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VESSEL SAFETY MODEL

VOLUME I
ANALYTIC DEVELOPMENT

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16. Abstract

A computer model which mathematically simulates the ship's movement in defined waterways is described. Volume I presents the capabilities and usefulness for ship traffic lane selection, alternate route selection, and safety analysis. The analytic development of the equations of motion and the collision and grounding probability analysis used in the computer program are also presented. Volume II of the report consists of a complete Users' Manual. Volume III is a self-contained Programmers' Manual.

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PREFACE

This final report, authorized under PPA No. CG401, is published by the Transportation Systems Center (TSC) as part of a safety analysis program initiated by the United States Coast Guard. The report covers a vessel safety simulation model which analyzes the effect on shipping safety with changes in ship design, navigational aids, or regulatory practices.

The authors are happy to acknowledge the extensive and substantial effort of Mr. Leon Tritter in the preparation of this document.

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1. INTRODUCTION

The increase in transoceanic shipping incidental to increased international trade has lowered the safety margin in ship navigation. As a natural consequence of the increased transoceanic shipping, ship traffic in restricted waterways such as rivers and harbor areas is considerably increased; this, in turn, greatly increases the possibility of collisions. When the north slope in Alaska becomes operational, and oil tankers sail from the port of Valdez south to ports along the western coast of the United States, the danger of collisions will further increase in these areas, and potential massive oil spills resulting from these collisions will endanger our environment.

In an effort to counteract the escalating dangers of ship collisions and groundings due to expected increase in traffic density, a simulation type vessel safety model has been designed and used for solution of problems created by this increase in traffic density. The model simulates an existing ship population, and then modifies the movement of ships in ways which will help determine optimum efficiency and safety. Specifically, the model recreates the existing traffic patterns, and then increases or decreases traffic density, ship separations, speeds, and lane widths. The data outputs from the model provide the necessary information for creating most efficient use of the harbor or waterway area consistent with the acceptable level of safety.

The possibility of oil spill accidents is evaluated, through simulation, under different rules-of-the-road regulations, ship design, and navigational aid capabilities. The vessel safety model can simulate accidents in situations of ships crossing, head-on or overtaking; situations of fog and reefs; traffic environment (heavy in- and out-going harbor traffic); human factors (skill levels of ship operators); different ship maneuverabilities; situations of different channel dimensions; and different operating ship speeds.

2. BASIC CONSIDERATIONS

2.1 MARINE SAFETY PROBLEMS

Movement of ships through defined waterways with a maximum degree of safety involves a number of factors relating to physical conditions, ship operations, and human adaptability. The conditions discussed below relate to harbor operation as well as to free spaces outside the harbor area.

2.1.1 Tide and Wind

Tide and wind affect the velocity and direction of a ship in its travel. Deviation in the velocity of a ship from that in still water and air must be measurable, as must the directional effect exercised by the wind with respect to the ship's heading.

2.1.2 Acceleration/Deceleration

The forces involving increase and decrease in a ship's velocity requires a knowledge of the relationship of the ship's mass with the existing conditions, whether in still water or air or under tide and wind turbulences.

2.1.3 Lane Assignments

When ships operate in restricted waterways, a problem exists as to how close together they may travel with mutual maneuvering capability. Inputs to the simulation model provide for movement of ships in predetermined tracks. Varying the tracks varies the distance between ships, and also the probability of collision between the ships. These variations in tracking data permit interpretation of the minimum allowable track displacement for a given probability of collision. With this information available, a rule of the road can be decided upon for lane assignment in designated areas.

2.1.4 Turn Maneuvers

Lane assignments and anticipated changes in ship's velocity during the turning maneuver must be known to maintain a safe relationship among adjacent ships. The radius of travel and the initial relative fore-and-aft positions of adjacent ships must also be considered during a turning maneuver. An important consideration exists with respect to a ship about to perform a turn maneuver. If a nearby ship travels near to the maneuvering ship and in a direction such that intersection between the ships' headings is possible and a potential collision hazard is indicated, then the maneuvering ship must begin its turn before it arrives within ship's reach. (Note: The ship's reach is the minimum distance between two ships at which the maneuvering ship must begin its turn to avoid the danger of colliding with the other ship due to its mass inertia.) Simulation of turn maneuvering with respect to ship's reach spacing between two ships provides a means for analyzing the situation and establishing the safe approach to this problem.

2.1.5 Speed

The speed of a ship determines its degree of maneuverability in changing from one state of travel to another. Initial speed must therefore be considered when relatively quick changes are to be accomplished. Maintenance of a given speed is also related to the area traversed, the population of ships in the area, and the number of anticipated maneuvers in the populated area.

2.1.6 Collision Probability

During a ship's travel in defined waterways, the ship can experience varying degrees of freedom from collision, depending on the density of ship traffic in these waterways. If the ship is surrounded by a significant number of ships, then each adjacent ship, in conjunction with the one whose safety is being assessed, can be considered as one of a pair whose chances of colliding are related to their proximity. Since each combination is equally relevant to such a probability, the total probability of collision avoidance for this condition requires an analytical solution quite different from one which would relate to a sparsely populated area. Also, since the number of ships in the waterways can vary with time, and since the probability of collision varies with the number of ships in the area, then the time spent in the waterways must be known in determining the total probability of collision.

2.1.7 Grounding Probability

The path traveled by a ship may create a potential for grounding the ship. A knowledge of the area being traversed or possible alternate routes are factors to be considered when marine safety is of concern. With proper inputs to the simulation model, the probability of grounding can be determined, and the basis for decision-making in this matter established.

2.1.8 Weather Conditions

Poor weather conditions call for precautions and rules of behavior on the waterways not called for during periods of calm and still waters and clear air. Fog, rain, tides, and winds are simulated, and the ship is caused to react with these inputs to the simulation model. The relationships for the respective inputs (or combinations of inputs) are indicated in the simulation model output.

2.1.9 Evasive Maneuvers

Under certain conditions, quick maneuvering of ships can free them from positions in which the probability of collision is high. The evasive maneuver capability of the model makes it possible to determine when collision can be avoided under given conditions of ship maneuverabilities, speeds, and bearings. By use of the model, and with initial range and initial time to closest point of approach known, the necessary maneuvers to avoid collision can be determined. This determination is possible when only one ship maneuvers, or when both ships maneuver. (For an example of the matter discussed here, refer to subroutine MISS in Table 3-1.) To the extent that turn maneuvers can be related to evasive procedures, the discussion in Section 2.1.4 is also pertinent in this instance.

2.2 NEED FOR SIMULATION

Simulation provides a mathematical equivalent for the conditions existing during the actual movement of ships through defined waterways. By this simulation, made possible by the design of the vessel safety model, arbitrary conditions are established in the model to represent those referred to in Sections 2.1.1 through 2.1.9. The conditions are mathematically analyzed, and equations are formulated for ship motion under the designated conditions. The equations and their solutions provide a direct analogy with actual physical conditions existing in the movement of ships through the waterways. Therefore, the equations serve equally well as analytical tools for those actual physical conditions for which the same mathematical equations also apply.

2.3 CRITERIA FOR SIMULATION MODEL

The simulation model performs its functions on the basis of certain criteria through which appropriate data are provided to the model. These criteria are listed and explained below.

2.3.1 Batch Mode Operation

For the condition that a number of ships operates in a defined waterway, the simulation model is provided with predetermined inputs of initial speed, position, and heading data for each ship in the group. Specific traffic routes or, alternatively, maneuver commands, are given for each ship. The ships then move in a specified track or according to the inserted commands, in each case corresponding to the equations for ship motion used in the simulation model. With the given data inserted in the equations, the resulting calculation indicates the next position of each ship for the next time step.

2.3.2 Probability Calculations

2.3.2.1 Probability of Collision - Calculation of estimated probability of collision is based on the uncertainty of a ship's position about its mean position. This uncertainty is in turn a function of the following:

a. On-Board navigational capability

b. Navigational aids for the area of interest

c. Capability of ship's crew.

Calculations based on the above factors are made for the total time spent in the area. The degree of safety for the particular traffic pattern chosen is related to the values obtained for the collision probability factor.

2.3.2.2 Probability of Grounding - Similar calculations are carried out for the probability of grounding for the total time spent in a given area. Calculations for this circumstance are based on the following:

a. Travel route taken by ship

b. Depth levels along the route

c. Proximity to reefs and islands.

Variations in the parameters for the above functions provide means for determining the safest route. (In an alternative approach, the traffic route is held constant, and the type of ship is varied to determine the greatest safety with respect to the probability of grounding.)

2.3.2.3 Probability Calculations for Varying Weather Conditions - Calculations for either collision or grounding probability are governed to a great degree by data in the simulation model representing different types of weather conditions. The solution to this type of problem with varying data varies as the simulated weather conditions presented to the model change from calm to adverse.

2.3.2.4 Evasive Action Capability of Simulation Model - Evasive maneuvers for avoiding collision can be programmed into the simulation model. The maneuvers are tested for their efficiency by referring either to the distance of closest approach between ships, or to the calculations for actual collision probability for various programmed inputs. The test results are provided from ship movement calculations and status reports for different time step intervals, depending on the traffic density in the area of interest.

2.4 USE OF MODEL

a. The vessel safety model presents a mathematical simulation of ships moving in defined waterways. When the model receives data inputs which simulate actual physical conditions, it provides analytical results identical to what could have been anticipated with the actual physical ship movement under similar physical conditions.

b. Simulated movement of ships in a defined area is accomplished in the model by means of instructions contained in the User's Manual, Volume 2 of this final report. Programs required to implement these instructions are provided in the Programmer's Manual, Volume 3 of this final report.

c. The vessel safety model is very flexible, and possesses strong analytical capabilities in its intended use. At the same time, its effectiveness is enhanced by judicious application of simulation experiments and tests.

3. MODEL OVERVIEW

3.1 SCOPE

3.1.1 Purpose

The vessel safety model provides simulated movement of ships through a defined waterway, and also furnishes information on the probabilities of collision and grounding. The ships are moved through the waterway in accordance with programmed ship motion equations, and the probabilities are printed out according to derived equations for the respective probabilities. The model is capable of a further printed output which indicates the closest point of approach (CPA) between any pair of ships in the waterway.

3.1.2 Input Data Requirements

Operation of the model with respect to ship movement and probability calculation is made possible by insertion, into the model, of specific inputs:

a. Ship characteristics

b. Harbor size and harbor depths

c. Initial positions and velocities of ships

d. Navigational and human factors accuracies

e. Ship's tracks or commands.

3.2 BACKGROUND OF VESSEL SAFETY MODEL

This model has experienced several major modifications and changes of emphasis since first developed in 1970 (as the IEC model) by the International Engineering Company of Arlington, Virginia.*

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*Vogler, W. K., All Weather Harbor Navigation Model, IEC Technical Report 70-04, International Engineering Company, Arlington, Virginia, 1970.

For example, the original purely deterministic model using the integrated equations of motion with predefined tracks for ship motion has been changed to a differential model with a step-by-step calculation of ship positions. This allows continual updating of a ship's position, either by a pre-programmed evasive maneuver based on the recognition of a potentially dangerous situation, or by operator interaction when the model is operating in the interactive mode. Further, the model was put on a CRT display with a user-operated zoom capability in order to be able to zero in on potential trouble areas. This demonstrated the capability of using the model for pilot- and harbor-controller training. Additionally, a probability analysis, a Monte Carlo capability, and a variable time-step feature have all be incorporated into the model. This new model has been used with actual traffic patterns and the physical layout of San Francisco Harbor with good, though incomplete, validation. The model is an extremely useful device for safety analysis, alternative traffic lane sections, and training purposes.

3.3 PROGRAM ELEMENTS

A list of program elements is provided in Table 3-1. The elements are grouped in accordance with specified functions, and are typed as main routines, subroutines, or functions.

3.4 FLOW CHART OF PROGRAM ELEMENTS

Figure 3-1 presents a flow chart which indicates in block form the program sequence for the program elements referred to in Section 3.3. The flow chart connects the programs, subprograms, and functions in a manner which shows their interrelationship in the model.

TABLE 3-1. PROGRAM IDENTIFICATION

Program Element	Type	Action
	Preliminary Programs	
LINES	Main	Reads waterway contour cards prepared by user and outputs data cards with contour data transformed for use in the model.
CHAR	Main	Reads 11 items of information about each ship and outputs punched cards with 20 ship characteristics used in the ship motion equations in the model.
SIMQ	Subroutine	Solves simultaneous equations for CHAR.
	Model and Required Subprograms	
MODEL	Main	Reads waterway and ship descriptions, initial conditions, ship commands, and user options. Controls execution of the simulation. Provides output printouts of input data, ship position summary, and closest points of approach (CPA's) at various times during simulation. Prepares plotter tapes of ship tracks and CPA frequency plot, if requested.
RDCNTR	Subroutine	Reads contour data directly as output by program LINES.
GETCOM	Subroutine	Obtains the next command in the command stack.
GETTIM	Subroutine	Obtains the time required to complete a command or phase of a command.
MOVE	Subroutine	Computes coordinates of ship at next time step.

TABLE 3-1. PROGRAM IDENTIFICATION (CONTINUED)

Program Element	Type	Action
Model and Required Subprograms (Cont.)		
INSERT	Subroutine	Computes CPA (closest point of approach) for pairs of ships, and accumulates the CPAs for frequency distribution.
ENDINS	Subroutine	Adds to the array OCCUR any distances that remain in the arrays at the end of the run. (array OCCUR contains the total number of CPAs accumulated for each interval between zero and the maximum distance apart experienced between two ships being checked for CPA).
GAUSS	Function	Computes integral function used in the tide and wind calculations
Subprograms Which Can Be Removed With Slight Program Modification		
RANDU	Subroutine	Generates uniform random numbers.
TIDWIN	Subroutine	Defines tide and wind conditions as a function of ship position in the waterway.
TRKRD	Subroutine	Converts track data inputs to ship commands.
For Probability of Collision Calculation		
ALPROB	Subroutine	Computes log of probability of no collision.
ELIPS	Subroutine	Computes standard deviations.
PCOLIJ	Function	Computes probability of collision between two ships.
PDLTD	Function	Calculates the probability that the distance between two ships is less than D, the distance between them before collision.

TABLE 3-1. PROGRAM IDENTIFICATION (CONTINUED)

Program Element	Type	Action
For Probability of Collision Calculation (Cont.)		
NDTR	Function	Computes an integral function
For Probability of Grounding Calculation		
ALPRG	Subroutine	Computes log of probability of no grounding.
PGNDN	Function	Computes probability of grounding.
ELIPS	Subroutine	Computes standard deviations.
EPSHL	Subroutine	Computes a new contour with respect to the main contour.
ANGLES	Subroutine	Computes angles from mean position of ship to the end points of the safe harbor lines.
ERINT	Function	Computes the value of the probability density curve of grounding for a given value of theta (angle from mean position of ship to the end points of the safe harbor lines).
R	Function	Used to compute length of line from mean position of ship to a point on a contour (or island) line.
EXPX	Function	Obtains exponential of X without overflow.
Example of a Collision Avoidance Rule (Optional)		
MISS	Subroutine	Used as a demonstration of a decision routine for avoidance of collision between two ships.
Subsequent Programs		
PLOT	Main	Plots ship's tracks or the frequency array as specified by the user.

TABLE 3-1. PROGRAM IDENTIFICATION (CONTINUED)

Program Element	Type	Action
Subsequent Programs (Cont.)		
PLOTS	Subroutine	Used for initialization and calling sequence with respect to main program PLOT.
SYMBOL	Subroutine	Used to plot a symbol for each ship being plotted.
SCALE	Subroutine	Provides scale factor for arrays XX (distance) and YY (frequency) used in plotting frequency curve.
AXIS	Subroutine	Used to draw the X and Y axes in process of plotting the frequency array.
LINE	Subroutine	Connects points determined in plotting of frequency array.
PLOT		Draws line between two points specified in inches.

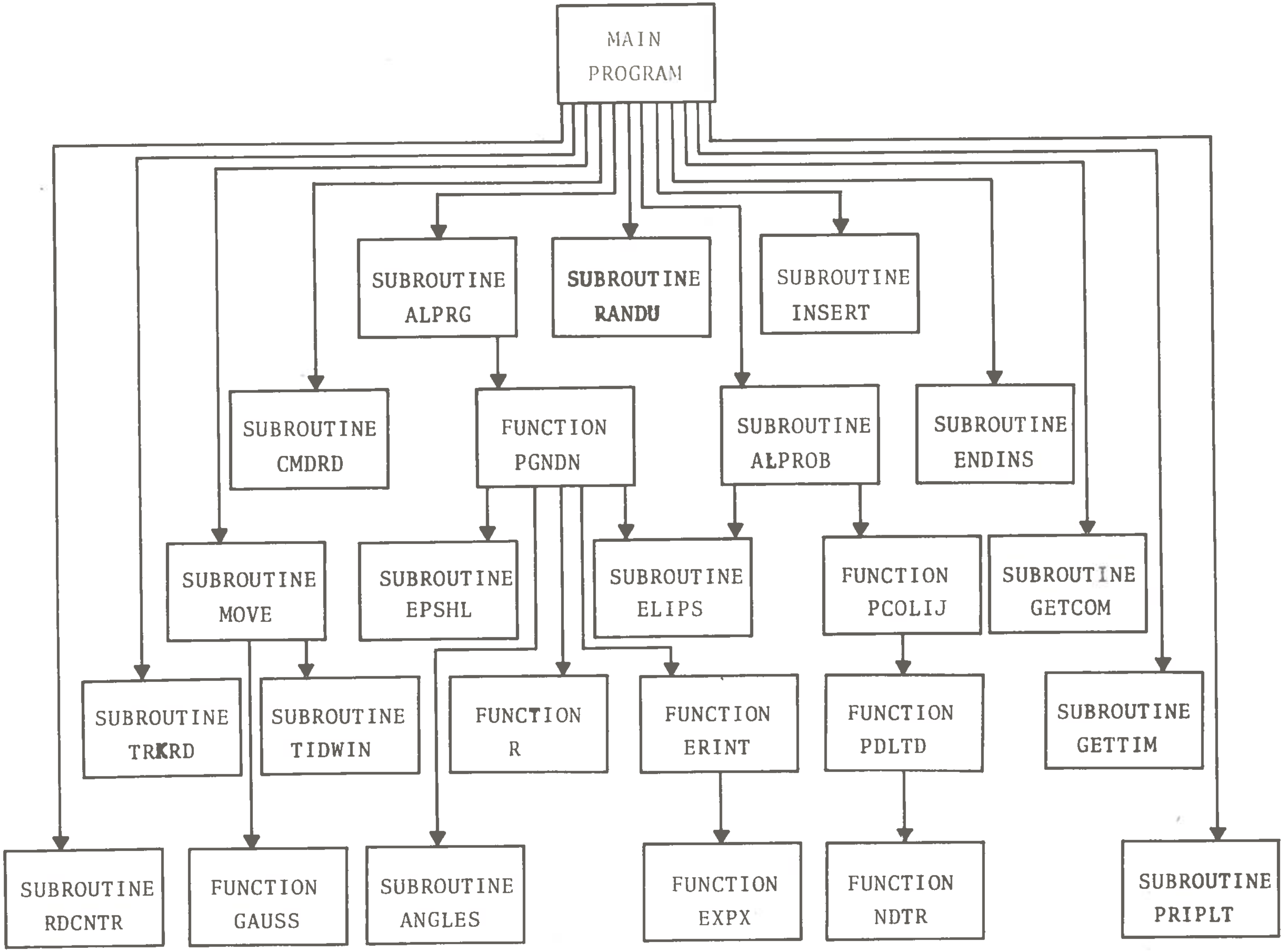


Figure 3-1. Flow Chart of Program Sequence for Program Elements

4. MODEL CAPABILITIES AND LIMITATIONS

4.1 MODEL CAPABILITIES

4.1.1 General Description

The analytic model described in this report mathematically simulates the movement of ships through defined waterways. The model is potentially useful for safety analysis, for choosing between alternative traffic lane ship routing, and for ship master and harbor traffic controller training. The model presently operates in batch mode, and therefore is most useful for such applications as safety analysis and traffic lane definition. However, the command stack concept used in the program permits conversion to the interactive mode and to computer scope/light pen display for training purposes. (Section 5.2.2f mentions zoom CRT capability.)

4.1.2 Batch Mode Operation

In its present batch-mode configuration, the model is geared toward running large varieties of possible traffic flow situations. The initial speed, position, and heading of each ship are put into the model. (If these are known only approximately, a range of values may be used; for example, an initial speed of $v \pm \Delta v$ knots). Specific traffic routes for each ship may then be given or, alternatively, a series of commands (such as particular acceleration and turn maneuvers) may be used. The ships will then proceed to move according to the specified track or commands given. This ship movement is based on the equations of motion used in the model, taking into account the existing wind and tide conditions inserted into the model. In this way, the next *position* of a ship is calculated at each time step, based upon current conditions influencing the ship at the time.

4.1.3 Model Printouts

Having programmed intended tracks for ship movement through the waterway, the model will print out status reports on the ship's speeds, positions, and headings (or on the ranges of speeds, positions, and headings if these are only approximately known) as calculated by the ship motion equations. The model also offers the option of printing out and plotting ship's closest points-of-approach during their traversal of the waterway. Distances closer than some specific amount are singled out, thus allowing for easy recognition of potentially dangerous near-misses. The resulting number of near-misses can then be compared for different traffic patterns used in the waterway. Safety and efficiency of different traffic patterns and lane usage can then be chosen based on the time needed to complete origin-to-destination trips and on the number of *close calls* encountered during the trip.

4.1.4 Probabilities of Collision and Grounding

a. The model also offers the option of calculating and printing out the estimated probabilities of collision and grounding. Collision probability is based on an uncertainty of a ship's position about its mean value. This position uncertainty may be a function of the on-board navigational capability, the harbor nav-aids, and on the crew's ability. In this way, a collision probability is calculated and accumulated for the total time the ship is in the area of interest (for example, in a harbor). Large values indicate that the particular traffic configuration leading to these large values is potentially unsafe, while small values indicate a safe harbor operation for the total time the ship was in the harbor.

b. Similarly for the probability of grounding, the model will calculate the probability of a grounding for the total time the ship remains within an area of interest. Different routes, various depth levels, and proximity to reefs and islands will yield different grounding probabilities, thus allowing comparisons of safe shipping operations to be made between specific routes. Additionally, the traffic routes may be held constant, and the type of ship may be varied for a study of design criteria as it relates to safety.

