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A SUMMARY OF OPTIMIZATION TECHNIQUES THAT CAN BE APPLIED TO SUSPENSION SYSTEM DESIGN

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FINAL REPORT

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16. Abstract Summaries are presented of the analytic techniques available for three levitated vehicle suspension optimization problems: optimization of passive elements for fixed configuration; optimization of a free passive configuration; optimization of a free active configuration. The techniques are applied to a heave dynamic model which includes gravity forces, random aerodynamic forces and random guideways making use of penalty functions which include vehicle acceleration, suspension displacement, gap variation, power requirements.

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PREFACE

The work described here was performed under the Multi-Modal Technology Program at the Transportation Systems Center sponsored by The Office of the Assistant Secretary for System Development and Technology of the U.S. Department of Transportation. This program is intended to provide for the development of advanced technology areas applicable to several different transportation modes. The work described in this report was performed by Dr. J. Karl Hedrick, Assistant Professor of Engineering at Arizona State University during a temporary appointment to the staff of the Transportation Systems Center during the Summer of 1972.

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I. Introduction

The possibility of high speed ground transportation has stimulated the study of new and improved suspension systems. At speeds of three hundred miles an hour conventional wheel-rail techniques will not be adequate. Air suspension and magnetically levitated systems have been proposed to provide passenger comfort and safety at these high speeds. Since these systems are in their developmental stage, there have been numerous analytical and experimental studies on how these suspension systems should be designed.

To date the air suspension concept has received the most effort in this country. The TACRV built by Grumman Aerospace is as about to be tested and the TACV to be built by the Rohr Corporation is being developed. MIT has been actively engaged in analytical as well as experimental investigations of air cushion supension design.

Magnetic levitation is presently being studied in this country and prototype models have been built in Germany.

A suspension system provides basically two functions:

1. Primary suspension--supports the weight of the vehicle and maintains safe clearance from guideway.
2. Secondary suspension--tries to isolate the vehicle from external disturbances thus providing comfort to the passengers.

These two functions can be provided by the same systems or by separate systems although the parameters of one system will be needed to design the other.

Two approaches can be taken in secondary suspension design. One is to try to isolate the vehicle from its environment by building an extremely smooth and contained guideway, for example by containing the vehicle in an enclosed tunnel to eliminate aerodynamic and other environmental disturbances. This approach although very effective is probably impractical due to the extreme cost. In general the cost of the guideway is the greatest cost of a transportation system. A second approach is to concentrate on the vehicle's capacity to isolate itself from disturbing input by a well-designed suspension system.

Suspension systems can be either active or passive in design. A passive system is defined as one which does not require a power supply. From a reliability and cost viewpoint a passive system is preferable. Thus, it is of interest to optimize a given passive system or to find the optimal passive system for a specific application.

Active suspension systems may be required to satisfy the performance requirements of high speed vehicles. This report will attempt to illustrate analytical design techniques that accomplish the following.

1. Optimization of the passive elements for a fixed configuration
2. Optimization of a free configuration passive system
3. Optimization of a free configuration active system

II. Optimization of the Passive Elements for a Fixed Configuration

Given a fixed configuration, how should one choose the parameters so that a specified performance index is minimized? The basic technique is to try to express the performance index in terms of the given parameters. This function can then be minimized. Except in the simplest of cases, a numerical algorithm will be required.

There are many ways to quantitatively express the performance of a suspension system. Comfort criteria are often stated in terms of RMS or peak acceleration levels. Complexity of the suspension systems is related to the RMS or peak relative displacement. Other criteria such as power expended and contact frequency are also important. For a system subject to random inputs the most appealing criteria from an analytical point of view is a weighted linear combination of mean square values.

To date most of the analytical studies on optimal suspension systems have been on the vehicles heave dynamics due to its simplicity and also due to the availability of statistical data for this motion. This mode will also be considered here.

Let y = vertical displacement

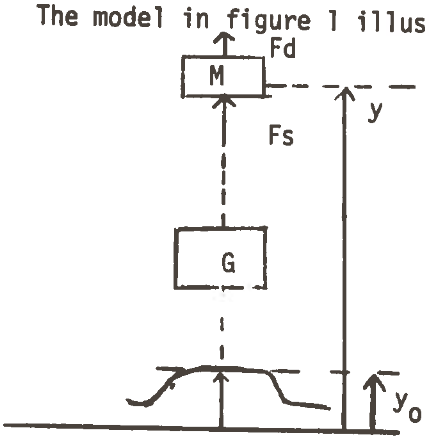
y_0 = guideway displacement

F_s = suspension force on mass

F_d = aerodynamic disturbing force

G = suspension system

The model in figure 1 illustrates these variables.



A mean square performance index would then be:

$$J = \overline{\ddot{y}^2} + \rho_1 \overline{(y-y_0)^2} + \rho_2 \overline{p^2}$$

FIGURE 1.

where $\overline{(\quad)}$ implies expected value, and where p = power required.

The equation of motion for the system shown is:

$$m \ddot{y} = F_s + F_d \ .$$

There are two basic methods for solving the fixed configuration optimization problem. The first involves expressing the mean square element of the performance index in terms of their Fourier transforms. These integrals are a function of the power spectral densities of the disturbances and the system parameters. (These do exist, [ref. 4] but even so this technique is rather awkward). Integrating these expressions can be extremely tedious even for simple systems.

A simpler method is to express the system dynamics in state variable form with white noise as the input. If the disturbing input are correlated rather than white then a shaping filter is needed to augment the system. If we have

$$\dot{X} = FX + W(t)$$

where F = state matrix

W = nx1 vector whose components are white noise and

$$E[W(t)W^T(t+\tau)] = Q\ \delta(t-\tau),$$

then it can be shown ref [1] that the mean square behavior can be described by

$$\dot{\underline{X}} = F\underline{X} + \underline{X}F^T + Q$$

where $\underline{X} = E[X\ X^T]$. For F constant and Q constant (stationary input) the steady state solution, if it exists, can be found by setting $\dot{\underline{X}} = 0$,

$$F\underline{X} + \underline{X}F^T + Q = 0$$

J, in general is a function of the elements of $\underline{X}$. Thus, the passive optimization of the problem can be stated as: Given

$$\dot{X} = FX + W(t), \quad E[ww^T] = Q\,\delta\,(t-\tau)$$

$$\min_{k} \{\sum \sum \rho_{ij} \underline{X}_{ij}\}$$

subject to $F\underline{X} + \underline{X}F^T + Q = 0$

where $F = F(k)$ and k are the passive system elements.

EXAMPLE I.

For example, given the system of Figure 2. Find k and b such that the performance index

$$J = \overline{\ddot{y}^2} + \rho\,\overline{(y-y_0)^2} \qquad \text{is minimized}$$

where y_0 represents the guideway irregularity.

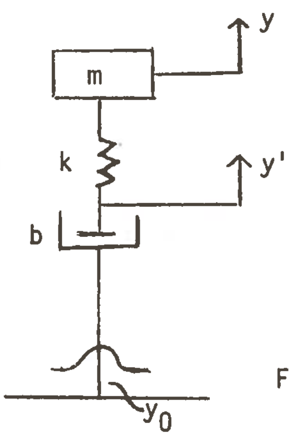


FIGURE 2.

