REPORT NO. DOT-TSC-FHWA-73-12

# SOME CONSIDERATIONS ON THE PROBLEM OF NON-STEADY STATE TRAFFIC FLOW OPTIMIZATION

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OCTOBER 1973
FINAL REPORT



Prepared for:
DEPARTMENT OF TRANSPORTATION
FEDERAL HIGHWAY ADMINISTRATION
Fairbanks Highway Research Station
McLean VA 22101

TECHNICAL REPORT STANDARD TITLE PAGE

| | | |
|---|---|---|
| 1. Report No.<br>DOT-TSC-FHWA-73-12 | 2. Government Accession No. | 3. Recipient's Catalog No. |
| 4. Title and Subtitle<br>SOME CONSIDERATIONS ON THE PROBLEM OF NON-STEADY STATE TRAFFIC FLOW OPTIMIZATION | | 5. Report Date<br>October 1973 |
| | | 6. Performing Organization Code |
| 7. Author(s)<br>David Kahn, Ronald Mintz | | 8. Performing Organization Report No.<br>DOT-TSC-FHWA-73-12 |
| 9. Performing Organization Name and Address<br>Department of Transportation<br>Transportation Systems Center<br>Kendall Square<br>Cambridge, MA 02142 | | 10. Work Unit No.<br>R3204 |
| | | 11. Contract or Grant No.<br>HW308 |
| 12. Sponsoring Agency Name and Address<br>Department of Transportation<br>Federal Highway Administration<br>Fairbanks Highway Research Station<br>McLean VA 22101 | | 13. Type of Report and Period Covered<br>Final Report<br>May 1973 |
| | | 14. Sponsoring Agency Code |
| 15. Supplementary Notes | | |

16. Abstract

This report contains our initial efforts aimed at extending the steady state freeway model for optimizing freeway traffic flow to a non-steady state model. The steady-state model does not allow reaction to continuously changing conditions which are often important. The non-steady state or dynamic model will allow this and is intended to be used whenever a metering rate which changes with time is needed. The dynamic modeling is accomplished by developing optimization procedures based on the principles of traffic dynamics, specifically, the continuum equations. In this initial effort only a tunnel roadway and a single lane freeway (but with exits and ramps) are considered.

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| 17. Key Words<br>Traffic Flow, Steady State, Dynamic, Tunnel, Freeway, Optimization, Discretization | | 18. Distribution Statement<br>DOCUMENT IS AVAILABLE TO THE PUBLIC THROUGH THE NATIONAL TECHNICAL INFORMATION SERVICE, SPRINGFIELD, VIRGINIA 22151. | |
| 19. Security Classif. (of this report)<br>Unclassified | 20. Security Classif. (of this page)<br>Unclassified | 21. No. of Pages<br>62 | 22. Price |

Form DOT F 1700.7 (8-69)

# PREFACE

The problem of the "Analysis of Shock Wave Phenomena for Freeway Control" was undertaken as part of an overall freeway-corridor traffic improvement program. Our part of this effort is to understand how disruptive shock phenomena occur on the freeway, what happens to vehicles as a result and to initiate a study into development of a traffic control system aimed at keeping traffic flow on the freeway optimal. Our efforts toward understanding disruptive shock phenomena was reported in a study done on the lane blockage problem, "Freeway Traffic Flow Following a Lane Blockage." In this report, we present our findings to date on the traffic flow modification part of the problem, namely on the problem of minimizing the total travel time in the freeway-corridor system.

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# 1. INTRODUCTION

The problem we address is that of maximizing the output flow in a freeway system or equivalently, minimizing the total travel time in the system. In Section 2 we discuss this optimization problem using a steady state assumption and point out its limitations, such as not allowing reaction to continuously changing conditions and not allowing a time varying ramp metering rate.

To allow for these, non-steady state or dynamic modeling must be considered. We do this in Section 3, first considering the single lane tunnel roadway using a continuum approach. The problem is to minimize the total waiting and travel time in the system, queue plus tunnel, during the time of interest. To do this the integral of the cumulative output flow must be maximized. The output flow is determined from the initial state of the tunnel and by the chosen entrance flow control.

First, the usual continuum equation with zero reaction time is used; then, the new non-zero reaction time model is used to obtain the vehicular output flow and vehicular concentration. It is shown that the zero reaction time model leads to unphysical results. It is therefore not used subsequently in the analysis. Instead the analysis proceeds on the basis of the finite reaction time model which takes account of non-instantaneous driver response. *The problem is to find a vehicular flow rate or concentration which maximizes the integral of the cumulative output and which can be practically used as an entrance control.* This problem is approached by discretizing the continuum equation, the criterion integral and by imposing certain required physical constraints on the flow.

A rather simple discretization is presented which allows the optimization to be done analytically and therefore, illustrates the ideas clearly. The continuity equation is discretized and solved subject to a constant entrance control. This yields the value of the concentration at the exit at any time in terms of the concentration which existed at the exit at the initiation of the control and in terms of the entrance control concentration itself. It is shown that if the entrance control concentration is less than the steady state optimal concentration, then the predicted concentration at the exit due to the institution of the control will approach the control concentration with time, as it should. More specifically, after one reaction time unit following the initiation of the control, the exit concentration approaches the control concentration by some specific amount. This is to be contrasted with the steady state result which predicts that the exit concentration becomes the same as the control concentration immediately.

For the case when the control concentration is greater than the steady state optimum concentration but is less than the initial exit concentration, there is an exponential increase in concentration indicating a stoppage, since the entrance flow in this case is larger than the initial exit flow.

The preceding rather simple discretization of the basic continuum equation is then replaced by a somewhat more accurate one in which an integral formulation of the continuity equation is presented. The equations are discretized by means of quadrature formulas and solved subject to given initial and boundary conditions. From this analysis we determine the interrelationships between different initial concentrations at the exit and different controls at the tunnel entrance and their effect on the stability of the traffic flow.

Following this non-steady state analysis of the optimization problem, we compare it with the steady state model. We indicate how the dynamic analysis may be considered as a higher order approximation to the steady state model.

Finally, in Section 4, the dynamic analysis is extended to a single lane freeway using an expanded version of the finite reaction time model which accounts for varying capacities along the roadway and which is formulated for inclusion of entrance and exit ramps. Diversion of traffic to the surrounding streets is taken into account.

## 2. STEADY STATE FREEWAY MODEL

The steady state freeway model[1] is a potentially useful model for the development of a peak period metering control system designed to keep total flows less than or equal to freeway capacities. The model assumes that the freeway can be divided into sections, in each of which the flow rate is constant over distance and time. It is assumed that section k has a fixed capacity $B_k$ which is the maximum flow rate possible through the section. If the flow rate into a section is less than the capacity of the section, congestion is assumed not to occur and steady state will prevail.

Using this steady state model, the optimization problem may be outlined as follows. We assume that the number of vehicles in the freeway does not change with time (in the absence of congestion) so that the total input rate of vehicles entering the freeway (from the queue) summed over all entrance ramps must equal the total output rate. If q denotes this rate, then the cumulative output at time t is qt, and we want to maximize

$$\int_0^T qt\ dt = \frac{1}{2}\,qT^2 \qquad (1)$$

which is equivalent to maximizing q,[3] (See also Appendix A).

Let $X_j$ be the metered input rate of flow (from queue at ramp onto freeway) at entrance j. We want to maximize

$$q = \sum_{j=1}^{n} X_j \ . \qquad (2)$$

Note that we are assuming the rate of arrival $D_j$ of vehicles in the queue at entrance j is invariant with respect to $X_1,\ldots,X_n$.

The optimization is done under several constraints, for example

$$0 \leq X_j \leq D_j \qquad (3)$$

which states that the metering rate cannot exceed the rate of arrival of vehicles at the ramp. Note that if $X_j < D_j$, there will be a steady increase in queue length on the j th entrance ramp during the time period in which the analysis is carried out. The analysis is still valid in such a case since the number of vehicles on the freeway remains constant, even though it is accompanied by an undesirable increasing queue.

Another set of constraints is necessary to prevent the flow into a section from exceeding the capacity of the section. Let $A_{jk}$ be the ratio of the number of vehicles which enter at ramp j and pass through section k of the freeway to the total number entering at ramp j. We can now express the flow through section k as

$$\sum_{j=1}^{n} A_{jk} X_j ,$$

where we sum over all entrance ramps from which vehicles can reach section k. The capacity constraint is then

$$\sum_{j=1}^{n} A_{jk} X_j \leq B_k, \; k=1,\ldots,m . \qquad (4)$$

Note that $A_{jk}$ is assumed constant in the time period in which the analysis is carried out, and would be determined by observation.

This constrained optimization problem can be solved by the methods of linear programming. Additional constraints could be added, as discussed in Wattleworth's article.[1] If the capacity of a section is reduced due to an accident, for example, the linear programming problem can be solved using the reduced steady state capacity resulting from the accident.

However, the important limitation of this method, in our opinion, is that it does not allow reaction to continuously changing conditions which require a ramp metering rate which varies with time. We therefore devote the rest of this report to considering an optimization method which is based on the principles of traffic dynamics.

## 3. TUNNEL OPTIMIZATION CONTINUUM MODEL

In treating the optimization problem dynamically, we first restrict our analysis to the single lane tunnel roadway and use a continuum model of the traffic dynamics.

We assume that the rate of arrival $D(t)$ of vehicles into the entrance ramp queue is invariant to the ramp metering rate, and denote the vehicular flow and the vehicular concentration at point $x$ and time $t$ by $q(x,t)$ and $K(x,t)$, respectively. The tunnel entrance is located at $x=0$, the exit at $x=L$, (see Figure 1).

We would like to minimize the total waiting and travel time in the system queue plus tunnel between time 0 and time T. Or as shown in Appendix A, we wish to maximize the integral of the cumulative output at time t, where

$$0(t) = \int_0^t q(L,t')dt' \tag{5}$$

$$\int_0^T 0(t)dt = \int_0^T \int_0^t q(L,t')dt'\, dt \tag{6}$$

The exit flow $q(L,t)$ is determined from the initial state of the tunnel at time 0 and from the chosen entrance flow $q(0,t)$.

The constraint on the choice of entrance flow $q(0,t)$ is

$$\int_0^t q(0,t')dt' \le n_o + \int_0^t D(t')dt' \;, \; t \ge 0 \tag{7}$$

which assures that no more vehicles enter the tunnel than have arrived at the entrance ramp. Here $n_o$ is the number of vehicles present at the entrance ramp queue at time $t=0$. The number of vehicles which have entered the queue between time 0 and time t is

$$\int_0^t D(t')\, dt'$$

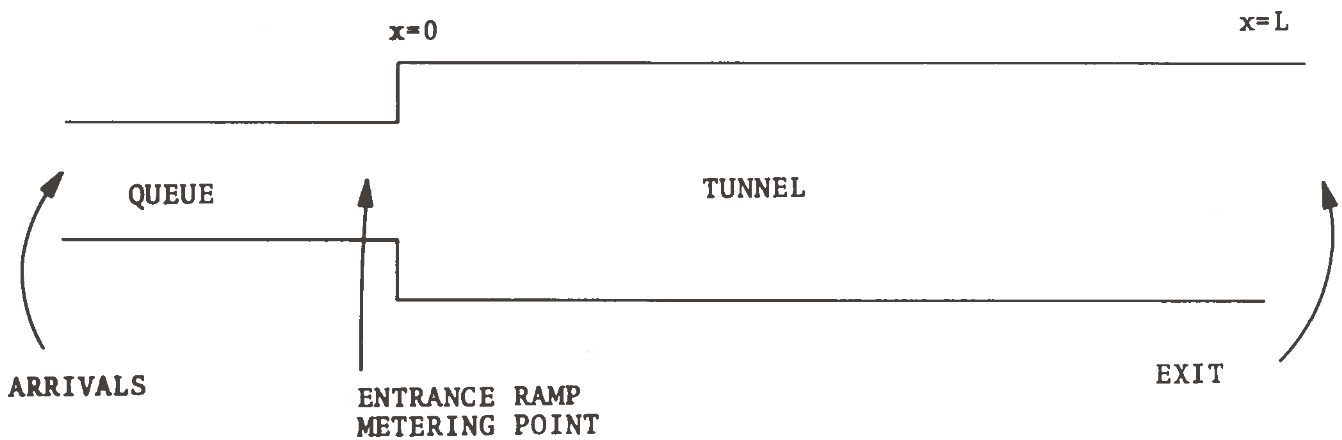


Figure 1. Schematic for Single Lane Tunnel Optimization Problem

while

$$\int_0^t q(0,t')dt'$$

is the number of vehicles which have entered the tunnel from the queue between time 0 and time t.

