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MODERN CONTROL ASPECTS OF AUTOMATICALLY STEERED VEHICLES

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16. Abstract

In the study of automatically steered rubber tired vehicles, little emphasis in the past has been placed on the steering control laws. This report examines the control law problem from the state variable point of view and it is shown that, except for possibly one velocity, the system is both controllable and observable allowing arbitrary system dynamics. It is also shown how optimal control theory may be used to select the feedback gains in order to minimize a cost function containing the square of the vehicle lateral acceleration.

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INTRODUCTION

Personal rapid transit and dual-mode vehicles capable of both manual and automatic operation, require automatic guidance systems. The guidance problem for such a vehicle divides into two parts - longitudinal guidance and lateral guidance. The longitudinal guidance system must keep the vehicle in its appropriate slot along the guideway from the initial point to destination. Its functions include routine speed control, inter-vehicle spacing, and collision avoidance schemes to cope safely with system failures. On the other side of the total guidance problem is lateral control which maintains the vehicle near the center of its guideway. Whereas the longitudinal system must of necessity involve relationships among many vehicles, the lateral control aspect involves each vehicle separately. The lateral guidance problem itself separates into two distinct, but often interrelated parts - sensing and the control action. Sensing involves the collection of information concerning the state of the vehicle with respect to the guideway reference system. This information is then used to initiate a control action. For example, in a railroad vehicle, deviations of a wheelset from the central position of the track (sensing) cause the wheelset, by means of the forces acting between wheel and rail, to turn in the proper sense to null the error (control action). Here, the sensing and control action functions are intimately coupled. In the case of a person driving an automobile, separation of the two functions is apparent. When the driver perceives his automobile to have deviated from the center of his lane (sensing), he steers the front wheels to return to a safe condition (control action).

To design and implement the automatic lateral guidance system for a rubber tired, non-tracked vehicle (dual-mode or personal rapid transit, for example) one need study both the sensing and control action sides of the problem. During the past fifteen years, several concepts have been proposed and studied for automatic highways[1,2,3,4,5,6,7]. Most of the emphasis has been placed upon the sensing half of the lateral control problem. This emphasis is justified because of the practical difficulties encountered. For example, reinforcing steel embedded in concrete highways distorts the magnetic field of a buried cable used as a reference and introduces errors into the state measurements. Very little effort, on the other hand, has been devoted to the control action part of the problem. About all that has been done is the feeding back of one or two of the state variables, varying the gains, and looking at the resulting root loci. It is true that these classical servomechanism analyses result in stable controllers, and one might accept the statement of Fenton[4],

> The design of a complete vehicular steering system consists of three main parts:
>
> 1. the design of a suitable roadway reference for guidance,
> 2. the design of appropriate sensors so that the position of the vehicle relative to the reference can be determined,
> 3. the design of the steering control.
>
> Since the design of the third item is easily accomplished, only the first two are discussed.

Fenton notwithstanding, the control action aspect does deserve a more nearly complete treatment from a modern control theory point of view. This treatment is not of academic interest alone, for it will be shown that a great flexibility in the system's dynamics can be achieved. In addition, optimal control techniques will be explored to determine the feedback gains which minimize a passenger comfort criterion, the square of the lateral acceleration. This is a first-cut analysis in the sense that neither system non-linearities nor noise in the measurements will be considered.

EQUATIONS OF MOTION

The linearized equations of motion for a rubber-tired vehicle will be derived with the aid of Figure 1. The vehicle mass center is located a distance y from the X-axis, and ψ is the angle between the longitudinal axis of the vehicle and the X-axis. β is defined to be the angle between the longitudinal axis of the vehicle and the velocity vector of the center of mass. This velocity is assumed to have a constant magnitude V. The steering angle of the front wheels (the control input) is δ, and a is the distance from the center of mass to the front and rear axles. All of the angles are assumed to remain small throughout the motion.