It has been shown in ref [4] that this suspension system is, in fact, the optimum passive system and that

$$\left(\frac{k}{m}\right)_{opt} = \sqrt{\rho}$$

$$\left(\frac{b}{m}\right)_{opt} = \frac{\sqrt{2}}{2}\,\rho^{1/4}$$

The guideway irregularity is often modeled, for a finite frequency range, by a power spectral density,

$$\bar{\Phi}\, y_0\,(s) = \frac{AV}{s^2}$$

If we define state variables:

$$x_1 \triangleq y - y_0 \qquad (1)$$

$$x_2 \triangleq y' - y_0 \qquad (2)$$

$$x_3 \triangleq \dot{y} \qquad (3)$$

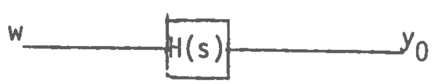
The equations for this system are:

$$\ddot{y} = \frac{k}{m}\,(y' - y) \qquad (4)$$

$$o = k(y' - y) + b(\dot{y}' - \dot{y}_0) \qquad (5)$$

As stated previously we must put this in the form $\dot{X} = FX + W$ where W is white noise. This can be done by using the following relation between power spectra [6]

$\bar{\Phi}\, y_0(s) = H(s)\, H(-s)\, \bar{\Phi}_w\,(s)$, where $\bar{\Phi}_w$ is the p.s.d. of the input.



if we let $\bar{\Phi}_w\,(s) = AV$ (white noise) and $H(s) = 1/s$ we obtain the required expression for $\bar{\Phi}\, y_0$.

Thus, we may model the random guideway by:

$$\dot{y}_0 = w(t), \quad E[w(t)w(t+\tau)] = AV\, \delta\,(t-\tau) \qquad (6)$$

substituting (1), (2), (3), and (6) into (4) and (5) we can derive the following first order differential equations.

$$\dot{X}_1 = X_3 - w(t)$$

$$\dot{X}_2 = \frac{k}{b}(X_1 - X_2)$$

$$\dot{X}_3 = \frac{k}{m}(X_2 - X_1)$$

$$J = \overline{\dot{X}_3^2} + \rho\, \overline{X_1^2} = \frac{k^2}{m^2}\left(\overline{X_2^2} - \overline{2X_2X_1} + \overline{X_1^2}\right) + \rho\, \overline{X_1^2}$$

$$\text{or } J = \frac{k^2}{m^2}\left[\,\underline{X}_{22} - 2\underline{X}_{12} + \underline{X}_{11}\right] + \rho\, \underline{X}_{11}$$

thus, in this case,

$$F = \begin{bmatrix} 0, & 0, & 1 \\ \frac{k}{b}, & -\frac{k}{b}, & 0 \\ \frac{-k}{m}, & \frac{k}{m}, & 0 \end{bmatrix}$$

$$\overline{w}(t) = \begin{pmatrix} -w\ (t) \\ 0 \\ 0 \end{pmatrix}$$

$$E[w\ \ (t)\ w^T(t + \tau\ )] = \begin{pmatrix} -w \\ 0 \\ 0 \end{pmatrix} (-w\ \ 0\ \ 0) = \begin{pmatrix} E(w \cdot w) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} AV & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \delta\,(t-\tau)$$

therefore

$$Q = \begin{pmatrix} AV & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

to find the $\underline{\bar{X}}_{ij}$ we must solve the following system of linear equations:

$$F\underline{\bar{X}} + \underline{\bar{X}}F^T + Q = 0$$

$$\overset{F}{\begin{bmatrix} 0 & 0 & 1 \\ \frac{k}{b} & -\frac{k}{b} & 0 \\ -\frac{k}{m} & \frac{k}{m} & 0 \end{bmatrix}} \overset{\underline{\bar{X}}}{\begin{bmatrix} \underline{\bar{X}}_{11} & \underline{\bar{X}}_{12} & \underline{\bar{X}}_{13} \\ \underline{\bar{X}}_{12} & \underline{\bar{X}}_{22} & \underline{\bar{X}}_{23} \\ \underline{\bar{X}}_{13} & \underline{\bar{X}}_{23} & \underline{\bar{X}}_{33} \end{bmatrix}} + \overset{\underline{\bar{X}}}{\begin{bmatrix} \underline{\bar{X}}_{11} & \underline{\bar{X}}_{12} & \underline{\bar{X}}_{13} \\ \underline{\bar{X}}_{12} & \underline{\bar{X}}_{22} & \underline{\bar{X}}_{23} \\ \underline{\bar{X}}_{13} & \underline{\bar{X}}_{23} & \underline{\bar{X}}_{33} \end{bmatrix}} \overset{F^T}{\begin{bmatrix} 0 & \frac{k}{b} & -\frac{k}{m} \\ 0 & -\frac{k}{b} & \frac{k}{m} \\ 1 & 0 & 0 \end{bmatrix}}$$

$$+ \overset{Q}{\begin{bmatrix} AV & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} = 0$$

These equations yield:

$$2\,\underline{\bar{X}}_{13} + AV = 0$$

$$\underline{\bar{X}}_{23} + \frac{k}{b}(\underline{\bar{X}}_{11} - \underline{\bar{X}}_{12}) = 0$$

$$\underline{\bar{X}}_{33} + \frac{k}{m}(\underline{\bar{X}}_{12} - \underline{\bar{X}}_{11}) = 0 \qquad (6)$$

$$2(\underline{\bar{X}}_{12} - \underline{\bar{X}}_{22})\,\frac{k}{b} = 0$$

$$(\underline{\bar{X}}_{23} - \underline{\bar{X}}_{13})\,\frac{k}{m} = 0$$

This leads to $\underline{X}_{13} = -AV/2$, $\underline{X}_{23} = -AV/2$, $\underline{X}_{33} = bAV/2m$

unfortunately $\underline{X}_{11}$, and $\underline{X}_{12}$ cannot be solved for independently but

$$\underline{X}_{11} = \underline{X}_{12} + \frac{bAV}{2k}$$

$$\underline{X}_{12} = \overline{X_1 X_2} = \overline{(y - y_0)\ (y' - y_0)}$$

one can show that for the variables defined

$$\underline{X}_{12} = \frac{\underline{X}_{13}\ \underline{X}_{23}}{\underline{X}_{33}} \qquad \text{i.e. } \frac{\overline{(y - y_0)\ \dot{y}} \cdot \overline{(y' - y_0)\ \dot{y}}}{\overline{\dot{y}^2}}$$

$$\text{thus, } \underline{X}_{12} = \frac{(\frac{-AV}{2})\ (\frac{-AV}{2})}{\frac{bAV}{2m}} = \frac{AVM}{2b}$$

$$\text{then } \underline{X}_{11} = \underline{X}_{12} + \frac{bAV}{2k} = \frac{AV}{2}(\frac{m}{b} + \frac{b}{k})$$