We now proceed with the optimization problem, first with the standard continuum equation that leads to the formation of shock waves, and then with the equations of traffic dynamics containing a finite driver reaction time.

### 3.1 ZERO REACTION TIME MODEL

The continuum equation is given as

$$\frac{\partial K}{\partial t} + c \ln(K_j/eK) \frac{\partial K}{\partial x} = 0 \tag{8}$$

where $K_j$ is jam concentration and Greenberg's log model has been assumed, with flow and concentration related by the equation

$$q(x,t) = c\,K(x,t) \ln\left[K_j/K(x,t)\right] . \tag{9}$$

The criterion to be maximized, therefore is

$$\int_0^T \int_0^t K(L,t') \ln\left[K_j/K(L,t')\right] dt'\, dt. \tag{10}$$

We must determine the dependence of K(L,t) upon K(0,t) and upon K(x,0). This may be obtained from the characteristic lines of the partial differential continuum Equation (8) determined from

$$\frac{dx}{c \ln(K_j/eK)} = dt \;;\; dK = 0 . \tag{11}$$

Hence, K is constant on a characteristic line and we have

$$x = \left[c \ln(K_j/eK)\right] t + D \tag{12}$$

as the set of characteristic lines, parametrized by constant K and D. The time $t_o$ of intersection of the characteristic through (0,0) with the line x=L is just the time it takes a control initiated at the tunnel entrance to affect traffic at the tunnel exit. For times $t<t_o$ the vehicular concentration at the tunnel exit, $K(L,t)$ would be determined by its initial value $K(x,0)$ while for times $t>t_o$, $K(L,t)$ is determined by the concentration at the tunnel entrance $K(0,t)$ which in turn depends upon the entrance ramp metering control. The optimum solution is for $K(L,t) = K_j/e$ for $t>t_o$; however, this would require vertical characteristic lines of the form x=L (see Eq. (12)), which cannot be generated by specifying boundary values at x=0. The optimal solution would then be

$$K(L,t) \to K_j/e \text{ as } t\to\infty \qquad (13)$$

with

$$K(0,t) = K_j/e \text{ for all } t > 0 \; . \qquad (14)$$

The characteristics which produce this optimal solution are illustrated in Figure 2. The slope of the characteristics dt/dx at (L,t) approaches $\infty$ as $t\to\infty$. This configuration is produced when the concentration at the tunnel entrance $K(0,t)$ is changed suddenly from $K(0,0)$ to $K_j/e$. But $K(L,t)$ for all $t>t_o$ is determined by these values of $K(0,t)$ at the entrance. For the behavior of $K(L,t)$ to be determined by the values of $K(0,t)$ over an infinitesmal interval of change from $K(0,0)$ to $K_j/e$ is not physically meaningful and hence it is unproductive to continue with this standard continuum approach which neglects the finite reaction time of drivers, leading to the unphysical result.

## 3.2 FINITE REACTION TIME MODEL

If the driver's reaction time is taken into account, a more meaningful result is obtained. Letting $\tau$ denote the reaction time, we use the following equation of state[2]

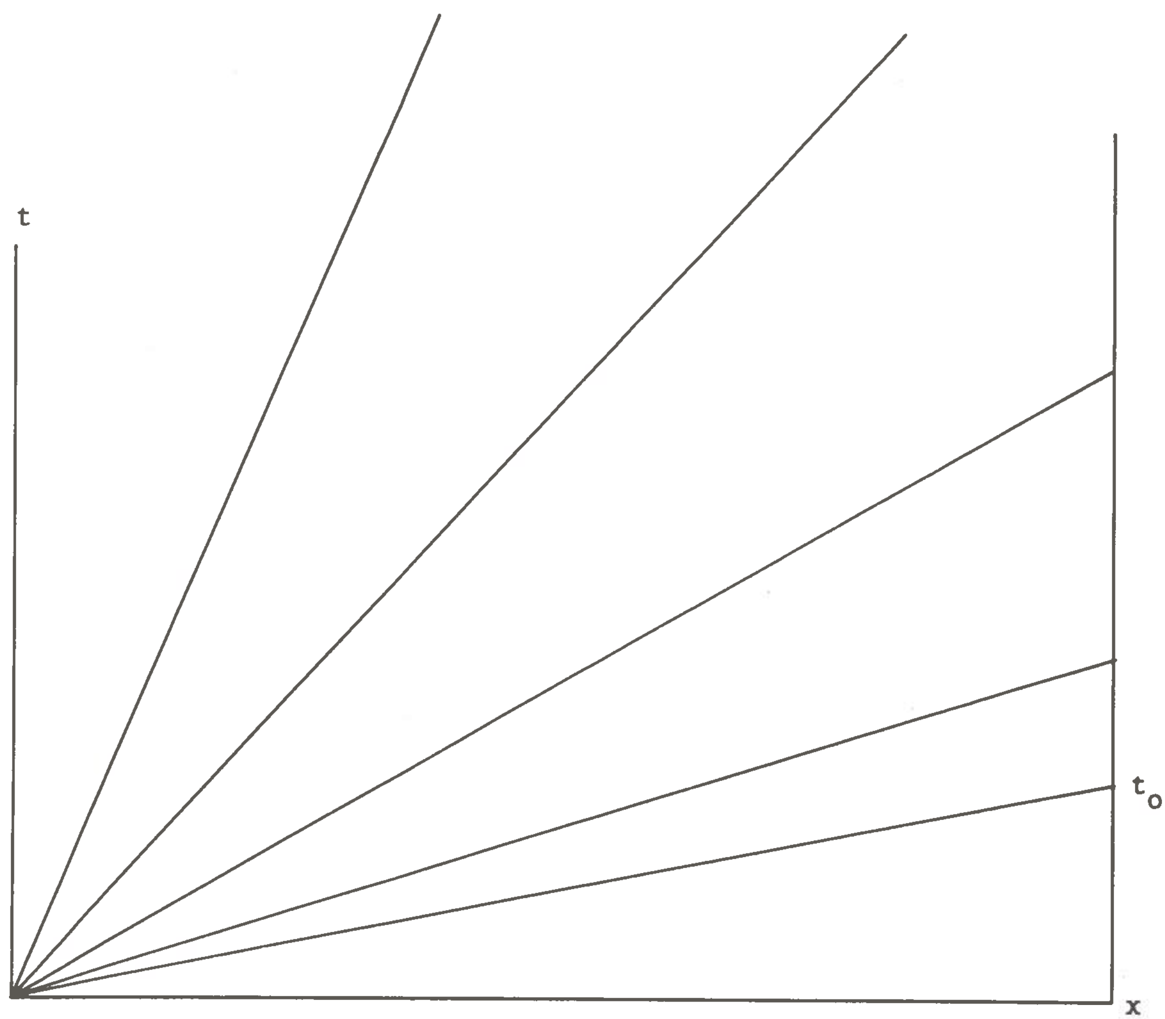


Figure 2. Sketch of Characteristics Producing Solutions Given by Equations 13 and 14

$$q(x,t) = c\, K(x,t)\, \ln\left[K_j/K(x,t-\tau)\right] . \tag{15}$$

Substituting this into the continuity equation

$$\frac{\partial K(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0 \tag{16}$$

we obtain

$$\frac{\partial K(x,t)}{\partial t} + c\left[\frac{\partial K}{\partial x}(x,t)\, \ln\left\{K_j/K(x,t-\tau)\right\} - \frac{K(x,t)}{K(x,t-\tau)}\frac{\partial}{\partial x}K(x,t-\tau)\right] = 0 \tag{17}$$

Because of the finite time lag in the equation, an initial condition must be specified over an interval of one reaction time. We specify the concentration $K(x,t)$ for positions anywhere along the roadway, $0 \leq x \leq L$, and for times $-\tau \leq t \leq 0$, which is the interval over which the condition must be specified.

The problem is to find an entrance control $K(0,t)$ for times $0 \leq t \leq T$ such that

$$\int_0^T \int_0^t q(L,t')dt'dt = c \int_0^T \int_0^t K(L,t')\, \ln\left[\frac{K_j}{K(L,t'-\tau)}\right] dt'dt \tag{18}$$

is maximized, subject to the constraint

$$\int_0^t q(0,t')dt' \leq n_o + \int_0^t D(t')dt', \quad t \geq 0 \tag{7}$$

which, again, states that the number of vehicles which have entered the roadway from the queue between time $t'=0$ and time $t'=t$ cannot exceed the number of vehicles that were in the queue at time $t'=0$ and the number which have arrived there since then.

In addition, we may also specify feasibility constraints on the concentration $K(x,t)$ such as

$$0 \leq K_{min} \leq K(x,t) \leq K_j$$

where the lower bound $K_{min}$ is used since according to the log model we are using, the velocity of the vehicles approachs infinity as $K \to 0$. $K_{min}$ is the lowest concentration for which the vehicles interact in accordance with the car following model.

We may also set an upper bound on the derivatives of K(x,t) since in reality the distance and time headways are finite, and a solution for K(x,t) which changes too rapidly with distance or time may not be physically realizable.

The problem of finding an optimal extrance control then can be solved by discretizing the partial differential equation, the criterion integral, and the additional constraints in accordance with some grid over time and space. Initial values for K(x,t) for $0 \leq x \leq L$ and $-\tau \leq t \leq 0$ would be specified at grid points based upon observational measurements.

### 3.2.1 Estimating K(L,t)

If we consider the following crude approximation to Equation (17), we can analytically determine the behavior of the resultant concentration at the exit, K(L,t)

$$\frac{\partial K(0,t)}{\partial t} + c\left[\frac{K(L,t) - K(0,t)}{L} \ln\left\{\frac{K_j}{K(0,t-\tau)}\right\} - K(0,t)\Big(K(L,t-\tau) - K(0,t-\tau)\Big) \Big/ K(0,t-\tau)L\right] = 0 \tag{19}$$

where we have used the approximation

$$\frac{\partial K(0,t)}{\partial x} \simeq \frac{K(L,t) - K(0,t)}{L}$$

If we solve this approximated continuity equation, subject to the boundary condition

$$K(0,t) = K_o = \text{constant} \tag{20}$$

which is used as the control at the entrance x=0 we obtain

$$K(L,t) = \frac{K(L,t-\tau)}{\ln(K_j/K_o)} + K_o\left[1 - \left\{\ln(K_j/K_o)\right\}^{-1}\right]. \tag{21}$$

This sequence is of the form

$$a_{n+1} = ra_n + s \;;\quad r = \frac{1}{\ln(K_j/K_o)} \;;\; s = K_o\left[1 - \frac{1}{\ln(K_j/K_o}\right]$$

$$a_n = K(L,n\tau) \tag{22}$$

The general term is

$$a_n = r^n a_o + \left(r^{n-1}+\cdots+1\right)s = r^n a_o + \left(\frac{1-r^n}{1-r}\right)s \tag{23}$$

which can easily be seen from the first few terms in the sequence. Hence, we have

$$K(L,n\tau) = \frac{K(L,0)}{\left[\ln(K_j/K_o)\right]^n} + \left[1 - \frac{1}{\left[\ln(K_j/K_o)\right]^n}\right]K_o \tag{24}$$

If,

$$K_o < K_j/e,\ 0 < \left[\ln(k_j/K_o)\right]^{-1} < 1 \tag{25}$$

then,

$$K(L,t+n\tau) \to K_o \text{ as } n \to \infty \tag{26}$$

The difference between the initial value of the concentration at the exit, $K(L,0)$ and $K_o$ decreases exponentially with time.

