-9 objects/km^3. Appendix A provides a table of the spatial density values from Figure 1.

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[1] It is assumed that all molecules in a gas have the same diameter, d. Therefore, when deriving the mean free path from the kinetic theory of gases, the simplification is made that the diameter of the molecule under consideration is 2d, and the impacting molecules are point masses.

[2] In reality, the number of objects in a region at any given time changes as objects in elliptical orbits pass between regions. For this reason, an average number of objects is used to describe the population in time invariant terms.

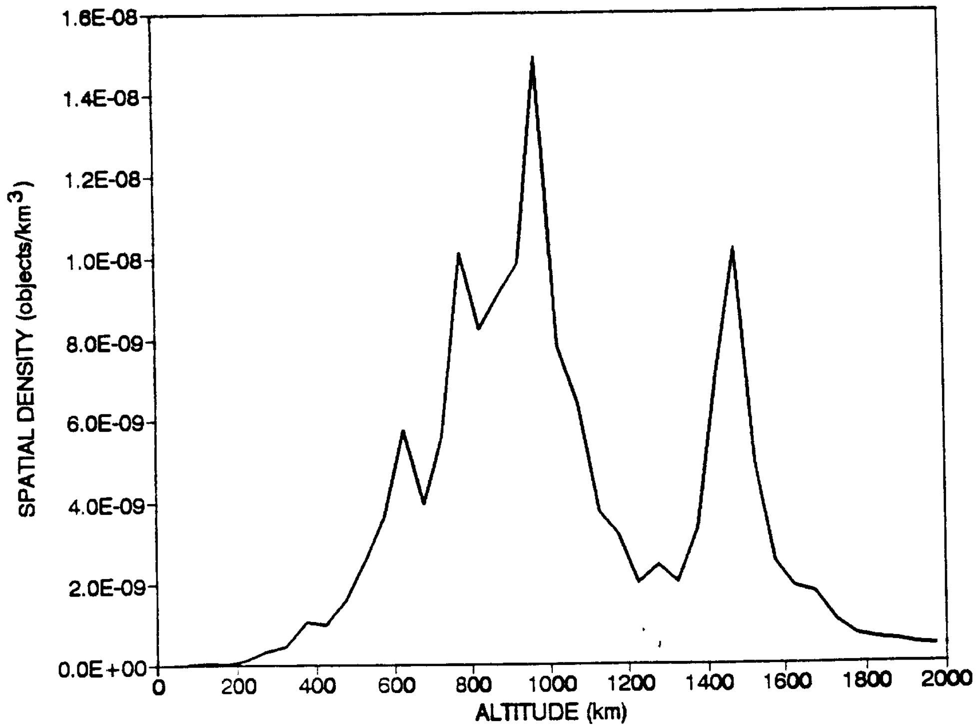

FIGURE 1. SPATIAL DENSITY CURVE CALCULATED FROM MARCH 1, 1992 SATELLITE CATALOG UP TO 2000 km.

Time (T): The time at risk is the length of time the object poses a risk to the population. If the object is to remain in orbit for a predetermined length of time, the time at risk is this value. If an object is in an orbit which is subject to decay by atmospheric drag, the lifetime of the object must be estimated. Assuming average solar activity, a first order estimate can be obtained from Appendix B for circular orbits below 925 km (525 nm).

Cross-sectional Area (AC): Cross-sectional area is the projected area of the object along the trajectory. Since this object could actually be tumbling through space (i.e., changing the projected cross-sectional area), calculating and using the largest possible cross-sectional area will provide the most conservative collision estimate. The largest cross-sectional area can be determined from the dimensions of the object. For example, if the object left in orbit is a cylindrical rocket stage, the largest cross-sectional area will be the length of the stage times the width or diameter.