4.1.5 Checks on Rules-of-the-Road

a. One particularly important use for these probability calculations is checking the safety of different rules-of-the-road under various conditions of position uncertainty. For example, one rule-of-the-road may be perfectly adequate in clear weather, but not so under reduced visibility. Probability calculations which take account of position uncertainty in fog should lead to more definitive answers to questions of adequate and safe separations.

b. Related to this is the evasive action capability of the model. If two ships are on a potential collision course, it is possible to program into the model any set of evasive maneuvers to avoid the collision. In this way, the efficacy of different evasive maneuvers may be tested by recourse to the distances of closest approach printed out during the encounter or by recourse to the actual collision probability as calculated by the model for the encounter. Situations such as these can be looked at in detail by calculating ship movement and calling for status reports at short-time intervals, say, at each second. On the other hand, the model has been constructed to allow variable time stepping, so that open water or light traffic density shipping can be followed at appropriately large time intervals, say, once every 5 minutes, with the resultant savings in computer time and cost. This variable time stepping (permissible even within one run) allows an efficient simulation of ship traffic both within the harbor complex and out to the open shipping lanes.

4.2 ADVANTAGES AND LIMITATIONS

Additional features which characterize the model's capabilities, together with the model's limitations, are described with respect to the specific items indicated below.

4.2.1 Waterways

4.2.1.1 Ship Positioning -

The position of a ship during its running time in a waterway is indicated by coordinate points. By means of these points, a ship can be positioned anywhere in its travel. This method of position identification is superior to the original method, which located a ship by means of separate grid areas within a mapped block area. The closest distance between two successive positions was 30 feet, whereas, with the coordinate point method, the ship can be positioned anywhere in a continuous fashion.

4.2.1.2 Scale Flexibility -

a. The time scale has been modified from fixed to variable. The resulting flexibility permits the model to determine a ship's position at time intervals related to the ship density in the waterway. For example, in a confined waterway populated by a large number of ships, the time scale can be reduced to an interval as short as one second. Conversely, for a relatively sparse ship population, the time scale can be lengthened, say, to five minute intervals for a number of ships.

b. Although the time scale flexibility is a decided improvement over the unit fixed time scale, a judicious choice must be made in determining the time step setting. This has to do with the possibility that during the interval between large time steps some incident can occur (such as collision or grounding) which will not be observed in the printout. Therefore, while the variable time scale is an asset, the above reservation must be reflected in the choice to be made.

4.2.1.3 Islands/Contours -

a. The IEC model provided depth contours which required very many grids to indicate depth at all points of the surrounding area. The coordinate point method, which superseded the grid concept, identified the same contour by the coordinates of the outlying points of the area in which depths were determined. Within the contour, islands were similarly identified by the coordinate points of their outer boundaries. When these outlying points were specified and an island thus identified, its area could be found, and the computer logic could determine where a ship was located with respect to the island.

b. The number of contours and the number of islands are related to the computer size. The maxima of five contours and five islands per contour are based on the core capacity of the computer used with the model, i.e., 32,000 words in memory. This number is flexible, and can vary upward with larger computers. (For example, the model has been run on a large-scale computer for as many as 100 ships.)

4.2.2 Vessels

4.2.2.1 Characteristic Parameters - The model is programmed for maximum values for ships and associated parameters to be used in model runs, and is indicated in the following listing. The item "types of ships" refers to ships which vary in characteristics such as length, beam, mean draft, horsepower, displacement, speed capability, and maneuverability.

Item	Maximum Allowed
Types of ships	15
Number of ships	15
Characters in ship's name	20
Different time steps per run	10
Intervals in CPA frequency plot	100
Ship commands per ship	25
Lines in each depth contour	150
Islands per contour	5

4.2.2.2 Limitations on Number of Ships in Waterway - The limitations on the number of ships which can be accommodated at one time in a waterway during a model run are based on a number of factors:

a. A trade-off is necessary when considering running time and cost of operation against the complexity of the model.

b. Input data are difficult to acquire, and many of the ships' characteristics are difficult to obtain. This factor limits the number and types of ships to those for which input data and ships' characteristics are readily attainable for insertion into the ship motion equations.

c. The running time is limited by the core capacity of the computer used with the model. An increase in the number of ships increases the running time, and the core capacity of the computer in use. For a maximum accommodation of 15 ships, the core capacity for the computer has been determined at 32,000 words in memory.

d. The track complexity also limits the number of ships in a given running time. Since a track input constitutes line segments having as parameters velocity, heading, and length, the number of ships in the waterway must be affected by the number of track segments and their complexity in terms of variations of their parameters.

4.2.3 Tracks/Commands

4.2.3.1 IEC Model (Track Concept) - The IEC model operated with input data for predetermined tracks. On the basis of the data provided, a ship was given information on velocity, heading, and length of track, and then, in accordance with the associated program routine, was started on its run. The limiting feature of the track type input was the fact that once begun, the run had to be completed in accordance with the original data input, and without the capability of changing any of the original data. Specifically, if, in performing the track run, the ship was on a course of collision with another ship, the track concept precluded the insertion of an evasive maneuver for avoidance of the collision.

4.2.3.2 Command Stack Concept - The command stack concept, introduced into the vessel safety model, overcame the disadvantage inherent in the track concept. With the new approach, stacks of cards with commands were inserted into the model program, and procedures were accomplished for certain lengths of time and/or for certain distances. Ship motion was not confined to particular tracks. Parameters such as velocity, acceleration, and deceleration were changed, and computer logic made possible decision-making such as introduction of evasive maneuvers to avoid collisions between ships.

4.2.3.3 Batch Mode to Interactive Mode - Because of the command stack concept, it is possible to supplement the command stack program so as to convert it from the batch mode to the interactive mode. The latter mode permits a user to monitor ship movement on an oscilloscope screen, and to type in changes to the program. In the conversion from the batch mode to the interactive mode, program modification involves additions to the existing command stack program rather than any rework of the existing program. Therefore, the change-over from batch mode to interactive mode is readily feasible.

4.2.4 Treatment of Time in Model

4.2.4.1 Time Designation - A time reference base is included in the formulation of equations for ship motion and probability of collision or grounding, as indicated in Appendix A through Appendix H. Besides the parameters which relate the equations to time variant physical situations, two constants, T and τ, are used to specify the time period within which a physical situation is observed, and the time increments chosen for determining ship positions in the waterway. These constants are defined as follows:

T = total time for ships in waterway

τ = time step (variable) determined by user in advance of run. During a model run, as many as 10 time steps can be applied in determining ship positions.

4.2.4.2 <u>Printout Frequency</u> - Output data on ship position are made available as printouts for respective values of τ. The printout frequency is equal to or a multiple of the value of τ. Since τ is variable to the user, printout frequency is likewise variable.

4.2.4.3 <u>Run Time/Real Time Ratios</u> - The ratios referred to here deal with the amount of time ships spend in the real world compared with the time the model must spend in the waterway in simulating the real world operation. The ratios will vary with run complexity; specifically, they will vary with the speed of the computer being used, the options applied (e.g., the probability of collision or of grounding, acceleration, deceleration, or turning), the number of ships involved in the runs, the number of runs, and the time steps (τ) during which computer printouts are obtained. Table 4-1 indicates computer times and real times relating to three examples of model runs under varying conditions of the type described above. The computer printouts for the three examples are provided in Volume II of this document, Appendix A through Appendix C. From the data provided and the results indicated in Table 4-1, the chief time consuming factor (and also the costliest) is the option of determining the probability of grounding.

4.2.5 Distance Measurement in Model

In circumstances concerning the probability of collision, a critical distance between ships, known as the CPA (closest point of approach), becomes a significant factor. The CPA is a measure of the distance between the centers of ships, and can be calculated by the model. In a series of runs, a different CPA is calculated for each run on the basis of conditions relevant to the particular

TABLE 4-1. RUN TIME/REAL TIME DATA FOR HARBOR MODEL

NUMBER OF SHIPS	NUMBER OF RUNS	(τ) (SEC)	NUMBER OF PRINTOUTS	PROBABILITY OF COLLISION	PROBABILITY OF GROUNDING	ADDITIONAL OPTIONS	REAL TIME (SEC)	COMPUTER TIME (SEC)
				Example 1	Input Data - Main			
2	1	10	23	✓	✓	Accelera-tion Decleration Turning	230	90.8
				Example 2	Input Data - Main 1			
2	2	Variable	10	✓	-	Turning	120	5.01
				Example 3	Input Data - Main 2			
1	1	10	17	- -	✓ -	Turning	170	52.21 3.13
				PLOT (Program Execution)				0.92

Notes: 1. The computer used for examples 1 through 3 was a DEC (Digital Equipment Corporation) Model PDP-10.

2. A check mark in either Probability column indicated that a probability measurement was made for the run (or runs).

run. Therefore, a distribution of CPA's is attainable with varying conditions (such as variations in ship velocity) for the respective runs. From this distribution, the observed CPA can be used as a criterion for determining the probability of collision, and a subroutine such as MISS can be used for decision making relative to application of evasive (turn) maneuvers toward avoidance of collision. (Note: The model has no capability for predicting the CPA. It can only calculate the CPA and print out the information. Also, the discussion of CPA has no application with respect to grounding.)

4.2.6 Mathematical Modes

There are three modes which find application in the vessel safety model mathematical operations. These are enumerated and discussed below.

4.2.6.1 Deterministic Mathematics - This refers to calculations based on predetermined parameters, definite and exact in nature. This type of calculation is exemplified by the ship motion equations for both track and command stack concepts.

4.2.6.2 Computed Probabilities - Computations relating to probabilities of collision and grounding take into consideration factors which, at best, can only be approximated as to their probable effect on ship motion with a given number of inputs. Included in this category are:

a. Deviation from a ship's course
b. Human error
c. Assumption that a probable event is situated around the ship motion curve.

4.2.6.3 Monte Carlo - This method of calculation involves the addition of random errors to the normal inputs to the model, performance of a number of runs with these inputs, and printout of the resulting outputs from the model. The procedure permits evaluation of the model's performance with input errors which range up to a maximum deviation from the normal.

Once the data for the Monte Carlo routine are inserted into the model and the program is applied, the Monte Carlo operation is automatic and fast. This may be contrasted with a table look-up technique, which, by comparison with Monte Carlo, reduces the available core space in the computer memory, and increases the computer time due to additional instructions on the data to be processed. (Note: All input parameters can be given random values in the Monte Carlo routine.)

4.2.7 Tide/Wind Variation with Position

Tide and wind affect ship's position. These variations are accounted for in the calculations of ship position. A user-supplied routine is needed to specify the velocities as a function of position in the harbor.

4.2.8 Single and Multiple Run Options

a. A user has the option of performing single or multiple model runs, with or without introduction of random errors to the input data. For example, a single run may be performed with exact data (i.e., zero error) input to represent a track run. The course traversed by the ship for this input is then established as the reference track run. Any random error introduced to the data for a succeeding run will provide a deviation in course from the reference track run.

b. A multiple run could be performed in a Monte Carlo routine, with random errors of plus or minus two percent, plus or minus five percent, and so on. The corresponding outputs would be observed, and the overall range of deviations from the reference would be determined for all the random error inputs.

4.2.9 Output Options

The user provides for specific outputs by inserting requests for them in the input. These output options are satisfied by initiating requests on input data cards associated with the main computer program. The listing on the next page describes the options introduced in the cards, and the resulting actions to be taken with reference to the requests. The attainment of any one of the alternatives included in the Action column depends on the conditional instructions inserted into the program, and on the numerical values obtained on execution of the instruction.

Option	Action
1. Generation of tape file containing ship position.	a. Tape is generated. b. Tape is not generated.
2. Action to be taken if probability of collision and/or probability of grounding becomes greater than a specified amount.	a. Take no action. b. Stop ship or ships involved. c. Delete ship or ships involved.
3. Calculation of probabilities.	a. Probabilities are calculated. b. Probabilities are not calculated.
4. Printing of distances	a. Print only distances between ships that are less than the maximum CPA. b. Print all distances between ships. (This action applies only to the condition that the number of ships used is not large. For a large number of ships, the option reverts to action for a) above.

4.2.10 Hardware

a. The computer used with the vessel safety model operates with a capability dictated by the maximum requirements listed in Section 4.2.2. (Actually, the computer core capacity determines the number of ships the model can move in a waterway large enough to contain the ships. In that sense, the vessel safety model has proved capable of runs, on a large-scale computer, for as many as 100 ships.)

b. Input equipment used by the computer included the card punch-card reader, and also the disc for data storage. By use of the disc and a suitable program, random access storage was made possible.

c. In addition to the use of the conventional printout equipment, the vessel safety model computer also used a pen plotter to plot, as output, data provided by the computer. The computer is programmed for the Calcomp type pen plotter. If a type of plotter different from the Calcomp were to be used, the program would have to be modified to accommodate the change. Such modification is generally not difficult, since most programs for pen plotters are analogous, and program modifications required to adapt to a change in type are relatively minor.

5. MODEL DESIGN APPROACH

5.1 IEC MODEL

During 1969-1970 the International Engineering Company of Arlington, Virginia developed and completed, for the United States Coast Guard, a mathematical (computer simulation) model known as the IEC model.

5.1.1 Purposes of Development

Development of a mathematical model was sought to permit movement of various types of ships through a harbor complex. Achievement of this primary objective made it possible to determine and specify evaluation parameters which would serve as navigational aids applicable to real world situations. Use of such a model offered a faster, more flexible, and more economical approach to the study of harbor complexes and systems than a method based only on collection and analysis of real world data.

5.1.2 Capabilities

The IEC model has these capabilities:

a. Moves ships according to a pre-determined pattern.

b. Receives input data on harbor features such as depth, obstructions, and navigational capabilities.

c. Maintains an account of harbor status.

d. Recognizes and makes available information concerning near and actual collisions (based on the touching of proximity boxes, calculated by adding to the length and width of a ship the lengths determined by navigational error estimates), potential and actual groundings and time spent in the harbor by each ship.

e. Computes time-area factor for each ship. This factor represents the cumulative sum of areas reserved for each ship at each clock time during its passage through the harbor.

5.1.3 Limitations

The IEC model has certain limitations, which are briefly discussed here. A detailed discussion of these limitations and consideration of methods of compensating for them are contained in Section IV of the IEC Technical Report 70-04, All Weather Harbor Navigational Model, dated 27 August 1970, prepared for the United States Coast Guard, Washington, DC.

5.1.3.1 Directionality of Navigational Contours - The IEC model has a call-up section in the memory which delivers accuracy indices for the harbor complex. These indices provide harbor navigation system accuracy contour intelligence, and are used as an absolute value by the computer main program. Use of an absolute value (say, harbor navigation accuracy of 60 feet) implies that it is applicable in all directions. In certain instances an absolute value may be detrimental. That is, for more realistic contour data, the model should have a directional capability with respect to navigation accuracy indices.

5.1.3.2 Harbor Complex Shape - The model is programmed to accept a 6 by 6 mile square as a harbor shape. In many cases a rectangular or odd shape is more convenient. In order to accommodate various harbor shapes and sizes, routines and operations have to be modified.

5.1.3.3 Time-Area Evaluation Index - The time-area index (referred to as merit figure) is computed for each ship in the harbor. The merit figure is a cumulative sum of the areas reserved for each ship at each clock time. The figure represents the means for measuring harbor efficiency or for providing the means for harbor evaluation. An ideal merit figure would apply to a relatively small reserved area which, at the same time, does not require an excessive time to be spent in the harbor by the ship. Since a ship can have different velocities at different clock times, the areas reserved for the ship will vary, and the time spent in the harbor will also vary. The limitations of concern refer to the fact that, while increased velocity will reduce the time spent in the harbor, the area reserved for that ship will be increased. Conversely, a reduction in velocity will reduce the required reserve area, but will increase the time spent in the harbor. The relationship between the merit figure and optimum harbor efficiency was not determined.

5.1.3.4 Model Execution Time - It has been found that when the computer simulation model is exercised, approximately four seconds of computer time are required for each ship for one second of harbor clock time. Translated to real world application, such as processing harbor information for controller usage, the computer would continually lag behind real time, making for an unrealistic and cumbersome situation.

5.1.3.5 Automatic Program Stoppage - If any element of a proximity box crosses the boundary of the 6 by 6 mile harbor square, the program automatically stops. (The proximity box refers to the area surrounding the ship which is reserved for the ship's use.) The harbor area is laid out in 6 by 6 mile square harbor blocks, and there does not exist any depth or navigation accuracy contour files for harbor blocks labeled zero, negative, or greater than 36. During operation of the model stoppages are undesirable. To correct this would require modification of the model.

5.1.3.6 Ship Characteristics - Operation of the model was limited by the fact that ship characteristic parameters for smaller ships were not available for insertion into the model. The model was therefore unable to develop the proper ship characteristics file upon which the computer main program depends for execution.

5.1.3.7 Tide and Wind Contours - In the IEC model, one vector is used for all areas in the harbor to represent wind and tide currents. While this is not a serious defect for the wind, it is a serious defect, in the case of the tide current, to use one vector through the entire harbor. This requires model modification to include call-up routines to deliver current vectors for different parts of the harbor.

5.2 TSC EFFORT

The initial efforts exercised by TSC (Transportation Systems Center) in the development of the vessel safety model were related to conversion and modification of the IEC model and improvements in its efficiency. The sequence of events and related objectives constitute the historical background leading to the development of the vessel safety model in its final form.

5.2.1 Overall Objective

The program objective was to develop and test a mathematical simulation of ship harbor traffic, and was treated in two phases. The first phase involved analyzing, verifying, and improving the IEC model. The second phase involved making several major additions to the model's capability, including true step-by-step calculations and interactive graphics for operating and control of the model.

5.2.2 Conversion and Modification of the IEC Model

a. The United States Coast Guard requirement for optimizing the execution speed of the IEC model was analyzed. An additional requirement handled by TSC called for modification of the program to allow loading of odd-shaped harbor descriptors and tide and wind variables.

b. The model, which was originally programmed to accept square harbor blocks, was modified to operate successfully with non-square harbor blocks.

c. Although inputs and outputs were still designated in the row and block format, calculations in the PROBO program (the routine for generating physical outlines and perimeters of the proximity box for all ships) were performed in the much more efficient x, y coordinate system. In a test run with one ship the program ran 40 minutes under the old model, and then ran 30 seconds under the modified model. The modified model provided an increase in speed of 80 to 1. With more than one ship involved, even greater increases in speed were possible.

For example, for five ships in open water there would be an increase in speed by a factor of 150.

d. The end-of-harbor mode was taken care of, in that the program continued to operate when ships left the defined harbor. The net effect of this action was the elimination from consideration of those ships which left the harbor.

e. A user's manual for the IEC model and a sensitivity analysis technical note were delivered to the United States Coast Guard. In this technical note the utility of the TAEI (time-area evaluation index) was investigated. It was shown that the TAEI was unreliable as a measure of the number of ship collisions and groundings. However, an alternative use for the TAEI was found. The TAEI was shown to be useful as a measure of the improvement of safe harbor utilization to be expected from a proposed increase in navigational aids.

f. Simplified Model with Zoom CRT Display

(1) The original harbor model was simplified to operate with the DDP-516 computer. Specifically, the PROBO program was completely eliminated, and the use of cartesian coordinates was instituted. The dimension statements of the main program and the turn, acceleration, and slowing equation subroutines were altered to permit only three ships with a maximum of five track segments each. Wind and tide variables were also eliminated.

(2) This simplified model was also provided with a routine which caused the CRT display to zoom from a 16 by 16 square mile block area down to any square area (and back). The light pen was used to choose any two diagonally opposite corners of the square block desired for display.

5.2.3 Vessel Safety Model

Continued modifications to the IEC model extended its capabilities to conform with the associated safety requirements of the vessel safety program. The following items were involved in the development of the vessel safety model.

a. The analysis for conversion to a step-by-step position calculation using differential equations of ship movement was developed. This changed the deterministic program, in which the paths of ships were completely prescribed, into a differential program, in which changes along a ship's path could be made as needed. This is particularly important in accounting for wind and tide effects along the ship's route.

b. The new differential equation formulation for ship movement was incorporated into the computer program.

c. The command stack concept was developed for the model. This gave the model the flexibility of inputting prescribed shipping routes as well as the ability to input instructions for ships to leave prescribed paths for any reason.

d. A method to determine the probability of ship collisions and grounding was developed. The analysis decided upon assumes that the model propagates the mean position of the ship. The actual position is known only approximately in the sense that the ships coordinates are jointly Gaussian distributed. The variance depends upon navigational capabilities on board ship and in the harbor. The model then calculates the probability of distances between ships for each time interval and a probability of grounding in the same time interval. Cumulative probabilities are then calculated for the total time the ship remains in the harbor.

e. Variable time stepping was incorporated into the model. Depending on the detail needed, the model could now follow ship movement accordingly: in a harbor with many ships, a small time interval would be chosen; in open water, a large time interval would be chosen; and in the transition from one to the other, different appropriate time intervals would be used.

f. The ability to input ship velocities, positions, and headings stochastically into the model was developed. This feature is particularly useful for testing ship movement along certain shipping lanes. With reasonable estimates of the uncertainties in ship starting velocities, headings and positions, a run with a large number of ships could be made to test traffic patterns for the lane. This would then be compared against other proposed traffic lanes for safe harbor operation.

g. A continuous contour formulation for determining depth levels was developed. This replaced the much less efficient point-discrete method for calculating ship grounding.

h. The model was tested using San Francisco harbor operations which were obtained from data cards of ship movement through the harbor by the Standard Watch in the radar room. The ability to use this readily available data in the model was demonstrated, and the general origin to destination path followed by the ships was reproduced in the model. While the model must still be verified with more refined data than were available from San Francisco during the time of this study, the limited validation thus far obtained should be considered encouraging, and should be pursued further. In addition to using the data cards, TSC demonstrated the ability of the model to reproduce several ship collisions and groundings as seen in photographs of the tracking radar scope.

