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ASSESSING FEASIBILITY OF PRIORITY OPERATIONS ON HIGHWAYS

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16. Abstract

This study was carried out as part of the BALANCE bus-priority-system evaluation. The objective of this study was to estimate the improvement in freeway passenger flow resulting from the reservation of priority lanes for buses and carpools. These estimates are based on the given conditions of the freeway before reservation, including observed speed-flow characteristics, auto- and bus-flow rates, auto-occupancy characteristics, bus-to-auto ratio, and total number of freeway lanes in one direction. The numerical results show the effects of the policy variables (number of lanes reserved and minimum carpool occupancy) on the passenger flow. An appendix includes an improved mathematical expression for the empirical speed-flow relationship

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CONTENTS

A. EXECUTIVE SUMMARY

To meet the rising demand for improvements in mass transportation, without having recourse to long-term high-cost systems, it is necessary to consider how improvements could be made in both use and efficiency of existing facilities. Under the Balance Project, for the DOT Office of Research and Development Policy (TST-10), various possibilities are investigated for the better utilization of existing highways through an appropriate preference procedure for buses or other high-occupancy vehicles.

The Balance Investigation contains three main thrusts:

1. A macroeconomic analysis whose purpose is to delineate the tradeoff between low-and-high capital-cost mass-transportation systems.

2. A computer simulation study to determine the costs, benefits, patronage, and impacts on traffic flow of a bus-priority transit system in a selected city (Seattle); it includes a comparison with the Seattle 1990 Plan.

3. An applicability analysis aimed at estimating if it is advisable to implement a preference procedure for buses and carpools on a facility under given conditions, and to determine what the optimum designation for a carpool should be in cases where it is advisable.

B. APPLICABILITY ANALYSIS -- SUMMARY

1. Objectives: The aims of the analysis are

   a. Under a given set of highway and traffic conditions, estimate the advisability of reserving one or more traffic lanes for such high-occupancy vehicles as buses and carpools.

   b. In cases where it is advisable to implement such a priority procedure, determine the appropriate designation of a carpool to achieve optimum results.

2. Inputs: The observable information consists of

   a. Highway Characteristics

      1) Number of highway lanes,

      2) Per-lane capacity.

   b. Traffic Characteristics

      1) Flow rate (vehicles per hour),

      2) Flow quality (subcapacity or congested),

      3) Bus-auto split.

   c. Occupancy Characteristics

      1) Mean bus occupancy,

2) Auto-occupancy distribution.

3. Method: The method involves the extrapolation of the performance under the priority operation from the data available under normal operation. This requires the utilization of appropriately designed speed-flow-density relations which are fully discussed in the appendix.

The procedure may be summarized in the following sequency of steps:

a. From the normal flow, we infer the corresponding speed and density for the normal operation.

b. Assuming that with the imposition of the priority regulations, the density is distributed on the two parts in proportion to the ratio of high- to low-occupancy vehicles occurring in the normal flow, we derive the respective densities for the reserved and unreserved lanes.

c. From each of these densities, we infer the corresponding speed and flow for each part under priority operation.

4. Outputs: In assessing the results of the priority operation, it is appropriate to exhibit the effects of the reserved-lane procedure as determined through an

appropriate measure of effectiveness.

By comparing the total passenger flow under both operations, we calculate <u>the relative increase/decrease of passenger flow per hour</u> resulting from the priority rules.

C. GLOSSARY (TECHNICAL SUMMARY)

1. <u>Facility Characteristics</u>

L = Number of lanes,

R = Number of reserved lanes,

C = Capacity in vehicles per hour per lane.

2. <u>Demand Characteristics</u>

a = Number of autos,

b = Number of buses,

α_j = Proportion of autos with exactly j occupants,

β = Mean bus occupancy,

$\alpha = \Sigma j\alpha_j$ = Mean occupancy per auto.

3. <u>Mathematical Formula</u>

$\rho = -e(1-m)\ln(1-m)$, $\lambda = (1 - \frac{1}{m}) \ln(1-m)$,

where

ρ = V/C - ratio (Normalized flow),

m = Normalized mean speed, $\frac{v}{v_0}$,

λ = Normalized density, $\frac{k}{k_*}$.

4. <u>Normal Operation (Steady State)</u>

v_N = Mean speed induced by normal flow,

D_N = Mean travel time per mile; i.e.,

$D_N = H \frac{a\alpha + b\beta}{v_N}$ passenger hours per mile, where H is the time period considered.

5. Priority Operation

n = Number of occupants per carpool,

$$\alpha_R = \sum_{j=n}^{5} \alpha_j, \quad \alpha_U = \sum_{j=1}^{n-1} \alpha_j,$$

$$p_n = \sum_{j=n}^{5} j\alpha_j \Bigg/ \sum_{j=n}^{5} \alpha_j, \quad q_n = \sum_{j=1}^{n-1} j\alpha_j \Bigg/ \sum_{j=1}^{n-1} \alpha_j$$

$$D_p = H \; \frac{a_R p_n + b_R \beta}{v_R} + \frac{a_U q_n}{v_U} \quad ,$$

where

v_R = Mean speed on reserved lanes,

v_U = Mean speed on unreserved lanes,

a_R = Auto flow on reserved lanes,

b_R = Bus flow on reserved lanes,

a_U = Auto flow on unreserved lanes

6. Effectiveness Measures

a. Passenger Flow

$$\Pi = \frac{a_R p_n + b_R \beta + a_U q_n}{a\alpha + b\beta} \quad .$$

b. Total Passenger Travel Time

$$\tau = 1 - \left[\frac{a_R p_n + b_R \beta}{a\alpha + b\beta} \, \frac{v_N}{v_R} + \frac{a_U q_n}{a\alpha + b\beta} \, \frac{v_N}{v_U}\right]$$

7. Sample Result

$L = 4, \quad R = 1$

$a = 6000, \quad b = 400$

n	Subcapacity Π (Passenger Flow)	Congested Π (Passenger Flow)
2	-0.35	R-J
3	0.04	0.24
4	0.07	0.43

D. INTRODUCTION

With the current decline in enthusiasm for highway construction, it is reasonable to anticipate that, in future transportation planning, there will be a continued shift of emphasis towards transit systems of high-passenger capacity. The large commitment of resources required for any comprehensive rail-type system makes it important to consider the possibilities for increasing the passenger-carrying capacity of existing facilities through a more systematic use of multi-passenger vehicles. The recognized adaptability of bus-systems as a convenient means of passenger transportation suggested an idea which, though previously discussed in a number of contexts, was given specific form in the 1963 address by Cherniak [1]; namely, that, during peak-traffic periods, a preferred claim be assigned to buses on the existing highway facilities. The idea has now been implemented in several locations, and its success has been noted in the GMC Progress Report [2]. It is now sufficiently established that it is quite conceivable that some form of buslane may become a feature of a significant part of future highway construction.

Depending on the particular circumstances, the preference procedure may take various forms, incorporating such features as priority assignment of certain entry and exit ramps, as well as the reservation of specified lanes either exclusively for buses or for some more flexible priority operation in accordance with prescribed regulations. The implementation of the reserved land designation, a version of which has been in use on the Shirley

Highway in suburban Washington, could also involve such features as the appropriation of one or more reverse lanes as has been done on the Fitzgerald Expressway in Boston. The utilization of a priority system on such impeded facilities as arterial and city streets might also require preference procedures at traffic-light intersections. Various operating techniques have been discussed by Goodman [3].

To be implemented, such priority operations must ultimately result in an overall benefit to the passenger population. The more efficient use of the facility should be reflected in an increased overall passenger flow.

A number of studies, aimed at assessing the effectiveness of such operations (and based on a comparison of passenger travel time), have been reported in the works of May [4], Stock [7], Morin and Reagan [5], and in the later extension by Sparks and May [6]. Besides the exclusive assignment of buslanes, these studies also consider priority operations wherein one or more lanes are reserved for buses and carpools, the latter being defined by the specification of a required minimum number of occupants. In fact, a valuable outcome of such systems analysis is the resulting decision table delineating the appropriate designation of a carpool to achieve optimum effectiveness. In a comprehensive report on work done for the Department of Transportation by Alan M. Voorhees & Associates, Inc. [8], a procedure proposed in earlier studies is applied to the I-90 Shoreway in Cleveland, for which a thorough parametric study is presented.

The present report, following the ideas both explicit and implicit in the work of May and others already cited, presents a method of analysis in a systematic and somewhat simpler form. The emphasis here is on the clarification of overall procedure so that the derived formula measuring effectiveness may be applied to any facility for which the necessary data on demand and occupancy are available. It should be emphasized, however, that our attention is confined to the estimation of the effectiveness of the priority operation on the facility, and does not consider such secondary features as the effect on waiting time at the entry points. It is reasonable to assume that whatever is the outcome of the operation on the facility, a corresponding effect will be felt at the entry points.

The assessment of feasibility involves a comparison of the performance, as reflected in the passenger flow under normal operation with the corresponding performance under the proposed preference systems. The procedure is described below.

With the given facility characteristics (capacity and number of traffic lanes), we associate the parameters (possibly time-varying) describing traffic demand and vehicle occupancy which constitute the inputs to the model. For any observed flow pattern, the corresponding speed is determined from the speed-flow relation characteristic of the facility, which may be assumed identical for all traffic lanes.

On the implementation of the priority operation, the reserved and unreserved lanes must be treated separately. The occupancy characteristics on the reserved lanes, as determined by the operating regulations, induces a different flow pattern which, in turn, alters the mean speed. A corresponding adjustment is experienced on the unreserved lanes.

Once the traffic flow has been determined for the two parts of the facility under priority operation, the total passenger flow can be immediately inferred. The comparison of the latter with the passenger flow under normal operation provides the appropriate measure of effectiveness in assessing the feasibility of the priority operation.

Tables are presented in section 4 which give the results in specific cases as determined from each of the two measures.

The ciritcal step in the analysis is the extrapolation of the priority performance from the known characteristics under normal operation. In previous studies, this has frequently been done by identifying the total flow for both cases, which, however, can imply both theoretical and practical contradictions. The method followed here is based on the identification of total density for both operations which has the merit of being at least theoretically self-consistent; however, the practical implications of this hypothesis may require closer examinastion.*

*In the previous studies, the total passenger travel time was generally used as the feasibility measure. Under the density-conservation hypothesis, it can be shown that this latter quantity remains constant.

A significant point in the computation is the use of an appropriate form of the speed-flow-density relation; on which, however, there appears to be no general agreement. In our quest for a form suited to the present application, we have been led to an analytic expression for the relation which appears more acceptable than those previously used. A more detailed analysis of this formula, together with a discussion of other representations previously proposed, is given separately in an appendix, where the speed-flow relation, together with the associated speed-density and flow-density relations are illustrated graphically.

E. MODEL DEVELOPMENT

1. System Parameters -- Normal Operation

We consider a facility having L traffic lane in either direction with an each-way capacity of C vehicles per lane per hour and on which the mean free speed is given by v_0.

At any point in time, the traffic in the direction under consideration is assumed to consist of a mix of autos and buses, the weight of vehicle demand being split between a autos and b buses per hour. We let β denote the mean passenger content of a bus; while for the occupancy classification of the auto distribution, we let α_j denote the proportion of autos with precisely j occupants so that with, $1 \leq j \leq 5$, we have

$$\sum_{j=1}^{5} \alpha_j = 1, \tag{1.1}$$

by assuming that only a neglibible fraction of autos have more than five occupants.

Taking the value 2 for the bus-equivalence factor, the traffic demand equivalent in vehicles per hour; namely,

$$V = a + 2b \tag{1.2}$$

may be written as the normalized fraction of the capacity in the form

$$\rho_N = \frac{a + 2b}{LC} , \tag{1.3}$$

where ρ_N denotes the V/C - ratio for normal operation. For this value of the flow demand, the corresponding value of the mean speed v_N can be determined from an appropriate form of the speed-flow relation, the optimum form of which would be provided by sufficiently accurate data from the particular facility under consideration.