<u>likewise</u>

$$\overline{\ddot{y}^2} = \frac{k^2}{m^2}\ [\ \underline{X}_{22} - 2\underline{X}_{12} + \underline{X}_{11}]$$

$$\underline{X}_{22} = \underline{X}_{12} = \frac{AVM}{2b}$$

$$\overline{\ddot{y}^2} = \frac{k^2}{m^2}\ [\ \underline{X}_{11} - \underline{X}_{12}] = \frac{k^2}{m^2}\ [\ \frac{m}{k} \cdot \underline{X}_{33}\ ] \quad \text{from (6)}$$

$$= \frac{k^2}{m^2}\ [\ \frac{m}{k}\ \frac{bAV}{2m}\ ] = \frac{kbAV}{2m^2}$$

$$\text{Thus } J = \overline{\ddot{y}^2} + \rho\ \overline{(y - y_0)^2}$$

$$J = \frac{kbAV}{2m^2} + \rho \frac{AV}{2} \left(\frac{m}{b} + \frac{b}{k}\right)$$

Now to find k and b optimal

$$\frac{\partial J}{\partial k} = 0 = \frac{b}{m^2} - \frac{\rho b}{k^2}$$

$$\Rightarrow \frac{k}{m} = \sqrt{\rho}$$

$$\frac{\partial J}{\partial b} = 0 = \frac{k}{m} + \frac{\rho}{k} - \frac{\rho m}{b^2}$$

$$\Rightarrow \frac{b}{m} = \frac{\sqrt{2}}{2} \rho^{1/4}$$

which agrees with the known result.

EXAMPLE II.

This technique can be applied to a more complex system with two inputs. Figure 3 shows a passive system subject to guideway irregularities and aerodynamic disturbing forces.

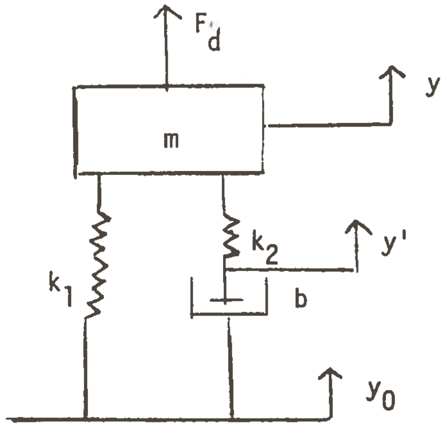


Figure 3.

The problem is to find the values of k_1, k_2, and b such that a performance index of the form

$$J = \overline{\ddot{y}^2} + \rho \, \overline{(y - y_0)^2}$$

is minimized.

The equations of motion for this system are:

$$\ddot{y} = \frac{k_1}{m}(y_0 - y) + \frac{k_2}{m}(y' - y) + \frac{Fd}{m}$$

$$0 = k_2 (y' - y) + b (\dot{y}' - \dot{y}_0) \tag{7}$$

The same statistical description will be used for the guideway as in the previous example, i.e.,

$$\dot{y}_0 = w_1(t), \quad E[\, w_1(t) \, w_1(t+\tau) \,] = AV\delta(t-\tau)$$

W. Young in his M. S. thesis [14] used a power spectral density model for the random part of the aerodynamic force:

$$\Phi_{sf_d}(\omega) = \frac{\beta\nu}{\nu^2 + \omega^2}$$

where $$F_d = cv^2$$

$$v = v_0 + v_r$$

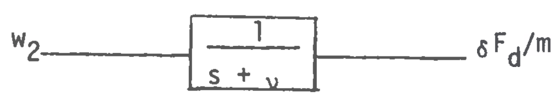
$$F_d = c\,(v_0^2 + 2v_0\, v_r + v_r^2) = F_{d_0} + \delta F_d$$

$$\delta F_d = c2v_0\, v_r + cv_r^2 \qquad \text{assuming } v_r^2 << 2v_0\, v_r$$

$\delta F_d \simeq 2cv_0 v_r$ where v_r represents the random component of velocity. If we consider $\frac{\delta F_d}{m}$ our statistical input into equation (7) then a power spectral density which correlates reasonably with experimental data is

$$\Phi_{\frac{\delta F_d}{m}}(\omega) = \frac{\beta'\nu}{\nu^2 + \omega^2} \quad \text{where } \beta' = \frac{4c^2\, v_0^2\, \overline{v_r^2}}{\pi m^2}$$

We can model this correlated noise as the output of a shaping filter whose input is white noise



where $E\,[\, w_2\,(t)\; w_2\,(t+\tau)\,] = \beta'\nu\delta\;(t-\tau)$

$$\text{Thus } \frac{d}{dt}\left(\frac{\delta F_d}{m}\right) = -\nu\,\frac{\delta F_d}{m} + w_2\,(t) \qquad (8)$$

Next we put these equations into state form

let: $z_1 = \ddot{y}$

$$z_2 = \dot{y}$$

$$z_3 = y - y_0$$

$$z_4 = \frac{F_d}{m}$$

Substituting these definitions into (7) and (8):

$$\dot{z}_1 = \frac{-k_2}{b} z_1 - \frac{1}{m}(k_1 + k_2)z_2 - \frac{k_2 k_1}{mb} z_3 + \frac{k_2}{b} z_4$$

$$+ w_2(t) + \frac{k_1+k_2}{m} w_1(t)$$

$$\dot{z}_2 = z_1$$

$$\dot{z}_3 = z_2 - w_1(t)$$

$$\dot{z}_4 = -\nu z_4 + w_2(t)$$

In matrix form we have

$$\dot{z} = Fz + \bar{w}$$

where

$$F = \begin{bmatrix} -K_2/b, & \frac{-(K_1+K_2)}{m}, & \frac{-K_2K_1}{mb}, & \frac{K_2}{b} \\ 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & -\nu \end{bmatrix}$$

and

$$\bar{W} = \begin{bmatrix} W_2(t) + \frac{1}{m}(K_1+K_2)W_1(t) \\ 0 \\ -W_1(t) \\ W_2(t) \end{bmatrix}$$

Thus $E[\bar{W}(t)\,\bar{W}^T(t+\tau)] = Q\delta(t-\tau)$

where

$$Q = \begin{bmatrix} \beta\nu+\frac{(k_1+k_2)^2}{m^2}AV, & 0, & \frac{-(k_1+k_2)}{m}AV, & \beta\nu \\ 0, & 0, & 0, & 0 \\ \frac{-(k_1+k_2)}{m}AV, & 0, & AV, & 0 \\ \beta\nu, & 0, & 0, & \beta\nu \end{bmatrix}$$

Next the equation

$$(9) \qquad F\underline{\bar{X}} + \underline{\bar{X}}F^T + Q = 0$$

must be solved where $\underline{\bar{X}} = E[X\,X^T]$.

Our performance index:

$$J = \overline{\ddot{y}^2} + \rho\overline{(y-y_0)^2} = \underline{\bar{X}}_{11} + \rho\underline{\bar{X}}_{33}$$

Thus we solve for $\underline{\bar{X}}_{11}$ and $\underline{\bar{X}}_{33}$ in terms of k_1, k_2, and b and then minimize J to find the optimal values of these parameters.

This problem is too complicated to solve in closed form but a simple steepest descent algorithm can be applied.