However, for

$$K_o = K_j/e \tag{27}$$

we have that

$$\ln(K_j/K_o) = 1 \tag{27a}$$

so that,

$$K(L,n\tau) = K(L,0) \text{ for all } n. \tag{28}$$

For

$$K_j/e < K_o < K_j,\ 0 < \ln(K_j/K_o) < 1,\ \frac{1}{\ln(K_j/K_o)} > 1 \tag{29}$$

we have

$$K(L,n\tau) = \frac{K(L,0) - K_o}{\left[\ln(K_j/K_o)\right]^n} + K_o \ . \tag{30}$$

The difference between K(L,0) and $K_o$ increases exponentially with time.

We thus see that this particular discretization is valid for $t\to\infty$ only when the control $K_o<K_j/e$, since unphysical results were obtained in the other cases. For the case when the control $K_o<K_j/e$, the discretization predicted that after one reaction time unit following the initiation of the control the tunnel exit concentration had become closer to the concentration $K_o$, specifically

$$K(L,\tau) = \frac{K(L,0) - K_o}{\ln(K_j/K_o)} + K_o \tag{31}$$

the initial difference between K(L,0) and $K_o$ has been divided by $\ln(K_j/K_o)$. While this is a gross overestimation of the rapidity of the change due to a control, it is still an improvement over a steady-state assumption which is equivalent to assuming that K(x,t) becomes $K_o$ immediately for all x for optimization purposes.

In the case where $K_j/e<K_o<K_j$, if $K(L,0)>K_o$, the exponential increase in concentration indicates a stoppage. This increase in K is physically meaningful, since when $K_j/e<K_o<K(L,0)$, $q(0,0)>q(L,0)$ since flow decreases as concentration increases above $K_j/e$. In other words, the fact that the initial concentratio is higher at the exit than at the entrance (and is greater than $K_m$ where $K_m$ is the value of the concentration for which the flow is maximum) means that the flow at the exit will be less than at the entrance, and therefore the number of vehicles in the tunnel will increase, causing K to increase with time.

For the case when the control $K_o > K_j/e$ but the initial concentration $K(L,0)$ is less than $K_o$, Equation (30) predicts that $K(L,t)$ will exponentially decrease without bound. This also has some physical meaning if $K(L,0)$ is not too small, since then there would be a higher flow at the exit than at the entrance, $q(L,0) > q(0,0)$ and more vehicles would leave the tunnel than enter. Consequently, K would decrease.

We note, however, that whenever $K_o > K_j/e$ and $K(L,0) \neq K_o$ the result for $K(L,t)$ cannot remain valid as $t \to \infty$ since in fact K does not increase or decrease without bound, rather $0 \leq K \leq K_j$. However, choosing $K_o > K_j/e$ would not be optimal in general due to the stoppage waves which could occur and in practice we would not consider this control.

### 3.2.2 An Alternative Discretization

The preceding method of discretizing Equation (17), is not necessarily the most accurate one; it was chosen because it was easy to work with analytically and made the application of the optimization procedure clear.

Another technique is to start with an integral formulation of the continuity equation. Consider a grid where the tunnel is partitioned by a set of points $x_o = 0, x_1, \ldots x_{n-1}, x_n = L$, and time is discretized by $\ldots, t_{-1}, t_o, t_1, t_2, \ldots$ The number of vehicles entering the section of the tunnel between positions $x_i$ and $x_{i+1}$ during the time interval $[t_j, t_{j+1}]$ is the integral of the flow at $x_i$

$$\int_{t_j}^{t_{j+1}} q(x_i, t)\,dt \; .$$

The number leaving the section is

$$\int_{t_j}^{t_{j+1}} q(x_{i+1},t)dt.$$

The net change in the number of vehicles in the section between time $t_j$ and $t_{j+1}$ is therefore

$$\int_{t_j}^{t_{j+1}} \left[q(x_i,t) - q(x_{i+1},t)\right]dt$$

This quantity can also be expressed as the number of vehicles in the section at time $t_{j+1}$minus the number at time $t_j$

$$\int_{x_i}^{x_{i+1}} K(x,t_{j+1})dx - \int_{x_i}^{x_{i+1}} K(x,t_j)dx.$$

Hence, the continuity equation can be written as

$$\int_{t_j}^{t_{j+1}}\left[q(x_{i+1},t) - q(x_i,t)\right]dt + \int_{x_i}^{x_{i+1}}\left[K(x,t_{j+1}) - K(x,t_j)\right]dx = 0. \tag{32}$$

If we divide by $(t_{j+1}-t_j)(x_{i+1}-x_i)$ and take the limit as $(t_{j+1}-t_j) \to 0$ and $(x_{i+1}-x_i) \to 0$, we obtain

$$\frac{\partial q}{\partial x} + \frac{\partial K}{\partial t} = 0.$$

However, it seems more direct to discretize the integrals directly by means of quadrature formulas than to reduce to differential form and approximate the derivatives by difference quotients.

Applying the trapezoidal rule to both integrals in Equation (32), we obtain

$$\left[q(x_{i+1},t_j) - q(x_i,t_j) + q(x_{i+1},t_{j+1}) - q(x_i,t_{j+1})\right](t_{j+1}-t_j)$$

$$+ \left[K(x_i,t_{j+1}) - K(x_i,t_j) + K(x_{i+1},t_{j+1}) - K(x_{i+1},t_j)\right](x_{i+1}-x_i) = 0. \tag{33}$$

Substituting the equation of state

$$q(x,t) = cK(x,t)\ \ln\left(K_j/K(x,t-\tau)\right) \tag{15}$$

into Equation (33) we obtain

$$c(t_{j+1}-t_j)\left\{K(x_{i+1},t_j)\ \ln\left[\frac{K_j}{K(x_{i+1},t_j-\tau)}\right] - K(x_i,t_j)\right.$$

$$\cdot \ln\left[\frac{K_j}{K(x_i,t_j-\tau)}\right] + K(x_{i+1},t_{j+1})\ \ln\left[\frac{K_j}{K(x_{i+1},t_{j+1}-\tau)}\right]$$

$$\left. - K(x_i,t_{j+1})\ \ln\left[\frac{K_j}{K(x_i,t_{j+1}-\tau)}\right]\right\}$$

$$+ (x_{i+1}-x_i)\left[K(x_i,t_{j+1}) - K(x_i,t_j) + K(x_{i+1},t_{j+1}) - K(x_{i+1},t_j)\right] = 0. \tag{34}$$

If the grid points are evenly spaced so that $x_{i+1}-x_i = h_x$ and $t_{j+1}-t_j = h_t$ for all i and j, we may write the equations as

$$K(x_{i+1},t_j)\left[\ln\left\{\frac{K_j}{K(x_{i+1},t_j-\tau)}\right\} - \frac{h_x}{h_t c}\right]$$

$$+ K(x_i,t_j)\ \left[-\ln\left\{\frac{K_j}{K(x_i,t_j-\tau)}\right\} - \frac{h_x}{h_t c}\right]$$

$$+ K(x_{i+1},t_{j+1})\ \left[\ln\left\{\frac{K_j}{K(x_{i+1},t_{j+1}-\tau)}\right\} + \frac{h_x}{h_t c}\right]$$

$$+ K(x_i, t_{j+1}) \left[ - \ln \left\{ \frac{K_j}{K(x_i, t_{j+1} - \tau)} \right\} + \frac{h_x}{h_t c} \right] = 0. \tag{35}$$

In order to solve this equation, suppose we first choose a coarse grid, $x_i = 0$ and $x_{i+1} = L$, note that the equation is satisfied by a constant, and take as our initial conditions

$$\left. \begin{aligned} K(0,t) &= K_o \\ K(L,t) &= K_1 \end{aligned} \right\} - \tau \leq t \leq 0. \tag{36}$$

The control provides the boundary condition

$$K(0,t) = K_o, \ t \geq 0. \tag{37}$$

Substitute Equations (36) and (37) with the grid $x_i=0$, $x_{i+1}=L$ into Equation (35). After some algebraic manipulation we obtain

$$K(x_{i+1}, t_{j+1}) = K(x_{i+1}, t_j)$$

$$+ \frac{2\left\{K_o \ln(K_j/K_o) - \frac{K(x_{i+1},t_j)}{2}\left[\ln\left(\frac{K_j}{K(x_{i+1},t_j-\tau)}\right) + \ln\left(\frac{K_j}{K(x_{i+1},t_{j+1}-\tau)}\right)\right]\right\}}{\ln\left(\frac{K_j}{K(x_{i+1},t_{j+1}-\tau)}\right) + \frac{h_x}{h_t c}}. \tag{38}$$

For $j=0$, $t_o=0$, $t_1=h_t$, we obtain

$$K(L,h_t) = K_1 + \frac{2\left[K_o \ln(K_j/K_o) - K_1 \ln(K_j/K_1)\right]}{\ln(K_j/K_1) + h_x/h_t c}, \quad 0 \leq h_t \leq \tau \tag{39}$$

as the concentration at the tunnel exit after an interval of time $h_t$. Now suppose

$$K_1 \ln(K_j/K_1) = K_o \ln(K_j/K_o) \tag{40}$$

so that $K(L,h_t)=K_1$. If $K_o \neq K_j/e$, two values of $K_1$ will satisfy this equation since the function $K \ln(K_j/K)$ is increasing when $K<K_j/e$ and decreasing when $K>K_j/e$

$$\frac{d}{dK}\left[K \ln(K_j/K)\right] = \ln(K_j/K) - 1 \left\} \begin{array}{l} > 0,\ K < K_j/e \\ < 0,\ K > K_j/e \end{array} \right. . \qquad (41)$$

Referring to Figure 3, we see that $K_1$ could be either of the two points indicated on the q-K curve, one when K exceeds $K_m$ and the other when it is less than it.

In order to solve the equation, we can now repeat the preceding steps, since we again have a boundary condition at x=0 of $K_o$ and $K(L,h_t)=K_1$, to obtain $K(L,2h_t)=K_1$ and so forth. Hence, we have $K(L,t)=K_1$ for all $t \geq 0$.

If $K_1=K_o$, the result is sensible: constant boundary conditions producing that same constant as solution.

If $K_1$ is the other root of Equation (40), however, the concentration at the exit remains $K_1$; it will never converge to $K_o$. While this would be unlikely, it does support the principle of conservation of vehicles applied to the whole tunnel as a unit (integral formulation of continuity equation).