In many cases, however, such data are either not available or are so limited in both range and accuracy that the use of a well designed mathematical formula would be no less reliable, as well as being far more convenient. Many formulas describing this speed-flow dependence have appeared in the literature. In the appendix, we discuss a number of them, and also, suggest a new formula that appears to give a more satisfactory response throughout the entire density range. In terms of the normalized flow and mean-speed variables; namely,

$$\rho = \frac{V}{C}, \quad m = \frac{v}{v_0} \quad , \tag{1.4}$$

the proposed relation has the form

$$\rho = -e(1-m)\ \ln(1-m), \tag{1.5}$$

in which e is the base of the natural logarithms. Corresponding to the flow demand (1.3), the associated normalized mean speed m_N is therefore to be determined from the equation

$$(1-m_N)\ \ln(1-m_N) = -\frac{1}{e}\ \rho_N, \tag{1.6}$$

from which the actual speed v_N is given by

$$v_N = v_0 m_N \ . \tag{1.7}$$

The reciprocal of this latter quantity represents the mean time per mile for passengers on the facility, under normal operation.

Equation (1.6) will have two roots in the range (0,1): the greater root, exceeding the capacity speed, corresponds to normal subcapacity flow, while the lesser root is implied for congested flow. The appropriate choice will be evident from the context since direct observation indicates which is the appropriate range.

If we use α to denote the mean occupancy per auto, so that

$$\alpha = \sum_{j=1}^{5} j\alpha_j \ , \tag{1.8}$$

then the (instantaneous) mean passenger flow is clearly given by

$$P_N = a\alpha + b\beta \tag{1.9}$$

passengers per hour. The associated (instantaneous) mean time per mile, normally experienced by passengers on the facility, may be written

$$T_N = \frac{a\alpha + b\beta}{v_N} \ , \tag{1.10}$$

measured as passenger-hours per mile per hour.

If the intensity factor (1.10) be integrated over the time period of operation, we obtain the total mean passenger travel time per mile on the facility. Considering the system in operation for a period of H hours, this overall quantity is therefore given by

$$D_N = \int_0^H \frac{a\alpha + b\beta}{v_N} dt, \tag{1.11}$$

measured as passenger-hours per mile. Over a period during which demand and mean occupancy may be assumed constant, we have

$$D_N = H \frac{a\alpha + b\beta}{v_N} \tag{1.12}$$

providing a simple basis of comparison with the corresponding factor evaluated under a different operation of the facility.

We recall that v_N is to be determined from relations (1.6) and (1.7). In interpreting formula (1.11) in the general case, it should be pointed out that both the demand and occupancy characteristics have to be determined from observations taken over time scales of about 15 minutes, while the total time for the considered operation (the peak-traffic period) is generally of about 2 or 3 hours.

Thus, an adequate evaluation of the integral can be made by a subdivision of the operating time into roughly ten segments, over every one of which the integrand may be assumed constant. Any pretension to greater precision would have to presume information, on the time variation of the integrand, finer than could be realistically expected within the practical limitations, and so would merit little practical interest.

In fact, if we consider the type of averaging inherent in the present analysis, it is likely that once the total volume and overall occupancy have been measured, there is no further loss in accuracy in assuming constancy for these quantities over a time period of the order of an hour.

2. Priority Operation

The priority rules envision the reservation of R lanes for the use of buses and carpools, the latter being designated by the requirement of a minimum number of occupants, n, in the range $2 \leq n \leq 5$. In accordance with this priority rule we let α_R and α_U, respectively, denote the proportion of autos to be assigned to the reserved and unreserved lanes, so that

$$\alpha_R = \sum_{j=n}^{5} \alpha_j, \quad \alpha_U = \sum_{j=1}^{n-1} \alpha_j, \tag{2.1}$$

where

$$\alpha_R + \alpha_U = 1 \ . \tag{2.2}$$

The reserved-lane traffic now consists of a mix of autos and buses while the unreserved lanes contain only autos.

A crucial step in the procedure is the estimation of the flow on the reserved and unreserved parts under priority operation from the known performance under normal operation. In a number of previous studies, this is done by a direct extrapolation, assuming that the flow on each part can be equated to the arrival rate of the designated vehicles, which implies that the total flow for both operations are identical. However, since this can lead to such anomalous requirements as a flow in excess of capacity on one part of the facility, it cannot be taken as an acceptable assumption. Moreover, it should be noted that, under this assumption of flow conservation, the total passenger flow is unaffected by the imposition of the priority operation.

A more reasonable method of estimation can be based on a procedure* which implies the identification of total traffic density for both operations. We have seen that from the flow under normal operation as given by (1.3), we can determine from relations (1.6) the corresponding normalized speed m_N, which may be read directly from the graphical representation of the speed-flow relation shown in figure A-1. Moreover, to each speed and flow, there corresponds a density k, in terms of which we define a normalized density λ by setting

$$\lambda = \frac{k}{k_*} \quad , \tag{2.3}$$

where k_* denotes the jam density. The implied speed-density and flow-density relations are shown in figures A-2 and A-3, respectively. From the flow-density curve in figure A-3, we can determine the normalized per-lane density λ_N corresponding to the flow ρ_N.

*This method of determining the priority performance was suggested by Calvin H. Perrine.

Under priority operation, there is a distinct per-lane density for each part. Assuming that on both the reserved and unreserved portions, the per-lane density is proportional to the fraction of normal flow designated for the respective parts, we set*

$$\lambda_R = \frac{a\alpha_R + 2b}{a + 2b} \frac{L}{R} \lambda_N, \tag{2.4a}$$

$$\lambda_U = \frac{a\alpha_U}{a + 2b} \frac{L}{L-R} \lambda_N , \tag{2.4b}$$

and the implied relation

$$L\lambda_N = R\lambda_R + (L - R)\lambda_U \tag{2.5}$$

expresses the identification of total density for both operations. From the flow-density relation depicted in figure A-3, we can read the normalized flow values ρ_R and ρ_U which correspond, respectively, to the reserved- and unreserved-lane density values λ_R and λ_U of (2.4). The flow values ρ_R and ρ_U will now be used to estimate the corresponding bus and auto flows on the respective parts.

The traffic on the unreserved lanes is composed of autos only, and so, if we let a_U denote the flow in cars per hour, it will be determined from the relation

$$a_U = (L - R)\, C\rho_U \ . \tag{2.6}$$

*As already noted in the Introduction, the practical implications of assumptions (2.4) may merit closer scrutiny.

On the reserved lane, we consider the flow as consisting of a mix of a_R autos and b_R buses per hour, which we assume to be in the same relative proportion as under normal operations, so that

$$\frac{b_R}{a_R} = \frac{b}{a\alpha_R} , \tag{2.7}$$

while the total reserved flow is given by

$$a_R + 2b_R = RC\rho_R \, . \tag{2.8}$$

If we use relation (2.7) in (2.8), we can determine the auto and bus flow separately in the form

$$a_R = \frac{RC\rho_R}{1 + 2\,\frac{b}{a\alpha_R}} , \quad b_R = \frac{RC\rho_R}{2 + \frac{a\alpha_R}{b}} \, . \tag{2.9}$$

It remains to determine the speeds on the respective parts. The quantities will be found from the speed-density relations depicted in figure A-2. Using the values (2.3) for the respective densities, we read from the speed-density curve the corresponding normalized speeds m_R and m_U from which the actual speeds v_R and v_U are, respectively, given by

$$v_R = v_0 m_R, \; v_U = v_0 m_U, \tag{2.10}$$

in which we assume that, on a freeway, the imposition of the priority operation leaves the free speed v_0 unaffected. A possible modification necessary on other types of facility will be discussed later.

Since we have assumed no shift in occupancy, we can write for the average passenger content of a carpool

$$p_n = \sum_{j=n}^{5} j\alpha_j \Big/ \sum_{j=n}^{5} \alpha_j , \tag{2.11}$$

while the mean occupancy of the nonpool autos is given by

$$q_n = \sum_{j=1}^{n-1} j\alpha_j \Big/ \sum_{j=1}^{n-1} \alpha_j . \tag{2.12}$$

In terms of the above notation, the quantities

$$P_R = a_R p_n + b_R \beta , \; P_U = a_U q_n \tag{2.13}$$

measure the mean passenger flow per hour on the respective parts, so that the total passenger flow under priority conditions is given by

$$P_P = a_R p_n + b_R \beta + a_U q_n , \tag{2.14}$$

corresponding to the quantity (1.9) for normal operation.

For the travel-time intensities on the reserved and unreserved parts, we have

$$T_R = \frac{a_R p_n + b_R \beta}{v_R} , \quad T_U = \frac{a_U q_n}{v_U} \tag{2.15}$$

measured in passenger-hours per mile per hour. If we evaluate the integral travel time in passenger-hours per mile on the respective parts, we obtain

$$D_R = \int_0^H \frac{a_R\, p_n + b_R\, \beta}{v_R}\, dt, \quad D_U = \int_0^H \frac{a_R\, q_n}{v_U}\, dt, \tag{2.16}$$

so that the sum,

$$D_P = D_R + D_U = \int_0^H \frac{a_R p_n + b_R\, \beta}{v_R} + \frac{a_U\, q_n}{v_U}\quad dt, \tag{2.17}$$

gives the total passenger travel time per mile under priority conditions, corresponding to the factor (1.11) for normal operation. Where the integrand may be assumed constant, we have

$$D_P = H\left[\frac{a_R\, p_n + b_R\, \beta}{v_R} + \frac{a_U\, q_n}{v_U}\right]\quad . \tag{2.18}$$

Again, as in the case of normal operation, whenever it is necessary to take account of the time variation in the integrand, the evaluation may be effected by subdividing the interval into segments over which the integrand may be assumed constant.

Addendum

To see that the total passenger travel time remains unchanged, we first note that, with a_R and b_R given by (2.9), we have

$$a_R p_n + b_R \beta = \frac{RC}{a\alpha_r + 2b}\, \rho_R(a\alpha_R p_n + b\beta), \tag{2.19}$$

so that

$$\frac{a_R p_n + b_R \beta}{v_R} = \frac{C}{v_0} \frac{R}{a\alpha_R + 2b} (a\alpha_R p_n + b\beta)\lambda_R \ , \tag{2.20}$$

where we have used the fact that

$$\rho_R = \lambda_R m_R \ . \tag{2.21}$$

If we now introduce λ_R, as given by (2.4a), into (2-20) and rearrange terms, we find

$$\frac{a_R p_n + b_R \beta}{v_R} = \frac{CL}{v_0} \frac{a\alpha_R p_n + b\beta}{a + 2b} \lambda_N \ . \tag{2.22}$$

Proceeding similarly with a_U given by (2.6), we find

$$\frac{a_U q_n}{v_U} = \frac{C}{v_0} (L - R) \, q_n \lambda_U \ , \tag{2.23}$$

which, on the introduction of λ_U from (2.4b) followed by a rearrangement, reads

$$\frac{a_U q_n}{v_U} = \frac{CL}{v_0} \frac{a\alpha_U \, q_n}{a + 2b} \lambda_N \ . \tag{2.24}$$

Combining (2.22) with (2.24), we obtain

$$\frac{a_R p_n + b_R}{v_R} + \frac{a_U q_n}{v_U} = \frac{CL}{v_0} \lambda_N \frac{a(\alpha_R p_n + \alpha_U q_n) + b\beta}{a + 2b} \ . \tag{2.25}$$

It can be easily checked that

$$\alpha_R p_n + \alpha_U q_n = \alpha \quad , \tag{2.26}$$

and if we also use the relation

$$\rho_N = \lambda_N \, m_N \quad , \tag{2.27}$$

and recall formula (1.3) for p_N, we find

$$\frac{a_R p_n + b_R \beta}{v_R} + \frac{a_U q_n}{v_U} = \frac{a\alpha + b\beta}{v_N} \; , \tag{2.28}$$

from which it follows that

$$D_P = D_N \tag{2.29}$$

showing the invariance of the time factor on the implementation of the priority operation.