Equation (9) leads to the following set of equations if we define the new parameters:

$$c_1 \triangleq k_1/m\ ,\ c_2 \triangleq k_2/m\ ,\ c_3 \triangleq b/m$$

$$\underline{\bar{X}}_{24} = \frac{\beta\nu\, c_3+\frac{c_2}{2\nu}}{c_2\nu + c_3(c_1+c_2) + \frac{c_1c_2}{\nu} + \nu^2c_3}$$

$$\overline{X}_{22} = \left(\nu + \frac{c_2}{c_3}\right) \frac{\overline{X}_{24}}{c_2} + \frac{\beta \nu c_3}{2c_2^2} + \frac{AV\,(c_1+c_2)^2 c_3}{2c_2^2} + \frac{c_1 AV}{2c_3}$$

$$\overline{X}_{11} = (c_1+c_2)\,\overline{X}_{22} - \frac{c_2 c_1 AV}{2c_3} - \frac{c_2}{c_3}\,\overline{X}_{24}$$

$$\overline{X}_{33} = \frac{1}{c_1}\left[\,\overline{X}_{22} + \frac{\overline{X}_{24}}{\nu} - \frac{AV}{2}\,(c_1+c_2)\,\frac{c_3}{c_2}\,\right]$$

Thus we want to minimize the function

$J = \overline{X}_{11} + \rho \overline{X}_{33}$ with respect to c_1, c_2, c_3

A program was written to do this on the IBM 7094 using a subroutine (FMFP) from the IBM S.S.P. This subroutine is a steepest descent algorithm.

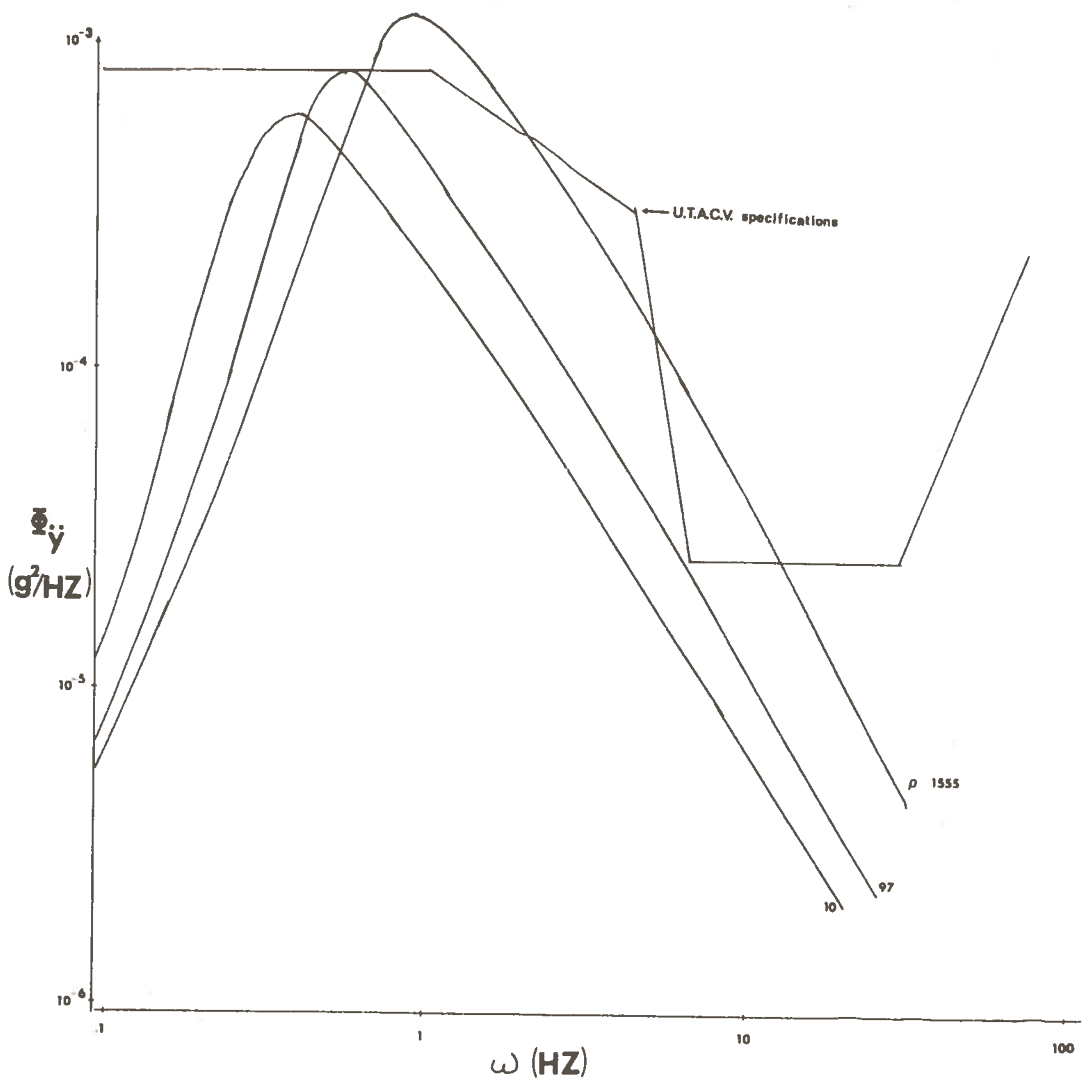
This program can be used to generate optimal values of k_1, k_2, and b for various values of ρ. Table 1 shows these parameters as well as mean square values of $\ddot{y}$ and Δy. This table shows the trend toward the softening of the suspension system as the weighting factor on Δy is decreased. This is to be expected since if $\rho = 0$ then an infinitely soft spring would be the optimum.

Figure 4 illustrates a power spectral density specification for the UTACV. The same computer program that was used to generate optimal k_1, k_2, and b can be used to compute the p.s.d. of the acceleration once these parameters are fixed. The p.s.d. for $\ddot{y}$ in terms of these parameters and the p.s.d.'s of the two inputs is given in equation (9).

$$\bar{\Phi}_{\ddot{y}}(\omega) = \frac{\omega^4 \left[\left\{ (c_1+c_2)^2 \omega^2 + \left(\frac{c_1 c_2}{c_3}\right)^2 \right\} \frac{AV}{\omega^2} + \left\{ \omega^2 + \left(\frac{c_2}{c_3}\right)^2 \right\} \frac{\beta\nu}{\omega^2+\nu^2} \right] \frac{2\pi}{(32.2)^2}}{\omega^6 + \left\{ \left(\frac{c_2}{c_3}\right)^2 - 2(c_1+c_2) \right\} \omega^4 + \left\{ (c_1+c_2)^2 - \frac{2c_1c_2^2}{c_3^2} \right\} \omega^2 + \left(\frac{c_1c_2}{c_3}\right)^2} \qquad (9)$$

Figure 4 shows several p.s.d. curves. All of the curves that are within the UTACV specification are acceptable solutions. Table 1 would illustrate to the designer what trade-off he was making in terms of $\ddot{y}rms$ and $\Delta yrms$.

The sensitivity of the suspension to changes in either input could be tested by varying Av, β and ν for fixed c_1, c_2, and c_3.