5.3 ADAPTATION OF TIME STEP CALCULATIONS TO MODEL

An analytical method for step-by-step calculation of position for the harbor model was adopted. This method uses differential equations for movement of ships instead of the predetermined track method used in the original IEC formulation. The usefulness of the model was limited in three respects because of the original IEC formulation:

a. The model did not allow influences on ship movement during its travel along the predetermined track to be taken into account.

b. The model did not provide an interrupt feature through which the ship motion might be changed.

c. The model was inherently limited in its method of accounting for the effects of wind and tide on ship motion. In the new differential equation formulation, the wind and tide effects are computed in a way which permits changes to be made anywhere along the path of the ship.

5.4 MAJOR REORIENTATION

a. The major modifications performed on the IEC model were those which changed the math model operational concept from basic harbor navigation to safety analyses. These modifications were in the main discussed in the second part of the discussion on the historical background of the TSC effort (see Section 5.2.3).

b. Three additional capabilities were incorporated into the model during the reorientation period.

(1) The batch mode of operation provided for insertion into the model of a concentrated amount of data together with instructions for performing diverse functions. The model then performed in accordance with the input commands, and operated automatically through a run to provide outputs related to the given input data. This mode of operation was more useful for the kinds of accident multiple runs desired by the Marine Safety Technology Division.

(2) Provision was made for the use of routines which allowed random errors to be inserted in the input parameters. This action, together with a significant number of runs with the various random error inputs, represents a feature of the model operation known as the Monte Carlo capability.

(3) In order to treat ship traffic in busy harbors as well as to be able to perform statistics efficiently, the model was modified to handle large numbers of ships.

5.5 TSC INITIATIVES

5.5.1 Validation Study

TSC performed a limited validation study of the model based on simulation runs with San Francisco operations data and in accordance with the approach described below.

a. Ships were given tracks similar to those taken by specific cargo ships using the harbor. The inbound routes of the tracks modeled were from the Golden Gate bridge in the east and south directions toward Oakland, and in the east and north directions toward Richmond. The outboard routes were from the Richmond and Oakland directions to the Golden Gate bridge, and in the direction of the main ship channel.

b. The simulation runs demonstrated the workability of the model and its ability to use certain kinds of traffic data. For example, these runs used the data cards prepared by the Standard Watch in the radar room using name of ship, time in, time out, point of origin, point of destination, visibility, and type of ship data. This information was then combined with the United States Coast Guard charts of the Harbor Traffic System to obtain reasonably accurate simulations of the harbor traffic. However, since ship speeds and turning movements were not known, the ship motion equations used by the model could not be validated.

5.5.2 Computed Probabilities

a. TSC developed a preliminary method for calculating collision and grounding probabilities fully described in Appendix C through Appendix H. The method which has been incorporated into the computer program as subroutines calculates the cumulative collision probability of a ship in a waterway. In doing so it assumes that the ship coordinates of position are jointly gaussian random variables distributed about mean values of position as determined by the ship motion equations. The distance between two ships is also a gaussian variable, and the probability of collision between two ships is determined by calculating the probability that this distance is less than an amount determined by the sum of the ship's length and the distance traveled in a time step. The variances in distance are determined by the crew's ability, ship navigational aids, and harbor navigational aids.

b. The probability of grounding of a ship while it is in the harbor was similarly calculated. In this case the probability distribution for the distance between ships was replaced by the probability that a ship's coordinates will remain in a portion of the harbor containing sufficient water depth for its draft.

5.5.3 Closest Point of Approach

TSC also incorporated into the model calculations of the closest point of approach (CPA) of two ships. (This is discussed in Volume II.) These calculations, when run in conjunction with the Monte Carlo capability, can also be used for collision analysis by running the model many times, and thus obtaining a distribution of CPAs.

6. MODELING OF ELEMENTS

6.1 DEGREE OF DETAIL

The degree of understanding of certain aspects of vessel safety varies from precise knowledge, as in the case of waterways, to imperfect understanding, as in the case of human factors. Correspondingly, simulation by the model can involve modeling of components from the standpoint of exact representation to simplistic characterization. The present discussion approaches the modeling of components in accordance with the categories listed below:

a. Physical influences on ship safety

b. Information aspects of ship control

c. Collision avoidance.

6.2 PHYSICAL INFLUENCES ON SHIP SAFETY

6.2.1 Modeling Harbor Geometry

The model is capable of providing the overall outline of a harbor by use of a pen plotter, such as the Calcomp plotter. Figure 6-1, which shows the outline of the San Francisco harbor, is a typical example of the harbor geometry which the model can provide.

6.2.2 Modeling the Ship

6.2.2.1 Dot Representation - Ships represented as dots in a waterway experience the least chance for realistic simulation by a model being used in a safety analysis for ships moving in the waterway. Very little (on a relative basis) can be accomplished with the dot presentation for safety analysis beyond general consideration of conditions prevailing because of the presence of the dots (ships) in the waterway. As a further example, it would be extremely difficult to attempt analysis of the probability of

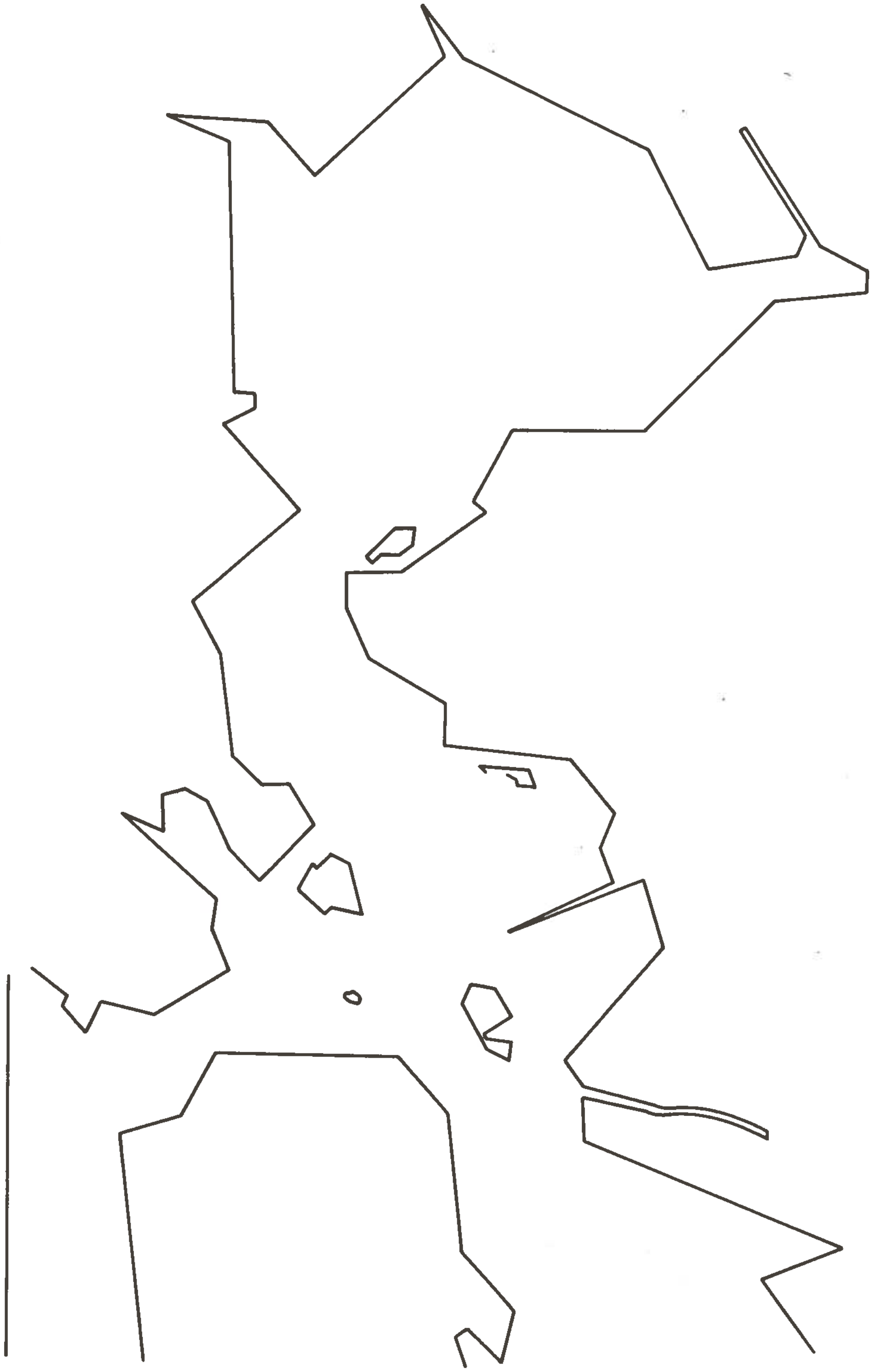


Figure 6-1. Outline of San Francisco Harbor

collision between two ships whose only means of identification happen to be adjacent dot representations in the waterway.

6.2.2.2 Outlines - A more reasonable approach provides the outlines of ships (length and beam). In this case, the ships can be considered with respect to relative space occupancy by the respective ships, and to the probability that collisions may occur between ships at any identifiable parts of their outlines.

6.2.2.3 Ship Characteristics - The vessel safety model employs the concept of ship modeling on the basis of a set of ship characteristics. This concept not only embraces the outline approach (by including the length and beam parameters in its list of ship characteristics), but also provides seven additional basic ship characteristics (mean draft, maximum velocity, ahead and astern horsepower, weight, wind sail, and reverse time) with which to permit supplementary maneuver capability in the model. The great increase in function potential justifies adoption of the ship characteristic concept in modeling of the ship.

6.2.3 Modeling Tide and Wind

Tide and wind effects are included in the ship motion equations, which are presented in Appendix A. The tide and wind effects are reflected in the derived equations for ship velocity and position. Additional information on tide and wind calculations is contained in Section A.2 of Appendix A.

6.2.4 Modeling Ship Motion

a. The ship motion parameters were modeled in accordance with the parameters of ship motion equations in the final report for the all-weather harbor navigational model, IEC Technical Report 70-04. This report which was furnished by the United States Coast Guard for use by TSC, provided the parameters as part of equations involving turning, acceleration, and deceleration maneuvers.

b. The ship motion equations are relatively easy to work with. Data needed for these equations are readily obtainable (at least for large cargo and tanker ships), and the equations are not excessively complex.

c. Accuracy and sensitivity data are directed to the following observations:

(1) The present equations do not consider shallow water effect on the turning rate of ships.

(2) The model is sensitive to certain vessel design details such as wind sail ratio (top side area to side-under-water area) and ahead and astern horsepower.

(3) The model is insensitive to hydrodynamic interactions between ships.

d. The TSC model is capable of accepting equations other than the equations currently being used. However, although the model is flexible in being able to accept the equations, the means whereby the model realizes the changeover are not easily achieved.

6.3 INFORMATION ASPECTS OF SHIP CONTROL

The vessel safety model can include effects of abundance of, or lack of data for navigational aids on-board ship or in the harbor, or for a crew's capability. This type of information is obtained from model inputs in the form of indices NAVC(I), HUFA(I), and NI(X).

6.3.1 On-Board Navigational Capability

This characteristic is indicated by NAVC(I), defined as a navigational capability index for ship I. It represents the positional error (in feet) for ship I due to lack of sufficient navigational aids aboard ship. This number is used with reference to computation of standard deviation for a ship from its true course.

6.3.2 Human Factors

The human factor index (e.g., a crew's capability) is referred to as HUFA(I), the human talent index for ship I, and is expressed in the form of a ratio:

$$\text{HUFA(I)} = \frac{\text{average crew's ability}}{\text{crew's ability for ship I}} \quad .$$

Therefore, the index for ship I will be a small number for a crew with high degree of ability, and a progressively higher number for a crew with lesser degree of capability. This factor is also used in the computation of the standard deviations from the ship's true course. (The human factor can also include potential aids such as those related to weather watch, radar watch, radar operation, and the ship's captain's input.)

6.3.3 Navigational Aids (Harbor)

The index NI(X) is a navigational index used as a function of the position of a ship in the harbor. The index is a measure of harbor capability at position X in the harbor, and reflects the adequate use of navigational aids such as buoys and lighthouses with respect to safe ship travel with a given ship population in the waterway.

6.4 COLLISION AVOIDANCE

a. The model has the potential for providing information on the relative merits of the existing, or projected, rules of the road for a congested waterway. The model can be given a set of inputs which will provide a number of ships with a specific number of traffic lanes, predetermined velocities, headings, and turn maneuvers in addition to simulated inputs for the autopilot concept and for the human factors such as pilot decision criteria. From these inputs, observations can be made concerning CPA, ship's reach, and probabilities of collision and grounding can be made, and decisions arrived at concerning optimum operation. With the possibility of providing numerous runs with variations in inputs, the output data can be analyzed as to the adequacy of existing rules of the road or to the desirability of modifying them or establishing new ones.

b. With respect to the inputs furnished the model, there exists the feasibiltiy of introducing into the main program a subroutine, as typified by a subroutine MISS. This subroutine, set up by TSC, is not included in the main program; rather, it is an example of what can appropriately be inserted when, on the basis of output data obtained from the model, information is obtained that a potential collision is at hand. The MISS subroutine inserts maneuver instructions, so that on their performance the probability of collision is minimized.

7. APPLICABLE TESTS WITH SIMULATION MODEL

7.1 GENERAL

a. The primary function of the vessel safety model is to simulate the movement of ships in a defined waterway area. It performs this function by means of input data which establish position, velocity, and heading, and by insertion of input commands which cause ships to track straight ahead, accelerate or decelerate, or make left or right turns. The basic operations are modified by varying the values and the nature of the input data in successive runs of the simulation model. The modifications represent changes under controlled conditions, and are readily duplicated in the movement of actual ships in a body of water.

b. Since the modifications referred to in "a" are controlled as to type and magnitude, they can be used as criteria for the movement of ships, probability of collision, probability of grounding, and attainment of optimum efficiency under various predetermined conditions.

c. The test applications of the following paragraphs indicate the potential approaches of which the simulation model is capable. The tests basically cause simulated ship movements under given conditions with specific inputs to the simulation model. From the indications provided for the given inputs, a mathematical analysis is obtained. Interpretation of the mathematical analysis is followed by steps to rectify any problem made evident by the simulation approach.

7.2 TEST CAPABILITIES

The following tests are accomplished in accordance with the same basic concept mentioned in Section 7.1. Inputs are provided to the simulation model, and simulated ship movement results. The ship movement is analyzed, and, if the inputs reflect a problem situation, a decision is made to modify the inputs and rectify the problem. Instructions on defining the conditions for simulation and for insertion of input data are implemented by programs set up for the immediate problem to be handled. The tests to be described require such programs.

7.2.1 Alternative Ship Control Subsystem Design

Inputs to the simulation model are varied to simulate changes in the ship's characteristics, and their effect on ship's control are observed and evaluated. The ship's characteristics which are considered in this test include:

a. Ahead horsepower

b. Astern horsepower

c. Maximum rudder angle

d. Maximum velocity

e. Turning velocity.

7.2.2 Alternative Rules-of-the-Road Formulation

a. This test is begun by making an initial ship's run under a given set of rules and calculating the probability of collision. The initial run is then followed by a sequence of many runs, each of which contains input parameter variations in the form of new rules of the road and track commands, and new probability of collision are calculated. The results of these calculations make possible a selection of rules which will minimize the probability of collision. Additional variations of input parameters for this type of test are diversified, and range from changes in weather conditions, navigational aids, and ship physical characteristics to changes in human capabilities and random errors.

b. An alternative approach toward collision avoidance involves changes in approach speeds, headings, and ship types. In addition, different rules-of-the-road are introduced and used under conditions identical to those for the original rules-of-the-road. A comparison is then made between the two sets of rules to determine the set more efficient in avoiding collisions.

7.2.3 Autopilot Concept

To the extent that the autopilot concept implies automatic ship control with extremely small error, the simulation model is provided with ship command inputs which have equally small errors, comparable with those allowed by an autopilot. Inputs representing error compatible with a pilot's capabilities and variations about a median human capability will provide information on the allowable deviation from the autopilot standard for pilot performance.

7.2.4 Automated Collision Avoidance System Designs

In this simulated test, ships are assumed initially to be traveling in parallel on assigned lanes in a defined waterway. Input commands to the simulation model cause variations in heading and velocity for either or both ships. If the heading and velocity of each ship and the lengths of assigned lanes are known, the probability of collision for any combination of headings and velocities can be determined. Conversely, if the bounding length of an assigned lane area is known, it is possible to determine the maximum total variation in heading and related velocity which will permit two ships in adjacent lanes to complete their travel over the total length of lane without collision. A decision based on the above computed data provides the equivalent of automated avoidance of collision. (Actual automated equipment operation corrects errors on either ship to maintain the heading and velocity within the collision avoidance limits determined through the simulation model.)

7.2.5 On-Board Navigational Capability

On-board navigational capability involves the use of communications equipment, radar, sonar, other instrumentation, and, in addition, the human capability. The simulation model assumes an initial output reference when average-condition input commands are simulated for radio communications, radar, instrumentation (e.g., ship's compass), and the human capability (e.g., a pilot's instructions). For test purposes, crew capability and navigational aid capability are varied, and the parameters are observed and evaluated for their effect on overall operation. The degree of influence on overall performance is analyzed for components which indicate significant errors, and their importance in maintaining safety standards is interpreted. Similar interpretation is possible when components become inoperative. On the basis of these interpretations, decisions can be made on the allowable downtime for the equipment, the priority of equipment maintenance, and the background requirements for satisfactory pilot performance.

7.2.6 Signaling Techniques

7.2.6.1 Visual/Aural Signaling - Ships traveling in waterways in which visual or aural means of signaling are stationed enjoy a greater degree of safety during their travel. Signal sources such as buoys carrying bells or lights or whistles, or lighthouse signal sources serve to alert the ship to waterway hazards and to the need for ship maneuvering in the vicinity of the signal sources. If the waterway areas known to be hazardous are mapped out, it is possible to test for the effectiveness of the navigational aids in these areas by use of the simulation model. A ship is assumed to travel on course along a given lane, and to approach an area where a buoy is located. By means of a simulated input representing the presence of a buoy at a mapped location, the simulated model output causes the ship to skirt the hazardous area. Supplementary programming returns the ship to its course after the hazardous area has been bypassed.

7.2.6.2 Ship-to-Ship Radio Communication - Ships can also communicate by radio. Because radio communication within relatively short distances is not affected by weather conditions, it has an advantage over the visual or aural methods. Ships which approach each other and come near to the CPA (closest point of approach without danger of collision) can alert each other to take measures to change course. With respect to the simulation model, programs can be set up for inputs representing ship separations close to the CPA. The other affected ships will be contacted, and appropriate maneuvering will be initiated to change course. The output data from the simulation model will be adaptable to actual physical conditions which approximate the conditions stipulated in the simulation model.

7.2.7 Traffic System Designs

7.2.7.1 Routing/Scheduling - Since the density of ship population in a defined waterway contributes to the total probability of collision for any one ship in that area, traffic control can serve a useful purpose in minimizing the probability of collision. Scheduling will control traffic with respect to the number of ships permitted in the waterway for a given time interval, and, for those ships permitted in the area, a routing of the ships can provide further means for keeping them outside the CPA (closest points of approach) and ship's reach limitations. The simulation model can be set up with a ship grouping for given time intervals and related routing arrangements to represent nominal inputs. These initial conditions can be varied as to the number of ships in the area and the paths taken by different ships. With these modifications, the probability of collision under different conditions can be evaluated, and the optimum set of conditions can be determined from the derived data outputs.

7.2.7.2 Lane Separation - Flexibility and safety of ship travel are affected by permissible lane separation between ships under existing rules-of-the-road. Different lane separations are indirectly related to the traffic density of the ship's population in the defined waterway. This is evidenced by a possible greater number of ships with reduced lane spacing and resulting increase in available lanes. Weather conditions such as fog, tide, wind, and storm also affect the determination of optimum lane separation for the greatest safety. Inputs to the simulation model for successive ship runs will modify original inputs representing an initial lane separation. Inputs representing fog, tide, wind, and storm will increase the hazard of collision between ships. Interpretation of the output data from the simulation model with respect to ship movement under the conditions indicated will provide information for optimum assignment of actual ship lanes.

7.2.7.3 Course/Speed Advisories Criteria - External direction by a central source of the course and speed of ships traversing a defined waterway may have a beneficial result on reducing the probability of collisions between ships. In the model, ship command inputs to the simulation model will vary the course of a ship in accordance with the following criteria:

a. Straight ahead
b. Acceleration
c. Deceleration
d. Left turn
e. Right turn.

These commands may be used to direct each ship in the waterway to operate at specific courses and speeds. With instructions pre-input into the model to change course and speed when certain conditions occur (such as ships approaching a specified distance from each other) the efficacy of external central direction may be simulated by the model.

7.2.7.4 Danger Warning Criteria - The basic methods for providing warnings of danger to ships include buoys, lighthouse signaling, and electronic communication (radio and radar). Inputs to the simulation model through the on-board and harbor navigational aids and the resulting outputs provide information as to the effectiveness of the danger warnings, and the basis for ultimate decision on their use. For additional comment on these applications refer to the discussion of signaling techniques in Section 7.2.6.

7.2.8 Effectiveness of Aids to Navigation

Navigational aids cover a number of categories which contribute to the smooth and safe flow of ship traffic in defined waterways. The overall navigational effectiveness is based on total evaluation of the items noted below. These items are discussed in the sections referenced in the listing.

