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FREEWAY TRAFFIC FLOW FOLLOWING A LANE BLOCKAGE

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16. Abstract

The theory of traffic flow following a lane blockage on a multi-lane freeway has been developed. Numerical results have been obtained and are presented both for the steady state case where the traffic density remains constant and the non-steady state case where the traffic density changes with time.

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PREFACE

The problem of "Analysis of Shock Wave Phenomena for Freeway Control" was undertaken as part of an overall freeway-corridor traffic improvement program. Our effort is an attempt to shed some light on the understanding of how disruptive shock phenomena occur on the freeway and what happens to vehicles as a result. This report presents our work on one important aspect of this problem, the problem of the traffic flow redistribution on a multi-lane freeway following the occurrence of a shock wave due to a lane blockage.

The related but separate problem of how to modify the traffic flow through the use of controls either to minimize the occurrences of shocks or to minimize their effects on the traffic stream is also an integral part of this freeway-corridor program. A preliminary investigation of this aspect of the problem was also undertaken and our findings to date will be released in a separate report to follow.

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1. INTRODUCTION

In this report we study the problem of traffic redistribution following the onset of a lane blockage on a multi-lane freeway. The onset of a lane blockage generates traffic density waves which may propagate either down or upstream of the blockage depending upon the traffic density. In the blocked lane where the density at the blockage is very high, a discontinuity or shock wave is generated which propagates upstream. When vehicles approach the blockage caused by an accident, a stalled car or a tire changer, for example, they are required to slow down, and in some instances will actually come to a halt. Because of the rates at which vehicles slow down, the slowdown will not be uniform throughout the line of cars in the blocked lane. As a result, there will be a large increase in the density near the blockage as cars begin to pile up and the familiar accordian effect will take place. This accordian effect can be described in mathematical terms as a shock wave or density discontinuity which propagates back from the point of the blockage and affects a number of cars approaching the blockage. The precise number of cars so effected depends upon the traffic density in the lane and on how quickly they can maneuver out of this lane. If the traffic density on the freeway is light such that the congestion in the blocked lane is able to dissipate through the other lanes, the discontinuity will eventually cease propagating further upstream and steady state conditions will be established. In other words, under light traffic conditions vehicles have enough room to either gradually slow down or to maneuver out of the lane to prevent a continued pile up of cars. In mathematical terms the shock wave stops propagating and the traffic density does not increase any further. We say that a steady state has been reached. This state may contain a higher density of cars than previously but there is no further increase in the traffic density due to the blockage. On the other hand, if the density on the unblocked lanes exceeds some critical density, steady state may never be achieved; the shock wave will continue to propagate upstream slowing down more and more vehicles, until the traffic density in which the shock wave travels falls below the critical density. There has to be enough spacing between cars, a density below some critical density which will prevent the pileup. Only when the spacing between cars becomes large enough (or the density small enough), will the pileup be averted. Until this is the case, the shock wave will continue to propagate and the density of cars (or pileup) will continue to increase.

In this report we treat, in detail, only the light density problem showing that after a finite time, steady state conditions become established. For this case, the resulting traffic density in each lane of the freeway is determined as a function of position behind the blockage and as a function of time since the blockage occurred. In this way, we hope to obtain a lane profile of the traffic redistribution following a lane blockage on a multilane freeway. The traffic redistribution is due to lane changing following the occurrence of the blockage. Thus, we will find the new traffic concentrations in each of the lanes of the freeway as a function of time following the occurrence of the blockage and compare these with the traffic densities in the lanes before the blockage occurred.

The analysis which follows is an extension of the work by Munjal, Hsu and Lawrence.[1] These authors assumed a constant value for the wave velocity c in the continuity equation. This implies that all traffic disturbances occurring on the freeway propagate in only one direction, either downstream or upstream. However, it is known that disturbances will propagate upstream only when the concentration is sufficiently high (above a density K_m at which the flow is maximum) and will propagate downstream when the density is below that level. Nonetheless Munjal et al[1] attempted to treat the case when the density in the unblocked lanes was less than the density at maximum flow, K_m, though the blocked lane contained a density which is greater than K_m near the blockage. But this requires that the wave velocity be positive (i.e. downstream) in the unblocked lanes and negative (i.e. upstream) in the blocked lane and thus the assumption of constant c cannot be valid for the problem.

In this report, the restriction of constant c in all freeway lanes is relaxed, instead, c will be allowed to take on positive or negative values depending upon the traffic density in the lane. Thus, for light traffic density in the unblocked lanes traffic disturbances will propagate downstream (c positive), while in the blocked lane near the blockage the propagation will be upstream (c negative), see Figure 1 and note that c (slope of q-k curve) is positive for $k<K_m$ and negative for $k>K_m$.

Independent of this requirement that the wave velocity c be allowed to take on both positive and negative values, we shall also distinguish between the steady state and non-steady state cases. Munjal's analysis assumed steady state conditions to hold and thus was not able to include the traffic density buildup following the occurrence of the blockage. The traffic density buildup during the initial several minutes (the precise time depending upon such things as the traffic densities in the lanes) following the blockage can be treated only by a nonsteady state analysis. In fact it is necessary to show that steady state will be achieved. If it is not, clearly any analysis which is based on the existence of steady state cannot be meaningful. We shall show that steady state is achieved under light traffic density conditions and calculate both the initial traffic density build up and the resulting steady state traffic densities after this initial buildup.

2. FLOW-CONCENTRATION ASSUMPTION

We shall assume, as in Munjal et al[1], that vehicular velocity is independent of vehicular concentration whenever the concentration is less than some value K_m and that for concentrations greater than or equal to K_m, the velocity V is related to concentration, k as shown below

$$V = q_m/K_m, \quad k < K_m \tag{1a}$$

$$V = \frac{q_m}{K_j - K_m}\left(\frac{K_j}{k} - 1\right), \quad k \geq K_m. \tag{1b}$$

where q_m is the flow at concentration K_m and where K_j is the jam concentration. The flow which is just the velocity multiplied by the concentration, then becomes a piecewise linear function of the concentration given by (see Fig. 1)

$$q = \frac{q_m}{K_m} k, \quad k < K_m \tag{2a}$$

$$q = \frac{q_m}{1 - K_m/K_j}\left(1 - \frac{k}{K_j}\right), \quad k \geq K_m. \tag{2b}$$

2.1 CONTINUITY EQUATION, CONSTANT c

The net flow across a lane boundary per unit distance of roadway is assumed to be proportional to the difference in concentration between lanes.

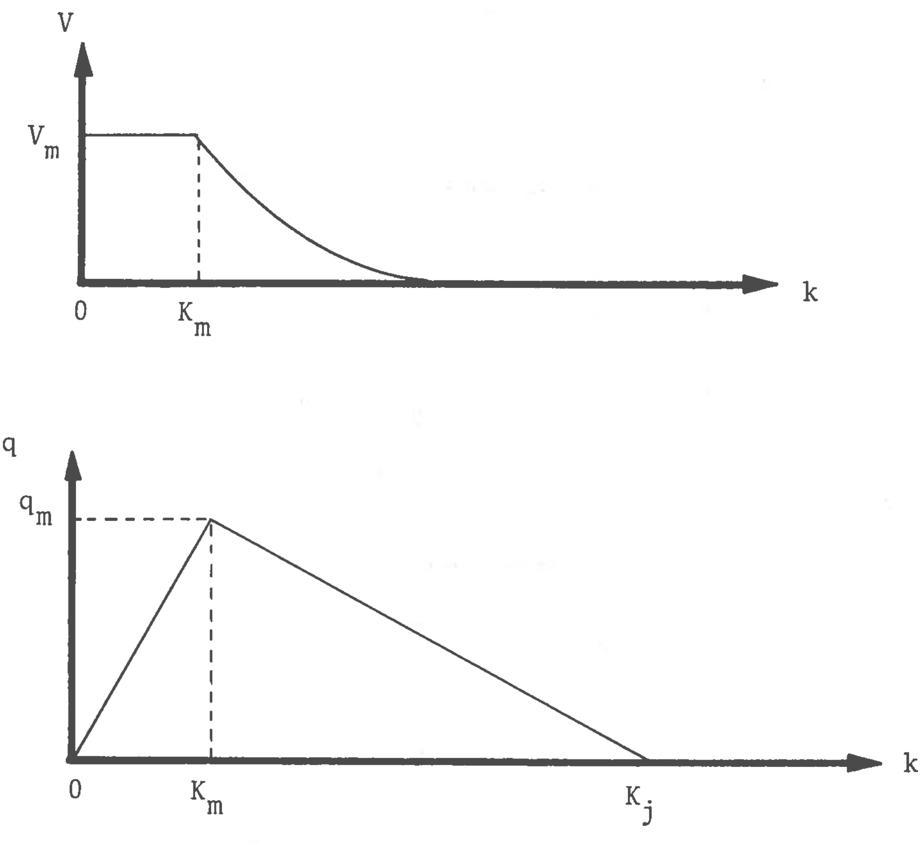

Figure 1. Schematic V-k and q-k Curves Based on Equations (1) and (2) Based on a Representation from Drake, Shofer and May[4]

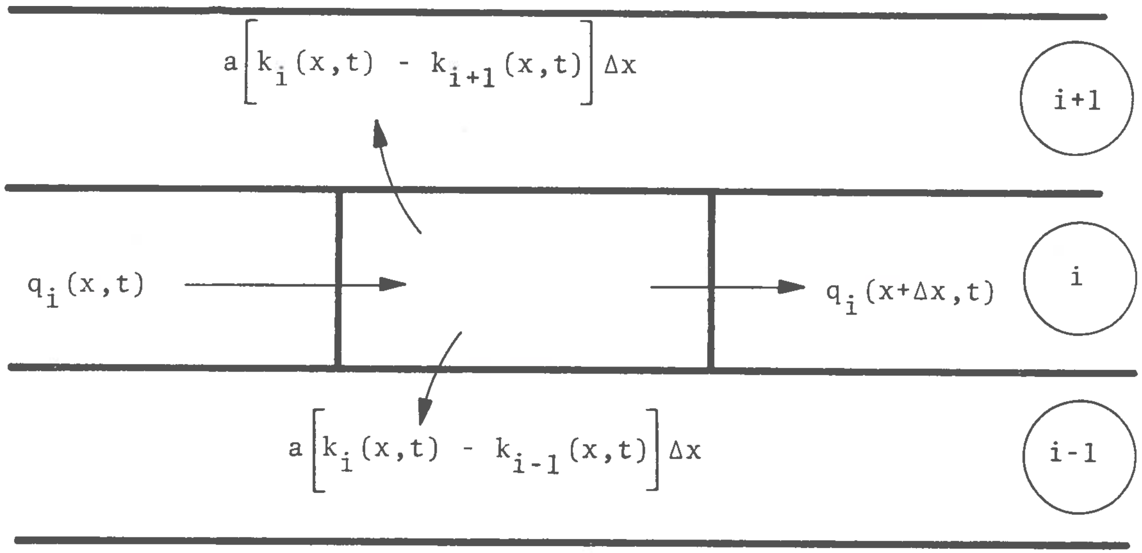

Figure 2. Schematic of Vehicle Flow for Derivation of Continuity Equations

Let $k_i(x,t)$ and $q_i(x,t)$ denote the concentration and flow, respectively in lane i at position x and time t. Figure 2 depicts the flow of vehicles from a segment of length Δx in lane i where adjacent lanes i-1 and i+1 receive (or give) vehicles to the segment depending upon the concentration difference between lanes. The net flow of the segment is

$$q_i(x+\Delta x,t) - q_i(x,t) + a\left[\left(-k_{i-1}(x,t)\right) + 2k_i(x,t) - k_{i+1}(x,t)\right] \Delta x$$

where a is a proportionality constant.

The net flow out of the segment must equal the rate of loss of vehicles from the segment

$$-\frac{\partial}{\partial t}\left[k_i(x,t)\Delta x\right].$$

Equating the two, dividing by Δx and taking the limit as $\Delta x \to 0$ gives

$$\frac{\partial q_i}{\partial x} + \frac{\partial k_i}{\partial t} + a\left(-k_{i-1} + 2k_i - k_{i+1}\right) = 0 \tag{3}$$

assuming the flows and densities to be continuous functions of x and t.

Since q_i is assumed to be a piecewise linear function of k_i, we have, from Equations (2a) and (2b)

$$\frac{\partial q_i}{\partial x} = \begin{cases} \dfrac{q_m}{K_m}\dfrac{\partial k_i}{\partial x}, & k_i < K_m \\ -\, q_m/(K_j - K_m)\, \dfrac{\partial k_i}{\partial x}, & k_i \geq K_m. \end{cases} \tag{4}$$

For an n lane freeway we have, with this substitution

$$\frac{\partial k_i}{\partial t} + c\,\frac{\partial k_i}{\partial x} + a\left(-k_{i-1} + 2k_i - k_{i+1}\right) = 0, \quad 2 \leq i \leq n-1 \tag{5a}$$

where

$$c = \begin{cases} q_m/K_m, & k_i < K_m \\ -\ q_m/(K_j - K_m), & k_i \geq K_m \end{cases} \qquad (5b)$$

and, for the outside lanes 1 and n, each having only one adjacent lane, we have

$$\frac{\partial k_1}{\partial t} + c\,\frac{\partial k_1}{\partial x} + a\,(k_1 - k_2) = 0$$

$$\frac{\partial k_n}{\partial t} + c\,\frac{\partial k_n}{\partial x} + a\,(-k_{n-1} + k_n) = 0 \qquad (6)$$

The continuity Equations (5) and (6) may be combined into vector form. Let

$$\vec{k} = \begin{pmatrix} k_1 \\ \cdot \\ \cdot \\ \cdot \\ k_n \end{pmatrix}$$

and

$$A = \left\{A_{ij}\right\} = \begin{bmatrix} 1 & -1 & \cdot & \cdot & & \\ -1 & 2 & -1 & \cdot & & \\ \cdot & -1 & 2 & -1 & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & -1 & 1 \end{bmatrix}$$

where

$A_{11}=1$; $A_{12}=-1$; $A_{n,n-1}=-1$; $A_{nn}=1$; $A_{i,i-1}=A_{i,i+1}=-1$; $A_{ii}=2$; for $2\leq i\leq n-1$; all other elements of A are zero.