3. Assessment of Effectiveness

In assessing whether the introduction of the priority operation would lead to beneficial results, it would appear that an increase in passenger flow would justify the priority operation, at least from the viewpoint of the Highway Authority. Since an increase in total passenger flow would require that

$$P_P > P_N, \tag{3.1}$$

it appears appropriate to introduce an associated measure Π by the formula

$$\Pi = \frac{P_P - P_N}{P_N} \quad , \tag{3.2}$$

so that favorable or adverse effects are reflected in whether Π is positive or negative. If we introduce forms (1.9) and (2.14) into (3.2), we obtain

$$\Pi = \frac{a_R p_n + b_R + a_U q_n}{a\alpha + b\beta} - 1 \tag{3.3}$$

for the explicit form of the measure reflecting the effect on total passenger flow.

4. Numerical Results

In this section, some numerical results will be presented to illustrate the kind of operational policy insights which may be gained from the approximate relationships derived earlier in the report. Two sample calculations will be described first in some detail to provide a better understanding of the numerical relationships to be presented later.

Consider a freeway of 4 lanes which has been shown to be able to flow a maximum of 2000 vehicles per lane per hour (C) at the optimum flow velocity. This freeway has also been shown to have a free-stream velocity (v_0) of 60 mph. During peak-hour operation, the freeway has been observed to become quite congested and vehicle flows are much less than maximum flow. Specifically, it has been observed that 2400 autos (a) and 240 buses (b) are able to pass a given point in a given hour during the period of congestion. Experimental evidence has also shown that the occupancy rates of automobiles to be 60 percent with 1 person, 30 percent with 2 persons, 8 percent with 3 persons, 2 percent with 4 persons and 0 percent with 5 persons. Buses have been determined to have an average passenger loading of 36 persons. Under these conditions, is reserving a single lane for buses and carpools a worthwhile operating policy?

To answer this question, the post-reservation conditions of the freeway under the proposed operating policy need to be computed. The given information is:

$$L = 4,\ R = 1,\ a = 2400,\ b = 240,$$

$$\alpha_1 = 0.6,\ \alpha_2 = 0.3,\ \alpha_3 = 0.08,\ \alpha_4 = 0.02,\ \alpha_5 = 0,\ \beta = 36.$$

The estimated post-reservation conditions are shown in the following table for the case in which carpools with 2 or more persons, and 4 or more persons, are permitted to use the reserved lane in addition to the buses.

n	ρ_N	m_N	λ_N	ρ_R	m_R	λ_R	ρ_U	m_U	λ_U	Π
2	0.36	0.144	0.92	0	0	>1	0.997	0.598	0.613	-0.51
4				0.961	0.524	0.674	0	0	>1	1.62

The normal conditions are shown in the second, third, and fourth blocks. The vehicle flows are initially only 36 percent of maximum flow condition (ρ_N). The traffic velocity before lane reservation is approximately 8.6 mph or 14 percent of free-stream velocity (m_N). The vehicle density is 92 percent of jam density.

For the case in which carpools with 2 or more persons are permitted to use the reserved lane (n = 2), the reserved lane becomes jammed ($\lambda_R > 1$, $\rho_R = 0$, $m_R = 0$).

The jamming of the reserved lane is caused by the excessive number of carpools using the reserved lanes with a policy of 2 or more persons per car being permitted to use the lane. The unreserved-lane conditions have greatly improved because of the absence of all cars with more than one person. Velocity has increased to 36 mph, or 60 percent of the free-stream velocity (m_U). The net result of this policy is 51 percent reduction in the number of passengers per hour flowing on the freeway (Π).

If the policy regarding carpools is changed so that only those cars with 4 or more persons are permitted on the reserved lane, then the results in the above table show that the unreserved lanes become jammed ($\lambda_U > 1$). This result is caused by the removal of one of the four lanes from an already congested freeway without the removal of many of the vehicles. The conditions in the reserved lane have improved such that the velocity is 31 mph, or 52 percent of the free-stream velocity (m_R). The passenger flow (Π) has increased by 162 percent even though the unreserved lane is at a standstill (m_U).

A less extreme, and more interesting, example is obtained by taking a considerably less congested case before reserving a lane. If, for example, the pre-reservation conditions were observed to be 4800 cars per hour and 480 buses per hour on the same 4-lane highway, then the results of reserving a single lane and allowing buses and cars with carpools greater than 2, 4, and 5 (no cars) persons, would be those shown in the following table:

n	ρ_N	m_N	λ_N	ρ_R	m_R	λ_R	ρ_U	m_U	λ_U	Π
2	0.72	0.325	0.815	0	0	>1	0.994	0.673	0.543	-0.75
4				0.999	0.615	0.598	0.504	0.209	0.887	0.54
5				0.994	0.673	0.543	0.438	0.178	0.906	0.62

The entries in the second, third, and fourth columns show that, in this case, the normal operation involves a flow at 72 percent of capacity (ρ_N), at a speed of almost 20 mph corresponding to 32.5 percent of the free speed (m_N), with a density 81 percent of jam density.

The admission of cars with two or more persons on the reserved lane again leads to a jam ($\lambda_R > 1$, $m_R = 0$, $\rho_R = 0$), while the accompanying lighter density (54 percent of jam density) on the unreserved lanes (λ_U) yields an increase in speed to 40 mph (67 percent of free speed) (m_U), and a flow close to capacity (ρ_U). The net result is a 75-percent reduction in the number of passengers per hour transported (Π).

When the operating policy restricts the reserved lane to cars with 4 or more persons, we see that it leads to a reserved-lane flow practically at capacity (ρ_R), where the speed is increased to 37 mph or 61 percent of the free speed (m_R) caused by a reduction in density to 60 percent of jam density (λ_R). This is achieved by the further congestion of the unreserved lanes on which the density is now increased to 89 percent of jam density (λ_U); this implies a reduction in flow to 50 percent of capacity (ρ_U), and a reduction in speed to 12.5 mph or 20 percent of the free speed (m_U). The net effect is a 54-percent increase in the overall passenger flow (Π).

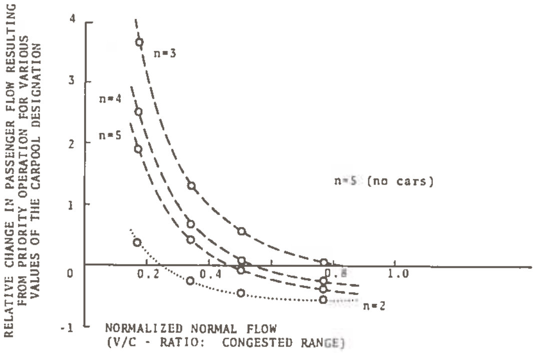
If the policy were carried one step further to the point where all cars are excluded from the reserved lane, we find that the improvement in passenger flow is further increased to 62 percent. The unreserved lanes are now even further congested to a density which is 91 percent of jam density, and the flow thereby reduced to 44 percent of capacity at a speed slightly less than 11 mph or 18 percent of the free speed. However, the accompanying increase in speed to 40 mph (67 percent of free speed) on the exclusive buslane is more than adequate compensation; in fact, the buslane density (54 percent of jam density) now implies a slightly sub-capacity flow on the reserved lane. The relatively high bus occupancy is a significant factor in the net gain in such cases.

The graphs indicate the general trend of such effects, showing how the passenger-flow measure varies with the a-priori flow (ρ_N = V/C - ratio) for a selection of values for the bus-auto ratio and with various designations of a carpool. A situation in which the introduction of the priority operation permits both lane systems to move is shown with a solid line, while the jamming of one or other lane system is indicated by a broken line. The dashed lines denote the jamming of the unreserved lanes, while the situations that would cause a particular priority operation to jam the reserved lanes are reflected in the dotted curves. The illustrations include results for three-, four- and five-lane highways.

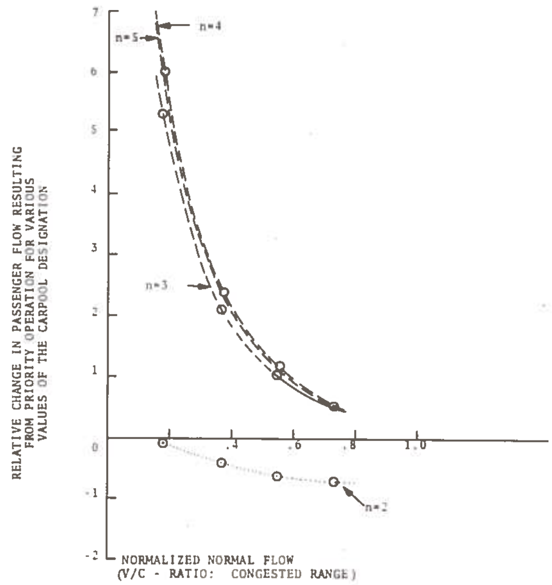
a. Three-Lane Highway

Figures 1a, 1b, 1c, and 1d give the results for a three-lane highway under congested conditions, where the reservation of one lane is contemplated. Figure 1a shows the expected outcome when the bus-auto ratio is as low as 1 percent; in that case, it appears that throughout the flow range, no priority operation would prove satisfactory since any operation leads to the jamming of one or other of the traffic lanes. The carpool designation n = 2 causes the jamming of the reserved lane and, except in the case of extremely high traffic density, also negative values for the passenger-flow measure. Any one of the carpool designations n = 3, 4, and 5 (the designation n = 5 signifies an exclusive buslane) causes the jamming of the unreserved lanes; however, it should be noted that despite this, the priority operation could still lead to significant improvements in the passenger flow, in the high-density range. It would appear that the designation n = 3 would give the most promising priority performance since it yields positive values for the passenger-flow measure throughout the flow range and extremely high values in the extremely congested range.