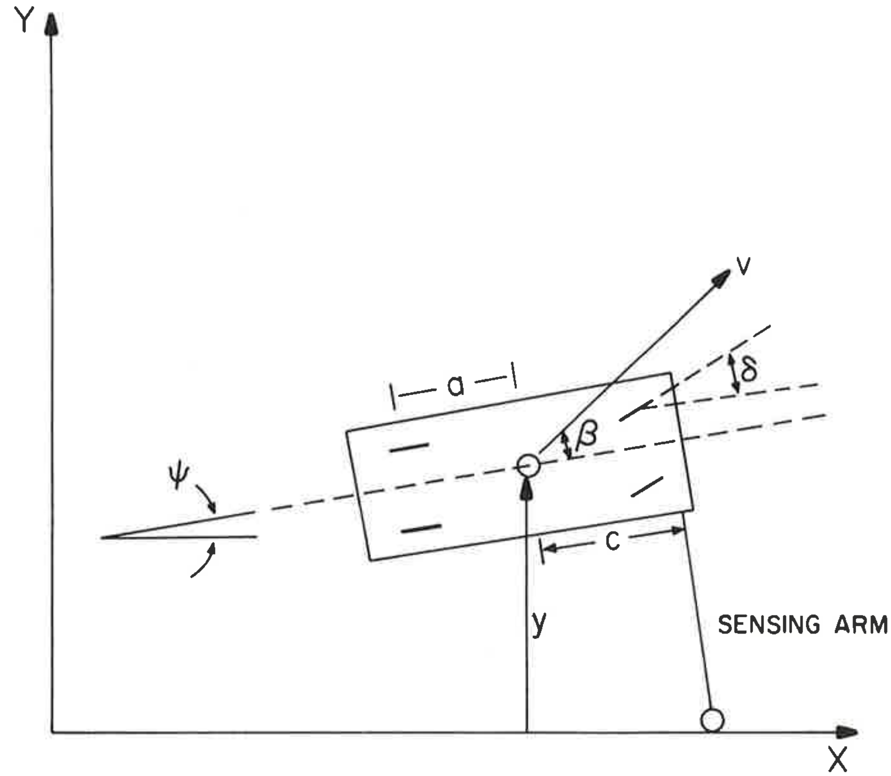

Figure 1. Vehicle Geometry

Pneumatic tires produce a side force on a vehicle which is a function of the tire slip angle, the driving or braking torque, properties of the guideway surface, etc. The slip angle is defined as the angle between the velocity of the wheel and the wheels's diametral plane. For small slip angles, the side force on the i^{th} wheel f_{yi} may be represented as being linearly related to the slip angle, α_i. That is,

$$f_{yi} = - cp_i \alpha_i$$

where cp_i is the constant of proportionality and is known as the "cornering power" of the tire. It will be assumed that the cornering power is the same for all four tires. The slip angles for the front and rear tires are, respectively,

$$\alpha_f = \beta + \frac{a}{V}\dot{\psi} - \delta$$

$$\alpha_r = \beta - \frac{a}{V}\dot{\psi}$$

The total force in the lateral direction is

$$f_y = -2cp(\alpha_f+\alpha_r) = -2cp(2\beta-\delta) = -4cp\,\beta + 2cp\,\delta$$

and therefore,

$$MV(\dot{\beta}+\dot{\psi}) = -4cp\,\beta + 2cp\,\delta$$

since $MV(\dot{\beta}+\dot{\psi})$ is the acceleration of the mass center in the y-direction for small angles. The moment of the tires along the Z - axis (out of the plane of Figure 1) about the vehicle's center of mass is

$$2cpa\,(\alpha_r-\alpha_f)$$

and thus

$$2cpa\,(-2\frac{a}{V}\dot{\psi}+\delta) = I\,\ddot{\psi}$$

where I is the vehicle's yaw moment of inertia (the Z - axis is assumed to be a vehicle principal axis of inertia direction). Rearranging, the relevant equations become