If the velocity c (note from Eq. (5b) that c indeed has dimensions of velocity) assumes the same value in all lanes (that is, if the concentration is either less than K_m in all lanes or is greater than K_m in all lanes), The continuity equations may be written as

$$\frac{\partial \vec{k}}{\partial t} + c \frac{\partial \vec{k}}{\partial x} + a A \vec{k} = 0 \qquad (7)$$

which is the continuity equation used by Munjal in Reference 1.

3. THE EFFECT OF LANE BLOCKAGE UNDER LIGHT TRAFFIC CONDITIONS

We assume the traffic density to be less than K_m in the unblocked lanes. Here the above continuity Equation (7) is not valid since the concentration will be above K_m in the blocked lane near the blockage and hence c will not be constant.

Instead let

$$c_1 = - q_m/(K_j - K_m) \tag{8a}$$

the value in the blocked lane, assumed to be the shoulder lane or lane 1 and

$$c_2 = q_m/K_m \tag{8b}$$

the value in the unblocked lanes, and let C be the diagonal matrix

$$C = \begin{pmatrix} c_1 & & & & \\ & c_2 & & & \\ & & c_2 & & \\ & & & c_2 & \\ & & & & \ddots & \\ & & & & & c_2 \end{pmatrix} \tag{8c}$$

such that $C_{11}=c_1$; $C_{ii}=c_2$ $2 \leq i \leq n$.

In the region near the blockage where $k_1 > K_m$ and $k_i < K_m$ for i=2,.....n, the continuity equation may be written as

$$\frac{\partial \vec{k}}{\partial t} + C \frac{\partial \vec{k}}{\partial x} + a A \vec{k} = 0. \tag{9}$$

This is a much more difficult differential equation to solve than Equation (7) in which the coefficient c is a scalar. In order to solve this equation, we first discuss the applicable boundary conditions of the problem and then, in Section 4, develop the equations which determine the propagation of the density discontinuity formed as a result of the traffic pile up in the blocked lane. Subject to these boundary conditions and discontinuity propagation, solutions to the system of continuity Equations (9) are developed in Section 5.

We assume that the concentration in all lanes would have some value K_o everywhere if the lane blockage did not occur, ($K_o<K_m$, light traffic conditions). The blockage is taken to occur at x=0 in lane 1 and remains there for all times $t\geq 0$. The analysis for the case when the blockage is removed at some time t=T will be considered in another paper. For times t<0 no blockage exists and

$$k_i(x,t) = K_o \tag{10}$$

for all i, all x, t<0.

At the point of blockage, on the upstream side, the vehicular velocity $V_1(x,t)$ in lane 1 goes to zero or

$$V_1(0,t) = 0 \ , \ t > 0 \ . \tag{11}$$

The velocity and concentration are related from Equation (1) by

$$V = q_m/K_m \ , \ k < K_m \tag{12a}$$

$$V = q_m/(K_j-K_m)\left(\frac{K_j}{k} - 1\right) \ , \ k \geq K_m \tag{12b}$$

so that we must have for the blocked lane 1 the condition

$$k_1(0,t) = K_j \ , \ t \geq 0 \ . \tag{13}$$

At the onset of the blockage, t=0, we have in lane 1 also that

$$k_1(x,0) = K_o \ , \ x < 0 \tag{14a}$$

$$k_1(0,0) = K_j \ . \tag{14b}$$

In words, in the blocked lane 1, a traffic density K_o exists before the time that the blockage occurs, t<0. (The same density K_o also exists in the other lanes prior to the occurrence of the blockage, Equation (10).) Later, at the time that the blockage first occurs, at time t=0, the density for all points upstream of the blockage, for $x<0$, remains at the K_o level since the effects of the blockage have not had time to propagate, (Equation (14a)); however, at the place of the blockage ($x=0$) the density increases to jam concentration values K_j at this time (Equation (14b)). The density remains at jam values K_j for all subsequent times $t>0$ (assuming the blockage is not removed (Equation (13)). Thus, a discontinuity, $(K_j - K_o)$ exists initially at $x=0$. The discontinuity, also known as a shock wave propagates upstream of the blockage. In the case of a single lane, the discontinuity would separate the vehicles which were stopped by the blockage from the unperturbed stream of vehicles as shown schematically in Figure 3.

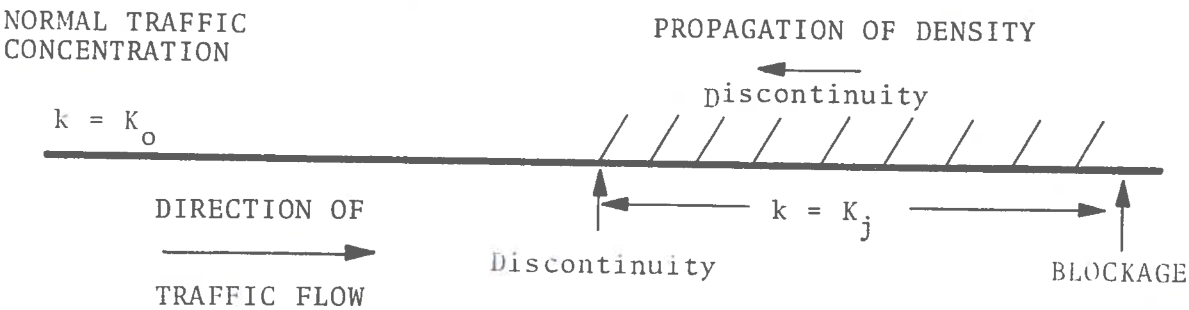

Figure 3. Schematic of a Single Lane Highway Blockage

In the multi-lane case, the vehicles between the upstream discontinuity and the blockage point would not be entirely stopped since vehicles would escape into the unblocked lanes, thus reducing the concentration in the blocked lane below jam concentration, K_j, (except at the blockage point). However, there would still be a discontinuity separating concentrations above K_m from the unperturbed concentration K_o. That is, the traffic concentration at the blockage $x=0$ diminishes from jam upstream of the blockage but never diminishes to the value K_o it had before the blockage occurred. Thus, an upstream discontinuity will exist which separates a concentration K_o from a higher one due to the blockage. This will be explained in the next section.

Before proceeding it should be noted that vehicles have been assumed to switch from the blocked lane only as a result of the increased concentration (or equivalently, the reduced velocity) there, but not because they see a blockage ahead. This assumption is made in the terms of the continuity equation which makes lane transfer proportional to the difference in concentration. Because of this assumption, it is not certain how closely the results can quantitatively be expected to agree with experimental data.

4. PROPAGATION OF THE DISCONTINUITY IN THE BLOCKED LANE

We now derive a relation between the traffic concentration existing at the discontinuity and the velocity of propagation of the discontinuity. This discontinuity has propagated upstream from the blockage located at x=0 and has reached location $x=\xi$ at time t. We wish to find the concentration associated with the discontinuity at its present location (at the blockage, at x=0, t=0, the concentration associated with the discontinuity was of course, just K_j).

Referring to Figure 4, let ξ (t) be the position of the discontinuity at time t. We define an interval $[x_1,x_2]$ containing

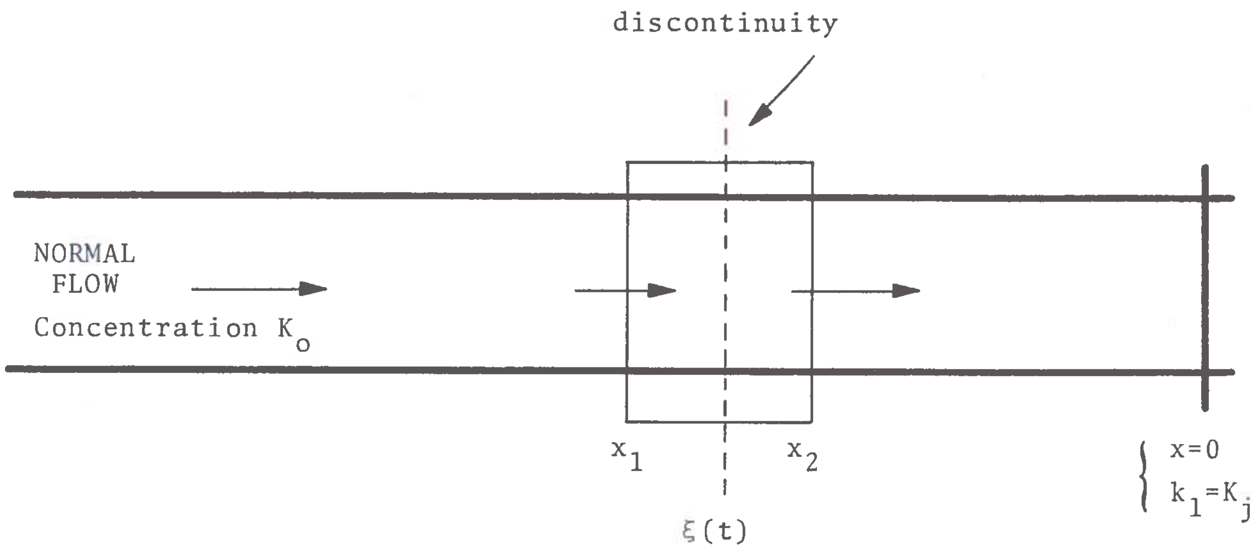

Figure 4. Schematic for Vehicular Flow Through Discontinuity

the discontinuity as shown in the figure and require that the number of vehicles be conserved so that the number of cars entering the interval equals the number leaving. We may note here that this concept as well as that of a traffic density requires the existence of a continuous function of space with all volumes of interest containing many vehicles. When we speak of, for example, a jam density at a blockage point we always mean volumes large enough to contain many vehicles so that a density can be defined. In this development as well as in any other using the concept of density, it helps to consider the vehicles some large distance below us. It is in this perspective with a "birds eye view" and a coarse resolution that the continuity equation and density concept becomes easily understood. With this in mind, in assuming that the interval[x_1,x_2] may be made arbitrarily small (as is always done in a mathematical analysis of a discontinuity) we really mean that the size of the interval may be made small enough so that the number of cars contained in it is small compared to the number of cars outside the region. The flow at x_1 is given by

$$q_1(x_1,t) = q_m\ K_o/K_m \tag{15}$$

since there is normal flow at this point. The flow at x_2 (which is on the blockage side of the discontinuity) is

$$q_1(x_2,t) = \frac{q_m}{1 - K_m/K_j}\left[1 - \frac{k_1(x_2,t)}{K_j}\right] \tag{16}$$

since we are assuming that $k_1(x_2,t)>K_m$.

The total number of vehicles in the interval $[x_1,x_2]$ at time t is

$$K_o\left[\xi(t) - x_1\right] + \int_{\xi(t)}^{x_2} k_1(x,t)\ dx.$$

This is changing at rate

$$K_o\frac{d\xi}{dt} - k_1\left[\xi(t),t\right]\frac{d\xi}{dt} - \int_{x_2}^{\xi(t)} \frac{\partial k_1(x,t)}{\partial t}\,dx$$

$$\cong \frac{d\xi}{dt}\left[K_o - k_1\big(\xi(t),t\big)\right] \quad . \tag{17}$$

The last term has been neglected since it is small compared with the others ($\partial k_1/\partial t$ is negligible in the interval $[\xi, x_2]$) to give the approximation shown. Equating the net flow into the interval (see Eqs. (15) and (16) to the rate of change of the number of vehicles in the interval, Equation (17), we have

$$\frac{q_m}{K_m} K_o - \frac{q_m}{1-K_m/K_j} \left[1 - \frac{k_1(x_2,t)}{K_j}\right]$$

$$- a \cdot \{\Delta K\} \cdot (x_2-x_1) = \frac{d\xi}{dt} \left[K_o - k_1\big(\xi(t),t\big)\right] . \quad (18)$$

where $\{\Delta k\}$ denotes the concentration difference between lanes. The last term on the left hand side denotes the vehicle flow from the blocked lane into the contiguous lane within the small distance $\Delta x = x_2 - x_1$. This term is completely negligible compared with the others so that, assuming Δx small, Equation (18) reduces to

$$q_m \left\{ \frac{K_o}{K_m} - \frac{K_j - k_1\big(\xi(t),t\big)}{K_j - K_m} \right\}$$

$$= \frac{d\xi}{dt} \left[K_o - k_1\big(\xi(t),t\big)\right] \quad (19)$$

where $k_1(\xi(t),t)$ is the concentration on the perturbed side of the discontinuity. We have thus found a relationship between the traffic concentration at the discontinuity k_1 $(\xi(t),t)$, and the velocity of the discontinuity, $d\xi/dt$.