Navigational Aid	Section
On-board navigational capability	7.2.5
Human factors	2.3.2a,7.2.3,7.2.5
Lane separation	2.1.3, 7.2.7.2
Traffic advisories	7.2.7.3
Rules-of-the-road	7.2.2
Autopilot concept	7.2.3
Signaling techniques	7.2.6
Traffic control	7.2.7

7.2.9 Effects of Weather

Weather conditions have a distinct effect on a ship's travel. A ship proceeding in a heavy fog faces problems different from those which confront a ship traveling in a high-visibility area. Similarly, under conditions of high tide and strong wind, a ship encounters possible changes in ship's velocity and ship's heading. Inputs to the simulation model representing these conditions will be reflected in model output data which compensate for these conditions.

7.2.10 Effect of Component Variations

Component failure on a ship affects the ship's safety in varying degrees. For example, a defective power component which causes the ship's speed to be reduced would offer a relatively small safety problem. On the other hand, a defective gyrocompass could give erroneous bearing indications and cause errors in the travel path, with accompanying increased probability of collision.

A simulation model with representative inputs simulating component failure can evaluate the relative total ship safety for different component failures.

7.2.11 Interactive Mode Test Application

a. This method of testing permits an operator to exercise external control of a program already being processed in the simulation model, and to observe on a CRT (cathode ray tube) display initial conditions as well as results obtained from input modifications. For example, a CRT display initially shows a ship traveling in a relatively free waterway area. A number of ships then converges into the area, presenting an increased probability of collision. A keyboard operator can input to the simulation model a turn command or a change course command in a direction away from the incoming ships. The result of either command will show on the CRT as a successful maneuver, depending on the adequacy of the input command for the given condition.

b. Because the interactive mode implies external control of active programs, the possibility exists for its use in human factor testing, and in determination of pilot and traffic controller qualifications.

(1) Human factor input can be represented by omission of a command or by delay in applying it, with corresponding results observable on the CRT display. A human factor element can also be evidenced by earlier insertion of a command about to be applied in the original program, or by application of a command which provides an output in excess of what was anticipated in the original program test run.

(2) Pilot commands are inserted into the simulation model with varying degrees of ship control. The effectiveness of these inputs in guiding a ship into a harbor area under different conditions of ship population is indicated on a CRT display.

(3) Traffic control can be simulated by introducing commands which will reroute ships, change the ship population for given intervals, control lane separation between ships, change course for selected ships, alter ship speeds, and set up danger warnings in the waterway. The results of these actions are also indicated on a CRT display. Interpretation and decision-making for actual physical conditions are based on these CRT displays for controlled inputs and resulting outputs.

8. MODEL VALIDATION

8.1 ISSUES RELATED TO VALIDATION

The considerations associated with model validation are discussed in this section according to the following topics:

a. Validation of modeling techniques versus validation of technical components.
b. Intent of validation and limitations.
c. Validation techniques used, and scope and results of validation tests.
d. Recommendations.

The extent and significance of the validation performed by TSC are based on the efforts and accomplishments of TSC, and on the factors which were present for the partial validation actually performed.

8.1.1 Technical Components Versus Modeling Techniques Validation

8.1.1.1 Technical Components - Overall validation of the model must include information on the accuracy of ship motion equations as well as specific data on aspects of ship control. Concerning TSC efforts with respect to the above, the ship motion equations were supplied by the United States Coast Guard, and have not been validated by TSC. Also, validation has not been performed on the human factor index, navigational capability (external to ship) index, or on-board-ship navigational capability index. (The potential for large variations in indices for different ships and waterways and time/cost considerations were factors involved.) In order that meaningful numbers can be assigned to the indices, the capabilities and limitations of the navigational aids (on-board and external to the ships) and the ships' crews must be investigated.

8.1.2 Modeling Techniques

a. The model was used in a simulation of ship motion (in the San Francisco harbor) along tracks of ships in accordance with radar photographs (Figures 8-1 through 8-13) furnished TSC by the United States Coast Guard.

Note, the figures show the paths taken by ships involved in a collision which occurred in November 1971 in the San Francisco harbor. With reference to the figures, the ships which collided are the SS Transchamplain and the SS Persepolis. The other two ships, SS Ryushu Maru and SS Louisiana, are incidental to the ship maneuvering involved during the time interval leading up to the collision. (A report of this collision is available as follows: Report No. 5943/C-155-72/WGW of 5 April 1972, obtainable from OCMI-Commanding Officer, USCG Marine Inspection Station B, Box 2029, San Francisco, California 94216.)

b. The simulation was performed without knowledge of the speeds or precise turning maneuvers of the involved ships. Therefore, this simulation cannot be considered an actual validation of the model. In following the ship tracks with arbitrary inputs of speed and turn maneuvers, the model has proved its ability to move ships along tracks which could be made to agree with real world data. In this sense, the procedure described resembles more a model calibration than a model validation.

8.1.3 Intent and Limitations

The intent of TSC was to obtain correlation between the real world and the model. The actual effort was limited by the lack of precise data and the lack of time and money which would have been required for achieving full validation.

8.1.4 Techniques Used, and Results Obtained

The approach adopted was the one described in Sections 8.1.1 and 8.1.2. The results obtained were partial, and at some variance with the original intent of validation. Specifically, because of

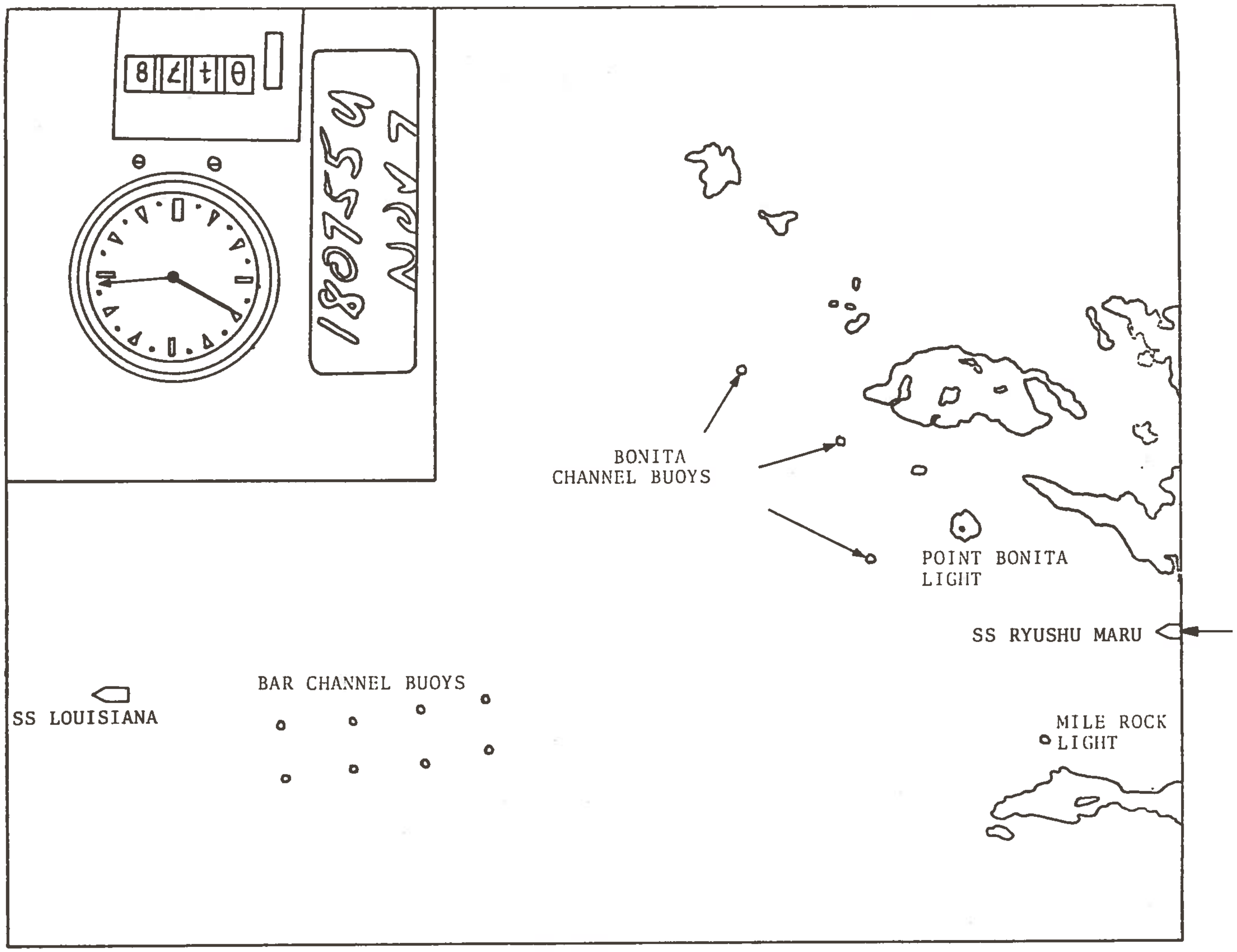


Figure 8-1. SS RYUSHU MARU Entering Picture at Arrow, Outbound

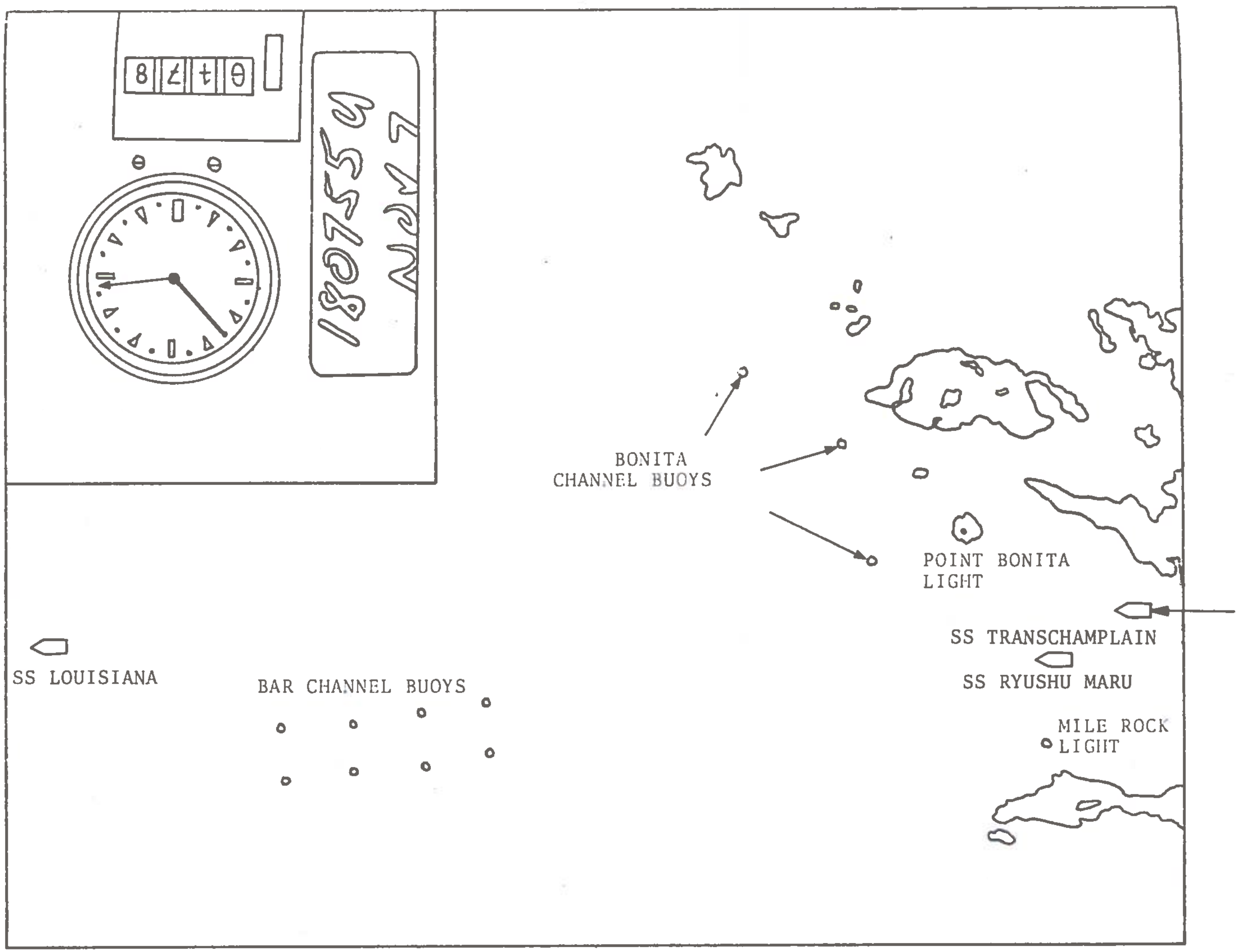


Figure 8-2. SS TRANSCHAMPLAIN Entering Picture at Arrow, Outbound, Following SS RYUSHU MARU

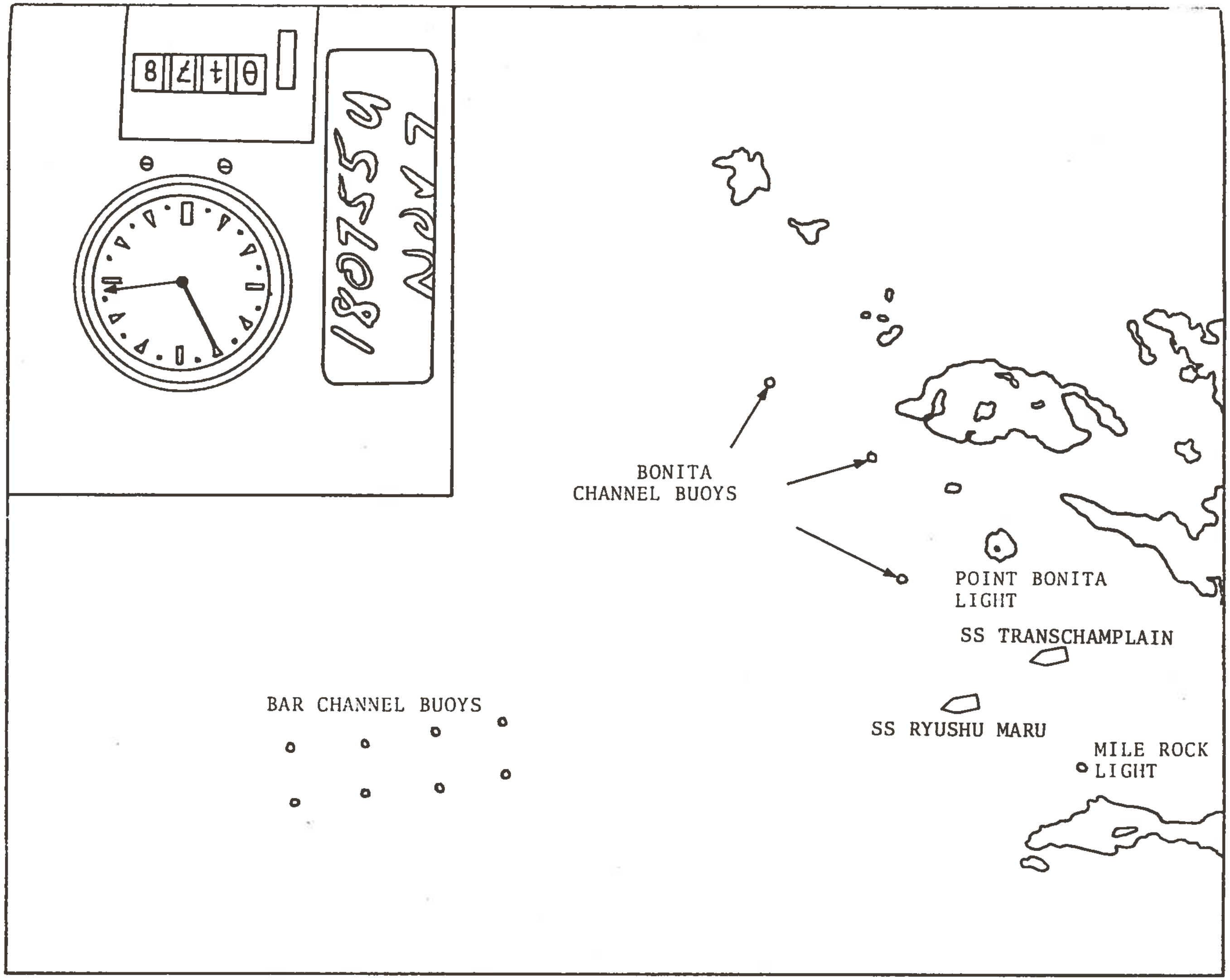


Figure 8-3. SS RYUSHU MARU and SS TRANSCHAMPLAIN Entering International Waters

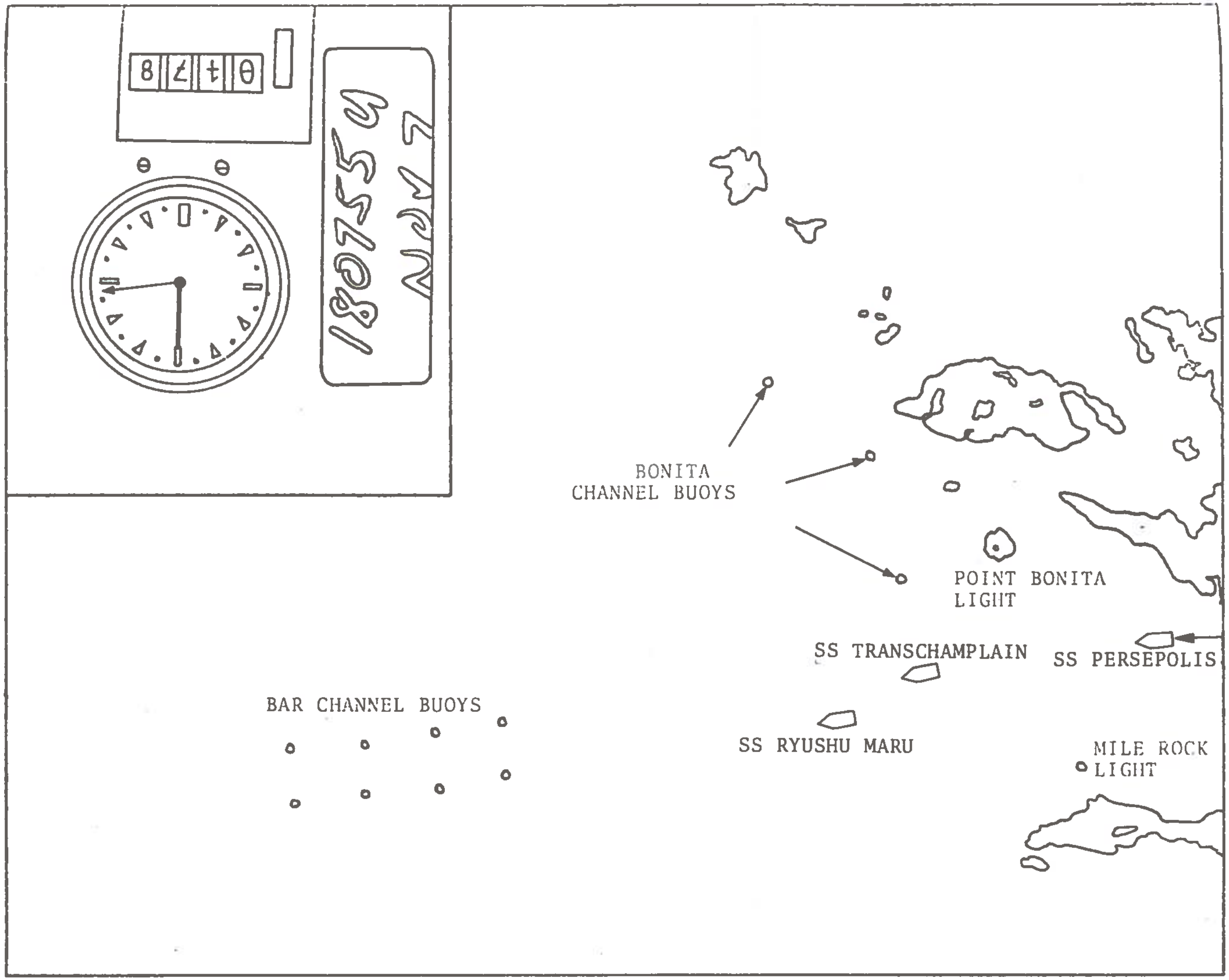


Figure 8-4. SS PERSEPOLIS Entering Picture, Outbound, at Arrow

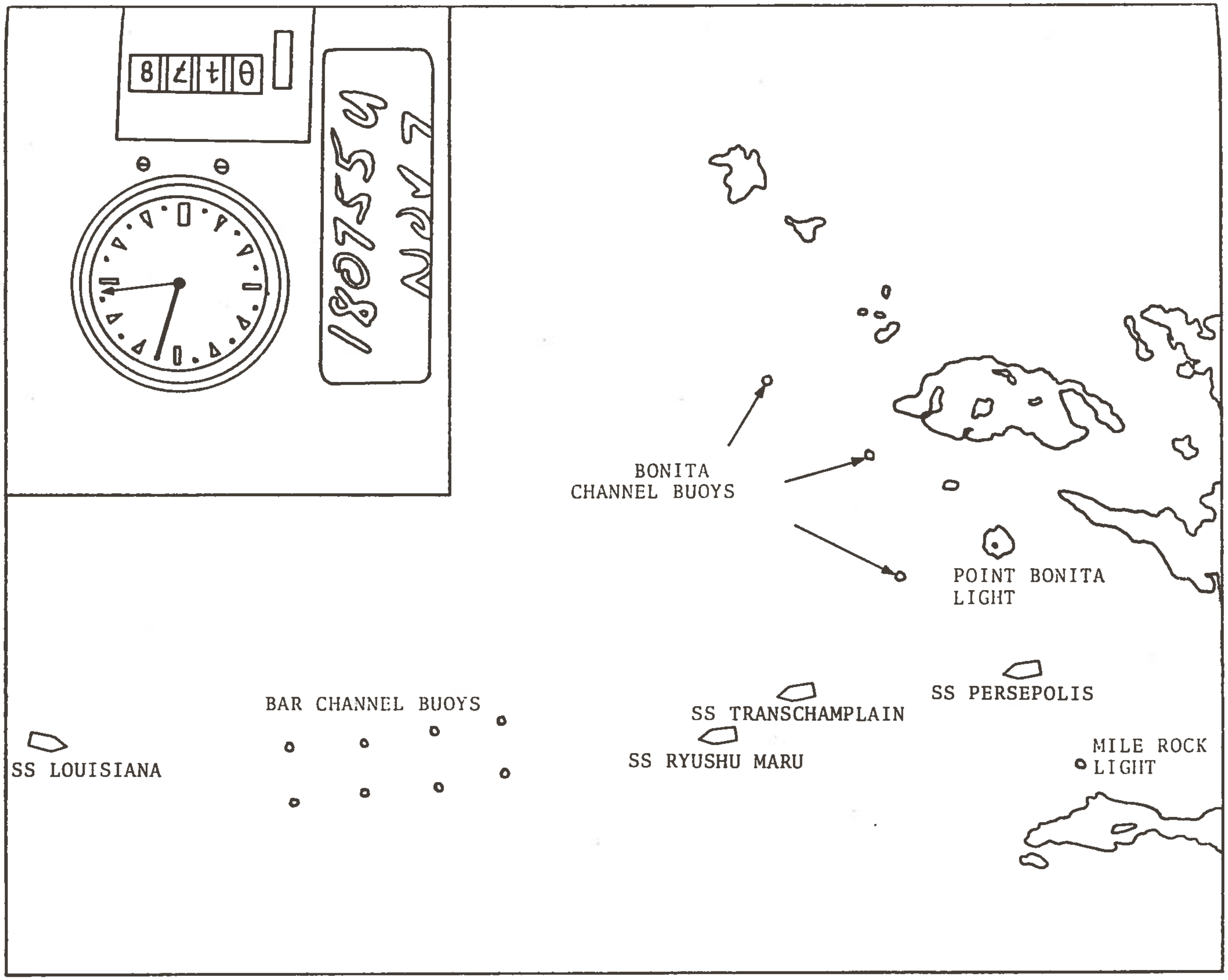


Figure 8-5. SS PERSEPOLIS Entering International Waters, Behind SS RYUSHU MARU and SS TRANSCHAMPLAIN