$$I\ddot{\psi} + \frac{4cpa^2}{V}\dot{\psi} = 2cpa\,\delta$$

$$MV\dot{\psi} + MV\dot{\beta} + 4cp\beta = 2cp\,\delta$$

In addition the lateral velocity is

$$\dot{y} = V(\psi+\beta)$$

It is convenient to put these equations into state variable form, i.e., a set of first order differential equations. To facilitate this, let

$$x_1 = \psi,\ x_2 = \dot{\psi},\ x_3 = \beta,\ x_4 = y$$

In terms of these new variables x, the equations of motion become

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{4cpa^2}{IV}x_2 + \frac{2cpa}{I}\delta$$

$$\dot{x}_3 = -\frac{4cp}{MV}x_3 - x_2 + \frac{2cp}{MV}\delta$$

$$\dot{x}_4 = Vx_1 + Vx_3$$

In matrix notation (x is the state vector),

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{4cpa^2}{IV} & 0 & 0 \\ 0 & -1 & -\frac{4cp}{MV} & 0 \\ V & 0 & V & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2cpa}{I} \\ \frac{2cp}{MV} \\ 0 \end{bmatrix} \delta$$

The system is thus in the standard form

$$\dot{x} = Fx + Gu$$

where $u = \delta$ (the control variable) in the case at hand. It is here assumed that the steering angle δ may be set instantaneously. This assumption is justified since the steering dynamics can be made sufficiently faster then the vehicle dynamics with a simple servomechanism.

It is interesting to note at this point that if the cornering power approaches infinity while allowing M and I to approach zero one obtains the kinematic relationships

$$\dot{y} = V(\psi + \frac{\delta}{2})$$

$$\dot{\psi} = \frac{V}{2a}\delta \cdot$$

These simplified equations are useful at low speeds when the inertial effects are small.

CONTROLLABILITY AND OBSERVABILITY

The desired motion of the rubber tired vehicle is along the X - axis (the reference axis of the guideway), i.e.,

$$\psi = \beta = y = 0 \quad (x_1=x_3=x_4=0)$$

Any deviations from this nominal condition must be nulled. Indeed, this is one purpose of the automatic lateral guidance system.* The steering angle $\delta(t)$ must, therefore, be chosen to bring the system to the null state from arbitrary initial conditions. This is, then, the classic regulator problem.

Clearly, in order to null initial conditions, there must be some knowledge of the state of the system and hence a measurement or measurements must be made. There are numerous measurements which can be made, e.g., location of the mass center (y), angle between the longitudinal axis of the vehicle and the X - axis (ψ), or even a linear combination of the state variables. A particularly convenient measurement is $y + c\psi$. This combination results when, for example, a magnetic field sensing coil, mounted a distance c ahead of the mass center, is used in conjunction with a buried cable to provide state information. It also arises when a sensing "feeler" tracks a curb on the side of the guideway as shown in Figure 1. For the remainder of this report it will be assumed that the combination $y + c\psi$ is available in real time. In terms of the state vector x, this measurement becomes

$$z = Hx = [c \ 0 \ 0 \ 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

With this measurement z, can one locate the roots of the system characteristic equation arbitrarily? This question can be answered only after two others are answered. First, if *all* the states are measured and available can the eigenvalues of the system be located arbitrarily? And if this is so, can all the states be inferred from the available measurement? The first question concerns controllability and the second, observability.

Formally, a system is said to be controllable if all initial states can be driven to the null state in a finite time. For constant, finite, dimensional, linear systems (as is the system under consideration), controllability is tantamount to the capability of locating the system roots arbitrarily with state variable feedback[8]. Let

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*Another function is forcing the vehicle to follow a curving guideway. This aspect will not be treated here.