We now show how the concentration at the exit changes with time for the given control strategy, $K(0,t) = K_o$, $t \geq -\tau$, and with an initial condition $K(L,t) = K_1$, $-\tau \leq t \leq 0$. We use Equation (39); let $\overline{K}_o$ denote the root of Equation (40) which is not equal to $K_o$. Suppose $K_o < K_j/e$. The situation is sketched in Figure 4 where $K \ln(K_j/K)$ is plotted against K. If $0<K_1<K_o$ or $\overline{K}_o<K_1<K_j$, we see that $K_1 \ln(K_j/K_1)$ is less than $K_o \ln(K_j/K_o)$, hence, $K(L,h_t)>K_1$. Note that the denominator of Equation (39) $[\ln(K_j/K_1) + h_x/h_t c] > 0$ for all $0 < K_1 \leq K_j$. This indicates that concentration at x=L will increase with time, which is physically meaningful because the flow at x=L is lower than that at x=0 and hence the average concentration in the tunnel must increase. Since the concentration is fixed at the entrance, x=0, the concentration at the exit, K(L,t), must increase. This direction of change is indicated by the arrows in Figure 4, $(K_1<K_o)$.

If $K_o < K_1 < \overline{K}_o$, $K_1 \ln(K_j/K_1) > K_o \ln(K_j/K_o)$, hence, $K(L,h_t) < K_1$ causing a decrease in the exit concentration, K(L,t). Therefore, if $0 < K_1 < \overline{K}_o$, we expect K(L,t) to converge to $K_o$ as $t \to \infty$.

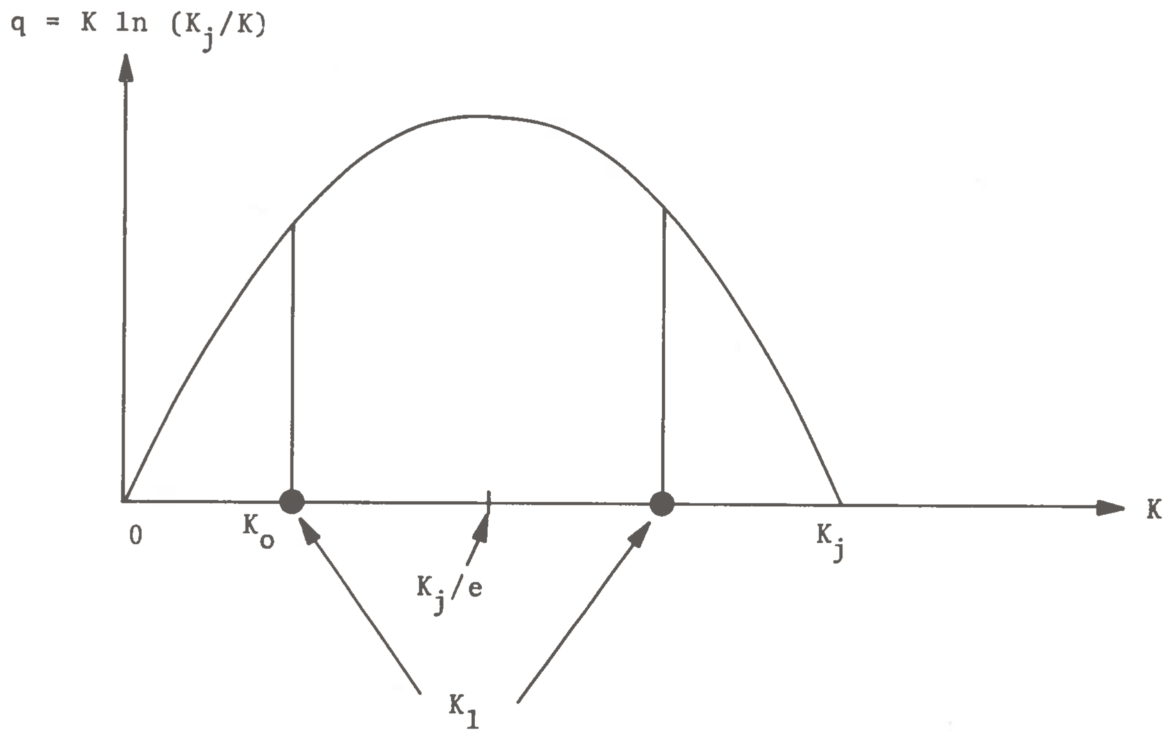


Figure 3. q-K Schematic Indicating the Two Solutions for the Concentration, One When $K_1$ is Greater than $K_j/e$ and One When it is Less than it.

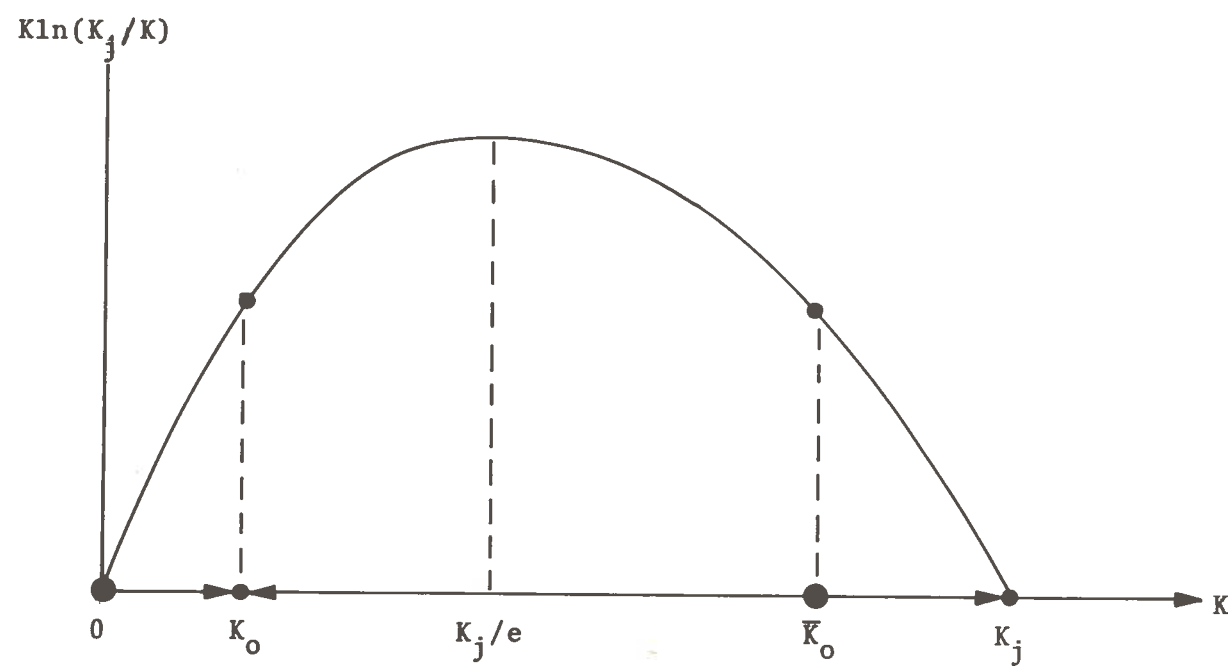


Figure 4. Schematic of $K \ln(K_j/K)$ Versus K Showing Direction of Increase or Decrease of $K(L,t)$ to $K_o$ for $0<K_1<\bar{K}_o$ and Increase of $K(L,t)$ to $K_j$ for $K>\bar{K}_o$

$K_1 = \overline{K}_o$ is a point of unstable equilibrium which is not physically meaningful arising because of our rather coarse approximation. For $K_1 > \overline{K}_o$, we would expect $K(L,t) \to \infty$ which indicates a stoppage. This case of positive feedback is physically meaningful because the excessive concentration at the exit reduces the tunnel output to below that of the input. Hence the number of vehicles in the tunnel increases, but since the entrance concentration is held constant at $K_o$, $K(L,t)$ must increase further.

In Appendix C another optimization method is presented in which the $K(L,t)$ which maximizes the criterion integral is determined and then a $K(0,t)$ is obtained.

## 3.3 COMPARISON WITH STEADY STATE METHOD

We may explain here in what sense the discrete models are to be considered as higher order approximations relative to the steady state model. In the steady state model, we approximate $K(x,t)$ by a constant. This constant is determined by the boundary condition $K(0,t)$. Since there is no need for a continuity equation, it is more convenient to work directly with the flow $q(x,t)$, which is directly related to the concentration, $K(x,t)$. The steady state model is optimized subject to the constraint that the flow into a section does not exceed its capacity. This must be done since the model cannot describe the congestion which would occur in such a case. In the simple case of tunnel optimization, the steady state model reduces to the problem of maximizing the flow in the tunnel, $q$, subject to the constraints $0 \leq q \leq$ demand rate, and $q \leq$ capacity of tunnel. The tunnel capacity would be $cK_j/e$, the maximum value of $q = cK \ln(K_j/K)$. Hence the optimal solution would be

$$q = \mathrm{MIN}\left[cK_j/e, \text{ demand rate}\right] .$$

In the discrete model following the integral formulation of the continuity equation, we approximated $K(x,t)$ by a piecewise linear function spatially, and approximated $q(x,t) = cK(x,t)$.

$\ln[K_j/K(x,t-\tau)]$ by a piecewise linear function with respect to time. This was implied when we discretized by the trapezoidal rule. If we divide the roadway length $[0,L]$ into n divisions, we need $(n+1)$ conditions to determine $K(x,t)$ as a function of x; for example, the control, $K(0,t)$ and the slope of $K(x,t)$ over each subinterval, assuming continuity at the end points of each subinterval. The boundary value, $K(0,t)$ specifies only one of these conditions. The other n conditions are specified by applying the integral form of the continuity equation to each subinterval. These equations assert that the net flow into the subinterval equals the net change of number of vehicles in the subinterval. It does not require that vehicles be conserved at every point of the subinterval as does the differential form of the continuity equation. This, of course, is not without its possible disadvantages, for example, if the grid spacing is large, it is possible that a jam may occur at a point within a grid space which would not be indicated by linear interpolation between end points. However, this would probably occur only when the initial conditions are inhomogeneous, and such jams would not be predicted at all by the steady state model which does not use a continuity equation.

Before leaving this comparison between the steady state model and the discrete models, we may note that $K$=constant is a solution of the continuity Equation (17) so that if the control applied at $x=0$ results in a solution which is so nearly constant over a section of roadway that it can be approximated by a constant, then discretizing the continuum equation and treating the entire section of roadway as one grid unit, will give at least as good an approximation as the steady state solution. However, the discrete equation can also allow for small variations.

In the presence of initial conditions in which K varies rapidly, the steady state approach is essentially a choice of constant control which optimizes the flow after a sufficiently long time interval has passed when K has become sufficiently close to a constant. The steady state approach disregards any immediate effects of this constant control. In reality, if we have the boundary condition $K(0,t)=K_o$=constant, we would expect $K(x,t)\to K_o$ as $t\to\infty$ for all x. If a discrete approximation to the continuum Equation (17) satisfies this, then it should be at least as good as the steady state model since it predicts the value of K correctly for sufficiently large t, and offers a better approximation for smaller t.

In Appendix E the steady state tunnel optimization problem is extended to allow for a fixed percentage of vehicles to be diverted to an alternate route.

## 4. SINGLE LANE FREEWAY

We now extend the optimization problem to the case of a single lane freeway* by means of the following modifications of the tunnel model:

1. There are n entrance ramps along the freeway
2. There are m exit ramps
3. To account for the fact that different sections of roadway may have different capacities, we assume

$$q(x,t) = C(x)\ K(x,t)\ \ln\left[K_j/K(x,t-\tau)\right] \tag{42}$$

   where C is now a function of x.

4. We assume that some percentage of vehicles arriving at an entrance ramp will divert to the surrounding streets as a result of queueing on the entrance ramp.

Wattleworth's steady state model[1] allows different sections of roadway to have different capacities of flow. If we assume the usual equation of state without the reaction time, or the new equation of state with the reaction time, the relationship of flow to concentration is the same for all sections of roadway. If we define capacity as the maximum possible flow over the section of roadway under steady state conditions, we have

$$q = C\ K\ \ln(K_j/K)\ . \tag{43}$$

Hence,

$$dq/dK = C\left[\ln(K_j/K) - 1\right] = 0,\ \text{when}\ K = K_j/e \tag{44}$$

and the capacity is therefore $CK_j/e$.