Power Spectral Density of Acceleration as a Function of Frequency

FIGURE 4

TABLE 1

OPTIMAL PARAMETERS AND R.M.S. VALUES AS

A FUNCTION OF ρ*

ρ	$\ddot{y}_{rms}$ (g's)	Δy_{rms} (in.)	K_1/m	K_2/m	b/m
10	.015	1.0	3.7	33.9	1.85
97	.019	.55	8.9	44.8	2.47
389	.024	.40	13.9	58.6	3.06
778	.026	.35	17.2	68.7	3.43
1555	.030	.30	20.8	83.1	3.99
3111	.035	.27	25.2	102.8	4.66
6222	.041	.23	30.3	130.0	5.50
12440	.049	.21	36.3	168.0	6.54

* V = 300 mph

$AV = 2.2 \times 10^{-3}$

$\beta = .236\ 16^2/slvg^2$

$\nu = 3$ rad/sec

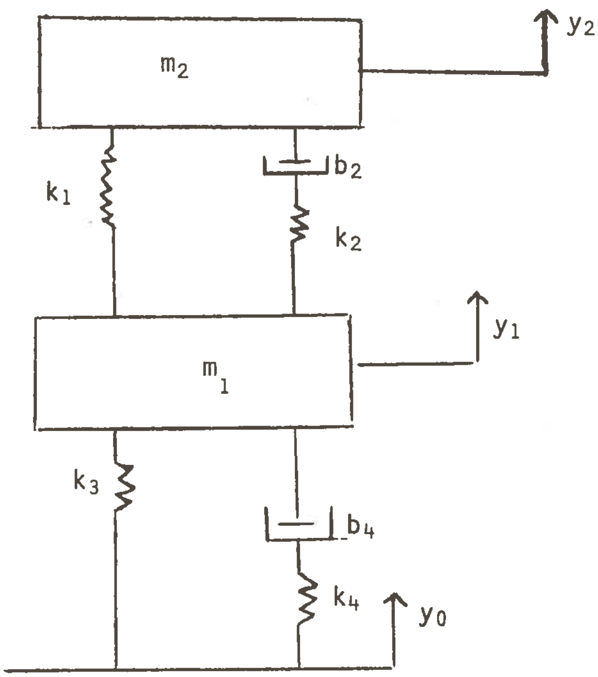


FIGURE 5

EXAMPLE III

The system shown in figure 5 represents a vehicle of mass m_2, a secondary suspension system, (k_1, k_2, b_2), a primary suspension system (m_1, k_3, b_4, k_4), and a guideway input (y_0). Ref [10] has shown that a simple but realistic model for a fluid suspension system is

$$\frac{\Delta F}{(y_1 - y_0)} = -k_s \frac{(\tau_1 s+1)}{(\tau_2 s+1)} \quad \text{where}$$

ΔF is the suspension force and (y_1-y_0) is the relative displacement.

Figure 6 shows a simple rigid plenum model

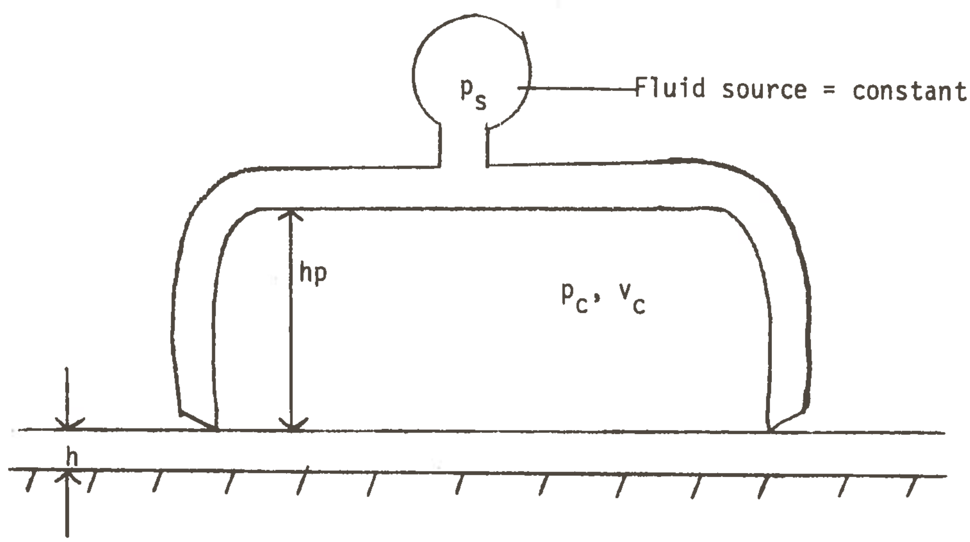


FIGURE 6

Ref [10] shows that for this system:

$$k_s = \frac{2mg}{h_e\left(1+\frac{Fe}{\delta F}\right)} \quad , \quad \tau = \frac{\rho c\ Ae\ he}{We}$$

$$\frac{\tau_1}{\tau_2} = \left(\frac{Aehe}{Aehp + Aehe}\right) \gamma \left(\frac{Pce}{Pce - Pa}\right) \quad \left(\frac{1 + Fe/\delta F}{2}\right)$$

where Fe = equilibrium force (mg)

δF = max. available increment = Ae (Ps - Pce)

We = equilbrium flow

Pce = equilibrium cushion pressure

Ae = total surface area of cushion

For this example we will assume $\frac{Fe}{\delta F} = 2$ and Pa<<Pce. Ref [4] assumes length of each pad = 20 ft., width = 10 ft., We = 140 lb/sec per pad

$$\Rightarrow K_s = \frac{2}{3}\frac{mg}{he} \quad , \quad \tau_1 = .0173 \text{ sec. (independent of he)}$$

$$\tau_2 = \left(1 + \frac{hp}{he}\right) \; .479\, \tau_1$$

for a first try lets assume hp≈o $\Rightarrow \tau_2 = .0087$ sec.

A passive system as shown in figure 7 has a transfer function

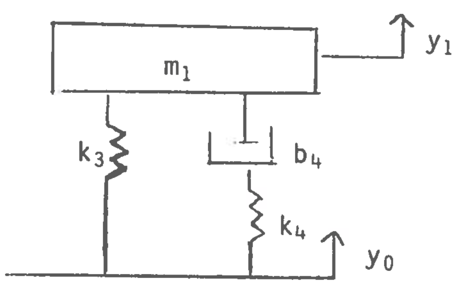


FIGURE 7

$$\frac{\Delta F}{(y_1 - y_0)} = -k_s \frac{(\tau_1 s+1)}{(\tau_2 s+1)}$$

where $k_s = k_3$

$$\tau_1 = \frac{b_4 \; (k_3+k_4)}{k_3+k_4}$$

$$\tau_2 = \frac{b_4}{k_4}$$

Thus comparing with the fluid suspension system:

$$k_3 = \frac{2}{3}\frac{mg}{he}$$

$$\frac{b_4}{k_4} = .0087 \text{ sec.}$$

$$\tau_1 = \frac{b_4}{k_4}\left(1 + \frac{k_4}{k_3}\right) = \tau_2\left(1 + \frac{k_4}{k_3}\right)$$

$$\frac{\tau_1}{\tau_2} = 2 = 1 + \frac{k_4}{k_3} \Rightarrow k_4 = k_3 \quad , b_4 = .0087\, k_3$$

Thus the system of figure 5 models a vehicle mass m_2 with a secondary suspension and a primary fluid suspension where the parameters k_3, k_4, and b_4 are related to the fluid suspension properties. The secondary suspension can be either fluid or mechanical but in this analysis we will assume that the parameters k_1, k_2, and b_2 are free.