$$\delta = - Cx$$

Where $$C = \left[C_1 C_2 C_3 C_4 \right]$$

Then $$\dot{x} = (F-GC)x$$

and if the system is controllable, the eigenvalues of F-GC can be placed in arbitrary locations. So the system being considered must be investigated to determine whether or not it is completely controllable. There are several ways of determining system controllability. One way is to look at the rank of the so-called controllability matrix,

$$\left[G \ FG \ \cdots \ F^{n-1}G \right]$$

where n is the dimension of F. If this matrix has rank n, then the system is controllable. Another way, one which sheds more light on the nature of the dynamical system, consists of finding a new coordinate system in which the equations adopt a simpler form. If a basis could be found in which the system matrix F were diagonal, then the equations in terms of the new state variables would be uncoupled and whether or not the system is controllable becomes transparent. In this case, the system is completely controllable if and only if the control enters each equation. Not all matrices can be diagonalized by means of a similarity transformation; however, all matrices can be put into the more general Jordan normal form. In the case at hand, the eigenvalues of F are

$$0, \ 0, \ \frac{-4cpa^2}{IV}, \ \frac{-4cp}{MV}$$

When $s = 0$ (s is the Laplace complex variable), the rank of $sI - F$ is 3, so the system does not have full degeneracy, i.e., F cannot be diagonalized. The Jordan form for F must be

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-4cpa^2}{IV} & 0 \\ 0 & 0 & 0 & \frac{-4cp}{MV} \end{bmatrix}$$

for $I \neq Ma^2$. Notice that the eigenvalues of F are along the main diagonal and that there is a one above the main diagonal. If

$$I = Ma^2,$$

then the Jordan matrix will have two ones above the main diagonal. One must now find the non-singular, constant matrix R which transforms F to the Jordan form. That is, define a new state vector Y to be a linear combination of the state variables x.

$$Y = Rx \; ; \; x = R^{-1} Y$$

Then

$$\dot{Y} = R\dot{x} = RFx + RGu$$

or

$$\dot{Y} = RFR^{-1}Y + RGu \qquad (1)$$

So,

$$RFR^{-1} = J$$

or

$$RF = JR \qquad (2)$$

Since F is 4 x 4, Equation 2 represents 16 equations in the 16 unknown elements of R. Solving these equations, one obtains

$$R = \begin{bmatrix} \frac{IV}{4cpa^2} + \frac{MV}{4cp} & 0 & \frac{MV}{4cp} & \frac{1}{V} \\ 1 & \frac{IV}{4cpa^2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{4cp}{MV} - \frac{4cpa^2}{IV} & 0 \end{bmatrix}$$

In terms of the new variables Y, the equations become

$$\begin{bmatrix} \dot{Y}_1 \\ \dot{Y}_2 \\ \dot{Y}_3 \\ \dot{Y}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-4cpa^2}{IV} & 0 \\ 0 & 0 & 0 & \frac{-4cp}{MV} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{V}{2a} \\ \frac{2cpa}{I} \\ \frac{2cpaM^2V^2 + 8(cp)^2 I - 8(cp)^2 Ma^2}{IM^2V^2} \end{bmatrix} \delta$$

In general, a system is controllable if and only if (1) no eigenvalue appears in more than one Jordan block, and (2) no zero row of RG is the last row of a Jordan block. Notice in our system that for $V \neq 0$*, all the rows of RG are non-zero unless

$$V^2 = \frac{4cp}{aM^2}\left[Ma^2 - I\right] \qquad \left[Ma^2 \neq I\right] \qquad (3)$$

---

*When V = 0, the system is clearly uncontrollable.