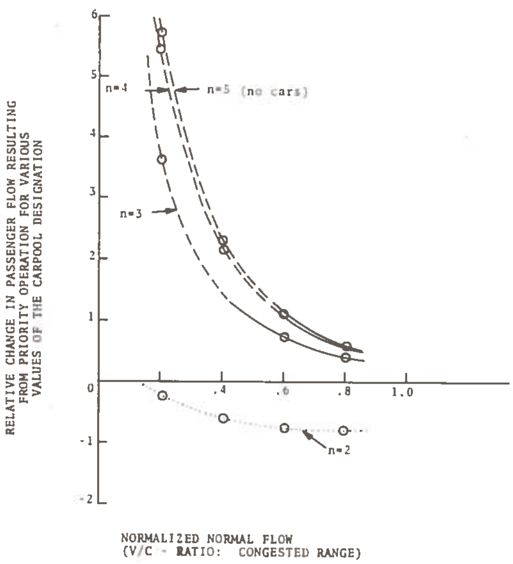
Figure 1b gives the corresponding results when the bus-auto ratio is increased to 5 percent. Here, the designation n = 2 again causes the jamming of the reserved lane and yields a negative value for the flow measure. The designations n = 3, 4, and 5 give roughly



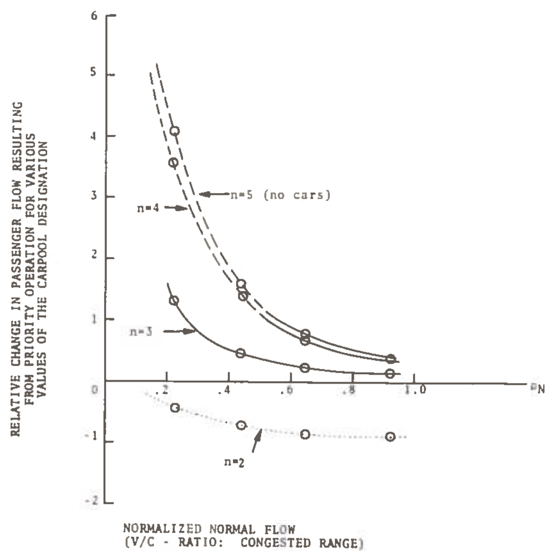
a. For a Bus/Auto Ratio of 1 Percent



b. For a Bus/Auto Ratio of 5 Percent



c. For a Bus/Auto Ratio of 10 Percent



d. For a Bus/Auto Ratio of 15 Percent

Note: Solid Curve indicates both reserved and unreserved lanes moving.
Dots indicate reserved lane jammed
Dashes indicate unreserved lanes jammed

Figure 1. One Lane Reserved on Three-Lane Highway

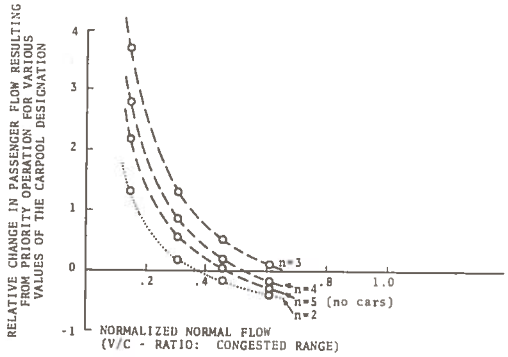
similar values to the flow measure; however, whereas the cases n = 4 and 5 imply the jamming of the unreserved lanes, the designation n = 3 would permit both lanes to move, at least in the moderately congested range.

The pattern is roughly the same when the bus-auto ratio is increased to 10 percent as shown in figure 1c. Again, the designation n = 2 indicates a negative result with jamming of the reserved lane, while the designations n = 3, 4, and 5 give positive values to the flow measure in spite of the possible jamming of the unreserved lanes in the high-density range. While the cases n = 4 and 5 give higher values to the flow measure than does n = 3, the latter permits a much wider range (i.e., density range) for the implementation of the priority operation, without leading to the jamming of the unreserved lanes. In this respect,the designation n = 3 may prove the most satisfactory.

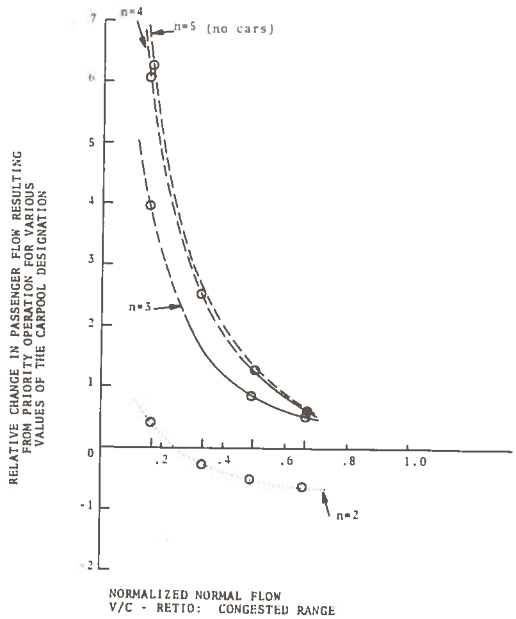
This pattern becomes even more pronounced when the bus-auto ratio is increased to 15 percent as illustrated in figure 1d. The designation n = 3 could now be applied over the entire density range leading to positive values for the flow measure without jamming any lane. The designations n = 4 and 5 yield much higher values for the flow measure but can cause jamming on the unreserved lanes in the range of high-density traffic.

b. Four-Lane Highway

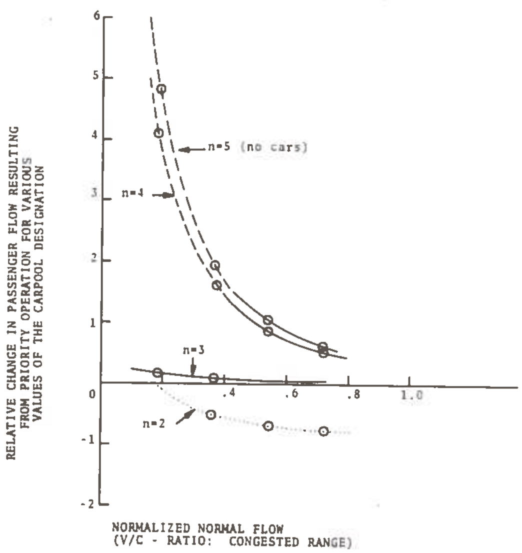
The results for a four-lane highway under congested conditions, where the reservation of one lane is contemplated, are shown in figures 2a, 2b, 2c, and 2d for the various values of the bus-auto ratio.



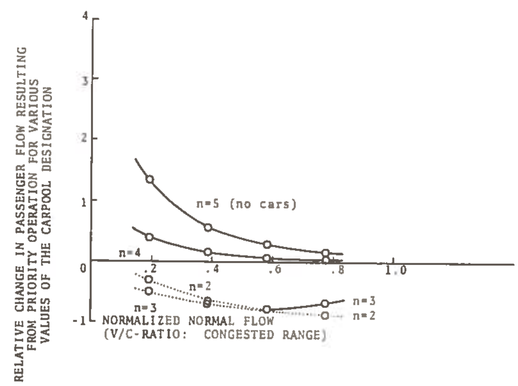
a. For a Bus/Auto Ratio of 1 percent



b. For a Bus/Auto Ratio of 5 percent



c. For a Bus/Auto Ratio of 10 percent



d. For a Bus/Auto Ratio of 15 percent

Note: Solid Curve indicates both reserved and unreserved lanes moving.
Dots indicate reserved lane jammed
Dashes indicate unreserved lanes jammed

Figure 2. One Lane Reserved on Four-Lane Highway

In the case of a low 1-percent bus-auto ratio as shown in figure 1a, the implementation of any priority operation will cause jam conditions. In the case of a carpool designation n = 2, the reserved lane is hammed, while in the other cases n = 3, 4, and 5, the unreserved lane is jammed. Since the designation n = 3 leads to the widest range of positive values for the passenger flow, and also to the largest values for the passenger flow at any point in the range, it would appear to offer the best potential.

The results for a bus-auto ratio of 5 percent are shown in figure 2b. The designation n = 2 again causes the jamming of the reserved lane and, except for the extremely high-density range, negative values for the passenger-flow measure. At the other extreme, the reservation of an exclusive buslane (n = 5) causes the jamming of the unreserved lanes but yields high values for the passenger-flow measure throughout the range. The intermediate values n = 3 and 4, while not giving such high values to the flow measure, do yield significant improvement without causing a jam in the region of moderate congestion. In fact, the designation n = 3 offering the widest range of jam-free priority operation would appear to offer the greatest promise.

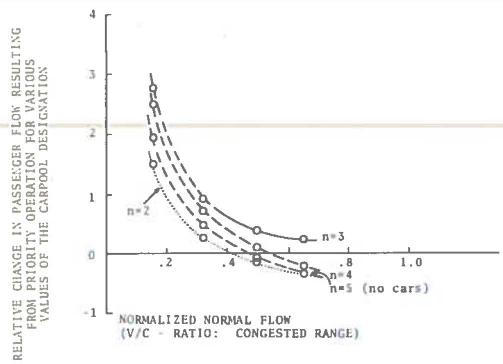
When the bus-ratio is increased to 10 percent (Fig. 2c), the designation n = 2 again gives quite unfavorable results. The designation n = 3 offers a very moderate improvement in passenger flow throughout the range without causing a jam on either part of the facility. The designations n = 4 and 5 give much greater improvements to the passenger flow; however, in the range of high congestion, this improvement is achieved at the expense of jamming the unreserved lanes. It would appear that for moderate congestion the best results are achieved by an exclusive buslane. In the range of intermediate congestion, it would be advisable to admit carpools of four persons. If the congestion is heavy, the best policy is probably to admit carpools of three or more persons on the reserved lanes.

In the case of high (15 percent) bus-auto ratio, the results shown in figure 2d are quite clear-cut. The designations n = 2 or 3 yield consistently negative values for the passenger-flow measure: this is accompanied by the jamming of the reserved lane for the case n = 2 in the entire range and also for the case n = 3 excepting the range of moderate congestion. On the other hand, the designations n = 4 and 5 yield significant improvements in passenger flow without jamming any of the lanes. It is of particular interest in this case that the exclusive buslane (n = 5) yields values for the flow measure more than double that obtained by admitting carpools of four persons on the reserved lane.

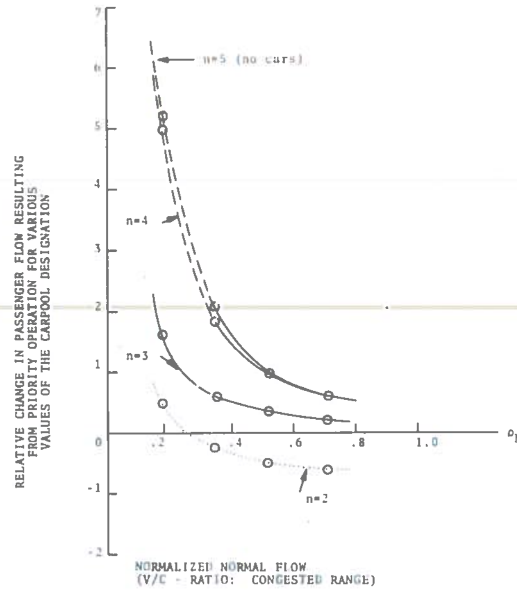
c. Five-Lane Highway

The results for a five-lane highway under congested conditions are illustrated in the sets of figures 3 and 4.

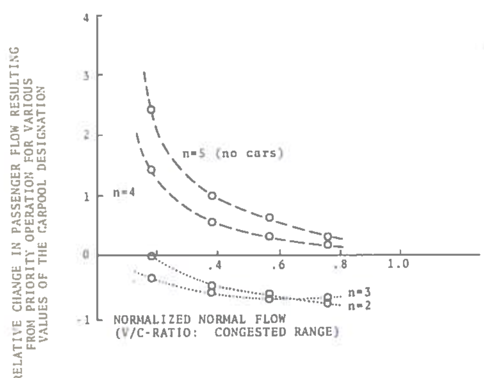
Figures 3a, 3b, 3c, and 3d show the expected outcomes when the operation is based on the reservation of one lane. The results in the case of low bus-auto ratio are shown in figure 3a. The curves for the respective carpool designation follow a definite pattern over the range of a-priori flow. However, where the designation n = 2 leads to a jammed reserved lane, the designations n = 4 or 5



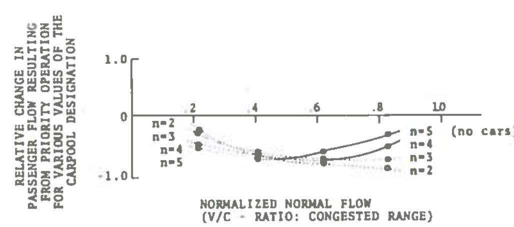
a. For a Bus/Auto Ratio of 1 percent



b. For a Bus/Auto Ratio of 5 percent



c. For a Bus/Auto Ratio of 10 percent



d. For a Bus/Auto Ratio of 15 percent

Note: Solid Curve indicates both reserved and unreserved lanes moving.
Dots indicate reserved land jammed
Dashes indicate unreserved lanes jammed

Figure 3. One Lane Reserved on Five-Lane Highway

cause jamming on the unreserved lanes. Again, the most promising designation is for carpools of three or more persons, which does not cause jamming except in the extremely high-density range and yields consistently positive values for the passenger-flow measure.