The velocity of propagation of the discontinuity is thus

$$\frac{d\xi}{dt} = \frac{q_m}{K_o - k_1\big(\xi(t),t\big)} \left\{ \frac{K_o}{K_m} - \frac{K_j - k_1\big(\xi(t),t\big)}{K_j - K_m} \right\} . \quad (20)$$

At $t=0$, $\xi(0)=0$, and $k_1\big(\xi(0),0\big)=K_j$, thus the discontinuity initially propagates at velocity

$$\frac{d\xi}{dt}\Big/_{t=o} = \frac{q_m}{K_o - K_j} \frac{K_o}{K_m} = - \frac{q_m K_o}{(K_j - K_o) K_m} < 0. \quad (21)$$

The negative value indicates upstream propagation. We also note that as k_1 $(\xi(t),t)$ increases, $d\xi/dt$ decreases (or, the absolute value increases, that is, the shock wave travels faster upstream) as we would expect. This may be seen by taking the derivative of $d\xi/dt$ with respect to $k_1(\xi(t),t)$ which turns out to be

$$\frac{\partial \dot{\xi}}{\partial k_1} = \frac{q_m \, K_j\left(K_o - K_m\right)}{K_m(K_j - K_m)\left(K_o - k_1\left(\xi(t),t\right)\right)^2} < 0 \ , \ K_o < K_m \ . \tag{22}$$

Under the assumption that K_o is so small that the concentration in the unblocked lanes does not exceed K_m even with the diffusion of vehicles from the blocked lane, we expect $\xi(t) \to \xi_o$ as $t \to \infty$ where ξ_o is a steady state location of the discontinuity. $d\xi/dt$ would increase asymptotically to zero, and $k_1\left(\xi(t),t\right)$ would therefore decrease from K_j at $x=0$ to a limiting value at ξ_o of

$$k_1(\xi_o) = K_j - \frac{K_o}{K_m}(K_j - K_m) = K_o + K_j \ (K_m - K_o)/K_m \tag{23}$$

which is obtained from Equation (20) after setting the velocity to zero. This is the value of k_1 which produces the same flow as at K_o, see Figure 5. Note that the concentration in lane 1, k_1, would steadily decrease from K_j at the blockage point to its limiting steady state value at the point of discontinuity ξ_o. This steady state value is larger than K_m so that the traffic concentrations between the blockage and steady state points will always be larger than K_m. The difference between the steady state value and K_m is

$$K_o + K_j\left(K_m - K_o\right)\Big/K_m - K_m = \left(K_m - K_o\right)\left(\frac{K_j}{K_m} - 1\right) > 0 \tag{24}$$

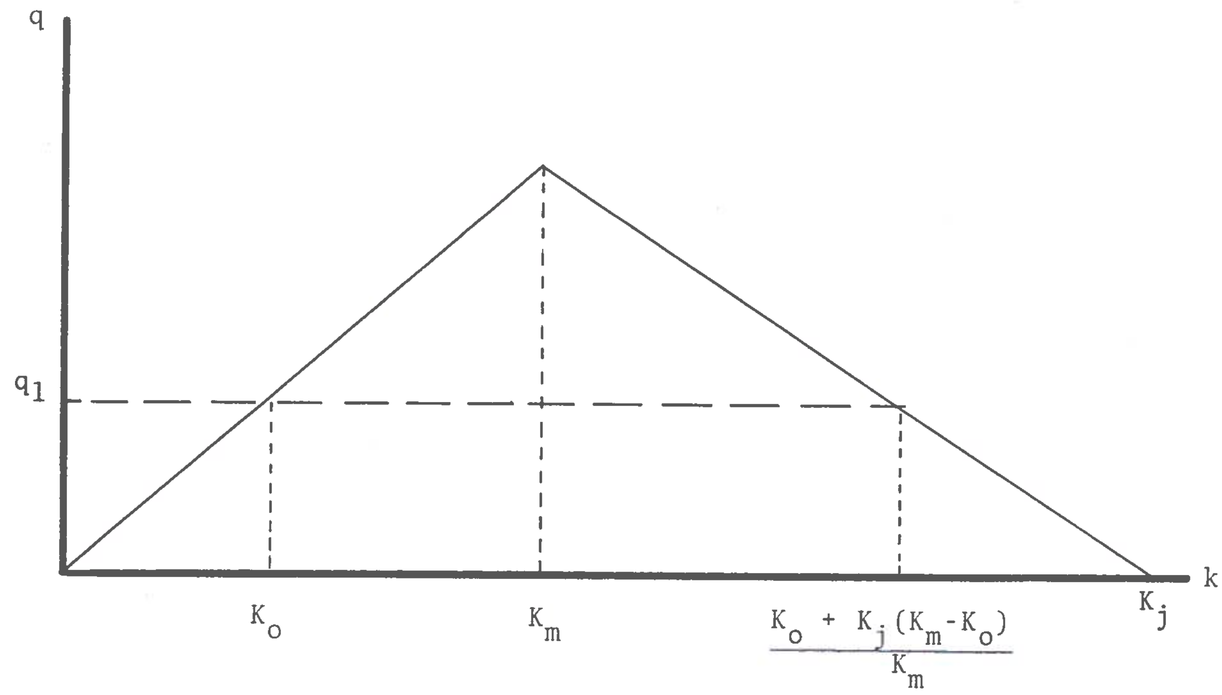

Figure 5. q-k Diagram Showing Concentrations K_o and $K_o + K_j\ (K_m - K_o)/K_m$ Producing Same Flows

With the development in this section of the boundary condition given by Equation (20) which relates the speed of the moving shock wave boundary to the traffic concentration, we are able to solve the Equations of continuity (9) for the concentration as a function of space and time along the freeway. This is done in the following section.

5. NUMERICAL SOLUTION OF CONTINUITY EQUATIONS FOR BLOCKAGE IN LIGHT TRAFFIC

In this section we formulate and develop a numerical solution to the problem of blockage of lane 1 under light traffic conditions, that is, for traffic densities k in the unblocked lanes which remain less than K_m even after diffusion of traffic from the blocked lane.

The boundary condition for the density in lane 1, k_1, is given at $x=0$, $t \geq 0$ as

$$k_1(0,t) = K_j \ , \ t \geq 0. \tag{25}$$

The boundary condition for the densities in the other lanes, $k_\ell, \ell=2,\ldots,n$ are given at the shock wave position $\xi(t)$ as

$$k_\ell\big(\xi(t),t\big) = K_o \ , \ t \geq 0 \ , \ \ell = 2,\ldots n \ . \tag{26}$$

This condition follows from the fact that the unblocked lanes are affected only by diffusion across lanes which is a continuous process. Thus vehicles in the unblocked lanes adjacent to the shock wave have not been adjacent to perturbed flow "for any length of time", hence the concentration remains at K_o at this point.

In other words, as the lane 1 shock wave propagates upstream, the concentration in the other lanes increases due to diffusion of vehicles from the blocked lane. However, since the diffusion is a continuous process this increase in concentration cannot occur at points adjacent to the shock wave but only after the shock wave has passed. The boundary curve $\xi(t)$ is not known explicitly. It is determined by the previously obtained condition for the speed of the discontinuity

$$\frac{d\xi}{dt} = \frac{q_m}{K_o - k_1\big(\xi(t),t\big)}\left[\frac{K_o}{K_m} - \frac{K_j - k_1\big(\xi(t),t\big)}{K_j - K_m}\right] \tag{27}$$

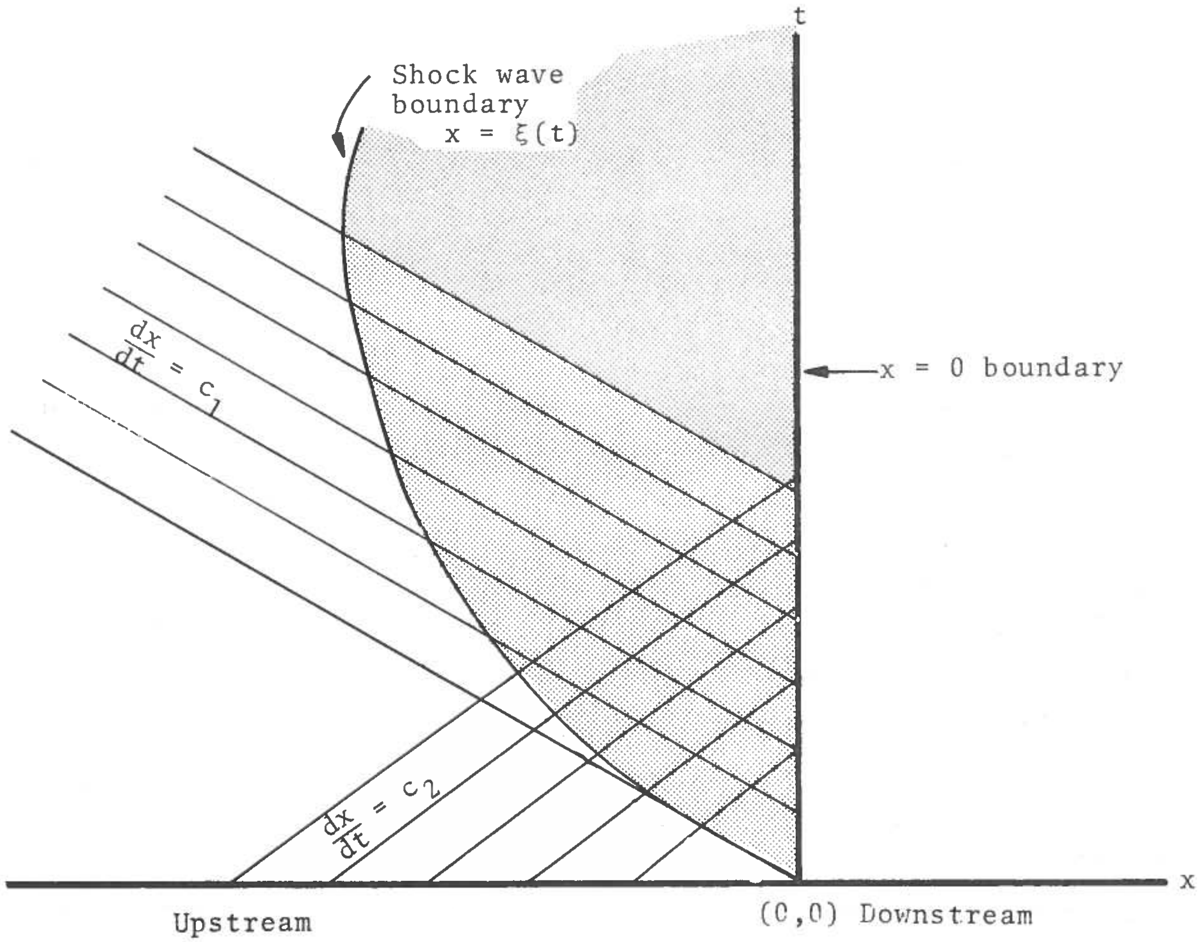

Figure 6. Space-Time Diagram for Lane 1 Indicating Characteristic Lines and Shock Wave Boundary Line. Outside of Shaded Region the Concentration Assumes the Unperturbed Value, K_o

which depends on the unknown quantity $k_1(\xi(t),t)$. We are solving the system of equations (see Section 3):

$$\frac{\partial \vec{k}}{\partial t} + C\frac{\partial \vec{k}}{\partial x} + aA\vec{k} = 0, \tag{28}$$

or,

$$\frac{\partial k_1}{\partial t} + c_1\frac{\partial k_1}{\partial x} + a\left(k_1 - k_2\right) = 0 \tag{29}$$

$$\frac{\partial k_i}{\partial t} + c_2\frac{\partial k_i}{\partial x} + a\left(-k_{i-1} + 2k_i - k_{i+1}\right) = 0 \tag{30}$$

$$2 \leq i < n$$

$$\frac{\partial k_n}{\partial t} + c_2\frac{\partial k_n}{\partial x} + a\left(-k_{n-1} + k_n\right) = 0 \tag{31}$$

We use the Forsythe and Wasow[2] method employing the powerful method of characteristics[3] to solve the set of differential equations. Because of space limitations it must be assumed that the reader has a basic textbook knowledge of the method of characteristics. If not, the reader is referred to the two references mentioned above for an excellent introduction to this method. Those readers who have neither the time nor the inclination to follow the mathematical solution of Equations (29)-(31), and (20), may skip the remainder of this section and proceed directly to Section 7.