If this is the case, then the equation for Y_4 is

$$\dot{Y}_4 = \frac{-4cp}{MV} Y_4$$

and hence the control δ can have no influence over the Y_4 mode. In this case, the system is not completely controllable. This fact *does not*, however, mean the system is unstable. It merely means that an eigenvalue of F (specifically, $\frac{-4cp}{MV}$) will also be an eigenvalue of F-GC for arbitrary C. This can be easily demonstrated. In terms of Y, we have

$$\dot{Y} = JY + RGu$$

and

$$u = -Cx = -CR^{-1}Y$$

So

$$\dot{Y} = (J - RGCR^{-1})Y$$

If the i^{th} row of RG is zero, then the i^{th} row of $RGCR^{-1}$ is also zero. Hence, the i^{th} row of $J - RGCR^{-1}$ will be the same as the i^{th} row of J. Only if the i^{th} row is the last row of a Jordan block are we interested. If this is the case, then there is at most one non-zero element of the i^{th} row of J (the i,i element) and $J-RGCR^{-1}$. Clearly, this term will be an eigenvalue of $J-RGCR^{-1}$. Thus, J and $J-RGCR^{-1}$ have an eigenvalue in common. One property of similarity transformations is that eigenvalues are left unchanged. Therefore, F and F–GC have an eigenvalue in common when the system is not completely controllable.

Fortunately, the eigenvalue which cannot be moved in this case has a negative real part; it is for this reason that the system can be stable at a velocity for which it is uncontrollable. Notice in Equation 3 that if $I > Ma^2$, then there is no real V for which the system is uncontrollable. In most rubber-tired vehicles, however, the wheels are near the ends of the vehicle and the radius of gyration is likely to be less than a, that is, I is likely to be less than Ma^2. Then there will be one positive real velocity for which the system is uncontrollable. If this velocity is not the desired operating velocity, then the lack of complete controllability is of little import.

In order to determine the row matrix C to produce the desired eigenvalues, the characteristic equation is

$$s^4 + \left[\left(\frac{4cpa^2}{IV} + \frac{4cp}{MV}\right) + \frac{2cpa}{I} C_2 + \frac{2cp}{MV} C_3\right] s^3$$
$$+ \left[\frac{16(cp)^2a^2}{IMV^2} + \frac{2cpa}{I} C_1 + \frac{8(cp)^2a}{IMV} C_2 + \left(\frac{8(cp)^2a^2}{IMV^2} - \frac{2cpa}{I}\right)C_3 + \frac{2cp}{M} C_4\right] s^2$$
$$+ \left[\frac{8(cp)^2a}{IMV} C_1 + \frac{8(cp)^2a^2}{IMV} C_4\right] s + \frac{8(cp)^2a}{IM} C_4 = 0$$

If the roots of this equation can be located arbitrarily, then each of the coefficients must be able to set arbitrarily. Fixing the constant term fixes C_4. With C_4 fixed, C_1 is determined by the coefficient of s. This then leaves C_2 and C_3 free to adjust the coefficients of s^3 and s^2. These coefficients can be adjusted independently only if

$$\begin{vmatrix} \frac{2cpa}{I} & \frac{2cp}{MV} \\ \frac{8(cp)^2a}{IMV} & \frac{8(cp)^2a^2}{IMV^2} - \frac{2cpa}{I} \end{vmatrix} \neq 0$$

Expanding this determinant, one obtains

$$V^2 \neq \frac{4cp}{aM^2}\left(Ma^2 - I\right)$$

which is the same condition for controllability as was found before from the Jordan form.

We have thus shown that except for possibly one velocity, the system is completely controllable, i.e., the roots of the characteristic equation may be located arbitrarily with state variable feedback. However, all the states are not directly available; indeed, there is but one measurement, $z = y + c\psi$. One must, therefore, investigate the possibility of estimating the state vector of the system. An asymptotic state estimator is a model of the system to be observed, the input to which is the measurement z and the control u, and the output of which is an estimate of the state. If the dynamical system is described by