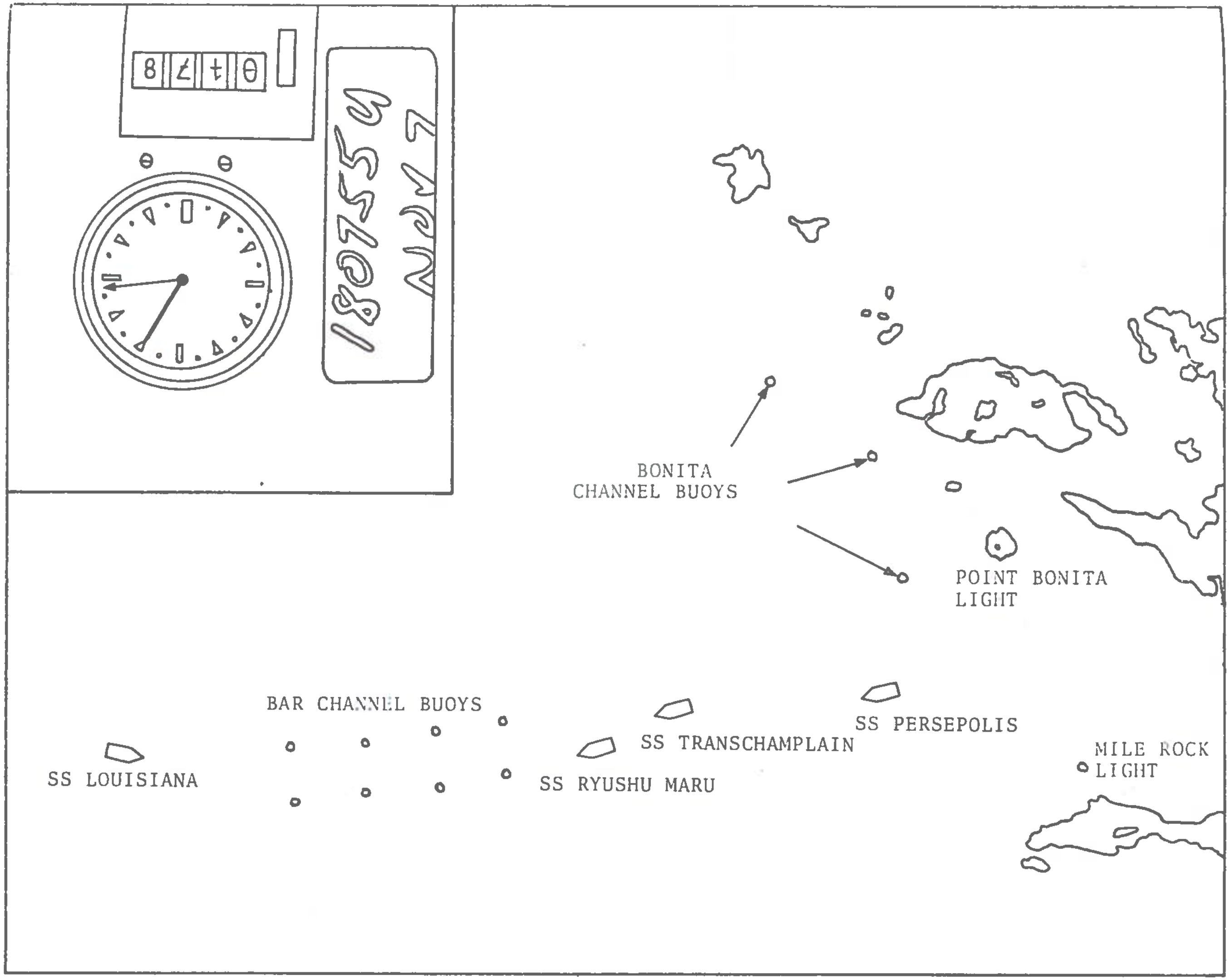


Figure 8-6. All Vessels Proceeding Towards Main Ship Channel, SS PERSEPOLIS Closing

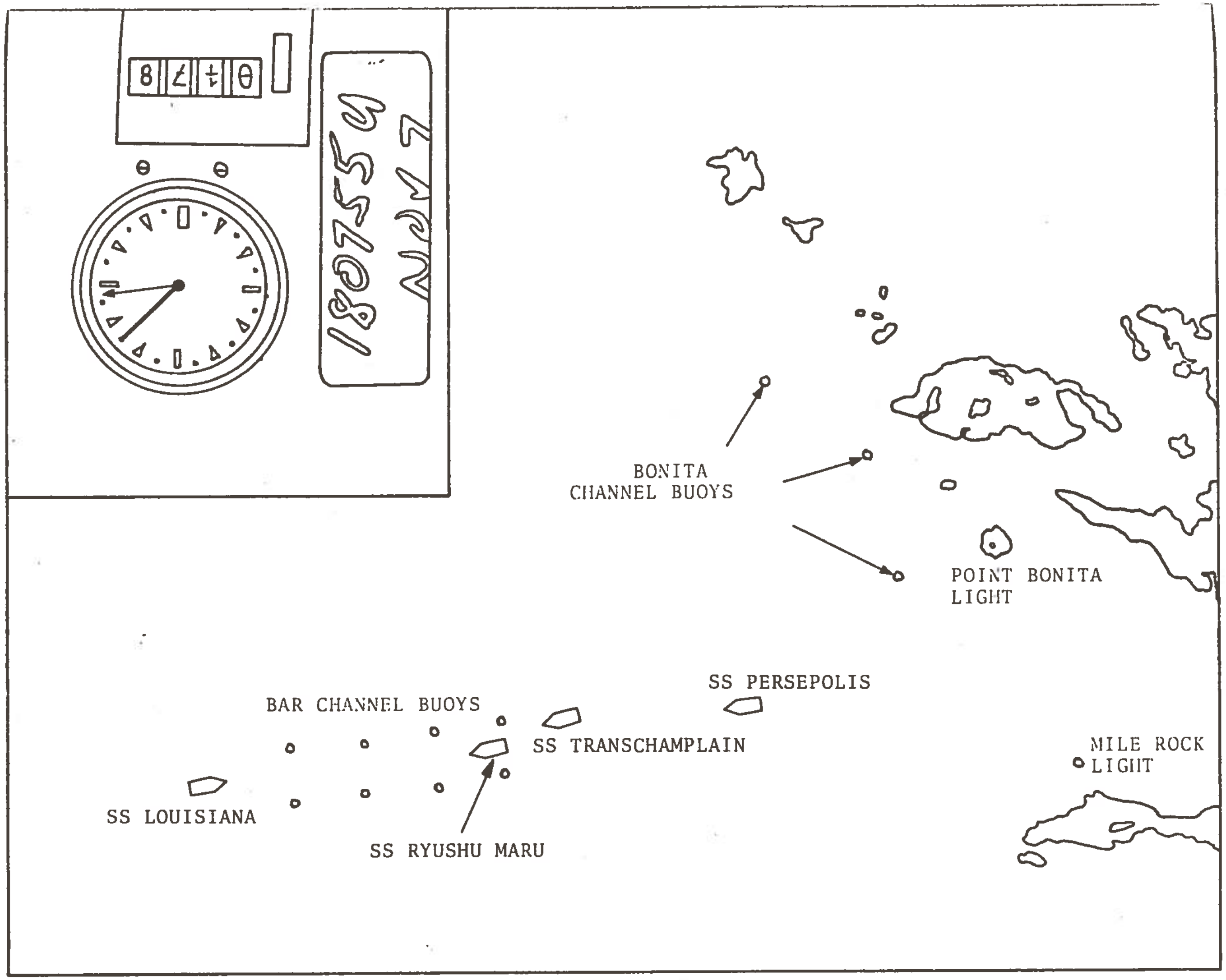


Figure 8-7. SS RYUSHU MARU Entering Main Ship Channel, Outbound,
SS LOUISIANA Inbound

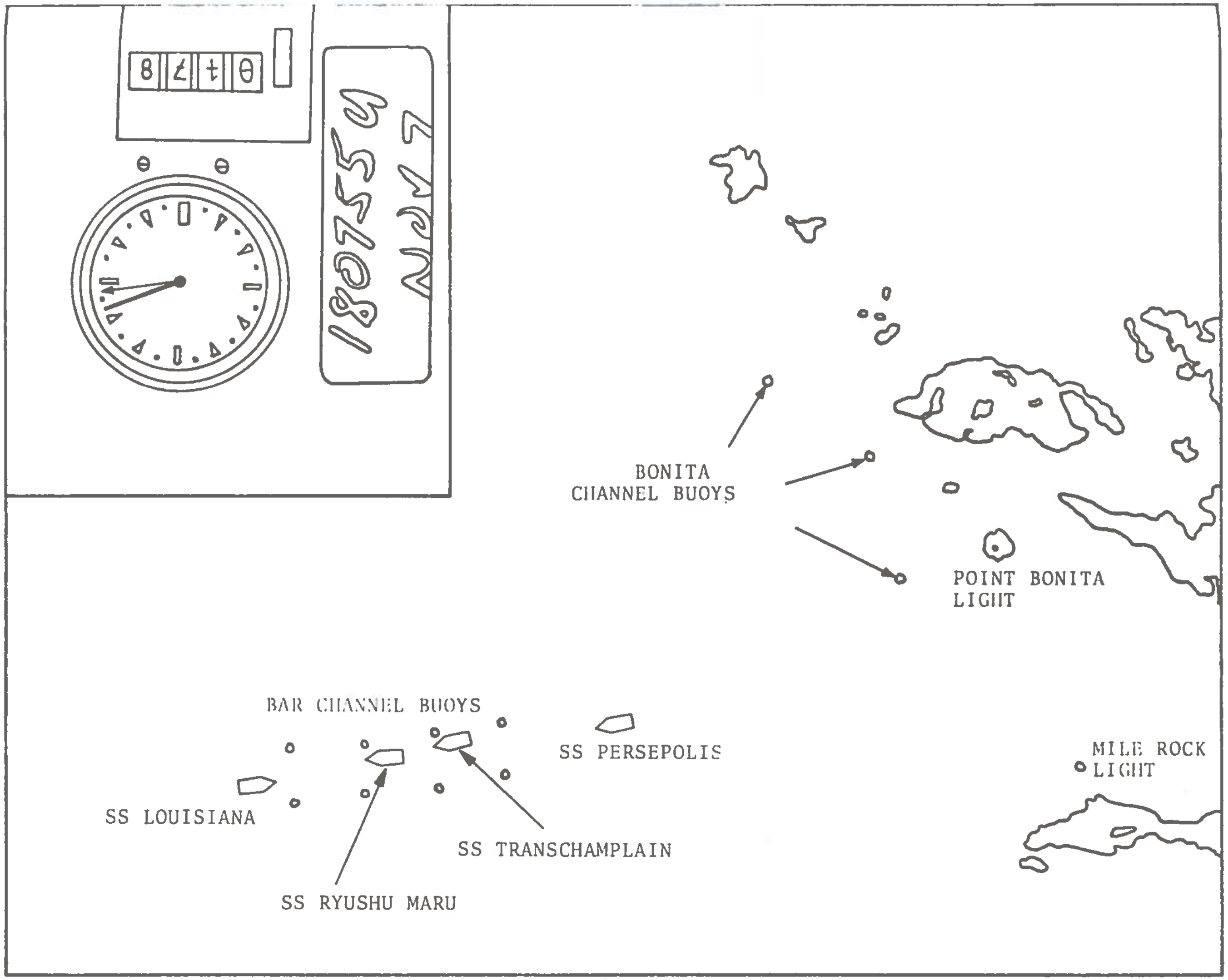


Figure 8-8. SS RYUSHU MARU and SS TRANSCHAMPLAIN in Channel, SS PERSEPOLIS Closing, SS LOUISIANA Inbound