Referring to Figure 6, we note that outside the shaded region bounded by the shock wave, the density in all lanes assumes the unperturbed value K_o (the shock has not reached these points outside the shaded region). Within the shaded region the concentration exceeds K_m and is at jam, K_j at the blockage location $x=0$ for all times $t \geq 0$. The figure further depicts the characteristic lines of the set of differential Equations (29)-(31) (see Ref. 3, for example, for the theory of characteristics). The characteristics on which $dx/dt = c_1$, result from the continuity Equation (29) for lane 1, and reflect upstream propagation from the blockage to the discontinuity. The characteristics on which $dx/dt = c_2$ result from the continuity Equations (30),(31) for all other lanes, and reflect downstream propagation from the discontinuity to the blockage in the other lanes. We choose the grid lines to lie along characteristics. This avoids certain restrictive conditions on the ratio of the horizontal to the vertical grid spacing otherwise necessary to insure that the solution to the discrete equations approaches the solution to the differential equation as grid spacing approaches zero. In order to solve the set of Equations (28) or (29)-(31) we first transform coordinates to α, β, where

$$\alpha = x - c_1 t$$

$$\beta = x - c_2 t \tag{32}$$

The characteristic lines are now of the form α = constant and β = constant. We use these as grid lines for the discretization. That is, for the "splitting up" of the continuous equation into discrete or finite parts so that it may be numerically solved. The differential Equations (29)-(31) transform to

$$\left(c_1 - c_2\right) \frac{\partial k_1}{\partial \beta} + a\left(k_1 - k_2\right) = 0 \tag{33}$$

$$\left(c_2 - c_1\right) \frac{\partial k_\ell}{\partial \alpha} + a\left(-k_{\ell-1} + 2k_\ell - k_{\ell+1}\right) = 0 \qquad 2 \leq \ell < n-1 \tag{34}$$

$$\left(c_2 - c_1\right) \frac{\partial k_n}{\partial \alpha} + a\left(-k_{n-1} + k_n\right) = 0 \tag{35}$$

where we have used the differentiation formuli

$$\frac{\partial k}{\partial t} = \frac{\partial k}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial t} + \frac{\partial k}{\partial \beta} \cdot \frac{\partial \beta}{\partial t} = -c_1 \frac{\partial k}{\partial \alpha} - c_2 \frac{\partial k}{\partial \beta}$$

$$\frac{\partial k}{\partial x} = \frac{\partial k}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial x} + \frac{\partial k}{\partial \beta} \cdot \frac{\partial \beta}{\partial x} = \frac{\partial k}{\partial \alpha} + \frac{\partial k}{\partial \beta}$$

Equations (33)-(35) represent the continuity equations in the (α, β) coordinate system. In this coordinate system, the region in which the differential equations are solved is shown in Figure 7 in terms of a schematic of the grid system that is used

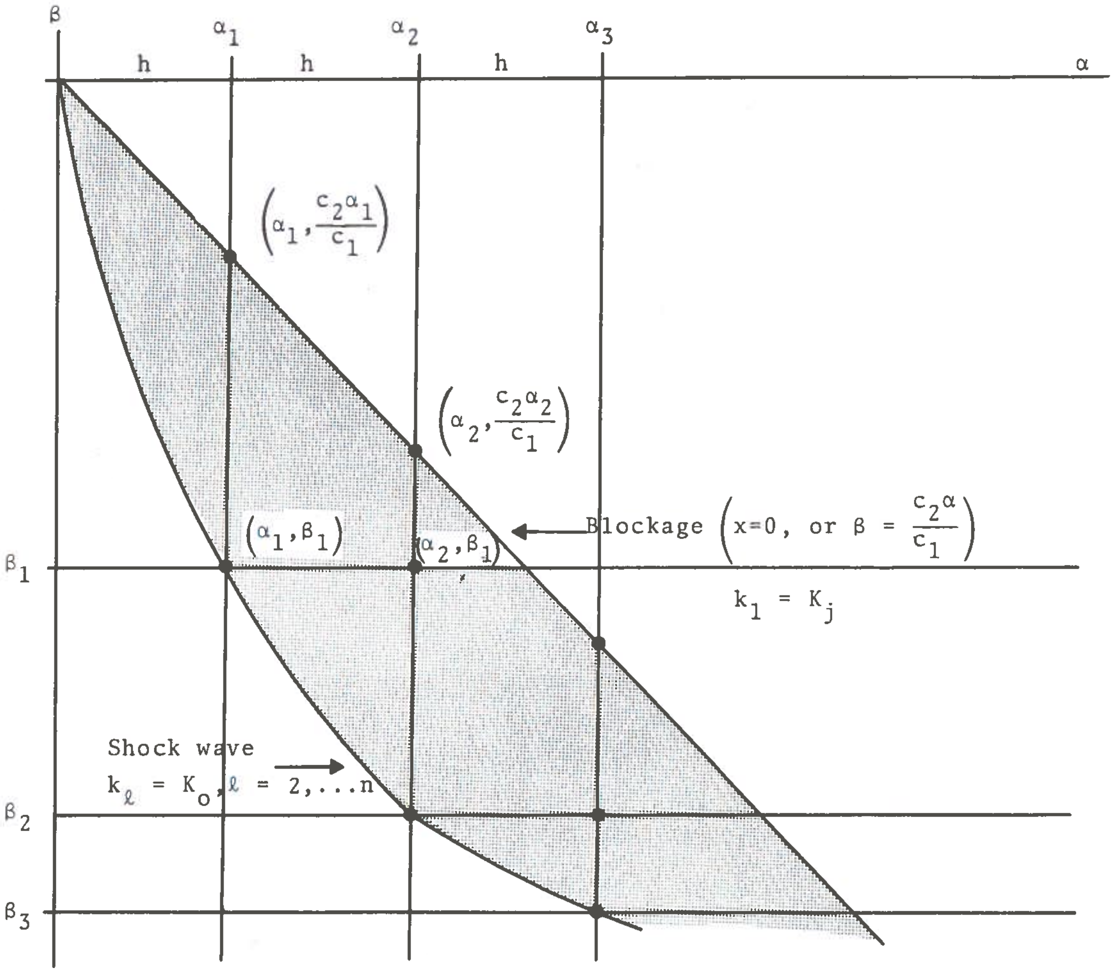

Figure 7. α, β Grid Lines Showing Shockwave-Blockage Boundaries

in the discretization. This region, as seen in Figure 7 is bounded by the blockage represented by x=0 or $\beta = c_2\alpha/c_1$ on one side and the shock wave on the other.

Vertical grid lines are of the form $\alpha=\alpha_i$ and are spaced uniformly at intervals of h, (α_i=ih). Horizontal grid lines are of the form $\beta=\beta_i$. These are not uniformly spaced. β_i is taken to be the value of β at the intersection of the line $\alpha=\alpha_i$ with the shock wave curve. The path of the shock wave is not initially known. We are to find these values as well as the values of the concentration at the grid points. The actual method of solution is left to Appendix 1. The numerical results of this solution of the continuity and shock boundary equations are presented in section 7. In the next section, we present the solution of these equations in the steady state limit.

6. STEADY-STATE SOLUTION FOR LANE BLOCKAGE IN LIGHT TRAFFIC

Before presenting the numerical results to the general case treated in the previous section where we solved the continuity and boundary condition equations for a lane blockage under moderately light traffic conditions, we develop a solution to these equations in the steady state limit.

In the steady state limit, the shock wave has ceased to propagate, the discontinuity remains in a fixed position and the traffic density does not change with time. These steady state conditions can be expressed as

$$d\xi/dt = 0; \quad x = d = \text{constant} \tag{36}$$

$$\frac{\partial k_\ell}{\partial t} = 0 \ , \ \ell = 1,\ldots n \ . \tag{37}$$

By setting $\partial\vec{k}/\partial t=0$ in the vector continuity Equation (9), we get the following vector ordinary differential equation

$$C \frac{d\vec{k}}{dx} + a \, A \, \vec{k} = 0 \tag{38}$$

where

$$\vec{k} = \begin{pmatrix} k_1(x) \\ \vdots \\ k_n(x) \end{pmatrix} \tag{39}$$

Let x=d be the steady state position of the discontinuity. This steady state position has also been denoted by ξ_o in Section 4.

Setting $d\xi/dt=0$ in Equation (20), we obtain the lane 1 boundary condition at d

$$k_1(d) = K_o + K_j\left(1-K_o/K_m\right) \tag{40}$$

which is the traffic density at position d upstream from the blockage in lane 1 after steady state has been established. We also have the boundary conditions for the densities in the other freeway lanes after steady state conditions have become established:

$$k_\ell(d) = K_o \ , \ \ell = 2,\ldots n \tag{41a}$$

Finally, the lane 1 boundary condition at the blockage point is

$$k_1(0) = K_j \tag{41b}$$

These (n+1) boundary conditions determine the n arbitrary scalar constants in the solution to the differential equation and the constant d.

Writing the differential equation as

$$\frac{d\vec{k}}{dx} = - a\ C^{-1}\ A\ \vec{k} \tag{42}$$

we let λ_i be the i th eigenvalue of $(-aC^{-1}A)$ and let $\vec{V}_i$ be the eigenvector associated with this eigenvalue. For arbitrary scalar constants $B_1,\ldots,B_n$,

$$\vec{k}(x) = \sum_{i=1}^{n} B_i \exp\left(\lambda_i x\right) \vec{V}_i \tag{43}$$

is a solution to the differential equation. To determine the constants d, $B_1,\ldots,B_n$, we use the boundary conditions and obtain the following set of simultaneous equations;

$$\sum_{i=1}^{n} B_i V_{i1} = K_j \qquad (44)$$

$$\sum_{i=1}^{n} B_i \exp\left(\lambda_i d\right) V_{i1} = K_o + K_j\left(1-K_o/K_m\right) \qquad (45)$$

$$\sum_{i=1}^{n} B_i \exp\left(\lambda_i d\right) V_{i\ell} = K_o \ , \ 2 \leq \ell < n \qquad (46)$$

where V_{ij} denotes the j th component of the i th eigenvector.

Applying the n Equations in (45) and (46), we can solve for $(B_i \exp[\lambda_i d])$ which is linear in $B_i \exp[\lambda_i d]$. We call this G_i, for $i=1,\ldots,n$, and substitute $B_i=G_i \exp[-\lambda_i d]$ into the first equation, to obtain

$$\sum_{i=1}^{n} G_i \exp\left(-\lambda_i d\right) V_{i1} = K_j \qquad (47)$$

This non-linear equation can be solved for d by standard methods, for example, iterations of Newton's method. Finally, having obtained d, we determine B_i from $B_i=G_i \exp[-\lambda_i d]$ which completes the solution, and gives the steady state concentrations in each of the freeway lanes. Numerical results are presented in the next section.

7. NUMERICAL RESULTS: LANE BLOCKAGE IN LIGHT TRAFFIC

In this section we present the numerical results obtained by solving the steady state problem discussed in the previous section and the non-steady state general case discussed in Sections 3, 4, and 5. In the steady state case, Equations (38), (40) and (41) were solved. Figure 8 illustrates this, showing the variation of traffic density in all four lanes with distance upstream from the blockage point for times sufficiently long for steady state to prevail. In the general case, the continuity Equation (28) together with the boundary conditions (25)-(27) were solved. Figures 9-13 illustrate this, showing the variation of traffic density with time as well as distance; the variation of shock velocity with time and distance; and the relationship between traffic density and shock wave velocity.

The numerical results were obtained using the following illustrative numerical values for the constants:

q_m = flow at K_m = 1 vehicle/second

K_j = .05 vehicles/foot (264 vehicles/mile)

K_m = .02 vehicles/foot (106 vehicles/mile)

K_o = .01 vehicles/foot (53 vehicles/mile)

a = .15 second^{-1}

N = 4 lanes

where "a" signifies the number of vehicles diffusing across lane boundaries per second per foot of roadway per concentration difference. The value a = .15 second^{-1}, for example, indicates that when the concentration difference is .05 vehicles/foot (jam concentration in one lane while the adjacent lane is empty), .75 vehicles per second will switch into the empty lane in a distance of 100 feet, $(a \cdot \Delta k \cdot \Delta x)$.