$$\dot{x} = Fx + Gu$$
$$z = Hx$$

then a state estimator for the system is

$$\dot{\hat{x}} = F\hat{x} + Gu + L(z - H\hat{x})$$

where $\hat{x}$ is the estimate of x and L is an arbitrary gain matrix. This presumes that the F, G, and H matrices are known exactly. In reality, they won't be known precisely and an error analysis must be performed. Such an error analysis will not be considered here. Now look at the error in the estimate $\dot{x} - \dot{\hat{x}} \triangleq \dot{\tilde{x}} = (F-LH)\tilde{x}$. That is, the error dynamics of the estimator is determined by the eigenvalues of F–LH. In analogy to the controllability problem, the eigenvalues of F–LH can be located arbitrarily if the system is observable. If a finite dimentional, linear system is not completely observable, then one or more eigenvalues of F will be eigenvalues of F–LH for arbitrary L. If the eigenvalue or eigenvalues which cannot be moved have negative real parts, then a stable state estimator can still be constructed, contrary to a statement by Kalman[9]. Tests for observability are analogous to those for controllability, and the system at hand is observable except when

$$V^2 = \frac{4(cp)\ a^2 c}{I^2}\left(I - Ma^2\right) .$$

Notice that the system is always observable for $Ma^2 > I$ (assuming $c > 0$). Remember that the condition for complete controllability involved the factor $Ma^2 - I$. Therefore the system cannot be completely observable and controllable for all velocities greater than zero ($I \neq Ma^2$, $c > 0$). It can be shown that the system is completely observable and controllable for all $V > 0$ when $I = Ma^2$. In addition, if $I > Ma^2$ and c is made negative, then the system again is completely observable/controllable. However, placing the sensor behind the mass center ($c<o$) is not desirable from the standpoint of tracking a curving guideway. When

$$V = \sqrt{\frac{4cpa^2 c}{I^2}\left(I - Ma^2\right)}$$

the eigenvalue common to F and F - LH is

$$s = -\frac{4cpa^2}{V}$$

and since it is negative, a stable observer can be constructed.

The total control system is

$$\dot{x} = Fx - GC\hat{x}$$

$$\dot{\hat{x}} = F\hat{x} - GC\hat{x} + L(z - H\hat{x})$$

where $\hat{x}$ (the estimate of x) is fed back since x itself is unavailable. In terms of x and $\tilde{x}$, the equations are

$$\dot{x} = (F-GC)x + GC\tilde{x}$$

$$\dot{\tilde{x}} = (F-LH)\tilde{x}$$

or

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} F-GC & GC \\ 0 & F-LH \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$$

Because of the triangular nature of this system matrix, the characteristic equation of the whole system is

$$\left|\left[sI-(F-GC)\right]\right| \left|\left[sI-(F-LH\right]\right|$$

So the 2n eigenvalues of the controlled system are the n eigenvalues of F—GC and the n eigenvalues of F—LH. When the system is both observable and controllable, these 2n eigenvalues can be chosen arbitrarily. The observer gain matrix L is likely to be chosen so that the observer roots are "faster" than the controller roots, i.e., so that the observer errors die out rapidly compared with the dynamics of the vehicle. Except for possibly the one velocity when the system is not completely controllable/observable, the roots of the system characteristic equation can be kept stationary as the vehicle proceeds from rest to its desired operating velocity. Although the system poles are stationary, the system is still time-varying since the zeros in general will change with velocity.

ROOT LOCATION SELECTION

The analysis in the previous section has shown that there is great freedom in locating the system poles. In order to synthesize a control strategy, the designer must have criteria for deciding where the poles should be. He might specify simply that the system be stable and well damped or that errors resulting from non-zero initial conditions die out sufficiently within a specified time. One of the gains might be specified, for example, by selecting the maximum steady state value that a state variable may take in response to a steady cross wind. Alternatively, one might use the calculus of variations to select gains which minimize a cost function selected by the designer. An important quantity from the standpoint of passenger comfort is the lateral acceleration of the vehicle. For mathematical convenience, a cost function will be selected which is the sum of the squares of the lateral acceleration and some of the state variables. State variables are included in the cost function to insure small deviations from the null condition. From the equations of motion, the lateral acceleration $\ddot{y}$ is