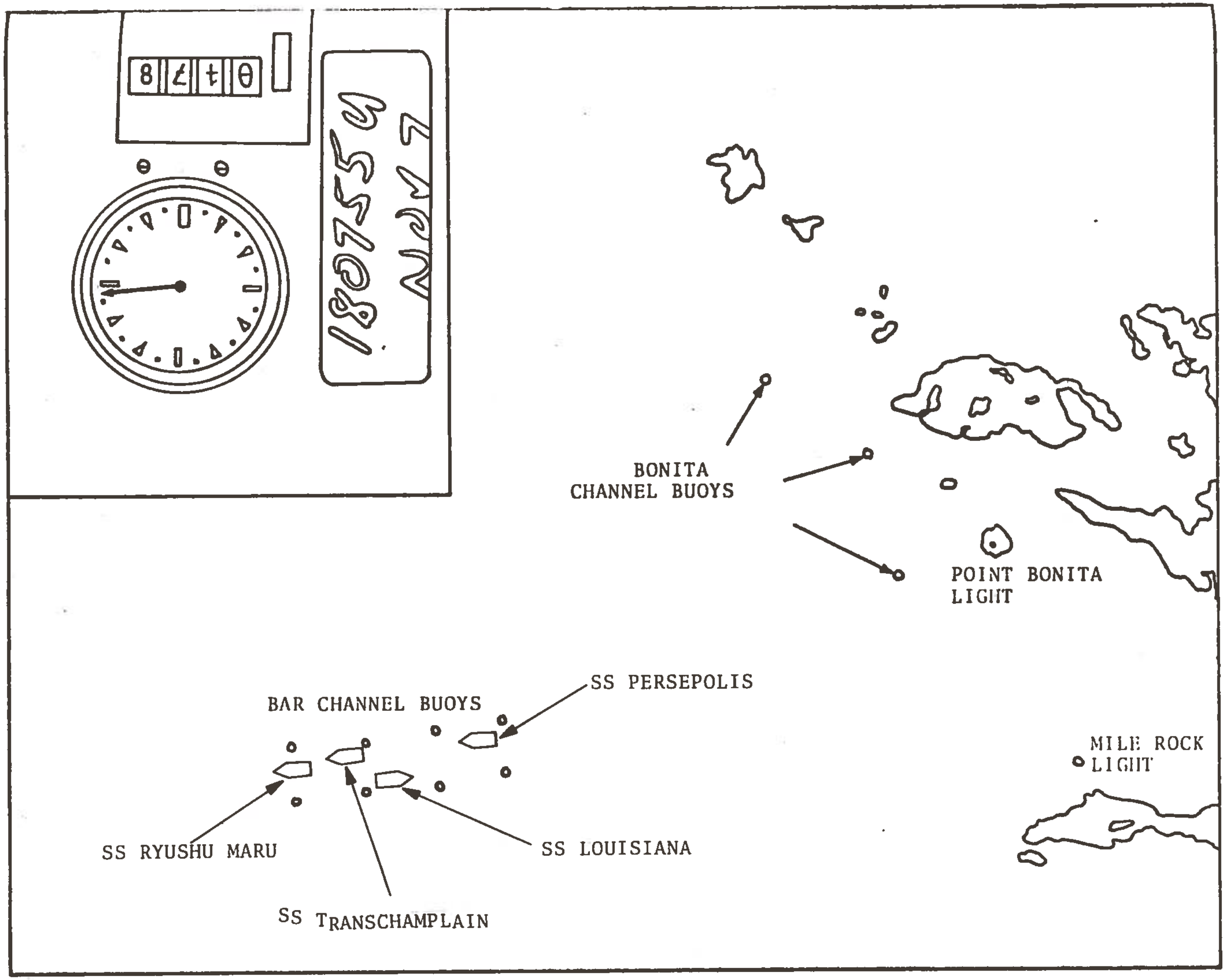


Figure 8-9. All Vessels in Main Ship Channel

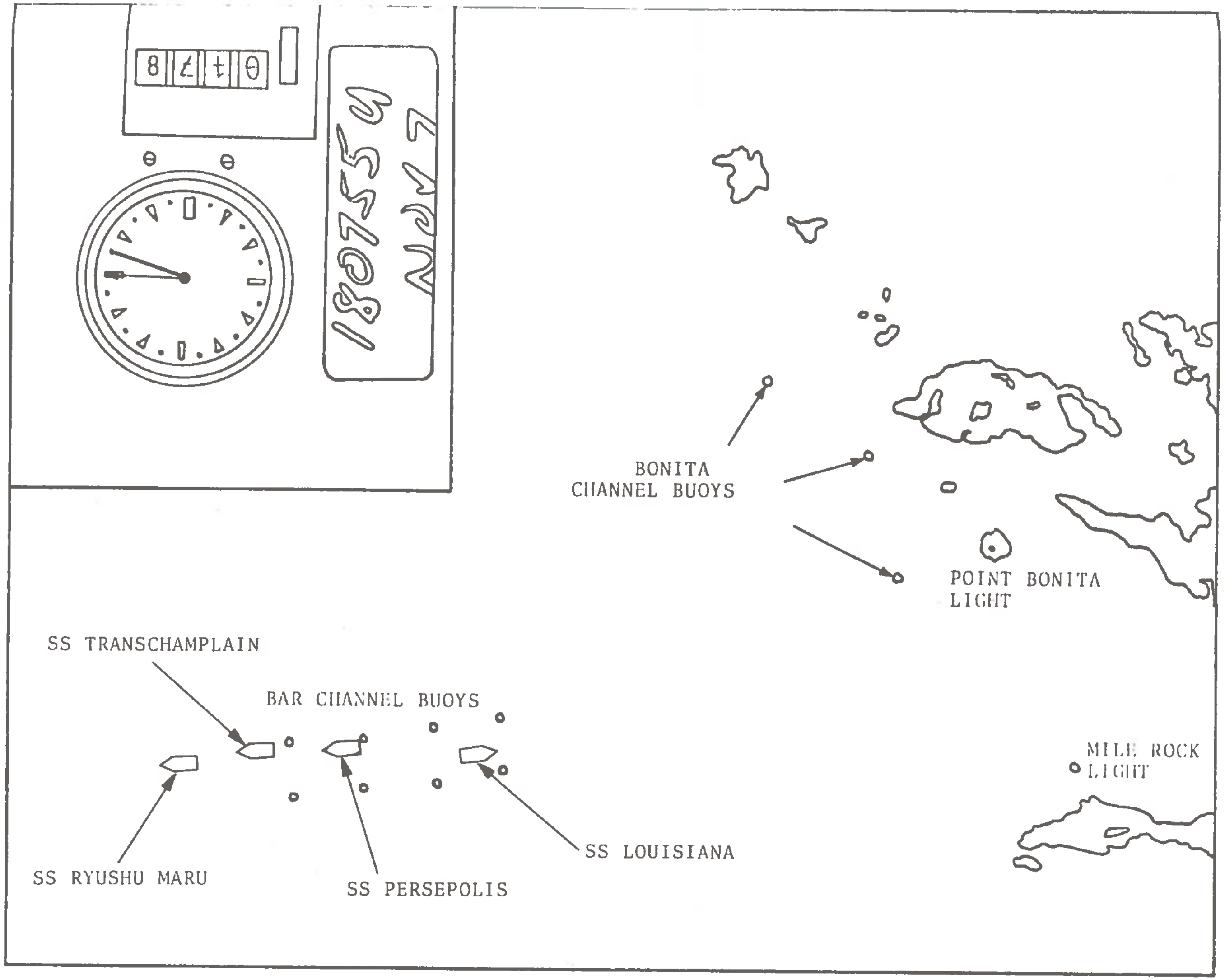


Figure 8-10. SS LOUISIANA About to Leave Ship Channel, Inbound

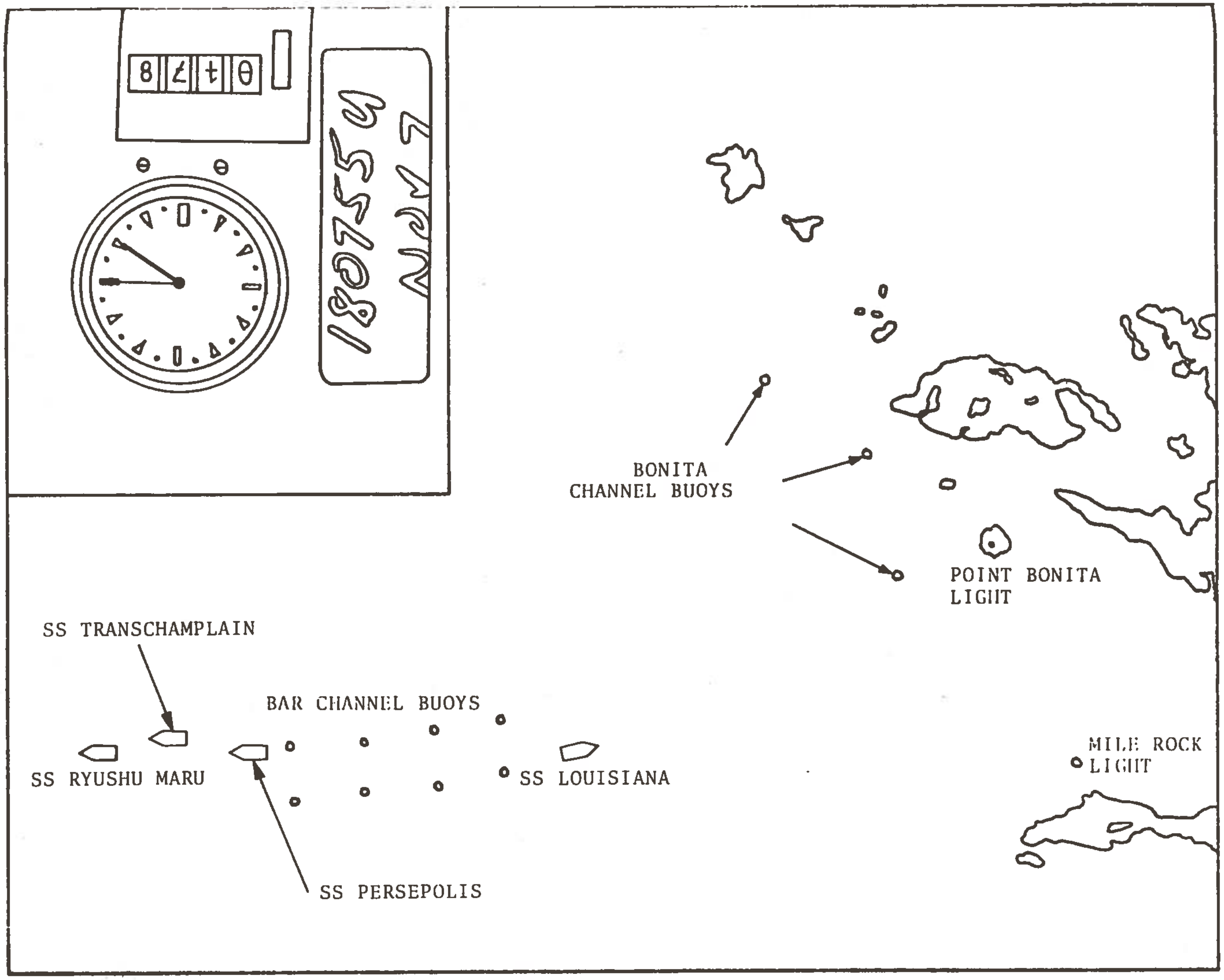


Figure 8-11. All Vessels Leaving Main Ship Channel

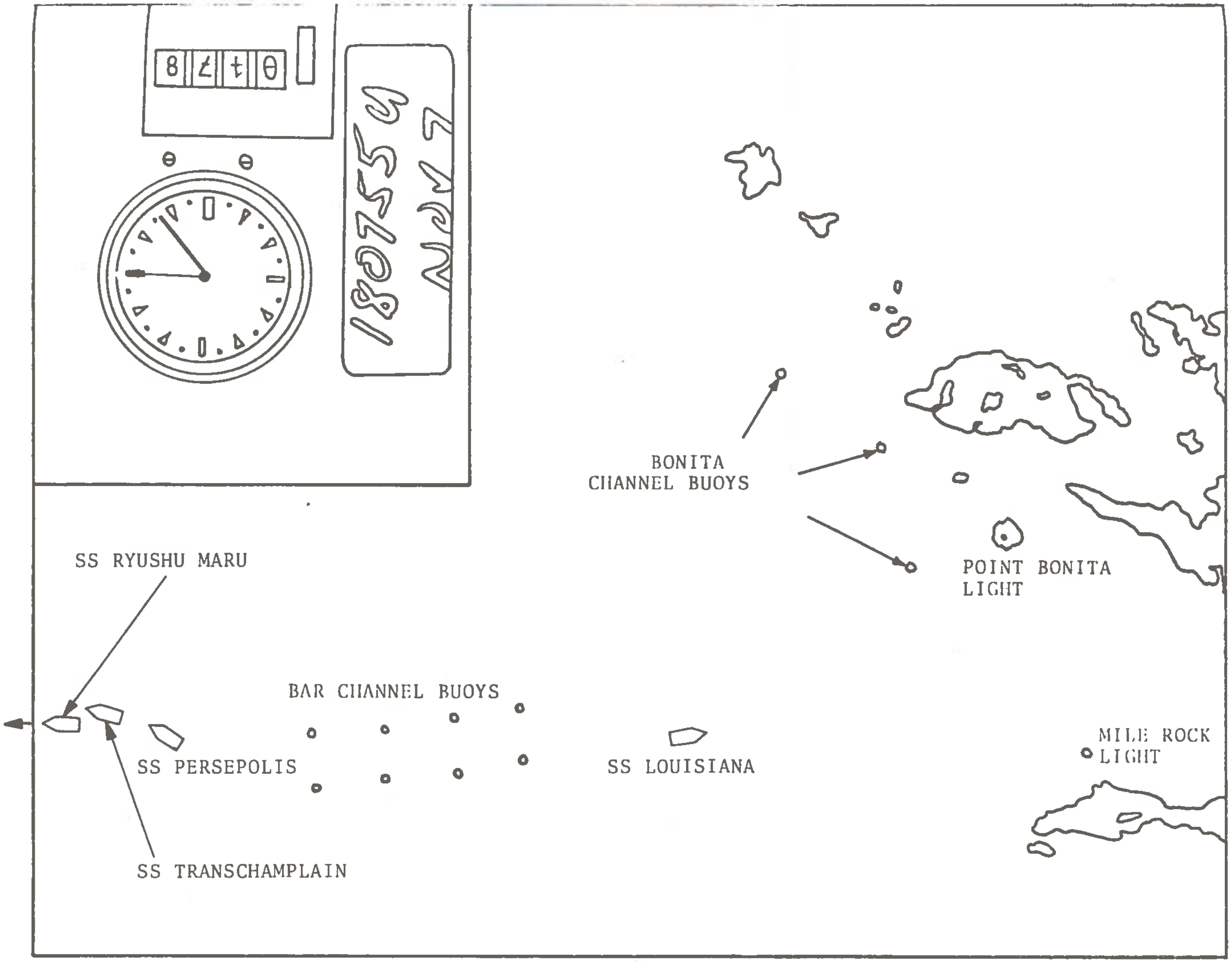


Figure 8-12. SS RYUSHU MARU Leaving Picture, Outbound, at Arrow

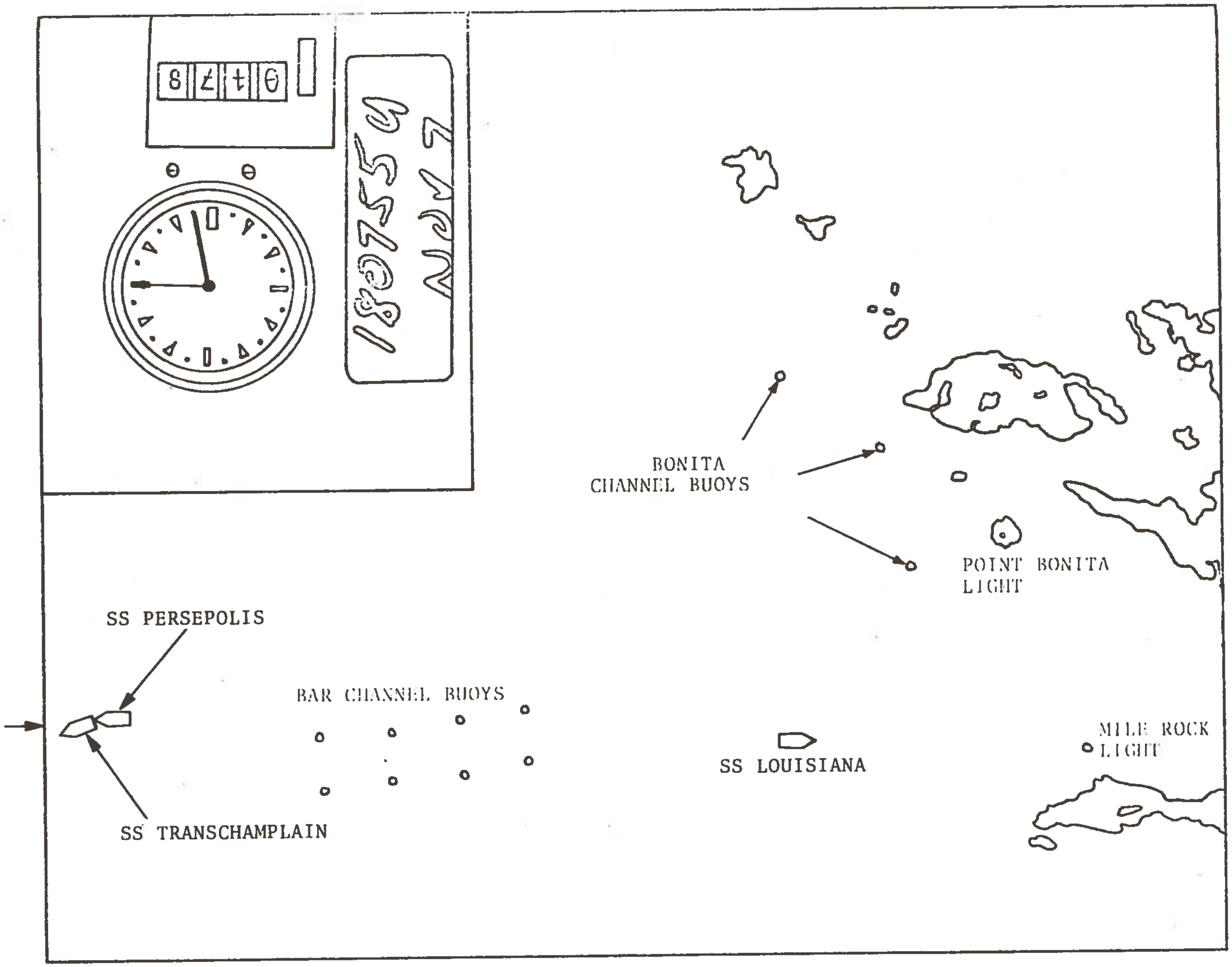


Figure 8-13. SS TRANSCHAMPLAIN and SS PERSEPOLIS at Arrow

the limitations already mentioned in Sections 7.1.1 and 9.1.2, the end result was in the nature of a model calibration rather than a model validation.

8.2 RECOMMENDATIONS

It is recommended that, as a near-term future undertaking, effort be made to establish a thorough validation of the model. A partial validation may be performed on San Francisco Harbor, where a large bank of data already exists because of the San Francisco Harbor radar and standard watch data. For example, the standard watch data cards contain such information as the date, the name of the ship, its origin and destination, and its time in and its time out of the harbor, as well as the existing visibility conditions. While this information does not provide enough information (for example, ships' speeds and turning maneuvers are not provided) for complete validation, enough information is available for at least a partial validation. To obtain even this partial validation, it will be necessary to obtain the ship characteristics for all ships listed on the data cards. While this type of information is not always easy to obtain for all ships, it is available, and is necessary as input for the model-validation test.

As a next step in the validation process, it is suggested that an attempt be made to reproduce mathematically the radar scope tracks of ships traversing the San Francisco Harbor area. A picture of the radar scope is taken automatically every three minutes, providing a very useful recording of the harbor traffic. Since the date of the event is recorded on the film strip, it is not difficult to obtain the information necessary to identify the ships recorded on the film strip. Partial model validation will then come from a successful comparison between the calculated and film recorded ship tracks. Full validation is possible if, in addition, speeds and turning maneuvers could be obtained.

The radar scope film provides an even more interesting possibility if the particular parts of the film which depict ship collisions, groundings, or near misses are used. Then, if the ship tracks and ship maneuvers recorded on the film can be successfully reproduced by the math model, the ability of the model to simulate mathematically the collision, grounding, or near miss will have been demonstrated. (TSC obtained its partial validation in this way. See Section 8.1.) It is suggested that this procedure be continued for further and more complete model validation.

Note, however, that even this validation is only partial because of the lack of sufficient resolution from the radar scope. Greater resolution could be obtained from a set of high-resolution aerial photographs of ships traversing the harbor. These would not be difficult to obtain, as experience in the area of aerial photography of vehicles on the highways could be called upon. Ship speeds and locations could be determined through data reduction of the photographs with very high accuracy; use of these would provide the higher resolution needed to set proper limits on the ability of the model to reproduce faithfully the given ship motion with its approximate ship motion equations.

With limits on the accuracy of the equations of motion obtained, the next important project is to establish specific values for the nav-aid and crew ability indices used in the model. It is important, for the collision probability and safety analysis, to know the uncertainty of a ship's position under various visibility, harbor and on-board nav-aid conditions. Since the prime use of the model is safety analysis, and since knowledge of ship position uncertainty is important for such analysis, it is desirable that an early effort be initiated to obtain these data.

For additional analytic effort, it is suggested that the following improvements be added:

a. A directional capability should be incorporated into the collision analysis. The collision probability calculation now assumes a three-sigma ellipse in which the uncertainty in position due to nav-aid deficiencies is directionally symmetric with respect to ship motion. This should be made more realistic to allow the model to differentiate between lateral and longitudinal ship position uncertainties. In order to do this, the ellipse of uncertainty must be rotated for the direction of the ship's motion. One way of doing this would be to define another coordinate system which moves with the ship, and to rotate the ellipse from the harbor coordinates to the frame of reference of the ship.

b. The model does not now distinguish between types, and, therefore, between severity of collisions (broadside, head-on collisions, etc.). For a more realistic assessment of accident damage, the ability to distinguish between types of collisions should be added to the model. Together with the existing probability of collision calculation, a good picture of total damage cost would then be obtained for the particular traffic pattern under investigation.

c. The method for calculating collision and grounding probabilities now used in the model should be considered simply as one of a number of different possible methods, and not necessarily the best one. It is suggested that other possible methods be analyzed for their relative merit with respect to the present method. Through such analysis it can be determined whether it is justified to insert one of them into the model as replacement for the present probability analysis.

In conclusion, the authors believe the present model to be a useful and efficient model for the mathematical simulation of ship movement through defined waterways, and to have potential use for safety analyses and other applications.

INTRODUCTION TO APPENDICES

This section of Volume 1 provides appendices which serve as mathematical supplements to the qualitative discussions of Sections 2 through 8. Mathematical expressions are derived for ship motion under various conditions of water and air, and for probability of collisions and groundings. When the proper data are applied to the equations formulated in these appendices, and the equations with the data are applied to the simulation model, the calculations performed in the model furnish the outputs for interpretation and decision-making frequently referred to in the first seven sections.

GLOSSARY OF PARAMETERS IN APPENDICES

A glossary is provided for parameters used in the ship motion equations derived in the appendices. The parameters are listed and defined (or referenced) under the same appendix and section headings in which they appear in the text. Therefore, only a portion of the glossary is consulted when a particular set of equations is being analyzed.

GLOSSARY

A.1 GENERAL CONSIDERATIONS

$[p_x(t), p_y(t)]$	x and y components of ship position at time t
$v(t)$	Ship velocity at time t
$\theta(t)$	Ship heading at time t
$v_x(t)$	x-component of ship velocity at time t
$v_y(t)$	y-component of ship velocity at time t
τ	Time-step interval
$[p_x(t+\tau), p_y(t+\tau)]$	Ship position at time (t+τ)
$v_{sx}(t)$	x-component of ship's velocity in still water and air
$v_{sy}(t)$	y-component of ship's velocity in still water and air
$v_{twx}(t)$	x-component of ship's velocity due to tide and wind
$v_{twy}(t)$	y-component of ship's velocity due to tide and wind

A.2 TIDE AND WIND CALCULATIONS

v_{win}	Wind velocity
θ_{win}	Wind heading
$v_{wx}(t)$	x-component of ship's velocity due to wind
$v_{wy}(t)$	y-component of ship's velocity due to wind
SC_8	Wind sail ship's characteristic
$\theta(t)$	Ship heading at time t
v_{tid}	Tide velocity
θ_{tid}	Tide heading

GLOSSARY (CONT'D)

A.2 TIDE AND WIND CALCULATIONS (CONTINUED)

$v_{tx}(t)$	x-component of ship's velocity due to tide
$v_{ty}(t)$	y-component of ship's velocity due to tide
$v_{twx}(t)$	See definitions in A.1
$v_{twy}(t)$	

A.3 MOTION OF SHIP IN STILL WATER AND AIR

$v_s(t)$	Ship's velocity in still water and air at time t
$\theta_s(t)$	Ship's heading in still water and air at time t
$v_s(s)$	Ship's velocity in still water and air at time s
$\theta_s(s)$	Ship's heading in still water and air at time s
$v_s(t+\tau)$	Ship's velocity in still water and air at time $(t+\tau)$
$\theta_s(t+\tau)$	Ship's heading in still water and air at time $(t+\tau)$
$v_{sx}(t+\tau)$	x-component of ship's velocity in still water and air at time $(t+\tau)$

A.3 MOTION OF SHIP IN STILL WATER AND AIR (CONTINUED)

$v_{sy}(t+\tau)$	y-component of ship's velocity in still water and air at time $(t+\tau)$
$\theta(t+\tau)$	Ship's heading at time $(t+\tau)$
cst	Ship's acceleration during time interval between t and $(t+\tau)$
T_R	Ship characteristic indicating time delay (during deceleration) associated with reversing ship's engine
R_e	A constant relating speed squared to the resistive force
M_e	Effective mass of ship
T_{HR}	Reverse thrust generated by ship's propeller during deceleration mode
con	Constant-equal to $(SC_{11} \cdot SC_{13})^{-1}$
SC_8	Ship characteristics or expressions derived from ship characteristics. (See Appendix B.)
SC_{10}	
SC_{11}	
SC_{13}	
SC_{15}	
SC_{16}	
SC_{17}	
SC_{19}	
SC_{20}	
θ_1	Integration constant determined by initial conditions. Appears in velocity expression for ship in deceleration mode.

GLOSSARY (CONT'D)

A.3 MOTION OF SHIP IN STILL WATER AND AIR (CONTINUED)

ADV	Distance, in feet, which a ship must travel before a turn is taken and the ship's rudder responds to the turn command
$v_s(t_i)$	Ship's initial velocity before performing turn maneuver
v_T	Turning velocity
R	Turning radius
$\theta_s(t+\tau)$	Ship's heading in still water and air at time $(t+\tau)$

NOTE

The following parameters occur frequently in Section A.3, and are defined under headings A.1 and A.2 of the glossary.

$p_x(t)$	See definitions in A.1
$p_y(t)$	
$\theta(t)$	
τ	
$p_x(t+\tau)$	
$p_y(t+\tau)$	
$v_{sx}(t)$	
$v_{sy}(t)$	
θ_{win}	See definitions in A.2
v_{tid}	
θ_{tid}	

APPENDIX A

SHIP MOTION EQUATIONS: STEP-BY-STEP CALCULATIONS

A.1 GENERAL CONSIDERATIONS

The ship motion equations are used in the simulation model for determining, from a ship's present position in its travel, the next position after a given time-step interval. For each of a number of ships traveling in a defined waterway, the position, initial speed, and heading are inserted into the simulation model to simulate an initial position in the ship's travel. Specific traffic routes, or alternatively, commands are given to the ships. The ships then move according to the specified track or commands. This ship movement is based on the equations of motion used in the model (and derived in the following analyses), taking into account the existing wind and tide conditions inserted into the model. In this way the next position of a ship is calculated, at each time-step, on the basis of current conditions influencing the ship.