These numerical values, in particular those chosen for the jam concentration K_j and for the density at which the flow is

$q_m = \begin{cases} 3600 \text{ veh/mile} \\ 1 \text{ veh/sec.} \end{cases}$ $K_o = \begin{cases} 52.8 \text{ veh/mile} \\ 0.01 \text{ veh/ft,} \end{cases}$ $K_j = \begin{cases} 264 \text{ veh/mile} \\ 0.05 \text{ veh/ft,} \end{cases}$ $K_m = \begin{cases} 105.6 \text{ veh/mile} \\ 0.02 \text{ veh/ft} \end{cases}$

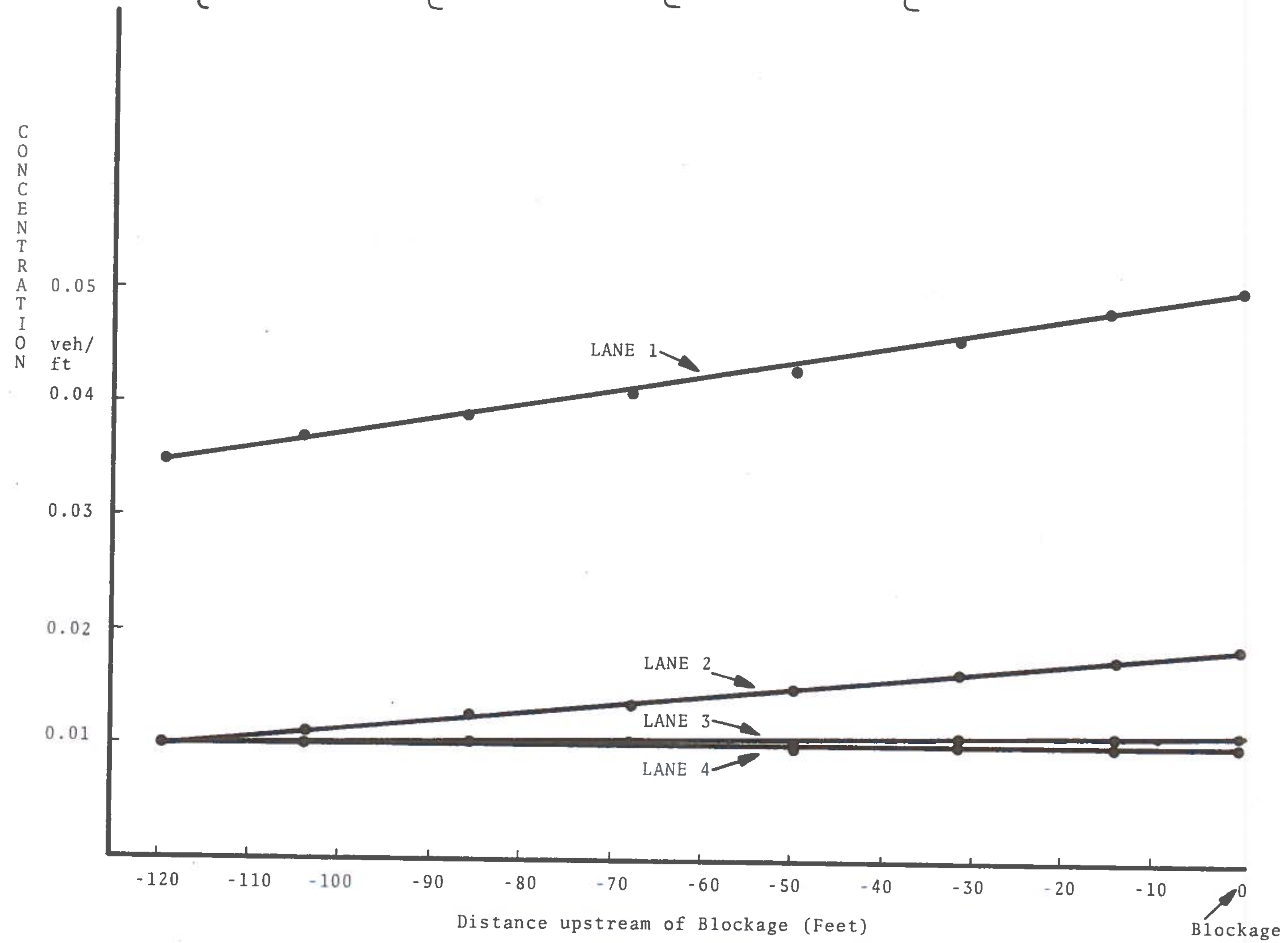

Figure 8a. Steady State Concentrations Per Lane as a Function of Position Upstream

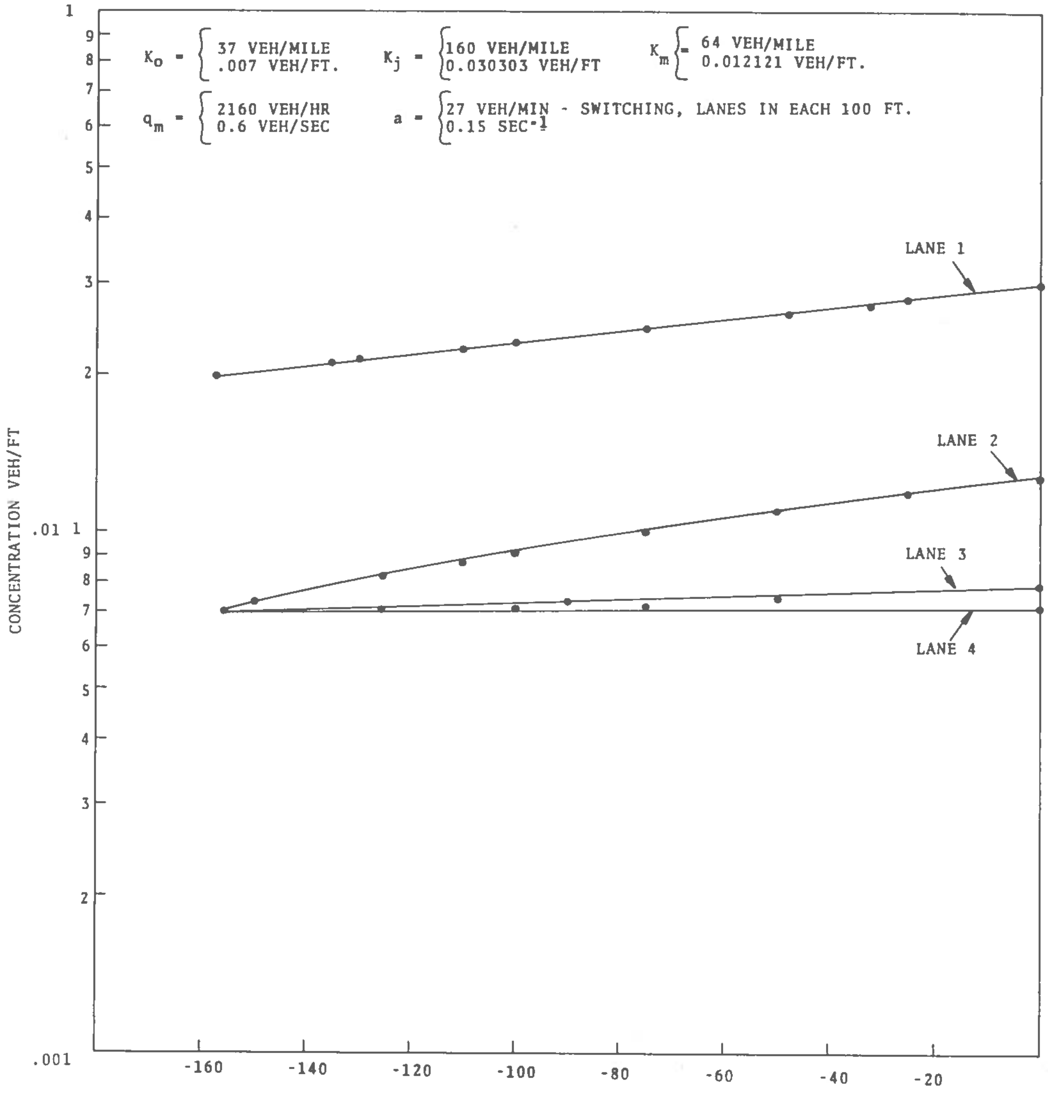

Figure 8b. Steady State Concentrations Per Lane as a Function of Position Upstream

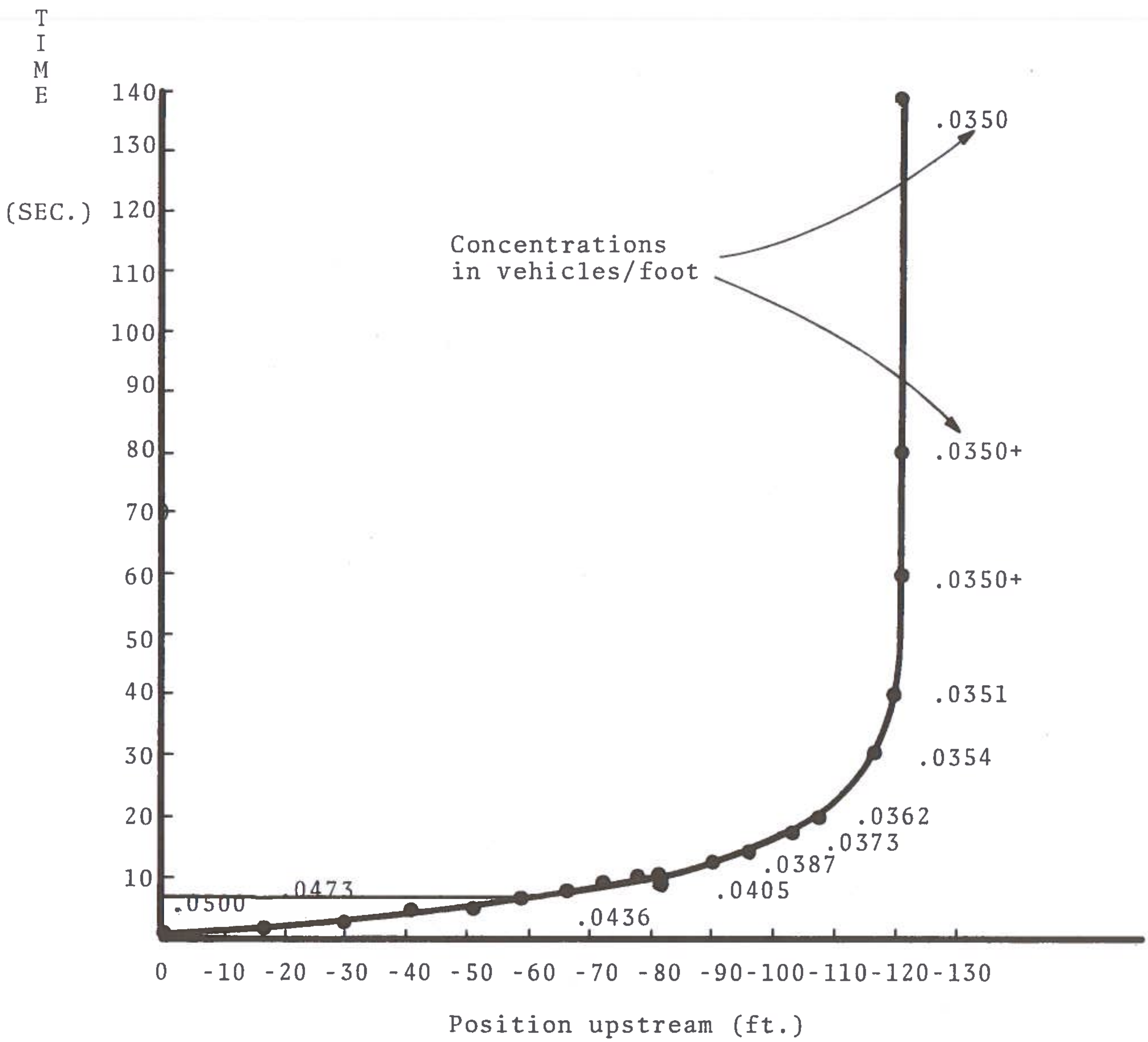

Figure 9. Space-Time Curve Showing Shock Trajectory and Traffic Densities in Lane 1

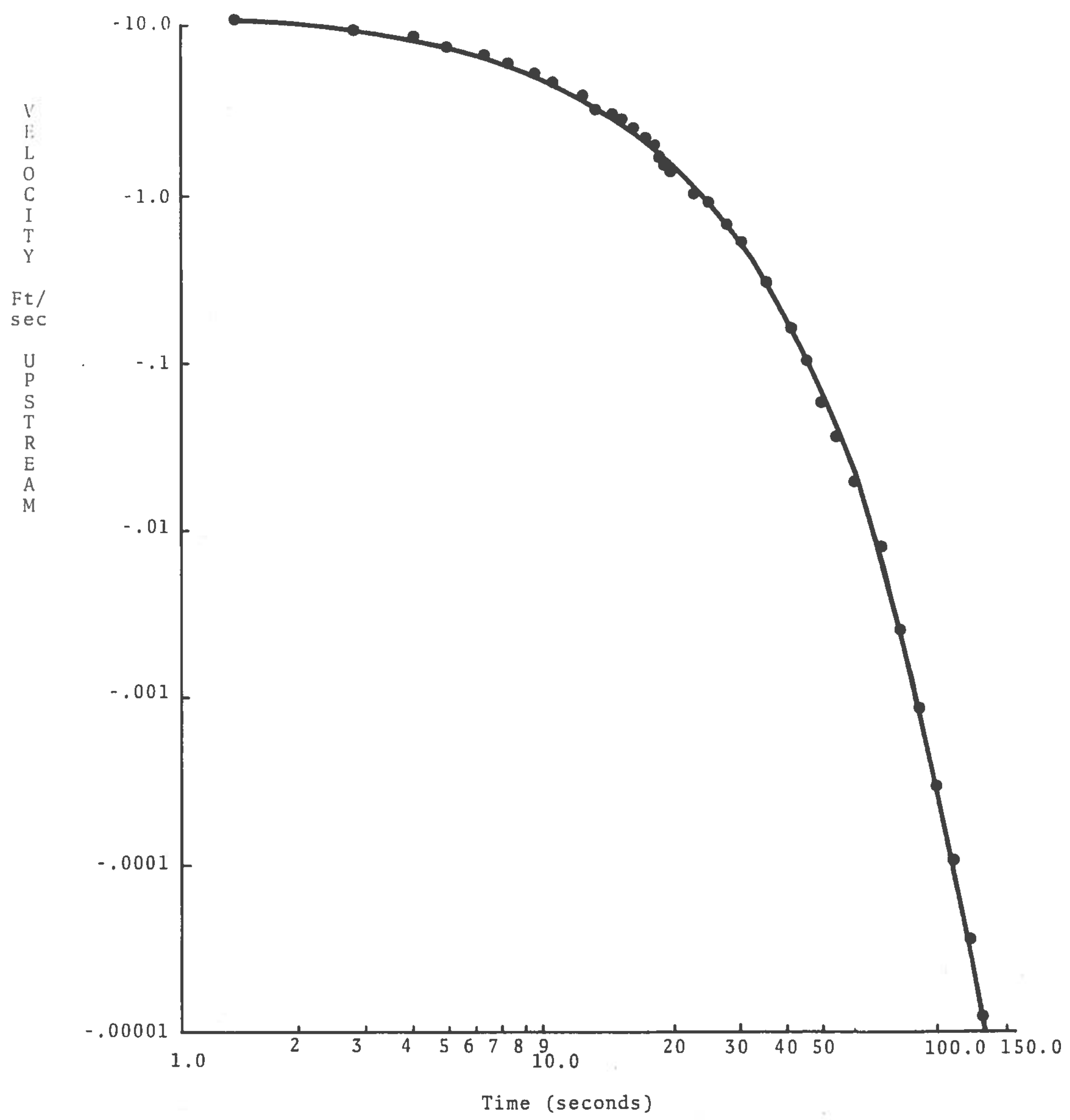

Figure 10. Velocity of Discontinuity in Blocked Lane as a Function of Time Since Onset of Blockage