*We note here that the problem treated in this section may be considered applicable also to the multilane freeway if flow, speed and density may be averaged over freeway lanes. To the extent that such averaging is efficacious for freeway control, the present section may be considered as addressing the multilane freeway problem.

Since $K_j$ and e are constants, one way to vary capacity along the road is to let C be a function of x as shown in Equation (42). While this might not accurately represent all reduced capacity situations, it can be handled without much change to the models. The variation of C(x) with x may be either continuous or discontinuous, depending on road conditions. Allowing C to be a function of x actually modifies the car following model[5] from which these equations may be derived, since we now have

$$V_n(t+\tau) = C(x_n) \ln\left[\left(x_{n-1}(t) - x_n(t)\right) K_j\right] \tag{45}$$

where $x_n(t)$ is the position of car n at time t. Differentiating, we obtain

$$\frac{d^2 x_n(t+\tau)}{dt^2} = C(x_n)\left(\frac{dx_{n-1}}{dt} - \frac{dx_n}{dt}\right)\Big/\left(x_{n-1}(t) - x_n(t)\right)$$

$$+ C'(x_n) \frac{dx_n}{dt} \ln\left[\left(x_{n-1}(t) - x_n(t)\right) K_j\right] \tag{46}$$

The additional term on the right hand side of the equation, not present in the usual models, reflects the fact that if the capacity of the road is decreasing, vehicles will be slowing down.

Let $q_k(x)$ be the flow rate of vehicles destined for exit k at point x, time t of the freeway.

Let $K_k(x,t)$ be the concentration of vehicles destined for exit k.

We are approximating the vehicles by a continuous 'fluid', which is actually a mixture of 'fluids' bound for different exits. Thus, we can subdivide flow and concentration according to destination. All vehicles at a point have the same velocity V(x,t) regardless of destination. The basic relations are:

$$q(x,t) = \sum_{k=1}^{m} q_k(x,t) = \text{total flow} \tag{47a}$$

$$K(x,t) = \sum_{k=1}^{m} K_k(x,t) = \text{total concentration} \tag{47b}$$

$$q_k(x,t) = V(x,t)\ K_k(x,t) \tag{47c}$$

4.1 EXTENDING THE TRAFFIC DYNAMIC EQUATIONS TO ALLOW C=C(x)

Since the set of vehicles which are destined for exit L are conserved, we may apply the continuity equation to this set of vehicles. Formulating this in integral form and discretizing by the trapezoidal rule, we have

$$\Big\{q_L(x_{i+1}),t_j) - q_L(x_i,t_j) + q_L(x_{i+1},t_{j+1}) - q_L(x_i,t_{j+1})\Big\}\ (t_{j+1}-t_j) + \Big\{K_L(x_i,t_{j+1}) - K_L(x_i,t_j) + K_L(x_{i+1},t_{j+1}) - K_L(x_{i+1},t_j)\Big\}\ (x_{i+1}-x_i) = 0 \tag{48}$$

The velocity of vehicles at (x,t) is given by

$$V(x,t) = C(x)\ \ln\left[K_j/K(x,t-\tau)\right] \tag{49}$$

where $K(x,t-\tau)$ is the total concentration of all vehicles. The flow is therefore,

$$q_L(x,t) = K_L(x,t)\ V(x,t) = C(x)\ K_L(x,t)\ \ln\left[\frac{K_j}{K(x,t-\tau)}\right] \tag{50}$$

and substituting into the continuity equation and combining terms gives

$$\begin{aligned} &K_L(x_{i+1},t_j)\Big\{C(x_{i+1})\ \ln\left[K_j/K(x_{i+1},t_j-\tau)\right] - h_x/h_t\Big\} \\ &+ K_L(x_i,t_j)\ \Big\{- C(x_i)\ \ln\left[K_j/K(x_i,t_j-\tau)\right] - h_x/h_t\Big\} \\ &+ K_L(x_{i+1},t_{j+1})\ \Big\{C(x_{i+1})\ \ln\left[K_j/K(x_{i+1},t_{j+1}-\tau)\right] + h_x/h_t\Big\} \\ &+ K_L(x_i,\ t_{j+1})\ \Big\{- C(x_i)\ \ln\left[K_j/K(x_i,t_{j+1}-\tau)\right] + h_x/h_t\Big\} = 0. \end{aligned} \tag{51}$$

## 4.2 PARTITIONING THE ROADWAY

There will generally be discontinuities of some sort at the freeway entrance and exit ramps. We, therefore, partition the freeway at these points. If x is such a point, there will be two values of concentration at x, a left-side value $K(x-,t) = \lim_{\xi \to x-} K(\xi,t)$ and a right-side value $K(x+,t) = \lim_{\xi \to x+} (\xi,t)$. We may further partition each section by a set of grid points for numerical computation. Equation (51) holds when neither $x_i$ nor $x_{i+1}$ is a point of discontinuity. When there is a discontinuity at $x_i$, flow into the interval $[x_i, x_{i+1}]$ will be $q(x_i+,t)$. When we interpolate concentration linearly, we use the right-side value at $x_i$, $K(x_i+,t)$. Thus, in Equation (48) $q_L(x_i,t)$ and $K_L(x_i,t)$ should be replaced by $q_L(x_i+,t)$ and $K_L(x_i+,t)$ respectively when there is a discontinuity at $x_i$. $q_L(x_i+,t)$ will be determined as a function of $K_L(x_i+,t)$ and $K_L(x_i+,t-\tau)$ as usual

$$q_L(x_i+,t) = C(x_i+)\ K_L(x_i\ +,t)\ \ln\left[\frac{K_j}{K(x_i\ +,t-\tau)}\right]. \tag{52}$$

Thus, when there is a discontinuity at $x_i$, but not at $x_{i+1}$, Equation (51) is replaced by

$$\begin{aligned}
&K_L(x_{i+1},t_j)\left\{C(x_{i+1})\ \ln\left[K_j/K(x_{i+1},t_j-\tau)\right] - h_x/h_t\right\}\\
&+ K_L(x_i+,t_j)\left\{-\ C(x_i+)\ \ln\left[K_j/K(x_i+,t_j-\tau)\right] - h_x/h_t\right\}\\
&+ K_L(x_{i+1},t_{j+1})\left\{C(x_{i+1})\ \ln\left[K_j/K(x_{i+1},t_{j+1}-\tau)\right] + h_x/h_t\right\}\\
&+ K_L(x_i+,t_{j+1})\left\{-\ C(x_i+)\ \ln\left[K_j/K(x_i+,t_{j+1}-\tau)\right] + h_x/h_t\right\} = 0.
\end{aligned} \tag{53}$$

We have now specified the equation for one grid spacing to the right (downstream) of a discontinuity. It is also necessary to specify an equation for the grid spacing to the left (upstream) of a discontinuity and to specify a relation between the concentrations on either side of the discontinuity, that is, between $K(x_i-,t)$ and $K(x_i+,t)$. As these relations will depend on the nature of the discontinuity, we will discuss each case separately.

### 4.2.1 Entrance Ramp

We define a number of parameters which are needed to discuss this case.

Let $y_j$ be the location of entrance ramp j, $(j=1,\ldots,n)$.

Let $e_j(t)$ be the metered input flow rate from entrance ramp j onto the freeway. This is our control variable, subject to the constraint that we cannot allow more vehicles onto the freeway than have arrived at the entrance ramp.

Let $e_{jk}(t)$ be the input flow rate of vehicles destined for exit k from entrance ramp j onto the freeway. Note that

$$\sum_{k=1}^{m} e_{jk}(t) = e_j(t) \quad . \tag{54}$$

The division of $e_j(t)$ according to destination is dependent on the relative demand rate for various destinations.

Let $D_{jk}(t)$ be the rate of arrival of vehicles at entrance ramp j which are destined for exit k, (demand rate). Not all of $D_{jk}(t)$ will enter the entrance ramp queue. Depending on queue length, a fraction of the demand will divert to the surrounding streets rather than wait in the queue.

Let $P_{jk}(L)$ be the percentage of demand at entrance j destined for exit k which diverts to surrounding streets when there are L cars in the queue. $P_{jk}(L)$ will generally depend on j and k, as well as L, since for short trips, vehicles will be less willing to wait in a queue than for long trips

Let $L_j(t)$ be the number of vehicles in queue at entrance ramp j at time t.

The actual rate at which vehicles destined for exit k enter the queue at entrance ramp j is

$$D_{jk}(t)\left[1-P_{jk}\left(L_j(t)\right)\right] \quad . \tag{55}$$

The total rate at which vehicles enter the queue at entrance ramp j is

$$\sum_{k=1}^{m} D_{jk}(t)\left[1-P_{jk}\Big(L_j(t)\Big)\right] \tag{56}$$

and the total rate at which vehicles leave the queue to enter the freeway is $e_j(t)$. Hence, queue length is determined by the differential equation

$$\frac{d\left[L_j(t)\right]}{dt} = \left\{\sum_{k=1}^{m} D_{jk}(t)\left[1-P_{jk}\Big(L_j(t)\Big)\right]\right\} - e_j(t) \tag{57}$$

where we assume $D_{jk}(t)$ and $P_{jk}(L)$ are known functions. We assume the initial condition $L(0)$ is known. We require

$$L_j(t) \geq 0 \text{ for all } 0 \leq t \leq T,\ j=1,\ldots,n$$

and $e_j(t) \geq 0$.

We can model the complete closure of ramp j by taking $e_j(t)=0$ and $P_{jk}(L)=1$ for all k if we have an initial condition of $L_j(0)=0$, since all vehicles would then be diverted to the streets.

Suppose vehicles arriving at entrance ramp j at time t enter the freeway at time $w_j(t)$. Then, the total number of vehicles entering the freeway from entrance ramp j between time t and $w_j(t)$ equals the number of vehicles in the queue at time t, since these are the vehicles ahead of the vehicle arriving at time t.

$$\int_{t}^{w_j(t)} e_j(t')\,dt' = L_j(t) \tag{58}$$

This may be written in differential from by differentiating with respect to t (the prime will indicate the derivative with respect to t),

$$e_j\left[w_j(t)\right] \; w_j'(t) - e_j(t) = L_j'(t) \; . \tag{59}$$

The proportioning of vehicles entering the queue at time t to various destinations will be the same as the proportioning of vehicles entering the freeway at time $w_j(t)$, since these are the same vehicles. Hence,

$$\frac{e_{jk}\left[w_j(t)\right]}{e_j\left[w_j(t)\right]} = \frac{D_{jk}(t)\left[1-P_{jk}\left(L_j(t)\right)\right]}{\sum_{k=1}^{m} D_{jk}(t)\left[1-P_{jk}\left(L_j(t)\right)\right]} \tag{60}$$

This relation determines $e_{jk}$ in terms of the control variable $e_j$, so that all parameters are now known for the problem. Equations (59) and (60) are discretized in Appendix D.

We now discuss the freeway traffic dynamics in the region of entrance ramp j. As sketched in Figure 5, flow destined for exit k upstream of the merging point is $q_k(y_j-,t)$ on the freeway and $e_{jk}(t)$ on the entrance ramp. Flow destined for exit k downstream of the merging point is $q_k(y_j+,t)$.

Since vehicles cannot pile up at the merging point, flow in must equal flow out, hence

$$q_k(y_j+,t) = q_k(y_j-,t) + e_{jk}(t) \; . \tag{61}$$

This is a continuity relationship, relating $q(y_j+,t)$ to $q(y_j-,t)$.