In order to evaluate the complete suspension shown in figure 5 we need a performance index. The one we shall use in this analysis is the following:

$$J = \overline{\ddot{y}_2^{\,2}} + \rho_1 \frac{\overline{(y_1-y_0)^2}}{h^2} + \rho_2 \overline{(y_2-y_1)^2} + \rho_3 h^2$$

The first term, $\overline{\ddot{y}^2}$ is the mean square acceleration of the passenger compartment and should be kept small. The second term,

$$\rho_1 \frac{\overline{(y_1-y_0)^2}}{h^2}$$

is a weighted percentage of the mean square gap height deviation to the squared equilibrium gap height. The value of this term corresponds to the number of contacts of the primary suspension with the guideway for a given operation time. It is desired to keep this term low to avoid wearing down the

primary skirt as well as applying an impulsive force to the system .
The third term, $\rho_2 \overline{(y_2-y_1)^2}$ is the means square deflection of the secondary suspension weighted by the constant ρ_2. This term is proportional to the cost of the suspension systems as well as the overall cost of the vehicle i.e. increased secondary stroke would require more frontal area etc. The last term, $\rho_3 h^2$, is proportional to the power that an actual system would expend.

From a design standpoint it would be nice to have the weighting factors weight each term according to their dollar costs. The first term is related to passenger comfort while the last three are related to the suspension cost. If these costs were known then the obvious question is how low a mean square acceleration can we get for a given cost.

In this analysis we will show how comfort varies as the relative cost factors are varied, i.e., if the cost of wearing down the skirt is doubled, how much is my performance degraded if I try to reduce my primary deflection?

The equations of motion for the system shown in Figure 5 are:

$$m_v \ddot{y}_2 = F_d + F_{s2}$$

$$m_p \ddot{y}_1 = F_{s1} - F_{s2}$$

where suspension forces F_{s1}, and F_{s2} are given by:

$$\frac{F_{s1}}{y_o-y_1} = K_{s1} \frac{(\tau_1 s + 1)}{(\tau_2 s + 1)}$$

$$\frac{F_{s2}}{y_1-y_2} = K_{s2} \frac{(\tau_3 s + 1)}{(\tau_4 s + 1)}$$

Define the following states:

x_1 = auxilliary state for F_{s2}

x_2 = auxilliary state for F_{s1}

$x_3 = y_1 - y_o$

$x_4 = y_2 - y_1$

$x_5 = \dot{y}_1$

$x_6 = \dot{y}_2$

$x_7 = F_d/m_v$

The equations of motion then become:

$$\dot{x}_1 = \frac{-x_1}{\tau_4} - x_4$$

$$\dot{x}_2 = \frac{-x_2}{\tau_2} - x_3$$

$$\dot{x}_3 = x_5 - w_1$$

$$\dot{x}_4 = x_6 - x_5$$

$$\dot{x}_5 = \frac{-K_{s1}}{m_v} \frac{m_v}{m_p} \frac{\tau_1}{\tau_2} x_3 + \frac{K_{s1}}{m_v} \frac{m_v}{m_p} \frac{\tau_1}{\tau_2} (1/\tau_1 - 1/\tau_2) x_2$$

$$\dot{x}_6 = x_7 \frac{-K_{s2}}{m_v} \frac{\tau_3}{\tau_4} x_4 + \frac{K_{s2}}{m_v} \frac{\tau_3}{\tau_4} (1/\tau_3 - 1/\tau_4) x_1$$

$$\dot{x}_7 = -\nu x_7 + w_2$$

where: $E\left[w_2(\tau)w_2(\tau+\tau)\right] = \beta\nu\delta\ (\tau-\tau)$

$E\left[w_1(\tau)w_1(\tau+\tau)\right] = AV\delta\ (\tau-\tau)$

For this problem we have τ_1, τ_2 constant and $\frac{K_{s1}}{m_v} = \frac{2}{3}\frac{g}{H}\left(\frac{m_p}{m_v} + 1\right)$.

The control parameters are $\frac{K_{s2}}{m_v}$, τ_3, τ_4, and H where H is the nominal primary gap height. These state equations can be put into state matix form:

$$\dot{x} = Fx + w(t)$$

It has been previously shown that the steady state covariance equation is

$$\underline{\overline{X}}F + F^T\underline{\overline{X}} + Q = 0$$

A numerical program has been written to minimize J subject to the previous equation by the choice of the control parameters. Some of these results are shown in Table 2 for various values ρ_1, ρ_2, and ρ_3.

A study was run to determine the effect of the mass ratio $\frac{m_p}{m_v}$ on the performance. These results are shown in Table 3. It is interesting to note that the optimal parameter values do not change as $\frac{m_p}{m_v}$ is varied from .001 to .5. The RMS values do vary slightly though. In general the primary stroke deviation and secondary stroke deviation increase as the mass ratio is increased.

TABLE 2

OPTIMAL PARAMETERS AND

RMS VALUES FOR DIFFERENT WEIGHTING VALUES.

ρ_1	ρ_2	ρ_3	a_{rms}(g's)	$(y_1-y_o)_{RMS}$ in.	$(y_2-y_1)_{RMS}$ in.	$\frac{K_s}{m_v}$	τ3	τ4	H ft.
100	150	36	.02	.29	.64	4.4	1.34	.32	.2
3600	4	36	.05	.14	2.76	.86	.94	.004	.33

$\frac{m_p}{m_v} = .1$,

TABLE 3

EFFECT OF MASS RATIO $(\frac{m_p}{m_v})$

$\frac{m_p}{m_v}$	a_{rms}(g's)	$(y_1-y_o)_{RMS}$ in.	$(y_2-y_1)_{RMS}$ in.	$\frac{K_s}{m_v}$	τ_3	τ_4	H ft.
.001	.02	.28	.64	4.4	1.34	.32	.2
.100	.02	.29	.65	4.4	1.34	.32	.2
.500	.02	.32	.66	4.4	1.34	.32	.2

$\rho_1 = 100$, $\rho_2 = 150$, $\rho_3 = 36$.

III. OPTIMIZATION OF A FREE CONFIGURATION PASSIVE SYSTEM

If a passive system is desired but the configuration is not specified beforehand the optimal passive system can be found by applying the Weiner filter technique. This method has been used by Wormley, Richardson, et al, ref [4], [14].

In applying the Weiner filter method a transfer function is needed to categorize the suspension system. In the MIT work the transfer function

$$G(S) = \frac{Fs(s)}{\Delta y(s)} \qquad (10)$$

was used to define the suspension system, where F_s is the suspension force and $\Delta y = y - y_0$. This transfer function, ref [4], assumes that F_s is a function only of Δy and its derivatives. This is true for a passive system with no unsprung mass. If an unsprung mass does exist then the optimal suspension will in general try to cancel its effects. Thus, (10) assumes we have a passive system with zero unsprung mass. Notice we have not assumed a form for G(S), the Weiner filter technique will provide the order and optimal parameter values.

The basic technique [14] is to formulate a block diagram relating the inputs to the outputs. For example, Young's problem [14] is shown in figure 8.