A.1.1 Calculation of Step-by-Step Position at Discrete Time-Steps

Consider the harbor of Figure A.1-1 in the following analysis:

Let $[p_x(t), p_y(t)]$ = ship position at time t

$v(t)$ = ship velocity at time t

$\theta(t)$ = ship heading at time t

$v_x(t)$ = x-component of ship velocity at time t

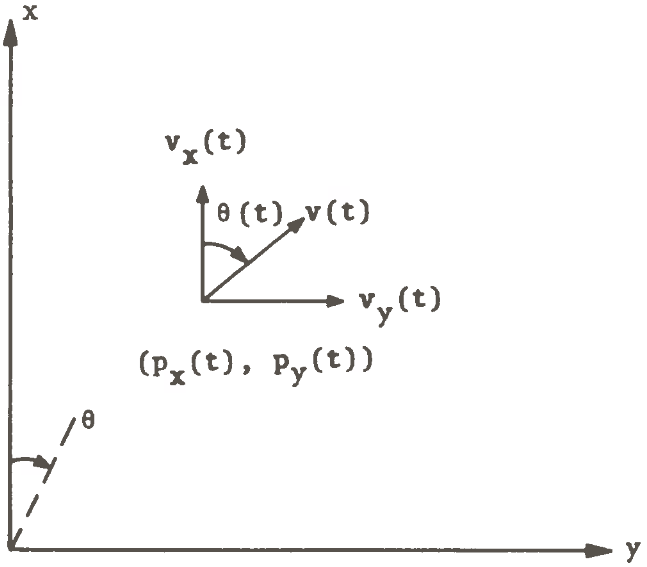
$v_y(t)$ = y-component of ship velocity at time t

τ = time-step interval from time t

$[p_x(t+\tau), p_y(t+\tau)]$ = ship position at time $(t+\tau)$

$v(s)$ = ship velocity at time s

$\theta(s)$ = ship heading at time s



NOTES:

1. ALL VELOCITIES ARE IN FT/SEC.
2. ALL ANGLES ARE IN RADIANS

Figure A.1-1. Ship Parameters in Harbor Area

From Figure A.1-1,

$$v_x(t) = v(t) \cos \theta(t) \quad \text{(A.1-1)}$$

$$v_y(t) = v(t) \sin \theta(t). \quad \text{(A.1-2)}$$

Then, for a new ship's position after a time-step τ of travel $(t+\tau)$, the following equations apply:

$$p_x(t+\tau) = \int_t^{t+\tau} v(s) \cos \theta(s)\, ds + p_x(t) \quad \text{(A.1-3)}$$

$$p_y(t+\tau) = \int_t^{t+\tau} v(s) \sin \theta(s)\, ds + p_y(t). \quad \text{(A.1-4)}$$

Since, by analogy with equations A.1-1 and A.1-2,

$$v(s) \cos \theta(s) = v_x(s) \tag{A.1-5}$$

$$v(s) \sin \theta(s) = v_y(s), \tag{A.1-6}$$

equations A.1-3 and A.1-4 are expressed as

$$p_x(t+\tau) = \int_t^{t+\tau} v_x(s)\, ds + p_x(t) \tag{A.1-7}$$

$$p_y(t+\tau) = \int_t^{t+\tau} v_y(s)\, ds + p_y(t). \tag{A.1-8}$$

A.1.2 Calculation of Ship Position Relative to Tide and Wind

The x- and y-components of ship's velocity at time t [$v_x(t)$ and $v_y(t)$] can further be subdivided to account for weather parameters as noted below.

Let $v_{sx}(t)$ = x-component of ship's velocity in still water and air

$v_{sy}(t)$ = y-component of ship's velocity in still water and air

$v_{twx}(t)$ = x-component of ship's velocity due to tide and wind

$v_{twy}(t)$ = y-component of ship's velocity due to tide and wind.

The two components in the respective x- and y-directions are considered additive. Therefore,

$$v_x(t) = v_{sx}(t) + v_{twx}(t) \tag{A.1-9}$$

$$v_y(t) = v_{sy}(t) + v_{twy}(t). \tag{A.1-10}$$

By use of the relationships in Equations A.1-9 and A.1-10 with respect to variable s, Equations A.1-7 and A.1-8 become

$$p_x(t+\tau) = \int_t^{t+\tau} \left[v_{sx}(s) + v_{twx}(s)\right] ds + p_x(t) \qquad \text{(A.1-11)}$$

$$p_y(t+\tau) = \int_t^{t+\tau} \left[v_{sy}(s) + v_{twy}(s)\right] ds + p_y(t). \qquad \text{(A.1-12)}$$

A.2 TIDE AND WIND CALCULATIONS

The combined effect of wind and tide on the ship's velocity, [$v_{twx}(s)$ and $v_{twy}(s)$ of Equations A.1-11 and A.1-12] in determining the position of a ship between times t and $(t+\tau)$ is converted to the sum of the respective individual effects. When this is done, the equations obtained for the position of a ship at time $(t+\tau)$ are expressed in accordance with the results of the following analysis.

a. Consider first the effect of wind velocity and wind heading on the ship's velocity in the x- and y-directions.

Let v_{win} = wind velocity
θ_{win} = wind heading
$v_{wx}(t)$ = x-component of ship's velocity due to wind
$v_{wy}(t)$ = y-component of ship's velocity due to wind

The ship's velocity in the x- and y-directions due to the wind velocity and wind heading is expressed in Equations A.2-1 and A.2-2.

$$v_{wx}(t) = \left[SC_8(0.0038)\cdot v_{win}^2\cdot\left|\sin(\theta_{win}-\theta(t))\right|\right]^{1/2} \cos\theta_{win}$$

(A.2-1)

$$v_{wy}(t) = \left[SC_8(0.0038)\cdot v_{win}^2\cdot|\sin(\theta_{win}-\theta(t)|\right]^{1/2}\cdot\sin\theta_{win} \qquad \text{(A.2-2)}$$

where SC_8 = wind sail ship's characteristic defined as the ratio of the topside area to the side underwater area. (Ship's characteristics are defined in Appendix B.)

b. Consider next the effect of tide velocity and tide heading on the ship's velocity in the x- and y-directions.

Let v_{tid} = tide velocity

θ_{tid} = tide heading

$v_{tx}(t)$ = x-component of ship's velocity due to tide

$v_{ty}(t)$ = y-component of ship's velocity due to tide.

The ship's velocity in the x- and y-directions due to the tide velocity and tide heading is expressed in Equations A.2-3 and A.2-4.

$$v_{tx}(t) = v_{tid}\cdot\cos\theta_{tid} \qquad \text{(A.2-3)}$$

$$v_{ty}(t) = v_{tid}\cdot\sin\theta_{tid} \qquad \text{(A.2-4)}$$

c. Assume that the wind and tide effects are additive. Then the combined effect of wind and tide on ship's velocity in the x- and y-direction is given by the following equations for $v_{twx}(t)$ and $v_{twy}(t)$.

$$\begin{aligned} v_{twx}(t) &= v_{tx}(t) + v_{wx}(t) \\ &= v_{tid}\cdot\cos\theta_{tid} + \left[SC_8(0.0038)\cdot v_{win}^2\cdot\left|\sin\left[\theta_{win}-\theta(t)\right]\right|\right]^{1/2} \\ &\quad\cdot\cos\theta_{win} \end{aligned} \qquad \text{(A.2-5)}$$

$$v_{twy}(t) = v_{ty}(t) + v_{wy}(t)$$

$$= v_{tid} \cdot \sin \theta_{tid} + \left[SC_8(0.0038) \cdot v_{win}^2 \cdot \left|\sin\left[\theta_{win} - \theta(t)\right]\right|\right]^{1/2} \cdot \sin \theta_{win} \qquad (A.2\text{-}6)$$

d. By substitution of (s) for (t) in the functional expressions of A.2-5 and A.2-6, the following equations are obtained:

$$v_{twx}(s) = v_{tx}(s) + v_{wx}(s) \qquad (A.2\text{-}7)$$

$$v_{twy}(s) = v_{ty}(s) + v_{wy}(s) \qquad (A.2\text{-}8)$$

The relationships in Equations A.2-7 and A.2-8 are now applied to Equations A.1-11 and A.1-12 for the terms $v_{twx}(s)$ and $v_{twy}(s)$ respectively. The resulting expansion of the latter equations provides expressions (Equations A.2-9 and A.2-10) for ship's position at time $(t+\tau)$ with due consideration for tide and wind effects.

$$p_x(t+\tau) = \int_t^{t+\tau} v_{sx}(s)\, ds + \int_t^{t+\tau} \left[SC_8(0.0038) \cdot v_{win}^2 \cdot \left|\sin\left(\theta_{win} - \theta(s)\right)\right|\right]^{1/2} \cdot \cos\theta_{win}\, ds + \tau \cdot v_{tid} \cos \theta_{tid} + p_x(t) \qquad (A.2\text{-}9)$$

$$p_y(t+\tau) = \int_t^{t+\tau} v_{sy}(s)\, ds + \int_t^{t+\tau} \left[SC_8(0.0038) v_{win}^2 \cdot \left|\sin\left(\theta_{win} - \theta(s)\right)\right|\right]^{1/2} \cdot \sin \theta_{win}\, ds + \tau \cdot v_{tid} \cdot \sin \theta_{tid} + p_y(t) \qquad (A.2\text{-}10)$$

In order to perform the integrations in Equations A.2-9 and A.2-10, expressions are needed for the velocity and heading of the ship in still water. Those expressions are derived in the next sections.

A.3 SHIP MOTION IN STILL WATER AND AIR

The following analysis provides equations for ship's position, ship's velocity, x- and y-components of the ship's velocity, and ship's heading for a ship traveling in still water and air. The equations to be derived are dependent on specific modes, listed below in the sequence in which they are discussed:

Mode 1: Straight line/constant velocity path
Mode 2: Straight line/acceleration path
Mode 3: Straight line/deceleration path
Mode 4: Turning path/constant velocity.

In the following discussions the ship's parameters for velocity and heading are defined as noted below:

$v_s(t)$ = ship's velocity in still water and air

$\theta_s(t)$ = ship's heading in still water and air

[Note: Wind and tide are assumed to induce no change in heading (angle at which the ship meets the wind); i.e., $\theta_s(t)=\theta(t)$ in all the following derivations.]

A.3.1 Mode 1: Straight-Line/Constant Velocity Path

A.3.1.1 Ship's Velocity and Heading Equations for Straight-Line/ Constant Velocity Path - For straight-line travel and constant velocity (i.e., for zero change in heading and zero acceleration),

$$\dot{v}_s(s) = 0 \qquad t \leq s \leq (t+\tau) \qquad \text{(A.3-1)}$$

$$\dot{\theta}_s(s) = 0 \qquad t \leq s \leq (t+\tau). \qquad \text{(A.3-2)}$$

Integrating Equations A.3-1 and A.3-2 provides the equations for ship's velocity and heading for straight-line/constant velocity path in still water and air.

$$v_s(s) = v_s(t) \qquad t \leq s \leq (t+\tau) \qquad \text{(A.3-3)}$$

$$\theta_s(s) = \theta_s(t) \qquad t \leq s \leq (t+\tau). \qquad \text{(A.3-4)}$$

By analogy, the following expressions apply for the x- and y-components of ship's velocity and for ship's heading in the time-step s in still water and air.

$$v_{sx}(s) = v_{sx}(t) \qquad t \leq s \leq (t+\tau) \qquad \text{(A.3-5)}$$

$$v_{sy}(s) = v_{sy}(t) \qquad t \leq s \leq (t+\tau) \qquad \text{(A.3-6)}$$

$$\theta(s) = \theta(t) \qquad t \leq s \leq (t+\tau) \qquad \text{(A.3-7)}$$

Since Equations A.3-3 and A.3-5 through A.3-7 are valid for values of s between t and $(t+\tau)$, then

$$v_s(t+\tau) = v_s(t) \qquad \text{(A.3-8)}$$

$$v_{sx}(t+\tau) = v_{sx}(t) \qquad \text{(A.3-9)}$$

$$v_{sy}(t+\tau) = v_{sy}(t) \qquad \text{(A.3-10)}$$

$$\theta(t+\tau) = \theta(t). \qquad \text{(A.3-11)}$$

A.3.1.2 <u>Ship's Position in Still Water and Air for Straight-Line/ Constant Velocity Path and After Discrete Time-Step τ</u> - Apply the relationships of Equations A.3-5 through A.3-7 to Equations A.2-9 and A.2-10 respectively. Then, by performing the indicated integration for the respective terms of the latter equations, obtain the position Equations A.3-12 and A.3-13 for the ship's position after time-step τ [i.e., at time $(t+\tau)$].

$$p_x(t+\tau) = \tau \left\{ v_{sx}(t) + v_{tid} \cos \theta_{tid} + \left[SC_8 (0.0038) \cdot v_{win}^2 \cdot |\sin(\theta_{win} - \theta(t))| \right]^{1/2} \cos \theta_{win} \right\} + p_x(t) \quad \text{(A.3-12)}$$

$$p_y(t+\tau) = \tau \left\{ v_{sy}(t) + v_{tid} \sin \theta_{tid} + \left[SC_8 (0.0038) \cdot v_{win}^2 \cdot |\sin(\theta_{win} - \theta(t))| \right]^{1/2} \sin \theta_{win} \right\} + p_y(t). \quad \text{(A.3-13)}$$

A.3.2 Mode 2: Straight-Line/Acceleration Path

A.3.2.1 Ship's Velocity and Heading Equations for Straight-Line/Acceleration Path - For straight-line travel and ship acceleration [i.e., for zero change in heading and uniform (constant) acceleration],

$$\dot{v}_s(s) = cst \qquad t \leq s \leq (t+\tau) \qquad \text{(A.3-14)}$$

where $cst = \left[2SC_{14}\right]^{-1}$

and SC_{14} is a ship characteristic which is a function of the ship's weight, its maximum velocity, its ahead horsepower and its propeller efficiency. (Ship characteristics are defined in Appendix B.)

$$\dot{\theta}_s(s) = 0 \qquad t \leq s \leq (t+\tau). \qquad \text{(A.3-15)}$$

Integrating Equation A.3-14 provides the ship's velocity due to the ship's acceleration cst. From Equation A.3-14,

$$v_s(s) = v_s(t) + (s-t)\ cst \qquad t \leq s \leq (t+\tau) \qquad \text{(A.3-16)}$$

which is the total ship velocity for any time s between t and $(t+\tau)$ for a constant ship acceleration cst and initial ship velocity $v_s(t)$.

Since the rate of change of heading is zero, the heading remains at its initial value $\theta_s(t)$. Therefore,

$$\theta_s(s) = \theta_s(t) \qquad t \leq s \leq (t+\tau) \qquad \text{(A.3-17)}$$

Expressing Equation A.3-16 in x- and y-components provides the following:

$$v_{sx}(s) = v_{sx}(t) + \left[cst \cdot (s-t) \cdot \cos \theta_s(s)\right] \qquad t \leq s \leq (t+\tau) \qquad \text{(A.3-18)}$$

$$v_{sy}(s) = v_{sy}(t) + \left[cst \cdot (s-t) \cdot \sin \theta_s(s)\right] \qquad t \leq s \leq (t+\tau) \qquad \text{(A.3-19)}$$

Since, from Equation A.3-17, $\theta_s(s) = \theta_s(t)$, and $\theta_s(t)$ is defined in paragraph A.3 as $\theta_s(t) = \theta(t)$, then Equation A.3-17 is expressed as

$$\theta_s(s) = \theta(t) \qquad t \leq s \leq (t+\tau) \qquad \text{(A.3-20)}$$

Also, Equations A.3-16 and A.3-20 are valid for values of s between t and $(t+\tau)$. Therefore,

$$v_s(t+\tau) = v_s(t) + \tau \cdot cst \qquad \text{(A.3-21)}$$

$$\theta_s(t+\tau) = \theta(t) \qquad \text{(A.3-22)}$$

A.3.2.2 Ship's Position in Still Water and Air for Straight-Line/Acceleration Path and After Discrete Time-Step τ - The ship's position equations for straight-line/acceleration path are derived from Equations A.2-9 and A.2-10 by solving for the integrals in those equations under the prevailing conditions of straight-line path and acceleration cst. These derivations require auxiliary operations on equations as noted in (1) and (2) below.

(1) In Equations A.3-18 and A.3-19 substitute $\theta(t)$ for $\theta_s(s)$. These equations now become

$$v_{sx}(s) = v_{sx}(t) + \left[cst\ (s-t)\ \cos\theta(t)\right] \qquad (A.3\text{-}23)$$

$$v_{sy}(s) = v_{sy}(t) + \left[cst\ (s-t)\ \sin\theta(t)\right] \qquad (A.3\text{-}24)$$

When $v_{sx}(s)$ and $v_{sy}(s)$ are integrated with respect to variable s between t and $(t+\tau)$, then

$$\int_t^{t+\tau} v_{sx}(s)\ ds = \tau \cdot v_{sx}(t) + \cos\theta(t) \cdot cst\left(\frac{\tau^2}{2}\right) \qquad (A.3\text{-}25)$$

$$\int_t^{t+\tau} v_{sy}(s)\ ds = \tau \cdot v_{sy}(t) + \sin\theta(t) \cdot cst\left(\frac{\tau^2}{2}\right) \qquad (A.3\text{-}26)$$

Equations A.3-23 and A.3-24 are now determined for $s = (t+\tau)$, and from the new equations an expression is found for τ.

$$v_{sx}(t+\tau) = v_{sx}(t) + \tau\cos\theta(t)\ cst \qquad (A.3\text{-}27)$$

$$v_{sy}(t+\tau) = v_{sy}(t) + \tau\sin\theta(t)\ cst \qquad (A.3\text{-}28)$$

Therefore,

$$\tau = \frac{v_{sx}(t+\tau) - v_{sx}(t)}{\cos\theta(t)\ cst} = \frac{v_{sy}(t+\tau) - v_{sy}(t)}{\sin\theta(t)\ cst} \qquad (A.3\text{-}29)$$

Substitute the expressions for τ in the second terms of Equations A.3-25 and A.3-26 respectively. This provides the desired relations for

$$\int_t^{t+\tau} v_{sx}(s)\ ds \qquad \text{and} \qquad \int_t^{t+\tau} v_{sy}(s)\ ds.$$

$$\int_{t}^{t+\tau} v_{sx}(s)\,ds = \tau \cdot v_{sx}(t) + \frac{\tau}{2}\left[v_{sx}(t+\tau) - v_{sx}(t)\right] \qquad \text{(A.3-30)}$$

$$\int_{t}^{t+\tau} v_{sy}(s)\,ds = \tau \cdot v_{sy}(t) + \frac{\tau}{2}\left[v_{sy}(t+\tau) - v_{sy}(t)\right] \qquad \text{(A.3-31)}$$

(2) In the second term of Equations A.2-9 and A.2-10 respectively substitute $\theta(t)$ for $\theta(s)$, since heading is constant, and equal to the initial heading at time t. Therefore, the entire second term, when integrated between t and $(t+\tau)$, is multiplied by τ.

With the above substitutions properly applied to Equations A.2-9 and A.2-10, new position equations are established for time-step τ beyond the initial time t [i.e., at time $(t+\tau)$]. These new equations are indicated below as Equations A.3-32 and A.3-33.

$$\begin{aligned} p_x(t+\tau) = \tau \Big\{ & \frac{1}{2}\left[v_{sx}(t) + v_{sx}(t+\tau)\right] + v_{tid} \cdot \cos\theta_{tid} \\ & + \left[SC_8 \cdot (0.0038) v^2_{win} \left|\sin(\theta_{win} - \theta(t))\right|\right]^{1/2} \cos\theta_{win} \Big\} \\ & + p_x(t) \end{aligned} \qquad \text{(A.3-32)}$$

$$\begin{aligned} p_y(t+\tau) = \tau \cdot \Big\{ & \frac{1}{2}\left[v_{sy}(t) + v_{sy}(t+\tau)\right] + v_{tid} \sin\theta_{tid} \\ & + \left[SC_8 \cdot (0.0038)\, v^2_{win} \left|\sin(\theta_{win} - \theta(t))\right|\right]^{1/2} \sin\theta_{win} \Big\} \\ & + p_y(t) \end{aligned} \qquad \text{(A.3-33)}$$

A.3.3 Mode 3: Straight-Line/Deceleration Path

This mode considers the ship's forward motion in two phases of its straight-line travel. The crossover from one phase to the other is related to T_R, a ship characteristic which indicates the time delay associated with reversing the ship's engine.

A.3.3.1 Ship's Velocity and Heading Equations for Straight-Line/Deceleration Path -

Phase 1

For $T_R/2$ seconds the ship travels in a straight line with no changes in speed.

Under these conditions,

$$\dot{v}_s(s) = 0 \qquad t \leq s \leq (t+T_R/2) \qquad \text{(A.3-34)}$$

$$\dot{\theta}_s(s) = 0 \qquad \text{(A.3-35)}$$

From this point, the analysis for the first $T_R/2$ seconds (after reversal of the ship's engine) corresponds to that for mode 1. Therefore, Equations A.3-3 through A.3-11 for ship's velocity and heading are equally applicable in this phase of mode 3.

Phase 2

After $T_R/2$ seconds the ship begins to slow down, still traveling in a straight line, in accordance with the relationships noted in Equations A.3-36 and A.3-37.

$$\dot{\theta}(s) = 0 \qquad \text{(A.3-36)}$$

$$M_e \cdot \dot{v}_s(s) = -R_e \cdot v_s^2(s) - T_{HR} \qquad \text{(A.3-37)}$$

where

M_e = effective mass of the ship

R_e = constant relating speed squared to resistive force

T_{HR} = reverse thrust generated by the propeller

Rearrange Equation A.3-37 to define $\dot{v}_s(s)$.

$$\dot{v}_s(s) = -\frac{R_e}{M_e} \cdot v_s^2(s) - \frac{T_{HR}}{M_e} \qquad \text{(A.3-38)}$$

In the all-weather harbor model these constants are given by:

$$\frac{R_e}{M_e} = \text{con} \cdot SC_{13}, \text{ and}$$

$$\frac{T_{HR}}{M_e} = \frac{\text{con}}{SC_{13}}$$

where

$$\text{con} = \left[SC_{11} \cdot SC_{13}\right]^{-1}$$

SC_{11}, SC_{13} = ship characteristics. SC_{11} is a function of the ships weight, its maximum velocity, its ahead horsepower, and its forward propeller efficiency. SC_{13} is a function of the ship's ahead and astern horsepower, its forward and reverse propeller efficiency, and its maximum velocity. (Ship characteristics are defined in Appendix B.)

Substitution of the above constant equivalents for $\frac{R_e}{M_e}$ and $\frac{T_{HR}}{M_e}$ in Equation A.3-38 provides the following expression:

$$\dot{v}_s(s) = -\text{con} \cdot SC_{13} \cdot v_s^2\ (s) - \frac{\text{con}}{SC_{13}} \qquad \text{(A.3-39)}$$

A solution to the differential Equation A.3-39 is given in Equation A.3-40:

$$v_s(s) = \frac{1}{SC_{13}} \cdot \tan\left[\theta_1 - (\text{con})s\right], \qquad \text{(A.3-40)}$$

where θ_1 is an arbitrary constant determined by initial conditions. Equation A.3-40 is verified as an actual solution of the differential equation by the following steps (1) through (3):

(1) Differentiate both sides of Equation A.3-40.

$$\dot{v}_s(s) = -\frac{con}{SC_{13}}\left\{\tan^2\left[\theta_1-(con)s\right] + 1\right\}$$

$$= -\frac{con}{SC_{13}}\cdot\tan^2\left[\theta_1-(con)s\right] - \frac{con}{SC_{13}} \qquad (A.3\text{-}41)$$

(2) Square both sides of Equation A.3-40.

$$v_s^2(s) = \left(\frac{1}{SC_{13}}\right)^2\cdot\tan^2\left[\theta_1-(con)s\right],$$

or

$$\tan^2\left[\theta_1-(con)s\right] = v_s^2(s)\cdot(SC_{13})^2 \qquad (A.3\text{-}42)$$

(3) Substitute the relationship of Equation A.3-42 in Equation A.3-41.

$$\dot{v}_s(s) = -\frac{con}{SC_{13}}\cdot v_s^2(s)\cdot(SC_{13})^2 - \frac{con}{SC_{13}}$$

$$= -con\cdot SC_{13}\cdot v_s^2(s) - \frac{con}{SC_{13}} \qquad (A.3\text{-}43)$$

which is identical to the original differential Equation A.3-39. Thus Equation A.3-40 is indeed a solution of the differential equation.