For accuracy, the average collision cross-sectional area[3] between the specified object and all other

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[3] The collision cross-sectional area between two objects comes from the impact parameter, b, in the kinetic theory of gasses. The impact parameter is the sum of the radii of the two molecules (i.e., $b = r_1 + r_2$) in a collision. The collision cross-sectional area between molecules is $\prod b^2$. Since radar cross-sectional areas of space objects are known, it is more convenient to

objects with which it could collide should be used in Equation (4) instead of the physical cross-sectional area. The average collision **cross-sectional area** is found by:

(5)
$$XC = \frac{1}{n}\sum ([AC]^{0.5} + [AC_i]^{0.5})^2$$

where: XC = Average collision cross-section
AC = Cross-sectional area of the object
AC_i = Cross-sectional area of $object_i$ in the population, and
n = Number of objects.

However, calculation of average collision cross-section can require significant effort. To simplify Equation (5), an assumption is made that all objects in the population are small compared to the object under consideration Thus, the collision cross-section can be substituted by the cross-sectional area of the object for a reasonable first order approximation.

Relative Velocity (VR): Relative velocity is a function of the relative angle of encounter of the object with another object and the velocity of the colliding objects. The relative angle of encounter varies markedly with the orbital inclination, right ascension, and eccentricity of both objects being considered. Consequently, many possible combinations of these orbital parameters make up the distribution of angles of encounter that any object may have during the time period considered. Thus, determination of relative velocity is very difficult to discretely calculate.

Current methods for obtaining accurate relative velocity values involve orbital simulations, or interpolation and extrapolation of empirical data. Relative velocity can theoretically range from 0 m/s to twice the orbital velocity. However, for simplification purposes, average relative velocity in LEO is commonly estimated at $\sqrt{2}$ times the orbital velocity or (approximately 10 km/s). This comes from the assumption that the average relative angle is 90 degrees

EXAMPLE PROBLEM

The following examples demonstrate the calculations that are required for: 1. an object of cross-sectional area 11 m^2 and a mass of 121 kg in a 525 km circular orbit left to decay by atmospheric drag; and 2. an object with cross-sectional area of 11 m^2 in an elliptical orbit with an apogee at 1185 km, and a perigee at 835 km that will remain in the same orbit for 10 years and then be removed.

For Object 1:

To calculate the lifetime of object 1, the ballistic coefficient of this object is needed. The ballistic coefficient is found by:

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work with area dimensions rather than radii. It can be shown that $\prod b^2$ is equivalent to $([\prod r_1 2]^{0.5} + [\prod r_2^2]^{0.5})^2$ This assumes that all objects are spheres with circular physical cross-sectional areas [Ref 6]

(6)

$$BC = \frac{Cd * A}{m}$$

where: BC = Ballistic coefficient,m^2/kg
Cd = Drag coefficient,2.0 to 2.2 [2.2]
A = Cross-sectional area, m^2, and
m = Mass, kg.

· SPD: For a circular orbit at 525 km, the spatial density is found using the altitude shell from 500-550 km in Appendix A to be 2 53E-9 objects/km^3.

· T: Given that the orbit is circular (e = 0) at an altitude of 525 km, and the ballistic coefficient is found from Equation (6) to be 0.2 m^2/kg, the lifetime is approximated from Appendix B to be 65 days.

· VR: Since the object is in LEO, the average relative velocity is assumed to be about 10 km/s.

· AC: The object's physical cross-sectional area is 11 m^2.

Using these values in Equation (4) provides a probability of collision of 1.56E-6 for the object's entire lifetime.

Note, however, that the object will not be at 525 km altitude for the entire lifetime; rather, its altitude will decrease as the orbit decays. In this case, the result can be considered the worst case probability of collision by assuming the object remains in this altitude for the entire time duration, rather than decaying into and through the lower altitude regions which are less populated, because the initial orbit's altitude has a higher spatial density than altitudes below it.

For Object 2:

This example is complicated by the fact that object 2 is an elliptical orbit that crosses several regions with different spatial densities.