The results for the higher bus-auto ratio of 5 percent are shown in figure 3b. The designation n = 2 causes jamming on the reserved lane and negative values for the flow measure except in the extremely congested range. The exclusive buslane (n = 5) leads to consistently significant improvements in the passenger flow, but is likely to cause jamming on the unreserved lanes in the high-density range; the admission of carpools of four persons on the reserved lane does not seriously affect the situation. When carpools of three or more persons are admitted to the reserved lane, the improvement in passenger flow is not as pronounced but the danger of jamming appears to be eliminated.

In the case of a bus-auto ratio of 10 percent (Fig. 3c), the results are quite clear-cut. The admission of carpools of either two or three persons jams the reserved land and yields consistently negative values for the passenger flow. On the other hand, the reservation of an exclusive buslane implies a consistent and substantial improvement in passenger flow over the entire range without any danger of jamming. The admission of carpools with four persons yields a qualitatively similar pattern though it moderates by roughly 30 percent the improvement in passenger flow.

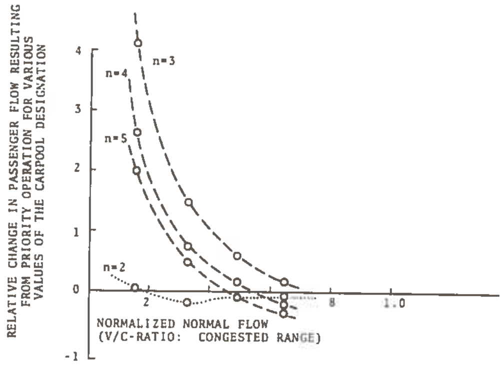
Figure 3d shows that when the bus-auto ratio is raised to 15 percent, no improvement in overall passenger flow can be expected from a priority operation. The reserved lane becomes jammed for the designations n = 2 and n = 3 and, except for the moderate congestion range, also for the case n = 4. Even the reservation of an exclusive buslane (n = 5) yields a consistently negative value for the passenger-flow measure, and in the high-density range, also leads to the jamming of the reserved lane.

The corresponding results when the reservation of two lanes is contemplated are shown in figures 4.

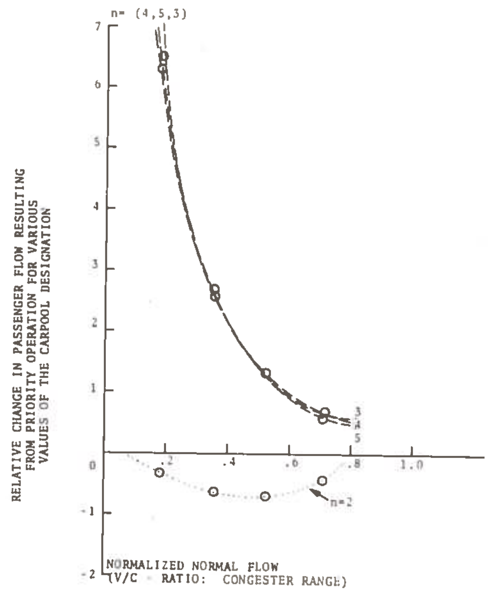
The predictions in the cases when the bus-auto ratios are 1 and 5 percent are shown in figures 4a and 4b, respectively. The designation n = 2 causes the jamming of the reserved lane in each case and (mostly) a reduction in the overall passenger flow. The designations n = 3, 4, or 5 cause the jamming of the unreserved lanes, while over most of the range they yield positive values to the passenger-flow measure.

When the bus-auto ratio is 10 percent (Fig. 4c), the designation n = 2 again leads to a jam on the reserved lane and negative values for the passenger-flow. If we admit only cars with three or more persons, then there is a consistent improvement in passenger flow with the tendency to jam the unreserved lanes confined to the high-density range. The reservation of an exclusive buslane (n = 5), or the admission of four-person carpools (n = 4) while increasing the passenger-flow measure, also tends to jam the unreserved lanes over the entire density range.

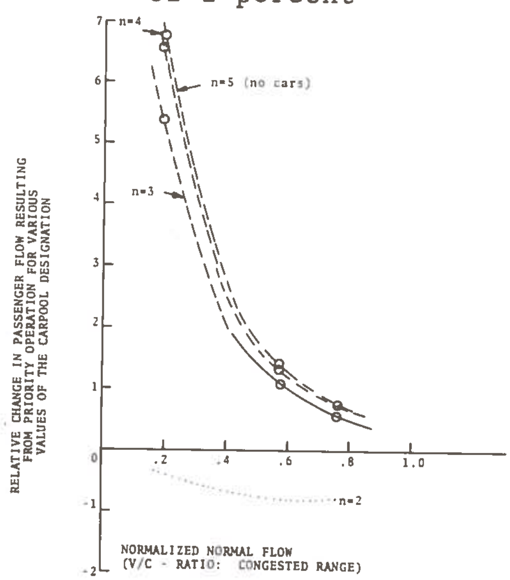
Greater potential for implementation is seen when the bus-auto ratio is increased to 15 percent (fig. 4d). The designation n = 2



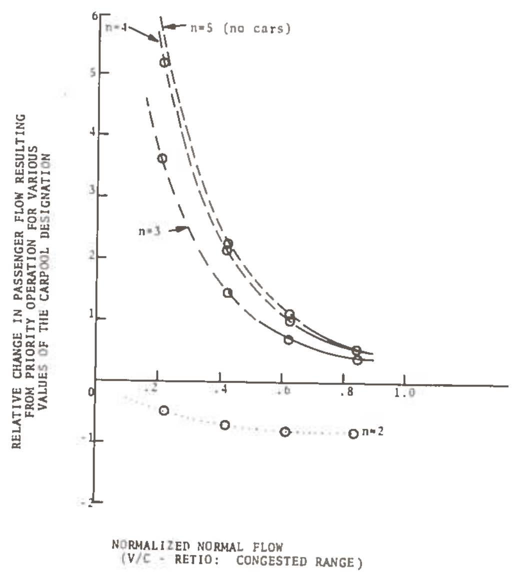
a. For a Bus/Auto Ratio of 1 percent



b. For a Bus/Auto Ratio of 5 percent



c. For a Bus/Auto Ratio of 10 percent



d. For a Bus/Auto Ratio of 15 percent

Note: Solid Curve indicates both reserved and unreserved lanes moving.
Dots indicate reserved lane jammed
Dashes indicate unreserved lanes jammed

Figure 4. Two Lanes Reserved on Five-Lane Highway

again jams the reserved lane and yields negative values for the passenger-flow measure. At the other end, the reservation of an exclusive buslane (n = 5) yields a positive value for the passenger-flow measure but causes jamming on the unreserved lanes, except when the congestion is quite moderate. The admission of carpools with four persons, while decreasing the passenger-flow measure, extends the range of nonjamming. The further admission of carpools of three persons continues this tendency; namely, the passenger-flow measure is further reduced but the range of satisfactory (nonjamming) operation is further extended.

5. Possible Extensions of Analysis

a. Highway Classification

The number of highway lanes characterizes the physical dimensions of the facility, and thereby, determines the capacity. The second dimension of classification which typically distinguishes between freeways and arterials is, at least partly, reflected in the parameter describing mean free speed. According as other factors, such as speed limits, become significant, the less likely does the mathematical formula approximate the speed-flow relation. The formula is most likely to be applicable on those facilities which share the principal features of a freeway. However, as long as we have at out disposal in some form a speed-flow curve describing the traffic pattern on the facility, the present method of analysis may be applied.

For the freeway, we assume that the imposition of the priority operation does not affect the mean free speed on either part.

On an arterial the increased restriction on the unreserved part may require a reduction of the mean free speed: accordingly, if we write v_0^* for this modified free speed, we have

$$v^*_0 = \lambda v_0,$$

where the quantity $\lambda \leq 1$ is to be chosen to give the appropriate adjustment. Typically, λ could be chosen to reflect the penalty for the unreserved lane in conceding priority to the reserved lane at traffic-light intersections.

In the case of city streets, there are available only very restricted data on the speed-flow characteristics. While a rough estimate could be obtained by using smaller values of λ, it is probably more reliable to use actual data from the facility no matter how crude and limited they may be. Moreover, it should be noted that when the bus frequency on a city street is sufficiently high to warrant consideration of priority rules, it is likely that a de-facto reserved lane is already in operation.

b. Reverse-Lane Operation

In certain cases, a more efficient use of the facility may be made by appropriating one or more reverse lanes for priority use. This would arise in a situation where the demand so predominates in one direction that such appropriation of lanes from the direction of lighter traffic would have little adverse effect on the flow in the latter direction.

In cases where such operations might be beneficial, the main difficulty arises in the necessity to design an effective operating procedure. Where the implementation appears practical, the preliminary quantitative assessment may be made by a straightforward extension of the procedures already outlined for a one-sided priority operation.

In the analysis of the extended system, it would be necessary to consider the demand and passenger flow in both directions, and thereby, estimate the overall reduction in travel time as well as the distinct effects on what are now three separate parts of the facility; namely, the side of dominant flow together with both the appropriated and nonappropriated parts of the other half. Such an operation is sure to improve the travel time for passengers in the dominant-flow direction. The purpose of the analysis would be to estimate if the improvement, as modified by the delay that may be thereby imposed on the opposite-direction traffic, is worth the cost of implementation.

Further modifications of the analytic procedure could be made to accommodate such factors as projected occupancy shifts that might result from the introduction of the priority operation.

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G. APPENDIX

THE FUNDAMENTAL FORMULA OF ROAD TRAFFIC

D. O'MATHUNA

INTRODUCTION

In the analysis of highway traffic, the quantities used to describe the kinematic features are

i. the flow, measuring the volume of traffic per unit time,

ii. the mean speed, representing an average vehicle speed, and

iii. the density, signifying the number of vehicles per unit length of highway.

The three factors are connected by the relation

Flow = Density x Mean Speed,

so that the specification of one further formula, connecting two of the three quantities, suffices for the complete characterization of the traffic flow. This complementary relation is called the Fundamental Formula of Road Traffic, and generally, is given a form expressing flow in terms of density. The corresponding forms of the speed-density and flow-speed relations can then be inferred.

Of the three parameters, the flow admits the greatest ease in measurement. Notwithstanding this and despite the fact that the density offers the greatest difficulty for observation, there has been a tendency to regard the latter as the independent variable. This may be at least partly caused by the fact that it has the most clearly defined maximum; namely, the jam density, which gives an unequivocal designation of the range. A contributing influence seems to arise from the analogy with fluid mechanics. In that context, the equation of state in gas dynamics would have its counterpart in traffic analysis in the flow-density relation, which is the usual manner of specifying the Fundamental Formula.

Various forms of the latter relation, or of its equivalent speed-density relation, appear in the literature, some of which have been derived from postulated models of traffic flow. On the basis of a car-following model, Pipes [12] proposed a linear flow-density relation, which, however, can have but limited validity. The logarithmic form given to the speed-density relation by Gazis, Herman, and Potts [4], was also derived from a car-following model: however, a different emphasis led Underwood [14] to an exponential form for the same relation. It is evident that the former could not be valid for free flow, while the latter could not reflect the real situation in the region of congested flow. Consideration of these factors in his analysis prompted Edie [3] to suggest two density ranges in the treatment of road traffic: the exponential form would be used for low-density flow, while the logarithmic form would be appropriate in the high-density range.

Other forms have resulted from more direct empirical considerations. The original linear form of the speed-density relation, proposed by Greenshields [7], still remains a strong candidate for attention. Apart from its simplicity, it has the further attraction of being asymptotic to both the exponential and logarithmic forms in their respective ranges of possible validity. The more complicated algebraic expression designed by Guerin [8] and Palmer [11] to fit actual data satisfies all the terminal requirements expected of the Fundamental Formula. Mention should also be made of the Gaussian formula, adopted by Drake, May, and Shofer [1], which, however, shares the drawback already noted for the exponential form.