$$\ddot{y} = V\dot{x}_1 + V\dot{x}_3$$

and since

$$\dot{x}_1 = x_2$$

$$\dot{x}_3 = \frac{-4cp}{MV} x_3 - x_2 + \frac{2cp}{MV} \delta ,$$

$$\ddot{y} = \frac{2cp}{M} (\delta - 2x_3)$$

Let the cost function K be

$$K = \lim_{t_f - t_o \to \infty} \frac{1}{2} \int_{t_o}^{t_f} \left\{ \frac{x_1^2}{x_{10}^2} + \frac{x_4^2}{x_{40}^2} + \frac{\left[\frac{2cp}{M}(\delta - 2x_3)\right]^2}{a_o^2} \right\} dt$$

where x_{10}, x_{40}, and a_0 are the maximum desirable values of x_1, x_4, and the lateral acceleration. Penalties could also be placed on x_2 and x_3 ($\dot{\psi}$ and β). Expanding the square of the lateral acceleration, one obtains

$$\frac{x_3^2}{\left(\frac{a_o M}{4cp}\right)^2} + \frac{\delta^2}{\left(\frac{a_o M}{2cp}\right)^2} - \frac{\delta x_3}{\left(\frac{a_o M}{4cp}\right)^2}$$

The cost function may be rewritten as

$$K = \lim_{t_f - t_o \to \infty} \frac{1}{2} \int_{t_o}^{t_f} \left[x^T \delta^T \right] \begin{bmatrix} A & N \\ N^T & \beta \end{bmatrix} \begin{bmatrix} x \\ \delta \end{bmatrix} dt$$

where

$$A = \begin{bmatrix} \frac{1}{x_{10}^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\left(\frac{a_0 M}{4cp}\right)^2} & 0 \\ 0 & 0 & 0 & \frac{1}{x_{40}^2} \end{bmatrix} ; \; N = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2\left(\frac{a_0 M}{4cp}\right)^2} \\ 0 \end{bmatrix}$$

$$B = \frac{1}{\left(\frac{a_0 M}{2cp}\right)^2}$$

In terms of A, B, and N, it can be shown[10] that the control which minimizes K is

$$\delta = - B^{-1} \left(N^T + G^T S\right) x, \tag{4}$$

i.e., a linear feedback relationship. The symmetric matrix S is the steady state solution (if it exists) of a Ricatti Equation

$$\dot{S} = 0 = - SF - F^T S + (SG+N) B^{-1} \left(N^T + G^T S\right) - A \tag{5}$$

The system at hand is fourth order, and because S is symmetric, Equation 5 represents ten simultaneous *non-linear* algebraic equations in the elements of S. Because the equations are quadratic, there may be extraneous roots which usually can be eliminated by the requirement that S be positive definite. Another technique for finding the steady state S is to numerically integrate Equation 5 backwards until $\dot{S} \approx 0$. Once S is known, Equation 4 gives the optimal gains.

Choosing the constants x_{10}, x_{40}, and a_0 does not guarantee that the state variables and lateral acceleration will be below these values. Simulations with typical initial conditions and/or disturbances must be run to determine whether or not the levels are acceptable. If the lateral acceleration obtained from a simulation is too high, for example, a_0 can be adjusted downward and a new set of gains computed. This rational approach guarantees that the lateral acceleration for the same initial conditions will decrease in the next simulation run. Without this technique, one is not likely to know which gains must be changed to alter the response in the desired way. The cost function selected is essentially a trade-off between quickness of response and lateral acceleration. If the penalty on lateral acceleration is larger than on the state variables, then the response will be more leisurely. The designer must thus compromise between response time and acceleration level. Once the suitable controller gains are selected by this quadratic synthesis approach, observer gains are chosen to provide quicker observer response.

CONCLUSION

The preceeding analysis has shown that there is a great amount of freedom available to the designer in selecting the closed loop dynamics of an automatically steered vehicle. Specifically, except for possibly one velocity, the system is completely controllable and observable allowing arbitrary characteristic equation design. A passenger comfort criterion such as the lateral acceleration can be implemented into a cost function. Optimal control theory then produces the optimal gains.

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