It is also necessary to relate velocity to concentration at this point. The velocity of vehicles downstream of the entrance will depend on the concentration downstream, hence

$$V(y_j+,t) = C(y_j) \; \ln\left[K_j/K(y_j+,t-\tau)\right] \tag{62}$$

$$\begin{aligned} q_k(y_j+,t) &= V(y_j+,t) \; K_k(y_j+,t) \\ &= C(y_j) \; K_k(y_j+,t) \; \ln\left[K_j/K(y_j+,t-\tau)\right] \end{aligned} \tag{63}$$

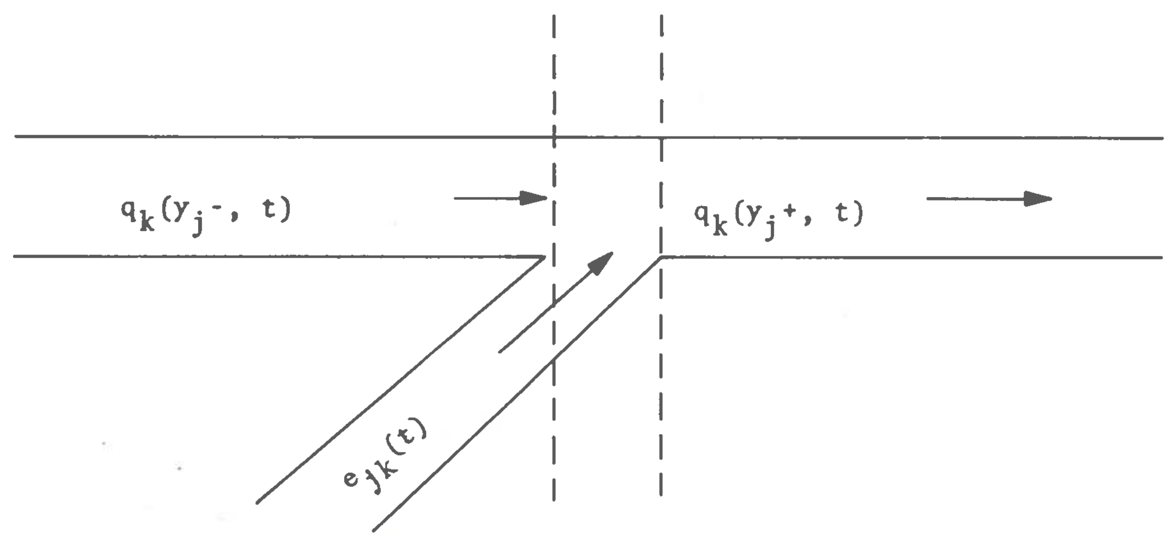


Figure 5. Sketch of Flow Near a Freeway Entrance Ramp

where

$$K(y_j+,t-\tau) = \sum_{k=1}^{m} K_k(y_j+,t-\tau). \tag{64}$$

However, vehicles upstream of the entrance ramp will generally perceive the higher concentration downstream and will adjust their velocity in accordance with this higher concentration. Hence,

$$V(y_j-,t) = C(y_j) \ln\left[K_j/K(y_j+,t-\tau)\right] \tag{65}$$

and

$$\begin{aligned} q_k(y_j-,t) &= V(y_j-,t)\ K_k(y_j-,t) \\ &= C(y_j)\ K_k(y_j-,t)\ \ln\left[K_j/K(y_j+,t-\tau)\right] . \end{aligned} \tag{66}$$

Substituting into Equation (61), the relation between $K_k(y_j-,t)$ and $K_k(y_j+,t)$ is

$$\left[K_k(y_j+,t) - K_k(y_j-,t)\right] C(y_j) \ln\left[K_j/K(y_j+,t-\tau)\right] = e_{jk}(t). \tag{67}$$

Note that flow rate and concentration have discontinuities at an entrance ramp. Velocity is continuous at an entrance ramp, since vehicles upstream and vehicles downstream both react to the downstream concentration. We may also note that with this model, the grid spacing along the freeway should not be too fine, since we assume that one grid space upstream of the entrance ramp, the drivers react to concentration at that point. If this point is too close to the entrance ramp, the drivers' reaction would in reality also be dependent on the concentration downstream of the entrance ramp. This fact, however, could be taken into account in the model by allowing velocity at points near the entrance to be dependent also on the velocity downstream of the entrance ramp in a manner reflecting drivers' deceleration practices.

We may now specify the equation for one grid space upstream of an entrance ramp when the grid space is sufficiently large so that velocity one unit upstream of the entrance is assumed dependent only on concentration at that point. Suppose the entrance ramp is at $x_{i+1}$. Applying the continuity equation to the interval $[x_i, x_{i+1}]$, we get Equation (48) with $q_L(x_{i+1},t)$ and $K_L(x_{i+1},t)$ replaced by $q_L(x_{i+1}^-,t)$ and $K_L(x_{i+1}^-,t)$, respectively. Substituting

$$q_L(x_{i+1}^-,t) = C(x_{i+1})\ K_L(x_{i+1}^-,t)\ \ln\left[K_j/K(x_{i+1}^+,t-\tau)\right] \tag{68}$$

and

$$q_L(x_i,t) = C(x_i)\ K_L(x_i,t)\ \ln\left[K_j/K(x_i,t-\tau)\right] \tag{69}$$

gives

$$\begin{aligned}
&K_L(x_{i+1}^-,t_j)\left[C(x_{i+1})\ \ln\left[K_j/K(x_{i+1}^+,t_j-\tau)\right] - h_x/h_t\right] \\
&+K_L(x_i,t_j)\left[-C(x_i)\ \ln\left[K_j/K(x_i,t_j-\tau)\right] - h_x/h_t\right] \\
&+K_L(x_{i+1}^-,t_{j+1})\left[C(x_{i+1})\ \ln\left[K_j/x_{i+1}^+,t_{j+1}-\tau)\right] + h_x/h_t\right] \\
&+K_L(x_i,t_{j+1})\left[-C(x_i)\ \ln\left[K_j/K(x_i,t_{j+1}-\tau)\right] + h_x/h_t\right] = 0
\end{aligned} \tag{70}$$

as the discretized dynamic equation governing the traffic flow near an entrance ramp.

### 4.2.2 Exit Ramp

We now consider traffic flow near an exit ramp.

Let $z_i$ be the location of exit ramp i. The highway flow on the downstream side of exit ramp i is related to the flow on the upstream side by the following equation

$$q_k(z_i^+,t) = \begin{cases} 0 \text{ if } k=i \\ q_k(z_i^-,t) \text{ if } k \neq i\ . \end{cases} \tag{71}$$

This is because all vehicles destined for exit i leave the highway, and all vehicles not destined for exit i are conserved and cannot pile up in an infinitesmal distance at the exit ramp.

We expect the total concentration of vehicles to be lower on the downstream side of the exit ramp. Vehicles on the downstream side will react to this lower concentration. Vehicles on the upstream side, however, will probably not accelerate in spite of the lower concentration ahead, because the immediate higher concentration retards them. Thus, we have

$$V(z_i-,t) = C(z_i)\ \ln\left[K_j/K(z_i-,t-\tau)\right] \tag{72}$$

$$V(z_i+,t) = C(z_i)\ \ln\left[K_j/K(z_i+,t-\tau)\right] \tag{73}$$

$$q_k(z_i-,t) = C(z_i)\ K_k(z_i-.t)\ \ln\left[K_j/K(z_i-,t-\tau)\right] \tag{74}$$

$$q_k(z_i+,t) = C(z_i)\ K_k(z_i+,t)\ \ln\left[K_j/K(z_i+,t-\tau)\right] \quad . \tag{75}$$

The velocity is discontinuous at $z_i$. This assumes that upon passing the exit ramp, vehicles instantaneously reach the higher velocity, and neglects the fact that vehicles cannot accelerate instantaneously as well as the fact that the driver may be continuing at the slower velocity for one reaction time before reacting to the decreased concentration ahead.

The condition relating concentration on the downstream side to concentration on the upstream side is therefore

$$K_i(z_i+,t) = 0 \tag{76}$$

$$K_k(z_i+,t)\ \ln\left[K_j/K(z_i+,t-\tau)\right] = K_k(z_i-,t)\ \ln\left[K_j/K(z_i-,t-\tau)\right] \quad . \tag{77}$$

The discretized continuity equation for one grid space $[x_i,x_{i+1}]$ upstream of the exit ramp, where $x_{i+1}$ is the location of the exit ramp is therefore just given by Equation (70) with $x_{i+1}+$ in the function $K(x_{i+1}+,t_j-\tau)$ replaced by $x_{i+1}-$.

### 4.2.3 Criterion Integral

The flow onto exit ramp i is $q_i(z_i-,t)$, since this is the flow of vehicles destined for exit i at a point immediately upstream of the exit ramp.

We want to maximize the integral of total cumulative output, summed over all exits

$$\sum_{i=1}^{m} \int_{o}^{T} \int_{o}^{t} q_i(z_i-,t')dt'\,dt$$

$$= \sum_{i=1}^{m} \int_{o}^{T} (T-t)\; q_i(z_i-,t)dt$$

$$= \sum_{i=1}^{m} C(z_i) \int_{o}^{T} (T-t)\; K_i(z_i-,t)\; \ln\left[\frac{K_j}{K(z_i-,t-\tau)}\right] dt \qquad (78)$$

which can be discretized as described in Appendix B.

### 4.2.4 Beginning of Highway

At the beginning of the highway, conditions are similar to that at an ordinary entrance ramp.

If $y_o$ denotes the location of this point, we have

$$q_k(y_o+,t) = e_{ok}(t) \qquad (79)$$

and we may take $q_k(y_o-t) = 0$ since there is no highway upstream of $y_o$.

Equation (67) reduces to

$$C(y_o)\; K_k(y_o+,t)\; \ln\left[K_j/K(y_o+,t-\tau)\right] = e_{ok}(t)\;. \qquad (80)$$

## 5. SUMMARY AND CONCLUSIONS

The important limitation of the steady state method of optimizing freeway traffic flow, in our opinion, is that it does not allow reaction to continuously changing conditions which require a ramp metering rate which varies with time. We, therefore, presented in this report a non-steady state optimization method which was based on the continuum equation of traffic flow.

We first considered a tunnel optimization continuum model. We wrote down the newly developed finite reaction time traffic flow equation and then the integral of the cumulative output, which is to be maximized, containing the concentration at the tunnel exit $K(L,t)$. We then determined the dependence of $K(L,t)$ on the initial exit concentration $K(L,t)$, $-\tau \le t \le 0$ where $\tau$ is the reaction time and on the entrance control concentration $K(0,t)$, $0 \le t \le T$, subject to physically meaningful constraints. We analytically determined this dependence from a simple discretization of the traffic flow equation under a constant control $K(0,t)=K_o$. We derived an equation for the concentration $K(L,t)$ which approached the control concentration $K_o$ with time whenever the control was less than $K_j/e$ (the concentration corresponding to maximum flow on the q-K curve). The difference between the initial value of the concentration at the exit and $K_o$ decreased exponentially for this case.

For a control $K_o > K_j/e$ and an initial concentration $K(L,0) > K_o$ an exponential increase in concentration indicating a stoppage was predicted.

An alternative discretization was also developed which was more accurate, in which the traffic flow equation was put into integral form and then discretized. The equation was solved subject to the initial conditions $K(0,t) = K_o$ and $K(L,t) = K_1$ for $-\tau \le t \le 0$, and used an entrance control $K(0,t) = K_o$ for $t \ge 0$. We obtained an equation which gave the concentration at the tunnel exit in terms of $K_o$, $K_j$ and $K_1$. This solution was discussed. For $K_1 = K_o$ we found that $K(L,t) = K_1$ as it should: if the initial concentration is equal to the control, no change occurs and the concentration remains the same.