From a statistical analysis of the traffic characteristics, Haight [9a, 9b] derived two further forms for the Fundamental Formula. The first may be considered a modified logarithmic relation, whereby the embarrassing singularity at low density is removed; the second generalizes Greenshields' formula, thereby allowing a further degree of freedom in the choice of parameters.

Finally, we note certain formulas deduced from the fluid-flow analogy. From this model, the logarithmic speed-density relation was obtained independently by Greenberg [6]. The procedure was later extended by Drew [2] who, thereby, derived a generalized form of Greenshields' formula, wherein the unit exponent was now replaced by a more general index. In this context, the formulas of Greenshields and Greenberg, as well as a third formula receiving special attention from Drew, all appear as particular cases.

From this brief survey, there emerges the curious fact that the formulas, derived from specific models of traffic flow, do not exhibit any particular advantage over the empirical formulas, obtained from more direct pragmatic considerations. This is not surprising since there appears to be no a-priori reason why any of the proposed models should be expected to reflect the dynamics of traffic flow throughout the entire density range. Either way, there is a valid case for approaching the problem pragmatically. When a satisfactory formula has been conceived, it would still be interesting to explore the inferences, and possible insights, for the various models; in particular, the implied equation of motion for the fluid analogy would merit attention.*

It is now appropriate - and perhaps overdue - to make explicit the features expected of the Fundamental Formula. Clearly, the speed-density relation should yield

1. a finite maximum mean speed (the free speed) at zero density,

2. a zero mean speed at jam density.

These constraints would imply the automatic satisfaction of the expected terminal conditions on the flow; namely,

3. a zero flow at zero density,

4. a zero flow at jam density.

There is a fifth terminal condition requiring that the Fundamental Formula reflect the fact that

5. on a nearly empty road the mean speed is unaffected by moderate changes in the density,

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*The interpretation of the various formulas in terms of the car-following model has been examined by a number of authors, including Gazis, Herman, and Rothery [5], Haight [9c], and Drew [2]. Citations to the work of other investigators may be found in these references, and also, in the book by Wohl and Martin [15], where various forms of the Fundamental Formula are compared from the viewpoint of applicable utility: a fuller discussion of the underlying hypothetical models is also included.

the reasonableness of which can hardly be questioned.* Besides these five conditions, there are other features which, though not logically necessary, are nonetheless desirable; namely,

6. the Fundamental Formula should be simple, and

7. it should lead to a clear definition of road capacity.

This last pair emphasizes two distinct aspects of what may be essentially a single intuitive concept. While simplicity may primarily be an esthetic feature, the unequivocal clarification of maximum flow, together with its associated speed and density, presupposes a certain simplicity in the Fundamental Formula.

As already noted, many of the formulas mentioned earlier violate one or other of the terminal conditions (1) and (2). Moreover, when we impose the fifth condition (5), only two candidates survive, namely

a. the Guerin-Palmer formula, and

b. a restricted form of Drew's formula,

where, in the latter, the restriction signifies that the index must be greater than unity. That Drew, in his analysis, gives only passing attention to this range, may be caused by the fact that it implies for the fluid model, on which the derivation is based, an equation of motion, in which the forcing term does not suggest an

*For a discussion of this and the other terminal conditions, see Haight [9c].

obviously plausible motivation. Returning to the Guerin-Palmer relation, we note that the formula was designed to fit actual data, bearing in mind the satisfaction of the terminal requirements, including the fifth condition, with commendable success. However, though the form is algebraic, it involves irrational factors which deprive it of analytic simplicity, as evidenced by the fact that the definition of capacity leads to a sixth-order algebraic equation with an undetermined coefficient. An assessment of the full merits of the Guerin-Palmer formula would require a more detailed examination of this latter equation.

In the present work, we propose a form of the Fundamental Formula which, besides satisfying the five terminal conditions, also meets the requirement of analytic simplicity, so that the capacity together with the associated speed and density is readily deduced Considering it reasonable to assume that the most reliable information lies in the data associating speed with flow, we find that an inspection of the graphs depicting this correspondence suggests a form that admits a relatively simple mathematical representation of the underlying relation. The derivation of the implied speed-density and flow-density relations is then a straightforward deduction, from which it emerges as a natural feature that density should play the role of the dependent variable. This perspective of the problem, although the most consistent with practical considerations, does not seem to have been explored heretofore.

Presented in its simplest form, the formula contains two independent parameters reflecting the jam density and the free speed, respectively. At first sight, it would appear to lack one disposable parameter available to the Guerin-Palmer formula, by means of which the latter may permit the independent specification of the capacity or of its associated density. The clarification of this question would require a fuller investigation of the latitude permitted in the choice of the undetermined coefficient in the Guerin-Palmer relation. Within the limitation already noted, the formula of Drew also has a third independent parameter permitting the independent designation of one of the three basic quantities at capacity. However, this freedom should be exercised with a cautious eye on how faithfully the associated graph portrays the shape of the data-based curves. We shall return to this point in the later discussion.

More significant for the form proposed here is the fact that it appears to offer fairly wide latitude for generalization, so as to admit the independent assignment of other road parameters besides jam density and free speed. Such factors might include capacity, or alternatively its associated mean speed or density, as well as other less obvious quantities. But there does not appear to be any general agreement as to what factors should be represented in the Fundamental Formula. This question can be decided only by systematic observation, designed with the specific aim of abstracting the specific characteristics. The observation and analysis of traffic wave propagation, suggested in the works of Lighthill and Whitham [10] and Richards [13] could be utilized to assess the relative validity of the proposed formula and its possible generalizations against the two alternatives discussed above. In particular, a detailed inspection of propagation speeds ought to indicate whether or not capacity should appear as an independent parameter in the Fundamental Formula. In the absence of adequate information permitting either an assessment of the dominant parameters or an appraisal of the relative merits, a full investigation of the possible generalizations of the formula presented here is hardly justified at the present time.

We conclude with a brief discussion of the implications for the fluid-flow analogy of the proposed formula. Since the Fundamental Formula determines the forcing term in the equation of motion; this could lead to the formulation of an appropriate hypothesis governing the acceleration pattern in traffic dynamics.

1. Formulation

In the analysis of road traffic, it is customary to let q denote the flow (vehicles per hour), while v and k represent the corresponding mean speed (miles per hour) and density (vehicles per mile), respectively. The three quantities are connected by the relation

$$q = kv, \tag{1.1}$$

which, it should be noted, is dimensionally necessary and so is not necessarily restricted in its relevance to vehicular movement on highways. For the particular application to this latter type of traffic, it is necessary to specify a further relation expressing the interdependence of the three quantities, characteristic of highway flow. This relation, called the Fundamental Formula of Road Traffic, may be conceived in its general form

$$q = q\ (v,k).$$

However, if we bear in mind that it is to be used in conjunction with relation (1.1), it is evident that this degree of generality is redundant. In fact, formula (1.1) can be used to eliminate any one of the three quantities from the characteristic relation, so that it suffices for the Fundamental Formula to express the mutual dependence of the remaining two. This dependence has generally been given one of the forms

$$q = q(k),\ v = v(k),$$

relating either the flow or the speed to the density. For the reasons already discussed, we shall rather consider the density to have been eliminated so that the Fundamental Formula takes the form

$$q = q(v), \tag{1.2}$$

expressing the dependence of flow on speed, both of which admit a fairly direct means of observation, and for which there is reasonably reliable data available.

The density has a clearly defined maximum; namely, the jam density, which we denote by k_*. Moreover, it is consistent with experience to associate with a vehicle unhindered by other traffic, a characteristic maximum speed called the free speed and denoted by v_0. If we further recall that jam-packed traffic is stationary, we see that the first pair of terminal conditions on the speed may be written

$$k = 0 \rightarrow v = v_0\ , \tag{1.3a}$$

$$k = k_* \rightarrow v = 0\ . \tag{1.3b}$$

These, in conjunction with relation (1.1), immediately imply the satisfaction of the terminal features expected of the flow; namely,

$$k = 0 \rightarrow q = 0, \tag{1.4a}$$

$$k = k_* \rightarrow q = 0. \tag{1.4b}$$

The fifth terminal condition, that on a nearly empty highway the speed is unaffected by modest changes in the density, may be incorporated by imposing the requirement

$$\lim_{k \to 0} \frac{dv}{dk} = 0, \tag{1.5}$$

which will receive more detailed attention later. We note that the five conditions have been formulated with respect to the terminal values of the density range. Since we shall consider the density as the dependent variable, and the Fundamental Formula in the form of a speed-flow relation, our procedure will necessitate a corresponding reformulation of these conditions.

As the fulfillment of requirements (1.4) follows automatically from (1.3), it will suffice to generate a formula of the form (1.2) which implies the satisfaction of conditions (1.3) and (1.5).

2. Normalization

It will prove convenient to introduce normalized density and speed parameters, λ and m, by setting

$$\lambda = \frac{k}{k_*}, \quad m = \frac{v}{v_0}, \tag{2.1}$$

so that with a dimensionless flow variable ϕ , defined by

$$q = k_* v_0 \phi , \tag{2.2}$$

relation (1.1) reads

$$\phi = \lambda m, \tag{2.3}$$

in which each of the quantities λ and m now has the range (0,1).

The terminal conditions (1.3) then become

$$\lambda = 0 \rightarrow m = 1, \tag{2.4a}$$

$$\lambda = 1 \rightarrow m = 0, \tag{2.4b}$$

while requirement (1.5) assumes the form

$$\lim_{\lambda \to 0} \frac{dm}{d\lambda} = 0 \quad . \tag{2.5}$$

From (2.3) and (2.4), the fulfillment of the dimensionless form of (1.4); namely,

$$\lambda = 0 \rightarrow \phi = 0, \tag{2.6a}$$

$$\lambda = 1 \rightarrow \phi = 0, \tag{2.6b}$$

will follow automatically.

The problem then is the determination of a formula expressing the dimensionless form of (1.2); namely,

$$\phi = \phi\ (m), \tag{2.7}$$

which, when taken in conjunction with relation (2.3) will satisfy conditions (2.4) and (2.5).

As a consequence of relation (2.3), we observe that

$$\frac{d\phi}{d\lambda} = m + \lambda \frac{dm}{d\lambda} \quad , \tag{2.8}$$

so that conditions (2.4a) and (2.5) also imply that

$$\lim_{\lambda \to 0} \frac{d\phi}{d\lambda} = 1, \tag{2.9}$$

to which it will be convenient to refer later.

3. The Fundamental Formula

Expressed in terms of speed and flow, the terminal features (2.4) and (2.6) clearly require that

$$m = 1 \to \phi = 0, \tag{3.1a}$$

$$m = 0 \to \phi = 0, \tag{3.1b}$$

while the fifth condition (2.5), if we note the limit requirement (2.9) and apply the chain rule, implies

$$\lim_{m \to 1} \left| \frac{d\phi}{dm} \right| = \infty \quad . \tag{3.2}$$

The formulation of the flow-density relation must meet the requirements (3.1) and (3.2).