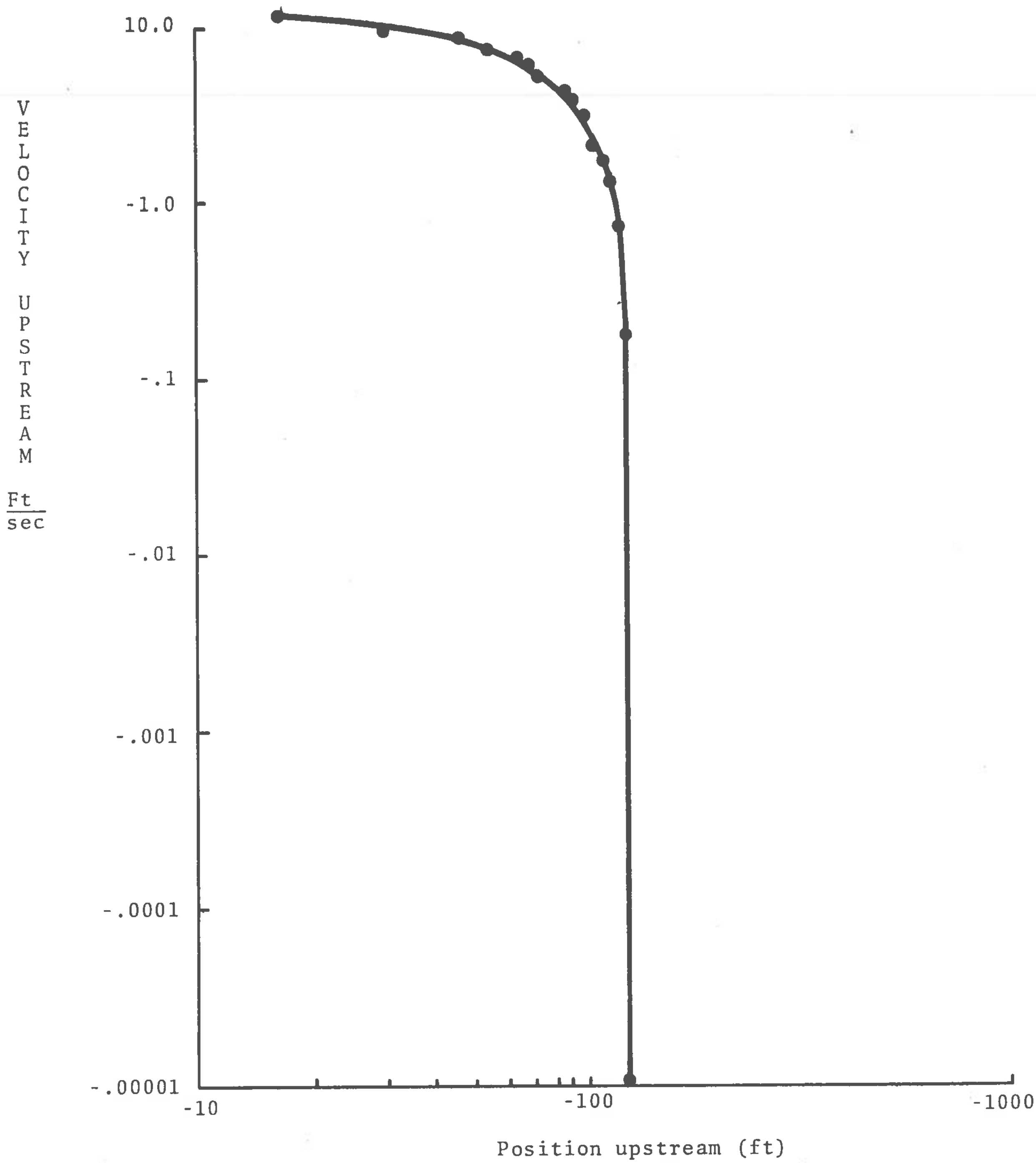

Figure 11. Velocity of Discontinuity in Blocked Lane as a Function of Position Upstream of Blockage

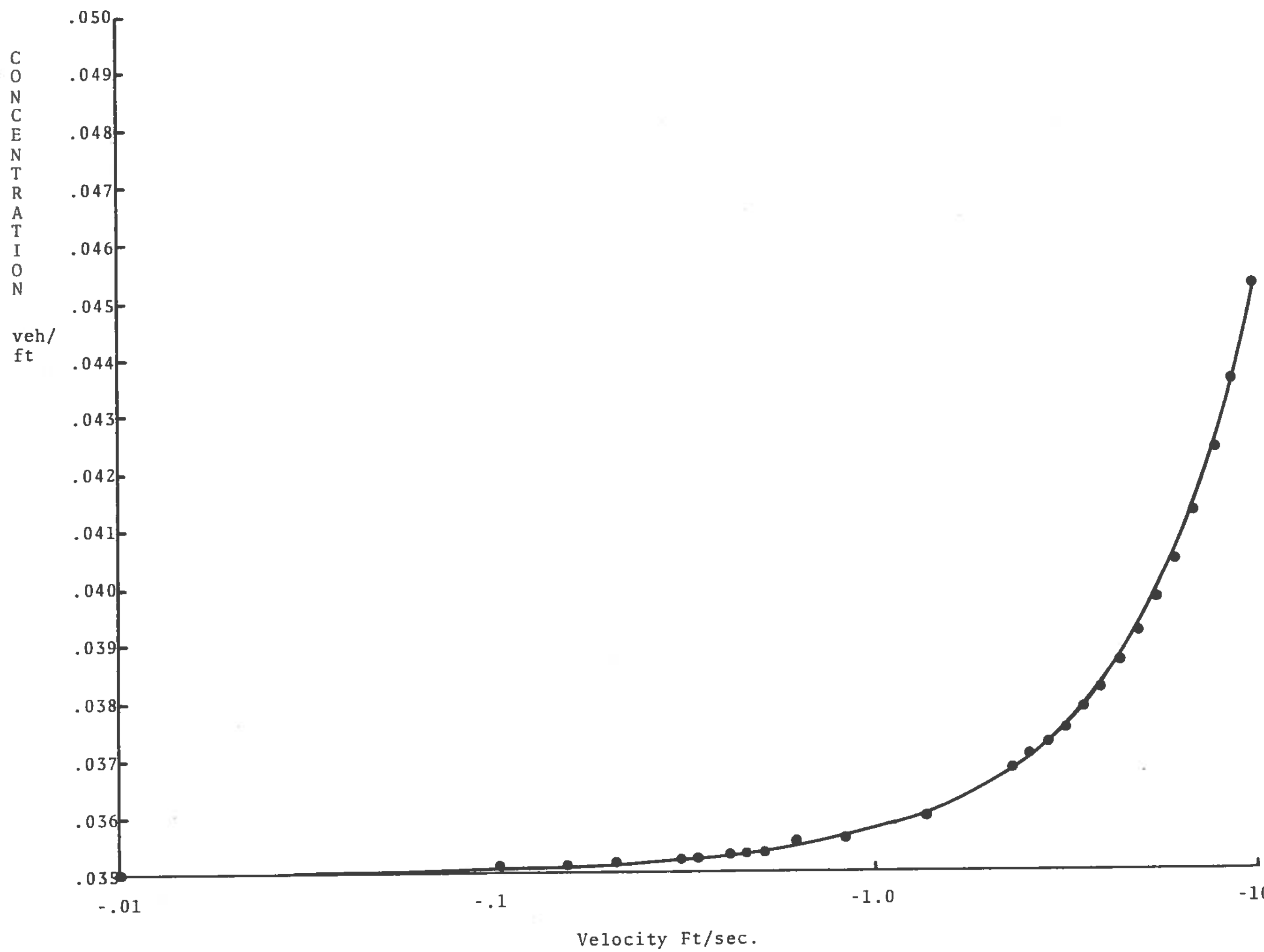

Figure 12. Concentrations in Blocked Lane as a Function of Shock Velocity

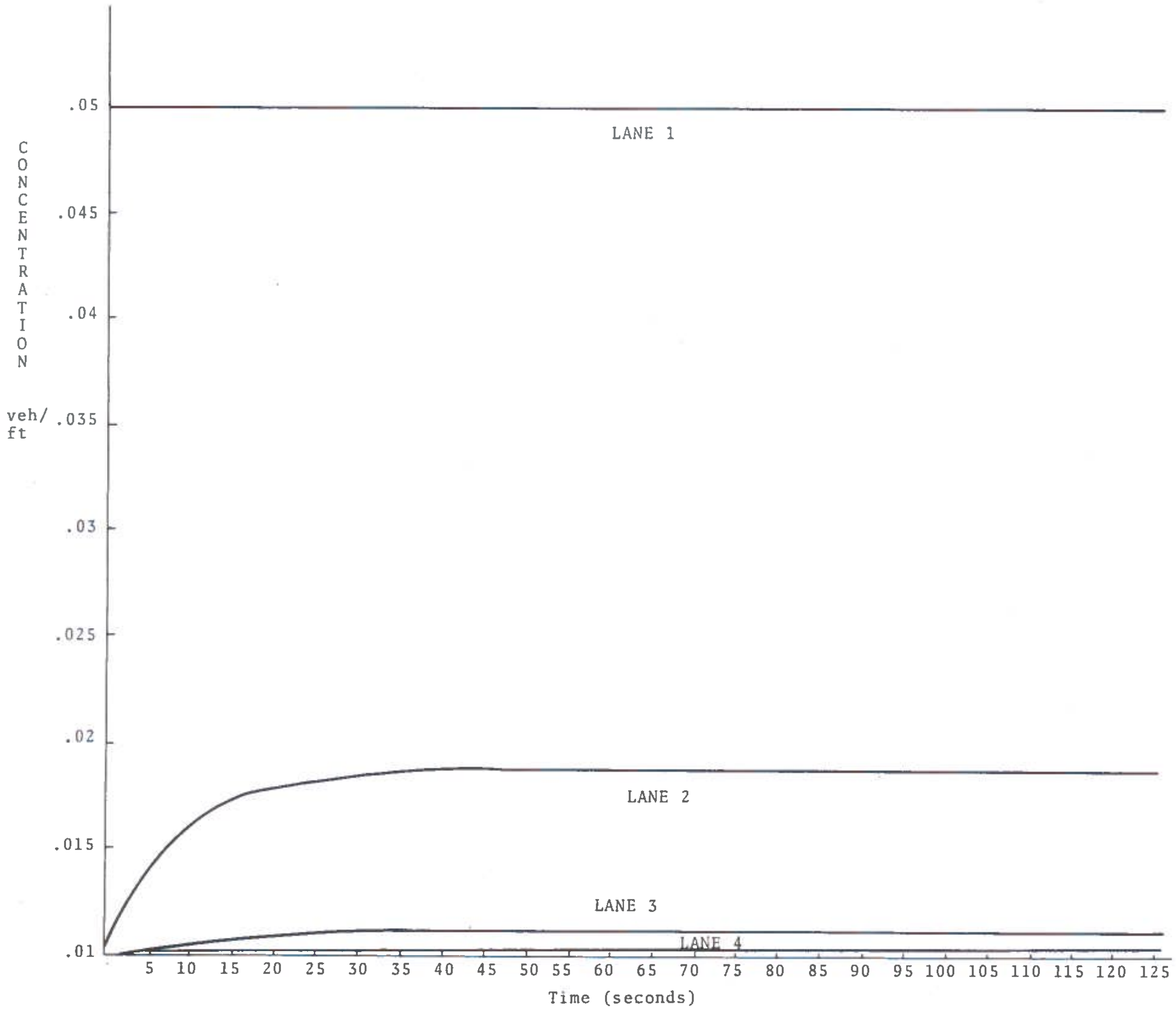

Figure 13. Concentration at Blockage Location as a Function of Time on a 4 Lane Freeway

maximum, K_m, are only illustrative. They undoubtedly are too high, though the ratio K_m/K_j is probably right. Because of this fact, we would expect all the qualitative features of the numerical solutions for the traffic density and shock wave velocity to agree with other values of K_j and K_m (say, K_j=160veh/mile and K_m=64veh/mile), however, we could not be sure of quantitative agreement with experiment until we knew what values were appropriate for the roadway under consideration. Figure 8b shows the difference when other values of K_j and K_m are used. As expected, the qualitative features do not change, though the absolute values of the densities, of course, change.