For a control $K_o < K_j/e$ and an initial concentration $K_1 < K_o$, the concentration increased with time because the flow at the exit was lower than at the entrance.

For a control $K_o < K_j/e$ and also less than the initial concentration $K_1$, the concentration decreased and converged to the control $K_o$ as $t \to \infty$. This was compared with the steady state model where the concentration at any point x and any time t, $K(x,t)$, is considered to be a constant. In the steady state approach the flow is optimized after a sufficiently long time interval has passed, when K has become nearly constant. In reality, $K(x,t) \to K_o$ as $t \to \infty$. The dynamic approach offers, therefore, an approximation for the concentration for earlier times (and allows for a time varying control).

We next extended the analysis to the case of a single lane freeway by including entrance and exit ramps and by modifying the traffic flow equation to allow for varying capacities along the roadway. We also allowed for a percentage of vehicles arriving at the entrance ramp to be diverted to the surrounding streets as a result of queueing on the entrance ramp. The dynamic equations for this problem were derived and discretized giving the basic relations for determining the traffic flow on the freeway system, near an entrance ramp and near an exit ramp. The criterion integral to be maximized for optimum flow was also derived.

## APPENDIX A
## INTEGRAL OF CUMULATIVE OUTPUT AND TOTAL TRAVEL TIME

We will show that minimizing the total travel time of all vehicles in some system between time 0 and T is equivalent to maximizing the integral of cumulative output

$$\int_0^T O(t)\,dt$$

where O(t) is the total number of vehicles which have left the system between time 0 and time t. This result holds only when the input rate is unaffected by whatever controls are being applied, and we are only considering controls applied after time 0. In the case of ramp metering at the entrance to a tunnel or freeway, we assume that arriving vehicles form a queue at the entrance ramp, and vehicles at the head of the queue are allowed into the tunnel at prescribed intervals. If we assume that the input rate into the queue is unaffected by the metering rate (frequency that vehicles are allowed to enter tunnel), then we may consider the tunnel and queue as the system. If the goal is to minimize total travel time in this system (which includes waiting time) during some fixed time interval, then an equivalent criterion is to maximize the integral of cumulative output. The latter criterion is particularly suitable if a continuum model of traffic dynamics is used.

We may now derive the above result (see also Reference 3).

Let

$$f_i(t) = \begin{cases} 1 \text{ if vehicle i is in system at time t} \\ 0 \text{ otherwise} \quad . \end{cases} \qquad \text{(A-1)}$$

The time spent by vehicle i in the system is just the integral of this

$$\int_o^T f_i(t)dt.$$

The total travel time of all vehicles in the system between times t=0 and t=T is the summation of the times each vehicle spends in the system during this time interval. Hence, we want to minimize

$$\sum_{i=1}^{N} \int_o^T f_i(t)dt = \int_o^T \left[\sum_{i=1}^{N} f_i(t)\right]dt \tag{A-2}$$

where the vehicles in the system between times 0 and T are numbered from 1 to N.

Let

$$S(t) = \sum_{i=1}^{N} f_i(t) = \text{No. of vehicles in system at time } t.$$

Let I(t) be the number of vehicles which have entered the system between time 0 and t (cumulative input).

Let O(t) be the number of vehicles which have left the system between time 0 and t (cumulative output).

Let $S_o$ be the number of vehicles in the system at time 0. Then,

$$S(t) = S_o + I(t) - O(t) \tag{A-3}$$

Hence, the total time in the system is given by

$$\int_o^T S(t)dt = S_oT + \int_o^T I(t)dt - \int_o^T O(t)dt \tag{A-4}$$

But, $S_o$ and T are constant with respect to controls applied after time 0, and I(t) is assumed to be invariant of these controls. Hence, minimizing the total time spent in the system is equivalent to maximizing

$$\int_0^T O(t)dt$$

the cumulative output.

The above analysis is valid both for freeways and single lane roadways under the given assumptions. Freeway output would be the sum of the outputs from all freeway exits.

# APPENDIX B
# DISCRETIZING THE CRITERION INTEGRAL

The integral of the cumulative output, or criterion integral is simply

$$\int_0^T \int_0^t q(L,t')dt'\ dt$$

which may be written as

$$\int_0^T (T-t)\ q(L,t)dt$$

and which we eventually want to maximize. Note that the weighting factor in the integral is (T-t). This gives a higher weight to the flow at earlier times than at later times. This makes it suboptimal to delay vehicles unnecessarily, lowering the flow initially and raising it later on since the flow at later times is given lower weight.

If we wish to be consistent with our assumption that q(x,t) is piecewise linear with respect to time over the grid $t_o=0,\ldots,$ $t_n=T$, we can use this assumption to discretize the integral. Because of the factor (T-t), the integrand would then be piecewise quadratic with a different parabolic arc within each subinterval, and therefore this approximation is less smooth than Simpson's rule which approximates the function smoothly over two subintervals. The resulting approximation may be written as

$$\frac{h_t}{2} \sum_{i=0}^{n-1} \left[ (T-t_i)\ q(L,t_i) + (T-t_{i+1})\ q(L,t_{i+1}) \right]$$

$$+ \frac{h_t^2}{6} \left[ q(L,T) - q(L,0) \right]$$

and we see that we have a trapezoidal approximation plus a correction term $(h_t^2/6)[q(L,T)-q(L,0)]$. This may be further simplified to

$$h_t \sum_{i=1}^{n-1} (T-t_i)\; q(L,t_i) + \frac{h_t}{2}\, T\, q(L,0)$$

$$+ \frac{h_t^2}{6}\left[q(L,T) - q(L,0)\right] \quad .$$

# APPENDIX C
## OPTIMIZING WITH RESPECT TO K(L,t) AND INTEGRATING BACK TO DETERMINE K(0,t) FOR THE TUNNEL MODEL

We may determine the function K(L,t), the concentration at the roadway exit, which maximizes

$$\int_{t_o}^{T}\int_{t_o}^{t} K(L,t')\ \ln\left[K_j/K(L,t'-\tau)\right]dt'\ dt$$

and then numerically integrate back, for example by Equation (35), to determine the control concentration at the entrance to the roadway, K(0,t). Here, the values of K at the grid points are not variables involved in the optimization, as they were in Section 3 of this report. Thus, a fine grid is feasible without using excessive computer time. However, after an optimal K(L,t) is determined, it is not guaranteed that the values of K(0,t) which are determined from K(L,t) and the initial conditions will be feasible. For example, they may not satisfy the demand constraint

$$\int_{0}^{t} q(0,t')dt' \leq n_o + \int_{0}^{t} D(t')dt' \text{ for } t \geq 0$$

where q(0,t) is determined from K(0,t); or K may fall outside the range $[K_{min},K_j]$, where the car following model is invalid for $K < K_{min}$; or q(0,t) may vary too rapidly to be physically attainable. Since these are inequality constraints, it should prove interesting to see whether or not an optimal K that we find does, in fact, satisfy these constraints.

The lower limit $t_o$ in the above criterion integral should ideally be the time t at which a perturbation in K(0,0) first causes a change in K(L,t), recalling that we have no control over K(L,t) for times $t < t_o$. If we use the equation[2]

$$\frac{\partial K}{\partial t} + C \ln\left(\frac{K_j}{eK}\right) \frac{\partial K}{\partial x} + C\tau \frac{\partial^2 K}{\partial x \partial t} = 0 \ , \qquad \text{(C-1)}$$

where functions of $K(x,t-\tau)$, from Equation (17), have been expanded in first order Taylor series approximations, the characteristics are horizontal and vertical lines and we therefore expect instantaneous propagation of perturbations in $K(0,t)$.

Let $K(t) = K(L,t)$ so that we are trying to find the function $K(t)$ which maximizes

$$I(K) = \int_{t_o}^{T} \int_{t_o}^{t} K(t') \ln\left(\frac{K_j}{K(t'-\tau)}\right) dt' \, dt \qquad \text{(C-2)}$$

Consider a small perturbation $\delta K(t')$ in the function $K(t')$. If $K$ maximizes $I$, we would expect $\delta I = 0$. Differentiating, we obtain

$$\delta I(K) = \int_{t_o}^{T} \int_{t_o}^{t} \delta K(t') \ln\left(\frac{K_j}{K(t'-\tau)}\right) dt' \, dt$$

$$- \int_{t_o+\tau}^{T} \int_{t_o+\tau}^{t} \frac{K(t')}{K(t'-\tau)} \delta K(t'-\tau) dt' \, dt \qquad \text{(C-3)}$$

where we have replaced the lower limits in the second pair of integrals by $t_o + \tau$ since $\delta K(t'-\tau) = 0$ for $t' < t_o + \tau$. The integrals may be transformed to somewhat simpler form. Let $s=t-\tau$ and $s'=t'-\tau$ so that

$$\delta I(K) = \int_{t_o}^{T} \int_{t_o}^{t} \delta K(t') \ln\left[\frac{K_j}{K(t'-\tau)}\right] dt' \, dt$$

$$- \int_{t_o}^{T-\tau} \int_{t_o}^{s} \frac{K(s'+\tau)}{K(s')} \delta K(s') ds' \, ds \ . \qquad \text{(C-4)}$$

Let

$$a(t) = \begin{cases} 1 \text{ for } t \leq T - \tau \\ 0 \text{ for } t > T - \tau \end{cases} \tag{C-5}$$

so that the second integrals can be written as

$$\int_{t_o}^{T} \int_{t_o}^{s} \frac{a(s)\ K(s'+\tau)}{K(s')}\ \delta K(s')ds'\ ds. \tag{C-6}$$

Combining and reversing the order of integration we have

$$\delta I(K) = \int_{t_o}^{T} \delta K(t') \int_{t'}^{T} \left[\ln\left\{\frac{K_j}{K(t'-\tau)}\right] - \frac{a(t)\ K(t'+\tau)}{K(t')}\right\}.\ dt\ dt' \tag{C-7}$$

Since we can choose $\delta K(t')$ arbitrarily at any $t' > t_o$, then if

$$\int_{t'}^{T} \left\{\ln\left[\frac{K_j}{K(t'-\tau)}\right] - \frac{a(t)\ K(t'+\tau)}{K(t')}\right\} dt \neq 0 \text{ for } t' \geq 0 \tag{C-8}$$

it must be non zero in some interval around this point because of continuity in $t'$, and therefore we could choose $\delta K(t')$ such that $\delta I(K) \neq 0$. Since $\delta I(K) = 0$ for all $\delta K(t')$, $0 < t' \leq T$ we must have

$$\int_{t'}^{T} \left\{\ln\left[\frac{K_j}{K(t'-\tau)}\right] - \frac{a(t)\ K(t'+\tau)}{K(t')}\right\} dt = 0 \quad t' \geq 0. \tag{C-9}$$

Since $t'$ is constant in this integral, the integral evaluates to

$$(T-t')\ \ln\left[K_j/K(t'-\tau)\right] - \frac{K(t'+\tau)}{K(t')} \int_{t'}^{T} a(t)dt = 0$$

$$\int_{t'}^{T} a(t)dt = \int_{t'}^{T-\tau} dt = T - t'-\tau \text{ for } t' \leq T - \tau \quad . \qquad \text{(C-10)}$$

The optimality condition is therefore

$$\frac{K(t'+\tau)}{K(t')}(T-t'-\tau) = (T-t') \ln\left[\frac{K_j}{K(t'-\tau)}\right]$$

$$\text{for } t_o \leq t' \leq T - \tau \qquad \text{(C-11)}$$

which may be written as a recursion relationship

$$K(t+\tau) = \frac{T-t}{T-t-\tau} K(t) \ln\left[\frac{K_j}{K(t-\tau)}\right], \; t_o \leq t \leq T - \tau \; . \qquad \text{(C-12)}$$

If we let $T \to \infty$, we obtain

$$K(t+\tau) = K(t) \ln\left[\frac{K_j}{K(t-\tau)}\right] , \; t \geq t_o \; . \qquad \text{(C-13)}$$

Starting with initial values for $K(t_o-\tau)$ and $K(t_o)$ which depends on the initial conditions of the tunnel, we can determine $K(t_o+n\tau)$ for all positive integers n.