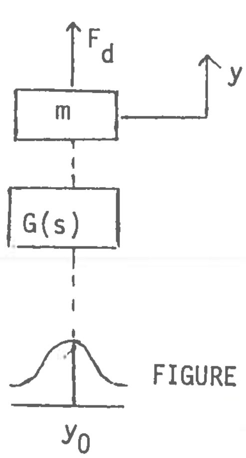


FIGURE 8.

$$G(s) \triangleq \frac{F_s(s)}{(y-y_0)} \qquad (11)$$

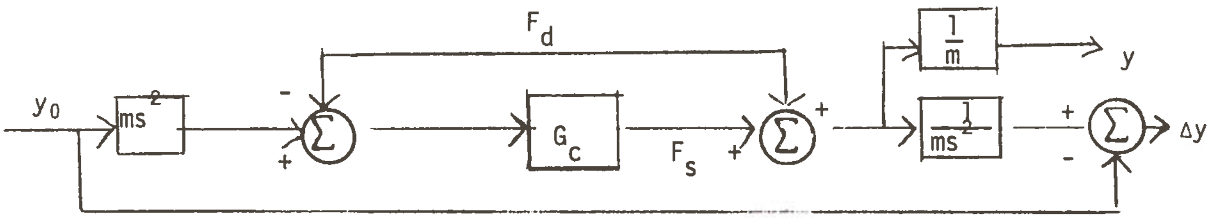
$$mys^2 = F_d + F_s \qquad (12)$$

equations (11) and (12) can be manipulated to give:

$$F_s(s) = G_c(s)\,[\,ms^2 y_0(s) - F_d(s)] \qquad (13)$$

$$\text{where} \quad G_c(s) = \frac{G(s)}{G(s)-ms^2} \qquad (14)$$

In block diagram form (ref [14] pg. 68)



Thus the inputs(y_0,F_d) are related to the outputs (y, Δy)

Assuming stationary inputs the Weiner filter method is used to minimize a performance index

$$J = \overline{\ddot{y}^2} + \rho\overline{(\Delta y)^2}$$

Once $G_c(s)$ is found then $G(s)$ is simply:

$$G(s) = \frac{G_c ms^2}{G_c - 1} \qquad (15)$$

Young's results (assuming $\Phi_{y_0} = \frac{-AV}{s^2}$, and $\phi_{F_d} = \frac{\beta\nu}{\nu^2 - s^2}$) can be realized by the passive system of figure 9.

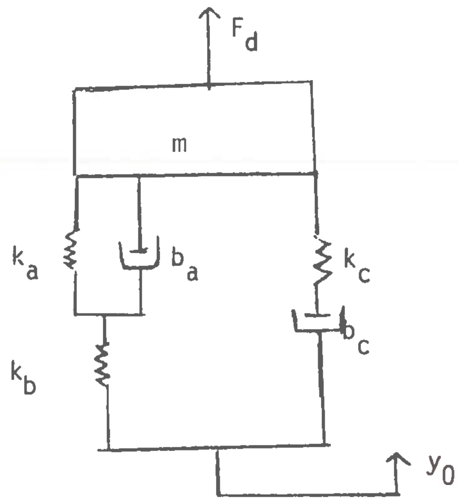


FIGURE 9.

The parameters k_a, b_a, k_b, k_c, b_c are given as a function of AV, β, ν, m, and ρ in Ref [14].

Hullender et al., Ref [4] also use this technique to show that the system of Figure 10 is the optimal passive system for a system subject to only guideway inputs.

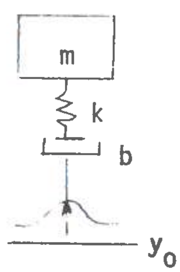


FIGURE 10.

$$J = \overline{\ddot{y}^2} + \rho\ \overline{\Delta y^2}$$

where

$$\frac{k}{m} = \sqrt{\rho}$$

$$\frac{k}{b} = \frac{\sqrt{2}}{2}\rho^{\frac{1}{4}}$$

IV. OPTIMIZATION OF A FREE CONFIGURATION ACTIVE SYSTEM

In this section we will approach the optimization from a modern control point of view. The technique to be used is that of quadratic synthesis i.e. linear constant coefficient differential equations forced with stationary white noise. We will choose the control that minimizes a quadratic form. A steady state regulator solution will be sought. In this section we will also assume that perfect and complete state measurements are available.

The advantage of this approach is that with no added difficulty we may assume that F_s (suspension force) is a function of absolute as well as relative displacement.

The suspension problems can be cast in the following form (ref [1])

$$\dot{X} = FX + GU + W(t) \tag{16}$$

$$* \; J = EXP \left[\frac{1}{2} \int_0^\infty \{ (X^T, U^T) \begin{bmatrix} A, & N \\ N^T, & B \end{bmatrix} \begin{pmatrix} X \\ U \end{pmatrix} \} \; dt \right] \tag{17}$$

The steady state solution is: (assuming perfect measurements and w (t) to be zero mean white noise).

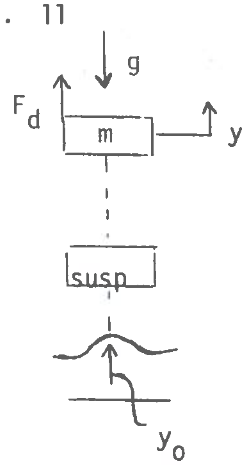
$$U = - B^{-1} [N^T + G^T S] X \tag{18}$$

$$0 = - SF - F^T S - A + (SG + N) B^{-1} (N^T + G^T S) \tag{19}$$

* EXP $\underline{\underline{\Delta}}$ EXPECTED VALUE

For our suspension problem we first consider the case of Figure 11.

Fig. 11



let

$$J = EXP\left[\tfrac{1}{2}\int_0^\infty \{\ddot{y}^{\,2} + \rho_1(y-y_o)^2\}dt\right] \qquad (20)$$

The equation of motion for the system of Figure 11 is:

$$m\ddot{y} = F_s - mg + F_d \qquad (21)$$

(21)

Since in this problem we are assuming perfect and complete measurements we can define our control as $U = \frac{F_s - mg + F_d}{m}$. (The more realistic case where F_d cannot be measured would require the use of an estimator, this problem will not be considered here.)

Thus we have: $\ddot{y} = U$ (22)

The same model for the guideway i.e.

$$\dot{y}_o = W(t), \quad E[W(t)\,W(t+\tau)] = AV\delta(t-\tau) \qquad (23)$$

will be used although a more realistic model with finite mean square value could be used e.g.

$$\dot{y}_o = -v' \, y_o + w' \, (t) \qquad E\,[w' \, (t) \, w' \, (t+\tau)] = AV\delta \, (t-\tau)$$

let $X_1 = y$

$X_2 = \dot{y}$

$X_3 = y_o$

Substituting into (22) and (23) yields

$$\dot{X}_1 = X_2$$

$$\dot{X}_2 = U$$

$$\dot{X}_3 = W(t)$$

$$J = EXP \, [\tfrac{1}{2} \int_0^\infty \{u^2 + \rho_1 (X_1 - X_3)^2 +\} \, dt]$$

In matrix notation

$$\dot{X} = FX + GU + \overline{W}$$

$$J = EXP \, [\tfrac{1}{2} \int_0^\infty \{u^T BU + X^T AX\} \, dt]$$

where $B = 1$, $A = \begin{bmatrix} \rho_1, & 0, & -\rho_1 \\ 0, & 0, & 0 \\ -\rho_1, & 0, & \rho_1 \end{bmatrix}$, $F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$G = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\overline{W} = \begin{pmatrix} 0 \\ 0 \\ \overline{W} \end{pmatrix}$