Mentioning agreement with experiment, it is well to point out that one disadvantage of this model of traffic dynamics (used in the paper by Munjal[1] as well as here) is that for $k<K_m$ (K_m being the value of concentration at which flow is maximized), the velocity is assumed independent of concentration (see Eq. (1), Fig. 1). How well this holds is not certain and must be determined by a validation test of the theory presented here.

A. Steady State Results-Figure 8

We first present the numerical results for the steady state case. The reader is referred to Section 6 for the theoretical development.

Lane 1 experiences a blockage at time t=0 at location x=0. We assume a sufficiently long time has elapsed so that steady state has been achieved and the density remains constant in time. We find the prevailing concentrations at various upstream points on the different lanes of the freeway, after the blockage has remained long enough for steady state conditions to be realized.

These steady state concentrations for each freeway lane for different positions upstream of the blockage are shown in Figures 8a and For the numerical parameters used in Figure 8a, steady state conditions are reached 2 minutes after the blockage has occurred. The concentrations shown in the figure are thus those which exist after the initial two minutes have elapsed.

Figure 8a shows the steady state concentration in lane 1 decrease from a jam concentration of 0.05 vehicles/foot or 264 vehicles/mile at the point of blockage to 0.035 vehicles/foot or 185 vehicles/mile some 40 yards upstream, an almost 30% decrease in traffic density between the blockage point and a point 40 yards upstream. The decrease is due, of course, to vehicles leaving the blocked lane. At this upstream point, according to the theory developed in this paper (see Section 4) a stationary discontinuity exists in lane 1 which separates the unperturbed concentration, K_o=0.01 veh./foot (53 veh/mile) of the upstream traffic from a traffic concentration which is higher than that. This, from Equation (23) is given by

$$K_o + K_j(K_m-K_o)/K_m = 0.035 \text{ veh./foot (185 veh./mile)}.$$

This higher value of the concentration produces the same flow as the unperturbed concentration K_o as previously described in Figure 5. Thus, practically, this means that a vehicle traveling in lane 1 which carries a traffic density K_o=53 veh./mile will, upon reaching a location 120 feet upstream of the blockage, drive into a traffic jam of density $K_o+K_j(K_k-K_o)/K_m$=185 veh./mile. His speed will be greatly reduced since the concentration is much higher though the flow is the same (see Figures 1 and 5). He will try to leave this lane before he gets to the blockage point since the traffic density is becoming worse the closer he gets to the blockage (at the blockage it is 264 vehicles per mile). In fact, a number of cars do escape as can be seen(Figures 8a and b) by the increase in the traffic density of the other lanes. These steady state concentrations in the other lanes at the location 40 yards upstream of the blockage, have the unperturbed values K_1 = 0.01 vehicles/foot. The concentrations increase to the values shown in the figure, 0.0186 vehicles/foot, 0.0113 vehicles/foot, and 0.0102 vehicles/foot, for lanes 2, 3 and 4, respectively, as the blockage at x=0 is reached due to vehicles from lane 1 leaving before the blockage is reached. Lane 2 closest to the blocked lane experiences a larger concentration buildup than the other lanes which are further away from the blocked lane.

Figure 8b is similar and is presented in order to show this qualitative similarity when different numerical values are used for the jam concentration and undisturbed traffic density.

B. Dynamic Results, Figures 9-13

Having presented the steady state results we now look at what happens from the moment the blockage occurs until steady state has been reached (for the numerical example given here, this happens after two minutes). We do this with the aid of a series of graphs showing the relationship between such parameters as shock speed and traffic density as well as by showing the buildup of traffic density with time until steady state is reached. The graphs were obtained from the solution of the continuity equations and boundary conditions, Equations (25)-(28).

1. Shock Wave Trajectory and Traffic Densities in Lane 1, Figure 9

Recall from the theoretical development (Sections 3-5) that following a blockage in lane 1 a discontinuity in traffic concentration or shock wave is created in lane 1 which travels upstream until steady state conditions are reached (assuming medium to light traffic densities prior to the blockage). This shock wave was shown to propagate upstream with velocity $d\xi/dt$ given by Equation (20). We also showed that the concentration of vehicles in lane 1 was strongly affected by the propagation of this shock wave, decreasing from a high at the blockage point of K_j to a steady state value at the upstream steady state position given by Equation (23). In Figure 9 a space-time curve has been plotted which shows the trajectory of the shock from the blockage point to the steady state position 120 feet upstream. We have also indicated what the traffic densities are along this space-time trajectory. With the given numerical values for the constants, the shock wave very quickly (after about seven seconds) travels half the distance between its starting and end points. Steady state conditions however, are not reached until some 113 seconds later. On the same graph, the traffic concentrations existing at different upstream space-time locations are shown. For example, at x=t=0, where and when the blockage first occurs, the traffic density is at jam concentration, 0.05 vehicles/foot. Further along the trajectory of the shock at x=-41 feet upstream and and t=4 seconds, the concentration in lane 1 diminsishes to about 0.0436 vehicles/foot. At the steady state position, x=-121.7 feet and t=120 seconds, the concentration ceases to change and remains at the steady state value of 0.035 vehicles/foot. Thus Figure 9 shows that the shock wave generated at the blockage at location x=0 at time t=0 at which location and time the traffic density is at jam concentration K_j=0.05 veh./foot, travels at first very quickly upstream, diminishing in velocity as it propagates, until finally the propagation ceases altogether at the steady state location. The traffic concentration is seen to reduce from the K_j=0.05 veh./foot value to the steady state value of 0.035 veh./foot given by Equation (23). The reduction in the traffic density as we move further upstream from the blockage is seen to reduce quite rapidly at first, from 0.05 veh./foot at time t=0 to 0.0436 veh./foot some 6 seconds later, to 0.0387 veh./foot in another 9 seconds. The concentration after that time diminishes only much more slowly, from 0.0387 veh./foot to 0.035 veh./foot in about another 105 seconds. Obviously, the greatest traffic density build up, coinciding with the greatest upstream shock wave propagation speeds occurs in the first 10 or 20 seconds from the start of the blockage. In general, even with different numerical values of the parameters, we would expect the greatest traffic density build up to occur within 15% or 20% of the total time measured from the onset of the blockage until steady state is reached when the density ceased to change.

2. Shock Wave velocity as a Function of Time, Figure 10

Figure 10 shows the velocity of the discontinuity or shock wave from Equation (20) as a function of time since the onset of the blockage. The figure clearly demonstrates that the shock wave velocity rapidly decreases with time approaching zero in about two minutes after the occurrence of the blockage. The figure clearly indicates that steady state is approached.

3. Shock Wave Velocity as a Function of Position, Figure 11

Figure 11 also shows the shock wave velocity, in this graph, however, as a function of position upstream of the blockage. The initial moderate decrease in the velocity of the discontinuity with increasing distance upstream followed by a sharp decrease to zero velocity further upstream clearly shows the approach to steady state where the velocity of the shock is zero.

4. Traffic Concentration as a Function of Shock Wave Velocity, Figure 12

In Figure 12 we plot the traffic concentration in the blocked lane as a function of the shock wave velocity. As expected, the traffic density in the blocked lane decreases from jam concentration K_j with decreasing shock wave velocity. It can be seen that the decrease in concentration is most rapid in the initial stages of the shock wave propagation, decreasing from 0.05 veh./foot to about 0.036 veh./foot as the shock wave slows down from about 10 feet/sec. to approximately 1.5 feet/sec. After that, the traffic density, slowly approaches a constant 0.035 veh./foot as the shock speed approaches zero and steady state is attained.

5. Traffic Density in all Four Freeway Lanes as a Function of Time, Figure 13

Finally, in Figure 13 we show the traffic concentration for each of the four freeway lanes at the x=0 location as a function of time.

If steady state had been reached, as in Figure 8, then the concentration in all four lanes would remain constant and four horizontal lines, one for each lane, would be drawn for this graph showing a constant density over time. This is because this graph shows the traffic density at one particular location on the road (here the x=0 location) and for steady state the traffic density does not change with increasing time. Since steady state, in fact, is not reached until two minutes after the occurrence of the blockage, the graph shows instead the increase of the traffic density at the x=0 location in each of the freeway lanes in the first two minutes since the occurrence of the blockage. The concentration in the blocked lane, lane 1, remains constant at jam concentration since the blockage has not been removed. The concentrations in the other lanes increase with time from their value before the onset of the blockage, K_o= 0.01 vehicles/foot to higher values as vehicles from the blocked lane diffuse into the other lanes. As expected, the greatest increase in concentration occurs in the lane closest to the blocked lane, (lane 2). The traffic density at x=0 in lane 2 increased 50% from its unperturbed value K_o=0.01 veh./foot at time t=0 to 0.015 veh./foot some 7 seconds later. The density levels off at 0.0186 veh./foot, its steady state value, after two minutes. The densities in lanes 3 and 4 also increase from their initial unperturbed values K_o=0.01 veh./foot but remain at lower traffic density levels than does lane 2 since they are farther from the blocked lane.

For other values of the jam concentration K_j, ambient concentration K_o and concentration at maximum flow K_m, other concentration-position-time curves would be generated which could be compared with experimental data when available. Since such data is not presently available, we shall be content with simply pointing out that the curves so generated would show the same functional dependence as the ones shown here, namely, the generation of a shock wave at the location of the blockage at time t=0 which propagates upstream with a rapidly decreasing velocity until steady state-zero velocity conditions are reached. The concentration in the blocked lane decreases very rapidly with both time and position upstream of the blockage initially and then more slowly until steady state conditions are reached. The concentration in the other lanes increases rapidly at first and then more slowly with the lanes closest to the blocked lane experiencing the greatest concentration increase.

8. CONCLUSIONS

We have treated the problem of the traffic redistribution following the onset of a lane blockage on a multi-lane freeway. Our work differed from previous ones in that we allowed for both up and downstream traveling waves, as one must, following a lane blockage on a multi-lane freeway.

It was shown that the onset of the lane blockage generated a concentration discontinuity which traveled upstream of the blockage while downstream traveling waves propagated in the other freeway lanes under moderately light traffic densities.

We distinguished between the steady and non-steady state cases, and solved the equations of motion to obtain the concentration in each of the freeway lanes both before and after steady state had been achieved. The theory developed here is ready for a validation study against actual freeway data to be taken following the occurrence of a blockage. Following a successful validation study, the theory may be applied to the implementation of a control system to minimize the deleterious effects that follow the onset of the blockage.

9. REFERENCES

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3. Courant and Hilbert, "Methods of Mathematical Physics," Interscience Publishers, New York, 1966, Chapters 1,2,3.

4. Drake, J.S., Schofer, J.L. and May Jr.,A.D., "A Statistical Analysis of Speed Density Hypotheses," Highway Research Record Number 154 pp 53-87.

APPENDIX 1. SOLUTION OF EQUATIONS (33) - (35) OF SECTION 5

In this appendix we outline the method of solution of Equations (33) - (35), the continuity equations in the α,β coordinate system. We repeat these equations here for convenience.

$$(c_1-c_2)\frac{\partial k_1}{\partial \beta} + a(k_1 - k_2) = 0 \tag{33}$$

$$(c_2 - c_1)\frac{\partial k_\ell}{\partial \alpha} + a\,(-k_{\ell-1} + 2k_\ell - k_{\ell+1}) = 0, 2\leq \ell < n-1 \tag{34}$$

$$(c_2 - c_1)\frac{\partial k_n}{\partial \alpha} + a\,(-k_{n-1} + k_n) = 0 \tag{35}$$

For convenience we also repeat Figure 7 which shows the α,β grid lines and the shockwave-blockage boundary lines.

The first step in the solution is to determine β_1 and $k_1(\alpha_1,\beta_1)$. These variables are determined by the lane 1 continuity equation

$$\left(c_1-c_2\right)\frac{\partial k_1}{\partial \beta} + a\left(k_1-k_2\right) = 0 \tag{33}$$

discretized over the interval connecting $(\alpha_1,\alpha_1 c_2/c_1)$ to (α_1,β_1) and by the shock wave speed boundary condition

$$\frac{d\xi}{dt} = \frac{q_m}{K_o-k_1\left(\xi(t),t\right)}\left[\frac{K_o}{K_m} - \frac{K_j-k_1\left(\xi(t),t\right)}{K_j-K_m}\right]. \tag{20}$$

To use this condition, we may point out that the slope of the curve $x = \xi(t)$ in (α,β) coordinates is given by

$$\frac{d\beta}{d\alpha} = \left(\frac{d\xi}{dt} - c_2\right)\Big/\left(\frac{d\xi}{dt} - c_1\right). \tag{1-1}$$

as can be seen from Figure 7 where the shock wave curve $x = \xi(t)$ is shown to the left of the blockage curve. The slope, of course, is just the derivative of β with respect to α. We also note that the average slope of the curve $\beta = \beta(\alpha)$ over the interval $[0, \alpha_1]$ is given by

$$\left(\beta_1 - c_2\ \alpha_1/c_1\right) \Big/ h$$

as can be seen in Figure 7 using the definitions of the slope to a curve. We equate this to the sum of the individual slopes at $\alpha=0$ and at $\alpha = \alpha_1$.

$$\frac{1}{2}\left\{\left(\frac{d\beta}{d\alpha}\right)_{\alpha=0} + \left(\frac{d\beta}{d\alpha}\right)_{\alpha=\alpha_1}\right\}$$

where $(d\beta/d\alpha)_{\alpha=0}$ is determined from the fact that $k_1(0,0)=K_j$ and $(d\beta/d\alpha)_{\alpha=\alpha_1}$ is expressed in terms of $k_1(\alpha_1, \beta_1)$. Condition (20) above is thus expressed in terms of the (α, β) grid and can therefore be discretized and hence numerically evaluated.