- SPD: A more complex method for calculating a weighted spatial density that is good for all eccentricities involves solving Kepler's equation and accounting for other factors to determine the time spent in each region. For simplification purposes, an average spatial density is calculated as shown in Table 1 for a first order approximation.

TABLE 1. AVERAGE SPATIAL DENSITY.

ALTITUDE REGION (km)	SPATIAL DENSITY (objects/km³)	WEIGHING FACTOR	AV. SPATIAL DENSITY CONTRIBUTION (objects/km³)
800 - 850	8.23E-09	0.3	2.47E-09
850 - 900	9.02E-09	1.0	9.02E-09
900 - 950	9.82E-09	1.0	9.82E-09
950 - 1000	1.49E-08	1.0	1.49E-08
1000 - 1050	7.81E-09	1.0	7.81E-09
1050 - 1100	6.36E-09	1.0	6.36E-09
1100 - 1150	3.74E-09	1.0	3.74E-09
1150 - 1200	3.19E-09	0.3	9.56E-10
	TOTALS	**6.6**	**5.51E-08**

TOTAL AVERAGE SPATIAL DENSITY = 5.51 E-08 / 6.6 = 8.35E-09 .

The spatial density contributions from the regions of apogee and perigee are weighted based on the approximate distance the object will travel into the region. For example, an object with a perigee at 835 km will travel 15 km into the 800-850 km region, thereby creating a factor of 15/50 or 0.3 to be multiplied by the spatial density to determine the contribution to the total density. Thus, given an elliptical orbit 835 x 1165 km, the average spatial density was found to be 8.35-09 objects/km^3. This method is good for low eccentricity, elliptical orbits in LEO, and will result in values within 10% of the more complex method that directly calculated the weighing factor for each region based on the time spent crossing the region.

- T: The object has a scheduled lifetime of 10 years; at which time it will be removed from orbit. It should be noted that by calculating the probability of collision over 10 years, an assumption is made that the population (SPD) does not change significantly over that time.

- VR: Since the object is in LEO, the average relative velocity is assumed to be about 10 km/s

- AC: The object's physical cross-sectional area is 11 m^2

Applying these values to Equation (4) gives a probability of collision of 2.90E-4 for the object's entire duration in this orbit.

CONCLUSION

This document provides the basic information needed to estimate a general probability of collision in LEO. Although the method described in this primer is a first order approximation, its results are reasonable. Furthermore, the methods described above can be the basis for developing more rigorous techniques for calculating collision probabilities through the use of the Poisson probability distribution