Before the presentation of the suggested form, it is appropriate to make some motivating remarks which shall be set in the context of the Guerin-Palmer and the Drew relations. The general formula of Drew, when transformed to a speed-flow relation, in the present notation reads

$$\phi = m(1-m)^{\frac{2}{N+1}}, \; N > -1,$$

which clearly meets requirements (3.1). Moreover, differentiation yields

$$\frac{d\phi}{dm} = (1-m)^{\frac{2}{N+1}} - \frac{2}{N+1} \, m(1-m)^{\frac{1-N}{1+N}},$$

which satisfies condition (3.2) only for $N > 1$.* In fact, for $N > 1$, we have

$$\left| \frac{d\phi}{dm} \right| = O\left(\frac{1}{1-m}\right)^{\frac{N-1}{N+1}}, \quad m \to 1 \; .$$

Even for values of N quite close to unity, this implies a rather sharply steepening curve in the region of low density, a feature also shared by the more complicated Guerin-Palmer relation, originally designed as a flow-density formula which, in the present notation, reads

$$\phi = \frac{\lambda \; (1-\lambda)^{1/2}}{a\lambda^2 + (1-\lambda)^{1/2}} \; ,$$

with a as a disposable parameter. However, there does not appear

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*This range received little attention from Drew [2].

to be any prior disposition for anticipating this rather abrupt pattern in the speed-flow relation.*

In fact, the existing data on the speed-flow correspondence, as produced, for example, in the Highway Manual, would seem to imply a curve steepening at a more moderate rate, in consideration of which a logarithmic growth for the speed-flow derivative in (3.2) immediately suggests itself. The formula which, besides meeting conditions (3.1), also leads to a logarithmic growth rate for the derivative in the free-flow (low-density) range, in its simplest terms, has the form

$$\phi = - (1-m) \ln(1-m) \ . \tag{3.3}$$

Regarding requirements (3.1), condition (3.1b) is obviously satisfied, as is also condition (3.1a), if we recall that

$$\lim_{x \to 0} x \ln x = 0 \ . \tag{3.4}$$

The satisfaction of condition (3.2) follows from noting that differentiation of formula (3.3) yields

$$\frac{d\phi}{dm} = 1 + \ln(1-m) \tag{3.5}$$

which also indicates the more modest growth rate for the flow-speed derivative; in fact, relation (3.5) was the motivation which led to formula (3.3).

* It should be emphasized that the judgment implicit in these and subsequent remarks is largely intuitive, the validity of which must be tested by more rigorous means.

Moreover, the curve follows an expected pattern; it has a unique maximum for a speed value m_c determined by setting the right-hand side of (3.5) to zero. Hence,

$$m_c = 1 - \frac{1}{e} \simeq 0.63 \tag{3.6}$$

(where e is the base of the natural logarithms), and the associated flow, ϕ_c, is given by

$$\phi_c = \frac{1}{e} \simeq 0.37 \tag{3.7}$$

The quantity (3.7) represents the dimensionless capacity flow, while (3.6) gives the associated normalized speed.

Since it is customary to measure flow as a volume-capacity ratio, it is convenient to introduce a normalized flow variable ρ, defined by

$$\rho = \frac{\phi}{\phi_c} = e\phi, \tag{3.8}$$

so that

$$\rho = - e(1-m) \ln(1-m). \tag{3.9}$$

This normalized speed-flow relation is shown in figure A-1.

As it contains no free parameter, formula (3.3) admits the independent specification merely of jam density and mean free speed. There is, as yet, no direct evidence, either way, whether such factors as capacity should be independently represented. In fact, it appears to be a valid question whether capacity could be an independent parameter since quite conceivably it, together with its associated density and mean speed, is completely determined by the jam density and free speed. Considered from another angle, as the physical dimensions of the facility are represented by the jam density, it is quite credible that the potential capacity be directly reflected in the free speed, at least within practical limits, as is implied by (3.7). Nevertheless, we shall return, in section 6, to a consideration of a more adaptable form of (3.3).

We remark that for near-capacity flow, the characteristic feature is the insensitivity of the flow to modest changes in the mean speed. Expressed in its converse form, this reflects the phenomenon experienced in the near-capacity range where the mean speed is extremely sensitive to slight changes in the flow.

We conclude this section by noting some limiting forms valid for congested flow. Recalling that, for m near zero, we have

$$\ln(1-m) \simeq - m - \frac{1}{2}m^2 - \frac{1}{3}m^3 - \ldots, \quad m \ll 1, \tag{3.10}$$

it follows that, in the congested range, formula (3.3) may be approximated by

$$\phi \simeq m - \frac{1}{2}m^2 - \frac{1}{6}m^3 - \ldots, \quad m \ll 1 . \tag{3.11}$$

In a similar manner, the asymptotic relation,

$$\frac{d\phi}{dm} \simeq 1 - m - \frac{1}{2}m^2 - \ldots, \quad m \ll 1, \tag{3.12}$$

follows from utilizing (3.10) in relation (3.5).

We now turn to a consideration of the relations of both speed and flow to density implied by the proposed formula.

4. Speed-Density Relation

In this section, we show that the speed-density relation implied by the Fundamental Formula does, in fact, meet the terminal conditions (2.4) and (2.5).

In considering the density, we refer to relation (2.3); written in the form

$$\lambda = \frac{\phi}{m} \tag{4.1}$$

this formula, on the introduction of ϕ from (3.3), yields the speed-density relation,

$$\lambda = \left(1 - \frac{1}{m}\right)\ln(1-m), \tag{4.2}$$

in implicit form.

Using the approximation (3.10) in relation (4.2), we obtain the asymptotic form

$$\lambda \simeq 1 - \frac{1}{2}m - \frac{1}{6}m^2 - \ldots, \quad m \ll 1, \tag{4.3}$$

valid in the region of congested flow. We can now derive the limiting relations

$$m = 1 \rightarrow \lambda = 0, \tag{4.4a}$$

$$m = 0 \rightarrow \lambda = 1, \tag{4.4b}$$

where the former follows from applying the limit formula (3.4) to relation (4.2), while the latter is evident from (4.3). That the satisfaction of the terminal conditions (2.4) is thereby established will follow from demonstrating the single-valuedness of the λ - m functional dependence in the range (0,1).

Taking the derivative of (4.2), we find

$$\frac{d\lambda}{dm} = \frac{1}{m}\left[1 + \frac{1}{m}\ln(1-m)\right] . \qquad (4.5)$$

Observing that for $1 \leq y \leq \infty$,

$$\ln y = \int_1^y \frac{1}{t}\, dt \geq 1 - \frac{1}{y},$$

with equality holding only at y = 1, it follows that for $0 \leq x \leq 1$,

$$-\ln x \geq 1 - x,$$

and hence, for $0 \leq m \leq 1$

$$\ln(1-m) \leq -m,$$

implying that

$$1 + \frac{1}{m} \cdot \ln(1-m) \leq 0, \qquad (4.6)$$

with equality possible only at m = 0. Moreover, by applying the expansion (3.10) near m = 0, we see that

$$1 + \frac{1}{m}\ln(1-m) \simeq -\frac{1}{2}m - \frac{1}{3}m^2 - \frac{1}{4}m^3 - \ldots, \quad m<<1, \qquad (4.7)$$

and hence,

$$\frac{1}{m}\left[1 + \frac{1}{m}\ln(1-m)\right] \simeq -\frac{1}{2}\left[1 + \frac{2}{3}m + \frac{1}{2}m^2 + ..\right], \quad m<<1. \tag{4.8}$$

From (4.6) to (4.8), it is clear that the derivative (4.5) is strictly negative throughout the range (0,1), which suffices to prove the single-valuedness of the speed-density relation (4.2); hence, the limiting relations (4.4) imply the satisfaction of the terminal conditions (2.4).

Taking the reciprocal of formula (4.5) for the derivative, we have

$$\frac{dm}{d\lambda} = \frac{m}{1 + \frac{1}{m}\ln(1-m)} \quad . \tag{4.9}$$

If we let $\lambda \to 0$, so that $m \to 1$, it is obvious that the right-hand side tends to zero, so that the fifth condition (2.5) is also satisfied. As was implicit in previous remarks, this logarithmic decay seems more attractive than the algebraic decay implied in both the Guerin-Palmer and Drew formulas.

The normalized speed-density relation is shown in figure A-2.

5. Flow-Density Relation

The flow-density relation can be inferred only indirectly through a combination of relations (3.3) and (4.1). Formula (4.1) gives the density as the ratio of flow to speed. To express the speed factor in terms of flow, we must rely on the implicit form of the speed-flow relation (3.3).

Though the relation must remain in this implicit form, it is still possible to make explicit the main qualitative features of the functional dependence. Applying the chain rule to the flow-density derivative, we find

$$\frac{d\phi}{d\lambda} = m \, \frac{1 + \ln(1-m)}{1 + \frac{1}{m}\ln(1-m)} , \tag{5.1}$$

in which we have used both relations (3.5) and (4.9). In the limit of the free-speed (low-density) range, we clearly have

$$\lim_{\lambda\to 0} \frac{d\phi}{d\lambda} = \lim_{m\to 1} \frac{d\phi}{d\lambda} = 1 \ . \tag{5.2}$$

On the other hand, if we employ the expansions (3.10) and (4.7) for the congested range, we obtain

$$\frac{d\phi}{d\lambda} \simeq \frac{m\left[1 - \frac{1}{2}m^2 - \ldots\right]}{-\frac{1}{2}m\left[1 + \frac{2}{3}m + \frac{1}{2}m^2 + \ldots\right]} , \ m<<1, \tag{5.3}$$

so that

$$\lim_{\lambda\to 1} \frac{d\phi}{d\lambda} = \lim_{m\to} \frac{d\phi}{d\lambda} = -2 \quad . \tag{5.4}$$

The denominator on the right-hand side of (5.1) is infinite only in the limit $m = 1$, in which case it is cancelled by the numerator to yield the limiting value (5.2) for the derivative. If we also observe the limiting value (5.4) for $m = 0$, we see that the only zero for the flow-density derivative is that of the numerator in (5.1), which has already been noted in section 3, as giving the capacity values for the speed (3.6) and the associated flow (3.7). From (4.1), we obtain the corresponding value of the density; namely,

$$\lambda_c = \frac{1}{e - 1} \simeq 0.58, \tag{5.5}$$

measured as a fraction of the jam density.

If we consider the normalized flow ρ, as defined by (3.8), then (5.2) and (5.4) imply the limiting values

$$\lim_{\lambda \to 0} \frac{d\rho}{d\lambda} = e, \qquad \lim_{\lambda \to 1} \frac{d\rho}{d\lambda} = -2e \quad . \tag{5.6a,b}$$

The normalized flow-density relation is shown in figure A-3.

The quantity defined by the derivative (5.1), usually denoted by c, is the wave speed; it measures the speed at which traffic disturbances are propagated through the flow medium. The ratio of wave speed to mean speed is given by

$$\frac{c}{m} \simeq \frac{1 + \ln(1-m)}{1 + \frac{1}{m}\ln(1-m)} \quad . \tag{5.7}$$

Since the denominator on the right-hand side of (5.7) exceeds the numerator throughout the range (except for equality at m = 1), the wave speed must always be less than the mean speed, reflecting the fact that traffic disturbances always propagate backward through the flow. Whether or not the waves travel forward or backward in space will depend on the flow being less or greater than capacity. These features have been treated at length in references [10] and [13].

Finally, relation (5.4) indicates that for the Fundamental Formula (3.3), the (backward) jam-shock wave speed is twice the (forward) mean free speed. This appears to be more realistic than the infinite value implied by the Guerin-Palmer formula.

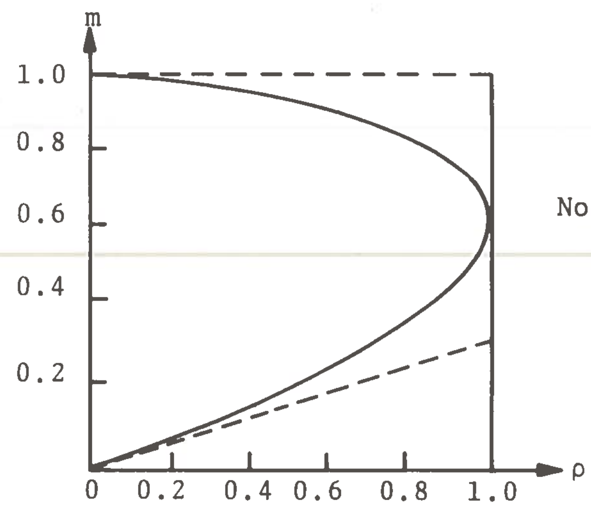

FIGURE A-1

Normalized Speed-Flow (m - ρ) Relation.