If $K(-\tau) = K(0) = K_j/e$, then $K(n\tau) = K_j/e$ for all n because inductively, if $K(t) = K(t-\tau) = K_j/e$, then $K(t+\tau) = K_j \ln(e)/e = K_j/e$.

Numerical computations were done which showed that K(t) in fact appears not to converge as $t \to \infty$, but oscillates indefinitely with a period of approximately $6\tau$. The period shows little dependence upon the initial conditions K(0) and $K(-\tau)$. The amplitude of oscillation does not change much, and is determined by the initial conditions. If $\tau$=1 second, then an oscillation of K with a period of 6 seconds would be predicted which would be very difficult to attain with discrete vehicles. Due to this high frequency oscillation, this solution is not a feasible solution. The fact that the amplitude of the oscillations does not decrease to zero and is dependent on the initial conditions is not physically meaningful; we would expect the optimal solution to depend significantly on the initial conditions only for a limited time, but after any existing congestion clears up, the steady state optimum $K_j/e$ should be reached.

It may be noted, if we wished to proceed with this method, we could apply constraints which would bound the derivative of $K(L,t)$ thus reducing the frequency of oscillation to more realistic values.

## APPENDIX D
## DISCRETIZATION OF EQUATIONS 59 AND 60 AND DETERMINING $E_{JK}(T)$ FROM $E_J(T)$ AND $W_J(T)$

The equations to be discretized are Equations (59) and (60), which are repeated here for convenience.

$$e_j\left[w_j(t)\right]\ w_j' - e_j(t) = L_j'(t) \tag{59}$$

$$\frac{e_{jk}\left[w_j(t)\right]}{e_j\left[w_j(t)\right]} = \frac{D_{jk}(t)\ \left[1-P_{jk}\left(L_j(t)\right)\right]}{\sum_{k=1}^{m} D_{jk}(t)\ \left[1-P_{jk}\left(L_j(t)\right)\right]} \tag{60}$$

The difficulty in discretizing these equations is that the function $L_j$ is evaluated both at t and at $w_j(t)$. If we let t assume values on a uniform grid, $w_j(t)$ need not be one of the grid points. We may therefore propose the following discretization.

Choose any convenient set of grid points $t_1,\ldots,t_n$ and let

$$e_{jkL} = e_{jk}(t_L)$$

$$\tilde{e}_{jkL} = e_{jk}\left[w_j(t_L)\right]$$

$$w_{jL} = w_j(t_L)\ . \tag{D-1}$$

We can relate $\tilde{e}_{jkL}$ to $e_{jkL}$ by linear interpolation, since $\tilde{e}_{jkL} = e_{jk}\left[w_j(t_L)\right]$. If $t_i \le w_j(t_L) < t_{i+1}$, we can interpolate $e_{jk}$ between $t_i$ and $t_{i+1}$ to obtain an expression for $\tilde{e}_{jkL} = e_{jk}\left[w_j(t_L)\right]$ in terms of $e_{jki} = e_{jk}(t_i)$ and $e_{j,k,i+1} = e_{jk}(t_{i+1})$.

We then substitute this into Equation (59) to eliminate $\tilde{e}_{jkL}$

$$\sum_{k=1}^{m} \tilde{e}_{jkL}\left\{\frac{w_{j,L+1} - w_{jL}}{t_{L+1} - t_L}\right\} - \sum_{k=1}^{m} e_{jkL} = \frac{L_j(t_{L+1}) - L_j(t_L)}{t_{L+1} - t_L} \tag{D-2}$$

Similarly, we can substitute in Equation (60) to obtain the desired discretization. (See the sketch shown in Figure 6.)

Another possible method of discretization would be to choose an irregular grid in the following manner (see Figure 7): We begin with an initial time $t_o$ and choose $t_o < t_1 < \ldots < t_L < w_j(t_o)$. Now let

$$
\begin{aligned}
t_{L+1} &= w_j(t_o) \\
t_{L+2} &= w_j(t_1) \\
&\ \ \vdots \\
t_{L+i} &= w_j(t_{i-1}) \qquad \text{(D-3)} \\
&\ \ \vdots
\end{aligned}
$$

For any grid point $t_i$, $w_j(t_i)$ is also a grid point. We can thus discretize Equation (59) and (60) straightforwardly. The disadvantage of this method is that the grid points $t_{L+1}, t_{L+2}, \ldots$ are variable since $w_j(t_i)$ is variable.

The feasibility of implementation of this method has not been completely analyzed though we did use unknown grid points in the lane blockage problem[4] successfully.

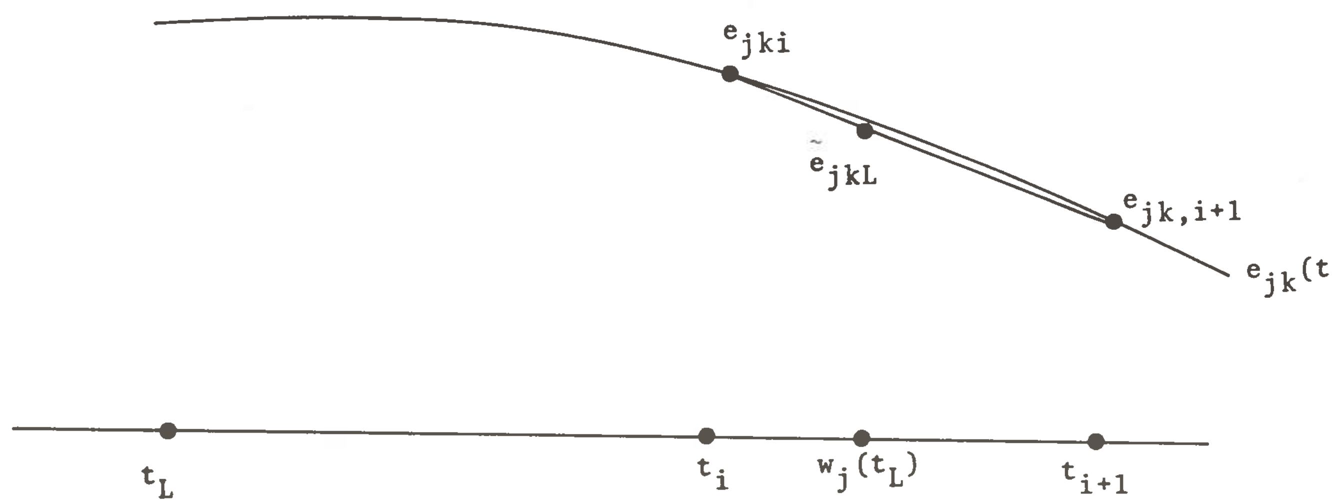


Figure 6. Sketch of Discretization of Equations 59 and 60

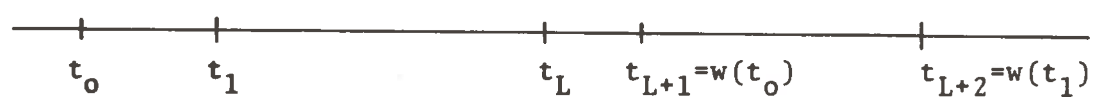


Figure 7. Sketch of Alternate Discretization of Equations 59 and 60

## APPENDIX E

## STEADY STATE OPTIMIZATION PROBLEM OF TUNNEL TRAFFIC WHEN A FIXED PERCENTAGE OF VEHICLES ARRIVING ARE DIVERTED TO AN ALTERNATE ROUTE (e.g., VIA THE SURROUNDING STREETS)

We assume that a constant demand rate of D vehicles per unit time arrive at the tunnel entrance and q vehicles per unit time are allowed to enter the tunnel. The flow q is the control variable, but it remains constant over time. Because we assume steady state conditions, q is the flow at any point in the tunnel at any time: $q \leq D$. The remaining (D-q) vehicles per unit time travel by an alternate route (e.g. the surrounding streets) where travel time is assumed to be a constant, $t_s$, independent of the number of vehicles diverted.

Let V be the constant velocity of vehicles in the tunnel and let K be the constant concentration of vehicles in the tunnel. We have the following relations between K, V, and q

$$V = c \ln(K_j/K) \tag{E-1}$$

$$q = KV . \tag{E-2}$$

We derive

$$K = K_j \exp(-V/c) \tag{E-3}$$

$$q = K_j V \exp(-V/c) \tag{E-4}$$

from the above equations.

Vehicles spend a time of L/V seconds traveling through the tunnel (of length L). Vehicles traveling via the alternate route take a time of $t_s$ seconds.

Since $q \cdot T$ vehicles enter the tunnel between time 0 and time T, and since (D-q)T vehicles travel via the alternate route, the total travel time of all vehicles is

$$q\ T\ (L/V) + (D-q)T\ t_s\ . \tag{E-5}$$

Expressing this in terms of V, we wish to minimize

$$f(V) = T\left[K_j\ L \exp(-V/c) + t_s\left(D - K_j\ V \exp(-V/c)\right)\right] . \qquad \text{(E-6)}$$

Setting the derivative of f to zero,

$$f'(V) = 0 \qquad \text{(E-7)}$$

we find that the minimum is achieved at a velocity of

$$V = L/t_s + c . \qquad \text{(E-8)}$$

The optimal flow is, therefore,

$$q = K_j(L/t_s+c) \exp\left(-\left[1+L/ct_s\right]\right) \qquad \text{(E-9)}$$

and the optimal concentration is

$$K = K_j \exp\left(-\left[1+L/ct_s\right]\right) = (K_j/e) \exp(-L/ct_s) . \qquad \text{(E-10)}$$

When no alternate route exists (let $t_s \to \infty$), the optimal concentration approaches $K_j/e$, as discussed previously in this report. Flow in the tunnel is maximized at $K = K_j/e$. When an alternate route exists, however, the optimal concentration decreases, being multiplied by a factor of $\exp(-L/ct_s) < 1$. This is because a lower concentration in the tunnel decreases the travel time of those vehicles which do enter the tunnel.

In the dynamic case, it may be profitable to have two ramp controls, one control forcing some vehicles to divert from the queue entrance, and the other control regulating flow from the queue onto the freeway. If drivers are allowed to make their own decisions about whether to enter the queue these decisions may not be optimal. Of course, it must be determined if a control on the queue entrance is feasible in practice.

## REFERENCES


1. Wattleworth, J.A., "Peak Period Analysis and Control of a Freeway System," Highway Research Record 157, PGS. 1-10, (1967).
2. Chin, G., Jordan, L., Kahn, D., Morin, S., Yoh, P., "Vehicular Traffic Flow Theory and Tunnel Traffic Flow Measurements," Report No. DOT-TSC-OST-71-7, Transportation Systems Center (1971).
3. Wattleworth, J.A. and Berry D.S. "Peak Period Control of a Freeway System - Some Theoretical Investigations," Highway Research Record 89, P. 4 (1965).
4. Kahn, D., Mintz, R., "Freeway Traffic Flow Following a Lane Blockage," Report No. DOT-TSC-FHWA-72-4, Transportation Systems Center, (1972).
5. Herman, R. and Rothery R.W. "Car Following and Steady State Flow." Proceedings of the Second International Symposium on the Theory of Traffic Flow, London, June 1963.