Equations (18) and (19) simplify to

$$U = {}^{-}CX \tag{24}$$

$$C = B^{-1} G^T S \tag{25}$$

$$0 = - SF - F^T S + SG B^{-1} G^T S - A \tag{26}$$

Equations (24) and (25) yield,

$$U = - S_{12} X_1 - S_{22} X_2 - S_{23} X_3$$

Equation (26) yields

$$S_{12} = \sqrt{\rho_1}$$

$$S_{22} = \sqrt{2} (\rho_1)^{1/4}$$

$$S_{23} = \frac{-\rho_1}{\sqrt{\rho_1}}$$

Thus $U = -\sqrt{\rho_1}\, X_1 - \sqrt{2}\,(\rho_1)^{1/4} X_2 +$

$$\frac{\rho_1}{\sqrt{\rho_1}} X_3$$

With $u = \frac{F_s - g + F_d}{m}$ we have

$$\frac{F_s}{m} = g - \frac{F_d}{m} - \sqrt{\rho_1}(y - y_0) - \sqrt{2}\,(\rho_1)^{1/4}\dot{y} \qquad (27)$$

where g is a constant and ρ_1 is the weighting factor on suspension relative deflection.

The constant force part of the optimal suspension could be provided for either actively or passively. Passively this could be done by a very long spring.

If we set $F_d = 0$ and equation (27) reduces to:

$\frac{F_s}{m} = g - \sqrt{\rho_1}(y - y_0) - \sqrt{2}\,\rho_1^{1/4}\dot{y}$ which is the result of

ref [4] and [14] except for the weight support. The passive system shown in Figure 2 has this suspension force, thus in this case a passive system is as "good" as an active device.

If F_d is not zero then equation (27) cannot be reduced to a form of an equivalent passive system.

Thus the suspension of Figure 9 is an optimal <u>passive</u> system. An active system of the form of equation (27) would provide "better" performance.

V. Conclusions

This report summarizes the analytical technique for solving three suspension optimization problems:

1. Optimization of the passive elements for a fixed configuration
2. Optimization of a free passive configuration
3. Optimization of a free active configuration

The first problem is approached via a state space method. The given configuration is formulated in a model of the form:

$$\dot{X} = FX + W(t)$$

where F is the state matrix containing the parameters to be chosen to minimize a performance index of the form:

$$J = \Sigma\Sigma\, \rho_{ij}\underline{\overline{X}}_{ij} \quad (1)$$

where the $\underline{\overline{X}}_{ij}$ are elements of the covariance matrix:

$$\underline{\overline{X}} = E\,[X(t)X^T(t)]$$

The steady-state covariance matrix (if it exists) can be shown to satisfy the following equation

$$\underline{\overline{X}}F + F^T\underline{\overline{X}} + Q = 0 \quad (2)$$

where $W(t)$ is white noise and $E\,[W(t)W^T(t+\tau)] = Q\delta(t-\tau)$

The elements are solved for from (2) and substituted into (1). J is then minimized numerically yielding the optimal parameters.

This method is applied to a fixed system subject to guideway and aerodynamic disturbances and design curves are presented.

The second problem (free passive configuration) is approached via a Weiner filter method. Results from the work done at MIT are summarized. Basically the suspension force is assumed to be a function of only the relative displacement and its derivatives. The Weiner method then yields the optimal transfer function and a passive system with an equivalent transfer function is sought.

The third problem (free active configuration) is approached via a steady-state regulator model using quadratic synthesis. At first perfect and complete measurements are assumed. The system equations are put in state form with a forcing function of white noise.

$$\dot{X} = FX + GU + W(t)$$

where U is the control (suspension force) and $W(t)$ is white noise. If the disturbing forces are not white a shaping filter forced with white noise is used to model the disturbance. If the performance index is put in the form

$$J = EXP\left[1/2 \int_{o}^{\infty} \{U^T BU + X^T AX\}\, dt\right]$$

then from ref [1]

$$U = -cx$$

$$c = B^{-1} G^T S$$

$$o = -SF - F^T S + SGB^{-1} G^T S - A$$

This method is applied to the complete heave dynamic model (mass subject to gravity, random guideway, random aerodynamic forces) and a penalty function is used that includes mean square acceleration and suspension deflection. The results show that only in certain cases can a passive system be used as an equivalent active system.

VI. REFERENCES

1. Bryson, A. E., Ho, Y. C., Applied Optimal Control, Blaisdell Publishing Company, Waltham, Mass., 1969.

2. Crandall,S. H., and Mark, W. D., Random Vibrations in Mechanical Systems, Academic Press, New York, 1963.

3. Harris, C. M., and Crede, C. E., Shock and Vibration Handbook, McGraw-Hill Book Co., New York, 1961.

4. Hullender, D. A., Wormley, D. N., Richardson, H. H., "A Preliminary Study of Actively Controlled Air Cushion Vehicle Suspensions" Engineering Project Laboratory, Report Number EPL-70-76110-11, June 15, 1970.

5. Meisenholder, S. G., Wang, T. C., "Dynamic Analysis of an Electromagnetic Suspension System for a Suspended Vehicle System" TRW Systems Group, Final Report to F.R.A., TRW Report No. 06818-6052-RO-00, January, 1972.

6. Newton, G. C., Gould, L. A. Kaiser, J. F., Analytical Design of Linear Feedback Controls, John Wiley & Sons, N. Y., 1961.

7. Osbon, W. O., and Putman, T. H., Engineering Design Study of Active Ride Stabilizer for the Department of Transportation's High Speed Test Cars. Westinghouse Research Labs., Pittsburgh, Pa., 1969, Clearinghouse No. PB 185008

8. Pasternack, S., Modern Control Aspects of Automatically Steered Vehicles, Transportation Systems Center, Cambridge, Mass. Report No. DOT-TSC-OST-72-3, 1972.

9. Paul, I. L., and Bender, E. K., Active Vibration Isolation and Active Vehicle Suspension. Engineering Projects Report No. 76109-1. M.I.T., Cambridge, Mass., 1966, Clearinghouse No. PB 173648.

10. Richardson, H. H., and Ribich, W. A., Dynamic Analysis of Heave Motion for Transport Vehicle Fluid Suspensions, Engineering Projects Laboratory, Report No. 76110-3, M.I.T., Cambridge, Mass., 1966, Clearinghouse No. PB 173685.

11. Richardson, H. H., Ribich, W. A., and Captain , K., Dynamics of Simple Air Supported Vehicles Operating over Irregular Guideways, Engineering Projects Laboratory, Report No. 76110-4, M.I.T., Cambridge, Mass., 1966, Clearinghouse No. PB 173655.

12. Sweet, L. M., "Optimal Linear Feedback Control and State Variable Estimation for Tracked Air Cushion Vehicle Suspensions" S. M. Thesis, Department of Mechanical Engineering, M.I.T., Cambridge, Mass. 1971.

13. Wilkie, D. E., "Dynamics, Control and Ride Quality of a Magnetically Levitated High Speed Ground Vehicle" Control System Department, Ford Motor Company, March 1972.

14. Young, J. W., "Optimization of Vehicle Suspensions Subject to Multiple Inputs", S. M. Thesis, Department of Mechanical Engineering, M.I.T., Cambridge, Mass., 1970.