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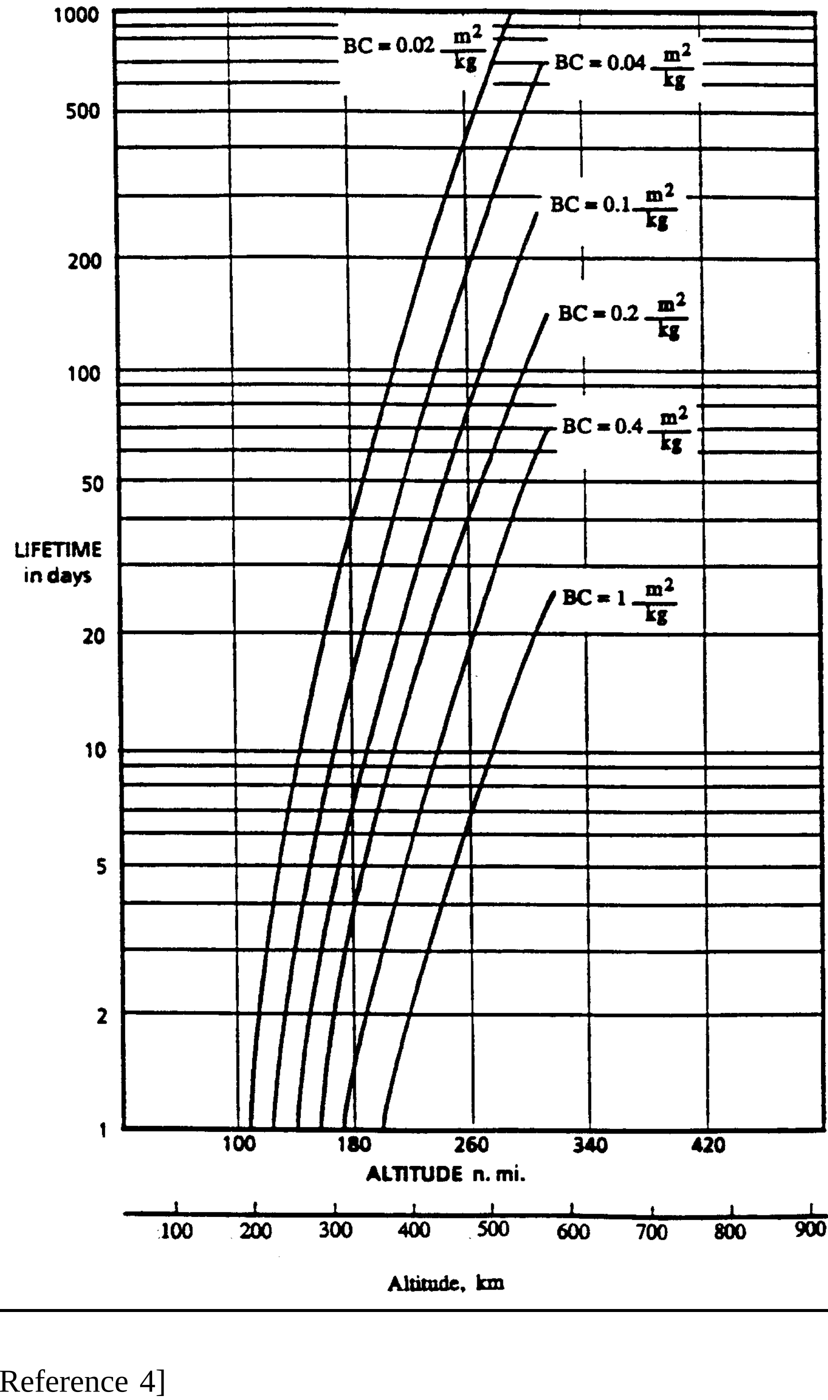
APPENDIX A

SPATIAL DENSITY OF EARTH ORBIT UP TO 2000 km CALCUALTED FROM DATA IN THE MARCH 1,1992 SATELLITE CATALOG.

Lower Altitude Boundary (km)	Upper Altitude Boundary (km)	Average Spatial Density (#/km^3)
50	100	8.03E-14
100	150	3.96E-11
150	200	2.65E-11
200	250	1.27E-10
250	300	3.40E-10
300	350	4.66E-10
350	400	1.06E-09
400	450	1.01E-09
450	500	1.56E-09
500	550	2.53E-09
550	600	3.66E-09
600	650	5.77E-09
650	700	3.94E-09
700	750	5.58E-09
750	800	1.01E-08
800	850	8.23E-09
850	900	9.02E-09
900	950	9.82E-09
950	1000	1.49E-08
1000	1050	7.81E-09
1050	1100	6.36E-09
1100	1150	3.74E-09
1150	1200	3.19E-09
1200	1250	1.98E-09
1250	1300	2.44E-09
1300	1350	2.01E-09
1350	1400	3.33E-09
1400	1450	7.06E-09
1450	1500	1.02E-08
1500	1550	4.94E-09
1550	1600	2.51E-09
1600	1650	1.91E-09
1650	1700	1.76E-09
1700	1750	1.06E-09
1750	1800	7.22E-10
1800	1850	6.12E-10
1850	1900	5.36E-10
1900	1950	4.70E-10
1950	2000	4.02E-10

APPENDIX B

CALCULATING AN OBJECT LIFETIME.



[Source: Reference 4]