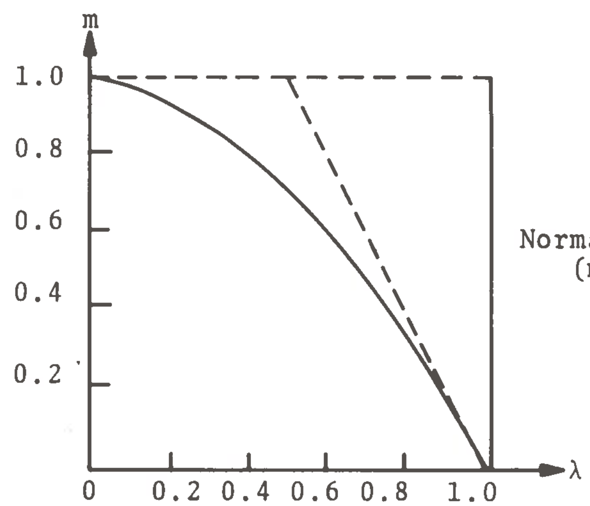

FIGURE A-2

Normalized Speed-Density (m - λ) Relation.

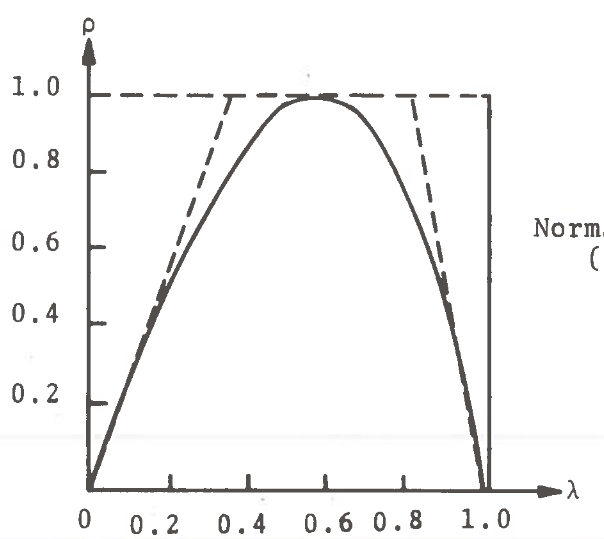

FIGURE A-3

Normalized Flow-Density (ρ - λ) Relation.

6. Generalizations

One could now consider an arbitrary function of the argument $(1-m)\ \ln(1-m)$, and investigate the conditions under which such a function would represent traffic flow within the stipulated requirements. However, we shall rather confine our attention to demonstrating that formula (3.3) is, in fact, the simplest example from a class of functions, which, under relatively mild restrictions, meet the requirements of the Fundamental Formula. We consider a more general speed-flow relation in the form

$$\phi(m) = -\ [(1-m)\ \ln(1-m)]\ .\ f(m), \tag{6.1}$$

and note the requirements on $f(m)$ in the interval $(0,1)$, under which the speed-flow pattern is preserved and the terminal conditions satisfied.

Clearly, $f(m)$ must be strictly positive. Moreover, for the satisfaction of conditions (3.1), it suffices that $f(m)$ be bounded (finite) on the interval. From the derivative

$$\frac{d\phi}{dm} = [1 + \ln(1-m)]f(m) - [(1-m)\ \ln(1-m)]f'(m), \tag{6.2}$$

it is evident that condition (3.2) is satisfied provided $f(1)$ is nonzero and $f'(1)$ is finite; also, the logarithmic growth for the derivative is thereby retained. We may further note that setting

$$1 + \ln(1-m) = \frac{f'(m)}{f(m)}\ [(1-m)\ \ln(1-m)] \tag{6.3}$$

gives the equation determining the capacity speed m_c, which, when substituted into (6.1), yields the capacity flow.

Turning to the speed-density relation, we combine (6.1) with (4.1) to obtain

$$\lambda = [\left(1 - \frac{1}{m}\right) \ln(1-m)]f(m), \tag{6.4}$$

so that, corresponding to relation (4.3), we have

$$\lambda \simeq [1 - \frac{1}{2}m - \frac{1}{6}m^2 - \ldots]f(m),\ m<<1, \tag{6.5}$$

and the terminal conditions (4.4) are clearly satisfied if we add the requirement

$$f(0) = 1\ . \tag{6.6}$$

Taking the derivative of (6.4), we obtain

$$\frac{d\lambda}{dm} = \frac{1}{m}[1 + \frac{1}{m}\ln(1-m)]f(m) - \frac{1}{m}[(1-m)\ln(1-m)]f'(m), \tag{6.7}$$

which remains strictly negative, provided f'(m) is nonpositive throughout the range. This latter condition ensures that the satisfaction of conditions (4.4) implies the satisfaction of conditions (2.4). Finally,

$$\frac{dm}{d\lambda} = \frac{m}{[1 + \frac{1}{m}\ln(1-m)]\ f(m) - [(1-m)\ \ln(1-m)]f'(m)}\ , \tag{6.8}$$

so that the logarithmic decay in the satisfaction of condition (2.5) is preserved if, as was already required for condition (3.2), f(1) is nonzero.

With the above restrictions on the generalized form, the pattern of every one of the three curves is also preserved.

Summarizing, we see that the two requirements

a. $f(m)$ is bounded, strictly positive, and has a nonpositive derivative on the interval $(0,1)$, and

b. $f(m)$ satisfies the terminal conditions

1) $f(0) = 1$,

2) $f(1) = a,\ 0 < a \leq 1$,

3) $|f'(1)| < \infty$,

are sufficient, though probably not necessary, to ensure that relation (6.1) preserves the features of the Fundamental Formula. Moreover, by combining (6.2) and (6.8), we obtain the generalized form,

$$\frac{c}{m} = \frac{[1 + \ln(1-m)]\ f(m) - [(1-m)\ \ln(1-m)]\ f'(m)}{[1 + \frac{1}{m}\ \ln(1-m)]\ f(m) - [(1-m)\ \ln(1-m)]\ f'(m)}\ , \qquad (6.9)$$

for the ratio of wave speed to mean speed.

After (3.3), which corresponds to $f = 1$, it appears that the formula next in order of simplicity would be the one implied by taking for f the linear form

$$f(m) = 1 - \alpha\ m\ , \qquad (6.10)$$

which, with $\alpha = 1 - a$, satisfies the above requirements, provided

$$0 \leq \alpha < 1\ . \qquad (6.11)$$

Moreover, equation (6.3) for the determination of capacity now has the form

$$1 + \ln(1-m) = - \frac{\alpha}{1-\alpha m} [(1-m) \ln(1-m)], \tag{6.12}$$

permitting, through the appropriate choice of α, the independent specification of either capacity or its associated speed. While this equation does not have an obvious solution, it does admit a simple approximate solution within the range of small values for α.

7. Fluid-Flow Analogy

In deriving the Fundamental Formula from the analogy with fluid mechanics, the usual procedure involves the application of both the equation of motion and the equation of continuity to the traffic stream.* The utilization of the dimensional relation (1.1) in the analog of the continuity equation leads to a partial differential equation, in which both speed and density appear as dependent variables.

To write the corresponding equation of motion in explicit form, it is necessary to postulate a specific law governing acceleration in traffic flow, which describes how the pattern varies with such factors as speed and density, including the distance-measured derivatives of these quantities. When combined with the yet-unspecified equation of state (Fundamental Formula), the equation of motion then yields a second partial differential equation governing speed and density. The requirement that the latter be consistent with the

*For more detail, see references [2] and [6].

one derived from the continuity relation leads to an ordinary differential equation, the integration of which yields the functional dependence expressed by the Fundamental Formula. Here we outline the procedure using a general form for the driving term in the equation of motion. From the ensuing consistency relation, it will be evident what the acceleration law should be in order that the implied Fundamental Formula should assume a pre-assigned form. In particular, we derive the acceleration pattern associated with the Fundamental Formula as expressed in the speed-flow relation (3.3).

We use t to signify the time variable, and let x denote the distance variable measured along the highway. At an arbitrary point in time and space, we consider the acceleration determined by the instantaneous traffic factors through an unspecified function, which, in view of the constraint (1.1), may, in its most general form, be written F(v,k). The equation of motion, therefore, reads

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = F(v,k) \quad . \tag{7.1}$$

If we consider the associated equation of state in the form of the general speed-density relation.

$$v = v(k), \tag{7.2}$$

then its introduction into the equation of motion (7.1) allows the latter to be written as

$$\frac{\partial k}{\partial t} + v\frac{\partial k}{\partial x} = \frac{1}{v'}F(v,k), \tag{7.3}$$

where we have used prime to denote differentiation with respect to k.

The corresponding equation of continuity has the form

$$\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0, \tag{7.4}$$

or, alternatively, in view of (1.1),

$$\frac{\partial k}{\partial t} + v\frac{\partial k}{\partial x} = -k\frac{\partial v}{\partial x} \, . \tag{7.5}$$

The latter may be rewritten as

$$\frac{\partial k}{\partial t} + v\frac{\partial k}{\partial x} = - kv' \frac{\partial k}{\partial x} \quad , \tag{7.6}$$

through the introduction of the functional dependence (7.2) into the right-hand side of (7.5).

The mutual consistency of equations (7.3) and (7.6) requires

$$F(v,k) = -kv'^{2}\frac{\partial k}{\partial x} \, , \tag{7.7}$$

so that the form of the driving term describing the acceleration pattern, in traffic, is determined by the functional dependence implied in the equation of state (7.2).

Conversely, if the acceleration law can be specified explicitly, equation (7.7) yields the implied equation for v', so that the determination of the equation of state is reduced to a quadrature. In fact, if we let

$$G = G(v,k) \tag{7.8}$$

denote an arbitrary function, dependent purely on the speed and density factors and independent of their space derivatives, and consider F in the form

$$F(v,k) = -k\frac{\partial k}{\partial x}\,[G(v,k)]^2 \tag{7.9}$$

(which, on reflection, appears quite reasonable), the equation for v' becomes

$$v' = G(v,k), \tag{7.10}$$

so that a straightforward integration yields the Fundamental Formula.

Returning to the more direct perspective of equation (7.7), we introduce the dimensionless variables (2.1), so that the form for F becomes

$$F = -v_0{}^2\lambda\frac{\partial\lambda}{\partial x}\left(\frac{dm}{d\lambda}\right)^2 \quad . \tag{7.11}$$

If we substitute for the dimensionless speed-density derivative from (4.9), we find

$$F = -v_0{}^2\lambda\frac{\partial\lambda}{\partial x}\,\frac{m^2}{[1 + \frac{1}{m}\ln(1-m)]^2} \quad . \tag{7.12}$$

Returning to dimensional variables, we obtain

$$F(v,k) = -\frac{1}{k_*^2}\,k\,\frac{\partial k}{\partial x}\,\frac{v^2}{\left[1 + \frac{v_0}{v}\ln(1 - \frac{v}{v_0})\right]^2} \tag{7.13}$$

for the law governing the acceleration pattern consistent with the assumed form (3.3) for the Fundamental Formula.

While the above form is considerably more complicated than either of those considered in references [2] and [6], it implies the relatively simple form (4.2) for the speed-density relation. We have already remarked on the desirability for simplicity in the Fundamental Formula. On the other hand, if we require that a single law describe the acceleration pattern throughout the entire density range, there is no reason to expect that such a law should have a simple formulation.

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