The lane 1 continuity equation

$$\left(c_1 - c_2\right) \frac{\partial k_1}{\partial \beta} + a\left(k_1 - k_2\right) = 0 \qquad (33)$$

itself was discretized by replacing $\partial k_1/\partial\beta$ by the difference quotient (e.g. Figure 7):

$$\frac{k_1\left(\alpha_1, \beta_1\right) - k_1\left(\alpha_1, c_2\alpha_1/c_1\right)}{\beta_1 - c_2\alpha_1/c_1}$$

where $k_1(\alpha_1, c_2\alpha_1/c_1) = K_j$ (since it is on the blockage line, Figure 7), while β_1 and $k_1(\alpha_1, \beta_1)$ are variables to be solved for. The term $a(k_1 - k_2)$ is evaluated at (α_1, β_1) since $k_2(\alpha_1, \beta_1) = K_o$ and $k_1(\alpha_1, \beta_1)$ is to be solved for.

We are now able to solve the resulting two simultaneous continuity and shock wave speed equations for $\underline{\beta_1 \text{ and } k_1(\alpha_1, \beta_1)}$. This was done.

Next, the continuity equations

$$\left(c_2-c_1\right) \frac{\partial k_\ell}{\partial \alpha} + a\left(-k_{\ell-1}+2k_\ell-k_{\ell+1}\right) = 0$$

$$2 \leq \ell \leq n-1 \qquad (34)$$

and

$$\left(c_2-c_1\right) \frac{\partial k_n}{\partial \alpha} + a\left(-k_{n-1}+k_n\right) = 0 \qquad (35)$$

are discretized by replacing $\partial k_\ell / \partial \alpha$ by the difference quotient (c.f. Fig. 7)

$$\frac{k_\ell\left(\alpha_2,\beta_1\right) - k_\ell\left(\alpha_1,\beta_1\right)}{h}$$

and evaluating the terms $a(-k_{\ell-1}+2k_\ell-k_{\ell+1})$ and $a(-k_{n-1}+k_n)$ at (α_1,β_1). Since we know $k_\ell(\alpha_1,\beta_1)$ for all ℓ (it is equal to K_o) we solve each equation ($\ell=2,\ldots.,n$) for $k_\ell(\alpha_2,\beta_1)$. Thus, we now have the values of $k_\ell(\alpha_2,\beta_i)$ for $\ell=2,\ldots.n$, and $i = 1,2$.

To determine $k_1(\alpha_2,\beta_1)$ and $k_1(\alpha_2,\beta_2)$ we use the lane 1 continuity equation

$$\left(c_1-c_2\right) \frac{\partial k_1}{\partial \beta} + a\left(k_1-k_2\right) = 0 \qquad (33)$$

which can be solved for $k_1(\alpha_2,\beta_1)$, knowing $k_2(\alpha_2,\beta_1)$ and using $k_1(\alpha_2,c_2\alpha_2/c_1) = K_j$.

We can now determine $k_1(\alpha_2,\beta_2)$ and β_2 by a method similar to that which was used to find β_1 and $k_1(\alpha_1,\beta_1)$.

We then find the values of k_ℓ along the grid line $\alpha=\alpha_3$ for lanes $\ell=2$ to n by the method with which we obtained these values along $\alpha=\alpha_2$. We can thus iterate to determine the values of the lane densities $k_1,\ldots..k_n$ from $\alpha=0$ to $\alpha=\alpha_m$, where m is as large as desired. In this manner the four lane freeway with a blockage in one of the lanes was solved to give the four lane densities, k_1, k_2, k_3, and k_4 resulting from this blockage.

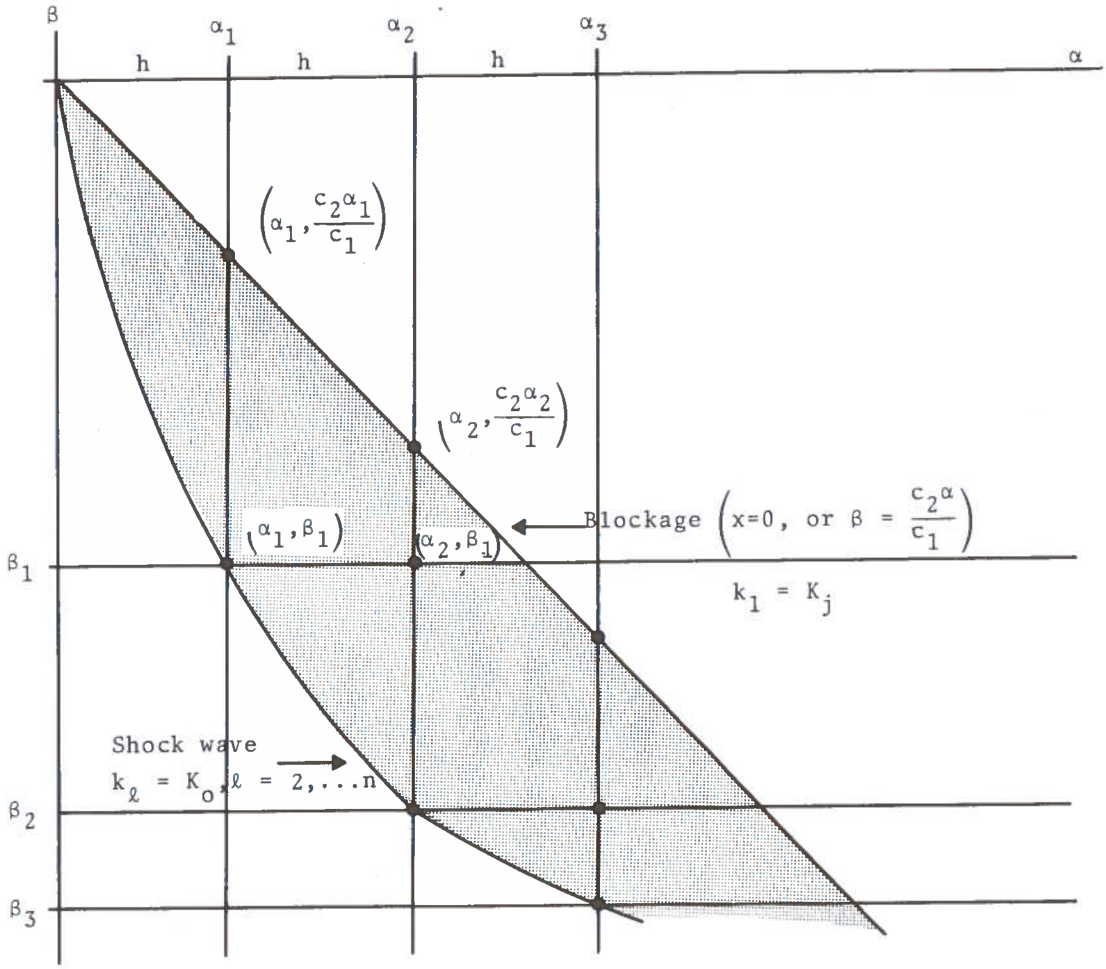

Figure 7. α, β Grid Lines Showing Shockwave-Blockage Boundaries

APPENDIX 2. WAVE PROPAGATION UNDER HEAVY TRAFFIC CONDITIONS

When traffic is heavier so that the concentration in the unblocked lanes exceeds K_m as a result of the diffusion of vehicles from the blocked lane, it is uncertain whether steady state would be achieved. This point as to whether or not steady state is achieved has not been analyzed here. It will be the topic of a future paper.

There are three cases of interest: The one treated in this paper which we call the moderately light traffic situation in which the density before the blockage occurred was $K_o<K_m$ which remains less than K_m in the unblocked lanes even after the blockage. The second case is when the density is $K_o<K_m$ before the blockage but which increases to values above K_m following the blockage in one or more of the unblocked lanes due to diffusion of cars from the blocked lane. Finally, the third case is the heavy density case in which the density K_o is greater than K_m even before the lane blockage.

For this latter case when traffic conditions are so heavy that $K_o>K_m$, then the concentration in all lanes is above K_m and Munjal's[1] solution can be applied. We have, from Munjal et al[1]

$$k(x,t) = MB(x)M^{-1}\, k(0,t-x/c_1) \qquad (2\text{-}1)$$

where B(x) is defined below and c_1 is the rate of change of flow, q, with respect to concentration, k for $k>K_m$, or

$$c_1 = -\, q_m/(K_j-K_m).$$

M is a matrix such that $M^{-1}AM=S$ where S is a diagonal matrix with the eigenvalues λ of A defined below. Thus the discontinuity caused by the sudden blockage of lane 1 would propagate upstream at constant rate c_1. As the wave propagates upstream, we now show that the limiting values of k in all lanes approach the same value, the average of the initial values, $k_j(0,t-x/c)$. Using Munjal's[1] solution given by Equation (2-1) we have B(x) defined as the diagonal matrix

$$B(x) = \begin{bmatrix} d_1(x) & & & \\ & d_2(x) & & \\ & & \ddots & \\ & & & d_n(x) \end{bmatrix}$$

where

$$d_i(x) = \exp\left[-\lambda_i \ ax/c_1\right]$$

For the 4-lane freeway case the eigenvalues are

$$\lambda_1 = 0$$

$$\lambda_i > 0, \ i = 2,3,4$$

Thus

$$d_1(x) = \exp\left[-\lambda_1 \ ax/c_1\right] = 1 \tag{2-2}$$

and

$$d_i(x) = \exp\left[-\lambda_i \ ax/c_1\right] \to 0 \text{ as } \frac{x}{c_1} \to +\infty \quad . \tag{2-3}$$

The particular values of λ_i are $\lambda_2=2-\sqrt{2}$, $\lambda_3=2$ and $\lambda_4=2+\sqrt{2}$. The M matrices are given by

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \sqrt{2} - 1 & -1 & -(\sqrt{2} + 1) \\ 1 & 1 - \sqrt{2} & -1 & 1 + \sqrt{2} \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2+\sqrt{2} & \sqrt{2} & -\sqrt{2} & -(2+\sqrt{2}) \\ 2 & -2 & -2 & 2 \\ 2-\sqrt{2} & -\sqrt{2} & \sqrt{2} & -(2-\sqrt{2}) \end{bmatrix}$$

Here $B(x) \to \text{diag.}\ [1,0,0,0]$ as $x/c_1 \to \infty$. Therefore, defining

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = M^{-1} \begin{bmatrix} k_1(0,t-x/c_1) \\ k_2(0,t-x/c_1) \\ k_3(0,t-x/c_1) \\ k_4(0,t-x/c_1) \end{bmatrix}$$

we have, using Equation (24)

$$k(x,t) \to \begin{pmatrix} \alpha \\ \alpha \\ \alpha \\ \alpha \end{pmatrix} \text{ as } \frac{x}{c} \to \infty \qquad (2\text{-}4)$$

where α is the first component of $M^{-1}k(0,t-x/c_1)$. Hence

$$\alpha = \frac{1}{4}\left\{k_1\left(0,t-\frac{x}{c_1}\right) + k_2\left(0,t-\frac{x}{c_1}\right) + k_3\left(0,t-\frac{x}{c_1}\right) + k_4\left(0,t-\frac{x}{c_1}\right)\right\}. \qquad (2\text{-}5)$$

Thus, as x/c_1 becomes large, travelling with the velocity of the wave so that $t-x/c_1$ remains constant, the concentration approaches the same value in all four lanes, the average over the four lanes of the perturbations at the blockage point. In other words, in this heavy traffic situation, the concentration for large times is the same in each of the four lanes. It is given in terms of the concentrations at the blockage location as indicated by Equation (2-5). We may also note that initially at t=0 we have jam concentration in lane 1

$$k_1(0,0) = K_j$$

and concentration K_o in the other lanes

$$k_i(0,0) = K_o,\ i=2,\dots n$$

so that traveling with the wave the concentration at $x=c_1t >> 0$ at time t approaches (see Equation (2-5).

$$k_\ell(c_1t,\ t) \rightarrow (1/4)[k_1(0,0) + k_2(0,0) + k_3(0,0) + k_4(0,0)]$$

$$= \frac{K_j + 3K_o}{4} \qquad \ell = 1,2,3,4$$

or, in general,

$$k_\ell(c_1t,t) \rightarrow \frac{K_j + (n-1)K_o}{n} \qquad \ell = 1,\ldots n$$

for an n lane freeway. One word of caution is in order here. In this heavy density case we may need to specify downstream conditions as well as upstream ones. This is because immediately downstream of the blockage in lane 1 the traffic density is zero. For some distance downstream of the blockage in that lane the density will remain below K_m (contrary to the original assumption that $k > K_m$) since cars cannot immediately return to this lane from the other lane after the blockage has been passed but will only diffuse gradually. The value of k_ℓ $(0,t)$ for $t\geq 0$ is known only for $\ell=1$, ($k_1(0,t)=K_j$ for $t \geq 0$). However, the values of k_ℓ $(0,t)$ for $t \geq 0$ for the other lanes could very well be affected by the downstream rate of diffusion of cars from these lanes to lane 1. This has not been investigated in this paper and is a subject for future research.