To derive the general equation for $v_s(s)$, first obtain θ_1 for initial condition at time t. From Equation A.3-40,

$$SC_{13}v_s(s) = \tan\left[\theta_1-(con)s\right] \qquad (A.3\text{-}44)$$

Then, for initial time t,

$$SC_{13}v_s(t) = \tan\left[\theta_1 - (con)t\right] \qquad (A.3\text{-}45)$$

and

$$\tan^{-1}\left[SC_{13}v_s(t)\right] = \left[\theta_1 - (con)t\right] \qquad (A.3\text{-}46)$$

which gives an expression for θ_1. Substituting into Equation A.3-40 yields an expression for $v_s(s)$ in terms of the initial velocity at time t.

$$v_s(s) = \frac{1}{SC_{13}} \cdot \left\{ \tan \left[\tan^{-1}(SC_{13}\ v_s(t)) - con(s-t) \right] \right\} \qquad (A.3\text{-}47)$$

$$t \leq s \leq (t+\tau)$$

Since the ship travels in a straight line, the heading equation is expressed as

$$\theta_s(s) = \theta_s(t) = \theta(t) \qquad (A.3\text{-}48)$$

The x- and y-components of the ship's velocity in the straight-line/deceleration mode are obtained from the relationships

$$v_{sx}(s) = v_s(s)\ \cos\ \theta_s(s) \qquad (A.3\text{-}49)$$

and

$$v_{sy}(s) = v_s(s)\ \sin\ \theta_s(s) \qquad (A.3\text{-}50)$$

Thus,

$$v_{sx}(s) = \frac{\cos\ \theta_s(s)}{SC_{13}} \cdot \tan \left\{ \tan^{-1} [SC_{13} v_s(t)] - con(s-t) \right\} \qquad (A.3\text{-}51)$$

$$v_{sy}(s) = \frac{\sin\ \theta_s(s)}{SC_{13}} \cdot \tan \left\{ \tan^{-1} [SC_{13} v_s(t)] - con\ (s-t) \right\} \qquad (A.3\text{-}52)$$

The general equation for ship's heading for straight-line travel is given by

$$\theta(s) = \theta_s(s) = \theta_s(t) = \theta(t) \qquad t \leq s \leq (t+\tau) \qquad (A.3\text{-}53)$$

A.3.3.2 Ship's Position in Still Water and Air for Straight-Line/Deceleration Path and After Discrete Time-Step τ - The ship's position equations for straight-line/deceleration path are derived from Equations A.2-9 and A.2-10 by solving for the integrals in those equations under the prevailing conditions of straight-line path, and deceleration along that path. Determination of position is divided into the same two phases which were discussed in paragraph A.3.3.1 relative to derivation of ship's velocity and heading equations. Phase 1 therefore considers discrete-time-step positions along a straight-line path with no deceleration, and phase 2 takes into account the deceleration factor.

Phase 1

For $T_R/2$ seconds the ship travels in a straight line with no change in speed. Therefore, this phase contains the same conditions as pertain to the ship's position in still water and air for straight-line/constant velocity path after discrete time-step τ. The derivation of position equations for the given set of conditions in this phase 1 is the same as that found in paragraph A.3.1.2 The resulting position Equations A.3-12 and A.3-13 of the section mentioned are equally applicable for this phase, and are here referenced for such use.

Phase 2

In this phase, $v_{sx}(s)$ and $v_{sy}(s)$ (Equations A.3-51 and A.3-52) must be integrated from t to $(t+\tau)$ for proper insertion into Equations A.2-9 and A.2-10.

(1) In Equations A.3-51 and A.3-52, substitute $\theta(t)$ for $\theta_s(s)$, and then integrate the equations with respect to variable s between t and $(t+\tau)$. The result of this integration is given by Equations A.3-54 and A.3-55.

$$\int_t^{t+\tau} v_{sx}(s)\,ds = \frac{\cos\theta(t)}{SC_{13}\cdot con} \ln \left\{ \cos \left\{ \tan^{-1} \left[SC_{13} v_s(t)\right] - con(s-t) \right\} \right\} \Bigg|_{s\,=\,t}^{s\,=\,t+\tau} \qquad (A.3\text{-}54)$$

$$\int_{t}^{t+\tau} v_{sy}(s)\ ds = \frac{\sin\theta(t)}{SC_{13}\cdot con}\ \ln\left\{\cos\left\{\tan^{-1}\left[SC_{13}v_s(t)\right] - con(s-t)\right\}\right\}\Bigg|_{s=t}^{s=t+\tau} \qquad \text{(A.3-55)}$$

(2) In the solutions to the definite integrals of Equations A.3-54 and A.3-55, perform the following substitutions to obtain the simplified format indicated below.

Let $k_1 = \frac{\cos\theta(t)}{SC_{13}\cdot con}$

$k_2 = \frac{\sin\theta(t)}{SC_{13}\cdot con}$

$k_3 = \tan^{-1}\left[SC_{13}\cdot v_s(t)\right]$

With these substitutions, the following equations are obtained:

$$\int_{t}^{t+\tau} v_{sx}(s)\ ds = k_1\left\{\ln\cos\left[k_3 - con(s-t)\right]\right\}\Bigg|_{s=t}^{s=t+\tau} \qquad \text{(A.3-56)}$$

$$\int_{t}^{t+\tau} v_{sy}(s)\ ds = k_2\left\{\ln\cos\left[k_3 - con(s-t)\right]\right\}\Bigg|_{s=t}^{s=t+\tau} \qquad \text{(A.3-57)}$$

Evaluate the solution of Equation A.3-56 between t and $(t+\tau)$.

$$\int_{t}^{t+\tau} v_{sx}(s)\ ds = k_1\left[\ln\cos(k_3 - con\ \tau) - \ln\cos k_3\right]$$

$$= k_1\left[\ln\left\{\frac{\cos(k_3 - con\ \tau)}{\cos k_3}\right\}\right] \qquad \text{(A.3-58)}$$

Reinsert the values for k_1 and k_3 in Equation A.3-58.

$$\int_t^{t+\tau} v_{sx}(s)\,ds = \left[\frac{\cos\,\theta(t)}{SC_{13}\cdot con}\right] \cdot \ln\left(\frac{\cos\left\{\tan^{-1}\left[SC_{13}\cdot v_s(t)\right] - con\ \tau\right\}}{\cos\ \tan^{-1}\left[SC_{13} v_s(t)\right]}\right) \tag{A.3-59}$$

By similar reasoning, Equation A.3-57 becomes

$$\int_t^{t+\tau} v_{sy}(s)\,ds = \left[\frac{\sin\,\theta(t)}{SC_{13}\cdot con}\right] \cdot \ln\left(\frac{\cos\left\{\tan^{-1}\left[SC_{13}\cdot v_s(t)\right] - con\ \tau\right\}}{\cos\ \tan^{-1}\left[SC_{13} v_s(t)\right]}\right) \tag{A.3-60}$$

(3) Equation A.3-47 for $v_s(s)$ is valid for all values of s between t and $(t+\tau)$ Therefore, for $s=(t+\tau)$, Equation A.3-47 can be expressed as:

$$SC_{13}\ v_s(t+\tau) = \tan\left\{\tan^{-1}\left[SC_{13}\ v_s(t)\right] - con\ \tau\right\} \tag{A.3-61}$$

Therefore,

$$\tan^{-1}\left[SC_{13}\ v_s(t+\tau)\right] = \tan^{-1}\left[SC_{13}\ v_s(t)\right] - con\ \tau \tag{A.3-62}$$

(4) The following statements apply for general parameters a and b.

$$\cos(\tan^{-1} a) = (1+a^2)^{-1/2} \tag{A.3-63}$$

$$\cos(\tan^{-1} b) = (1+b^2)^{-1/2} \tag{A.3-64}$$

$$\ln \frac{\cos(\tan^{-1} a)}{\cos(\tan^{-1} b)} = 1/2\ \ln \frac{1+b^2}{1+a^2} \tag{A.3-65}$$

The above identities, together with Equation A.3-62, are used to obtain the final integrated expressions for $v_{sx}(s)$ and $v_{sy}(s)$ to be used in Equations A.2-9 and A.2-10.

$$\int_t^{t+\tau} v_{sx}(s)\,ds = \frac{\cos\,\theta(t)}{2\cdot SC_{13}\cdot con}\,\ln\left[\frac{1 + SC_{13}^2 v_s^2(t)}{1 + SC_{13}^2 v_s^2(t+\tau)}\right] \tag{A.3-66}$$

$$\int_t^{t+\tau} v_{sy}(s)\,ds = \frac{\sin\,\theta(t)}{2\cdot SC_{13}\cdot con}\,\ln\left[\frac{1 + SC_{13}^2 v_s^2(t)}{1 + SC_{13}^2 v_s^2(t+\tau)}\right] \tag{A.3-67}$$

(5) The required information for obtaining position equations for straight-line/deceleration path is now available. Substitution in Equations A.2-9 and A.2-10 of the values for

$$\int_t^{t+\tau} v_{sx}(s)\,ds \text{ and } \int_t^{t+\tau} v_{sy}(s)\,ds \quad \text{(Equations A.3-66 and A.3-67)}$$

and $\theta(t)$ for $\theta(s)$ (Equation A.3-53) provides the required position equations as follows:

$$\begin{aligned} p_x(t+\tau) = {} & p_x(t) + \frac{\cos\,\theta(t)}{2\cdot SC_{13}\cdot con}\,\ln\left[\frac{1 + SC_{13}^2 v_s^2(t)}{1 + SC_{13}^2 v_s^2(t+\tau)}\right] \\ & + \tau \cdot v_{tid} \cdot \cos\,\theta_{tid} \\ & + \tau\left[SC_8\ (0.0038)\ v_{win}^2 \left|\sin(\theta_{win} - \theta(t))\right|\right]^{1/2} \cos\,\theta_{win} \end{aligned} \tag{A.3-68}$$

$$p_y(t+\tau) = p_y(t) + \frac{\sin\theta(t)}{2\cdot SC_{13}\cdot con} \ln \left[\frac{1 + SC_{13}^2 v_s^2(t)}{1 + SC_{13}^2 v_s^2(t+\tau)}\right]$$

$$+ \tau \cdot v_{tid} \cdot \cos\theta_{tid}$$

$$+ \tau\left[SC_8\ (0.0038)\ v_{win}^2 \left|\sin\left(\theta_{win} - \theta(t)\right)\right|\right]^{1/2} \sin\theta_{win}$$

(A.3-69)

A.3.3.3 Summary of Mode 3 Equations -

Phase 1 $(t \leq T_R/2)$

$$v_s(t+\tau) = v_s(t)$$

$$\theta_s(t+\tau) = \theta_s(t)$$

$$v_{sx}(t+\tau) = v_{sx}(t)$$

$$v_{sy}(t+\tau) = v_{sy}(t)$$

$$\theta(t+\tau) = \theta(t)$$

$$p_x(t+\tau) = \tau\Big\{v_{sx}(t) + v_{tid}\cos\theta_{tid}$$

$$+ \left[SC_8(0.0038)\cdot v_{win}^2\left|\sin(\theta_{win} - \theta(t))\right|\right]^{1/2} \cos\theta_{win}\Big\}$$

$$+ p_x(t)$$

$$p_y(t+\tau) = \tau\Big\{v_{sy}(t) + v_{tid}\sin\theta_{tid}$$

$$+ \left[SC_8(0.0038)v_{win}^2\left|\sin(\theta_{win} - \theta(t))\right|\right]^{1/2} \sin\theta_{win}\Big\}$$

$$+ p_y(t)$$

Phase 2 $(t > T_R/2)$

$$v_s(t+\tau) = \frac{1}{SC_{13}}\left\{\tan\left[\tan^{-1}\left(SC_{13}v_s(t)\right) - con\ \tau\right]\right\}$$

$$\theta_s(t+\tau) = \theta_s(t)$$

$$v_{sx}(t+\tau) = v_s(t+\tau)\ \cos\ \theta_s(t+\tau)$$

$$v_{sy}(t+\tau) = v_s(t+\tau)\ \sin\ \theta_s(t+\tau)$$

$$\theta(t+\tau) = \theta(t)$$

$$p_x(t+\tau) = p_x(t) + \frac{\cos\ \theta(t)}{2 \cdot SC_{13} \cdot con}\ \ln \left[\frac{1 + v_s^2(t)SC_{13}^2}{1 + v_s^2(t+\tau)SC_{13}^2} \right] + \tau \cdot v_{tid} \cdot \cos\ \theta_{tid} + \tau \left[SC_8(0.0038) \cdot v_{win}^2 \left| \sin(\theta_{win} - \theta(t)) \right| \right]^{1/2} \cos\ \theta_{win}$$

$$p_y(t+\tau) = p_y(t) + \frac{\sin\ \theta(t)}{2 \cdot SC_{13} \cdot con}\ \ln \left[\frac{1 + v_s^2(t)SC_{13}^2}{1 + v_s^2(t+\tau)SC_{13}^2} \right] + \tau \cdot v_{tid} \cdot \sin\ \theta_{tid} + \tau \left[SC_8(0.0038) \cdot v_{win}^2 \left| \sin(\theta_{win} - \theta(t)) \right| \right]^{1/2} \sin\ \theta_{win}$$

A.3.4 Mode 4: Turning Path/Constant Velocity

This mode considers the ship's maneuverability in two phases. The change from the first phase to the second occurs after the ship has traveled a distance of ADV feet. (ADV refers to the distance, in feet, a ship must travel before a turn is taken and the rudder responds to the turn command.)

4.3.4.1 Ship's Velocity and Heading Equations for Turning Path/Constant Velocity -

Phase 1

For ADV feet the ship travels in a straight line at constant speed, with the same heading and velocity it had when entering the turn.

$$\dot{v}_s(s) = 0 \qquad t_i \leq s \leq ADV/v_s(t_i) \qquad \text{(A.3-70)}$$

$$\dot{\theta}_s(s) = 0 \tag{A.3-71}$$

$$ADV = \frac{SC_{15} v_s^2(t_i)}{2\left[SC_{10} + SC_{16} \cdot v_s^2(t_i)\right]} \ln \left[\frac{v_s^2(t_i)}{v_T^2}\right] \tag{A.3-72}$$

where $v_s(t_i)$ = initial velocity

v_T = turning velocity

$\left.\begin{array}{l} SC_{10} \\ SC_{15} \\ SC_{16} \end{array}\right\}$ = ship's characteristics

SC_{10} is a function of the ship's forward propeller efficiency, its ahead horsepower and its maximum velocity. SC_{15} is a function of the ship's weight. SC_{16} is a function of the ship's length, its mean draft, and its rudder angle. (Ship's characteristics are defined in Appendix B.)

Since phase 1 of mode 4 involves straight-line travel and constant velocity, the conditions are the same as those discussed in Section A.3.1.1. Therefore, Equations A.3-3 through A.3-11 for ship's velocity and heading are equally applicable in this phase of mode 4.

Phase 2

After distance ADV has been traveled, the velocity is changed to v_T. This is given in terms of the ship's initial velocity $v_s(t_i)$ according to Equation A.3-73

$$v_T = v_s(t_i) - SC_{18} - SC_{19} \cdot v_s(t_i) - SC_{20} \cdot v_s^2(t_i) \tag{A.3-73}$$

where SC_{18}, SC_{19}, and SC_{20} are the ship's turning velocity constants.

In practice various turning velocities v_T are assumed and the corresponding initial velocities are found. The equation that is used is:

$$\frac{v_T}{v_s(t_i)} = 1 - e^{-(0.62\ R/L)} \qquad \text{(A.3-74)}$$

where

$R = 57.3\ v_T/r\delta$ = turning radius

$r = 0.025\ v_T^2/\ {}^2_{max}$ = turning rate

δ = rudder angle

The model assumes ten values of the turning velocities (from zero to maximum initial velocity). An initial velocity is calculated for a given turning velocity; then, the difference between the turning velocity and initial velocity is calculated. This is repeated ten times and a least squares method is used in which the following equations are solved for a, b, and c.

$$10a + b\Sigma\ v_s(t_i) + c\Sigma\ v_s^2(t_i) = \Sigma\big(v_T - v_s(t_i)\big) = B(1) \qquad \text{(A.3-74a)}$$

$$a\Sigma\ v_s(t_i) + b\Sigma\ v_s^2(t_i) + c\Sigma\ v_s^3(t_i) = \Sigma\ v_s(t_i)[v_T - v_s(t_i)] = B(2) \qquad \text{(A.3-74b)}$$

$$a\Sigma\ v_s^2(t_i) + b\Sigma\ v_s^3(t_i) + c\Sigma\ v_s^4(t_i) = \Sigma\ v_s^2(t_i)[v_T - v_s(t_i)] = B(3) \qquad \text{(A.3-74c)}$$

A subroutine SIMQ solves the system of three simultaneous equations and replaces the array B(N) with the solutions for the constants a, b and c, and these are entered into the SC_{18}, SC_{19} and SC_{20} variables appearing in Equation A.3-73.

From then until the end of the turn,

$$\dot{v}_s(s) = 0 \qquad \text{(A.3-75)}$$

$$\dot{\theta}_s(s) = \pm\frac{v_T}{R} \qquad \text{(A.3-76)}$$

where R = turning radius.

In equation A.3-76, turning radius R is identified by the following relationship:

$$R = SC_{17}/v_T \tag{A.3-77}$$

where SC_{17} = ship's characteristic.

The plus or minus sign in Equation A.3-76 signifies the direction of ship's turn. The plus sign refers to the ship's clockwise turn, and the minus sign indicates the ship's turn in the counterclockwise direction. Integration of differential Equations (A.3-75) and (A.3-76) provides the velocity and heading equations for this phase.

$$v_s(s) = v_s(t) = v_T \qquad t \leq s \leq (t+\tau) \tag{A.3-78}$$

$$\theta_s(s) = \theta_s(t) \pm \frac{v_T}{R} (s-t) \qquad t \leq s \leq (t+\tau) \tag{A.3-79}$$

The x- and y-components of the ship's velocity in the turning path/constant velocity mode are obtained from the relationships

$$v_{sx}(s) = v_s(s) \cos \theta_s(s) \tag{A.3-80}$$

and

$$v_{sy}(s) = v_s(s) \sin \theta_s(s) \tag{A.3-81}$$

Therefore, when the relationships from Equations A.3-78 and A.3-79 are substituted in Equations A.3-80 and A.3-81, the expressions for $v_{sx}(s)$ and $v_{sy}(s)$ become:

$$v_{sx}(s) = v_s(t) \cos \left[\theta_s(t) \pm \frac{v_T}{R} (s-t)\right] \tag{A.3-82}$$

$$v_{sy}(s) = v_s(t) \sin \left[\theta_s(t) \pm \frac{v_T}{R} (s-t)\right] \tag{A.3-83}$$

Since Equations A.3-78 through A.3-81 are valid for values of s between t and $(t+\tau)$, the following equations apply to this mode:

$$v_s(t+\tau) = v_s(t) = v_T \tag{A.3-84}$$

$$\theta_s(t+\tau) = \theta_s(t) \pm \frac{v_T}{R} \tau \tag{A.3-85}$$

$$v_{sx}(t+\tau) = v_s(t+\tau) \cos \theta_s(t+\tau) \tag{A.3-86}$$

$$v_{sy}(t+\tau) = v_s(t+\tau) \sin \theta_s(t+\tau) \tag{A.3-87}$$

Also, from the note added to the definition of $\theta_s(t)$ in Section A.3, $\theta_s(t) = \theta(t)$. Therefore,

$$\theta_s(t+\tau) = \theta(t+\tau) \tag{A.3-88}$$

A.3.4.2 Ship's Position in Still Water and Air for Turning Path/Constant Velocity and After Discrete Time-Step τ - The ship's position equations for turning path/constant velocity are derived from Equations A.2-9 and A.2-10 by solving for the integrals in those equations under the prevailing conditions of turning path and constant velocity. Determination of position is divided into the same two phases which were discussed in Section A.3.4.1 relative to derivation of ship's velocity and heading equations. Phase 1 therefore considers discrete-time-step positions along a straight-line path with constant velocity, and phase 2 takes into account the turning path factor.

Phase 1

For ADV feet the ship travels in a straight line with constant velocity. Therefore, position Equations A.3-12 and A.3-13 for ship's travel in still water and air for straight-line/constant velocity path apply to this phase of mode 4, and are referenced here for such application.

Phase 2

In this phase (after ADV feet), $v_{sx}(s)$ and $v_{sy}(s)$ (Equations A.3-82 and A.3-83) must be integrated from t to $(t+\tau)$ for proper insertion into Equations A.2-9 and A.2-10. Perform steps (1) through (3) below to obtain the desired expressions.

(1) Integrate both sides of Equations A.3-82 and A.3-83 between limits t and $(t+\tau)$.

$$\int_{t}^{t+\tau} v_{sx}(s)\, ds = \int_{t}^{t+\tau} v_s(t) \cos \left[\theta_s(t) \pm \frac{v_T}{R}(s-t)\right] ds \qquad \text{(A.3-89)}$$

$$\int_{t}^{t+\tau} v_{sy}(s)\, ds = \int_{t}^{t+\tau} v_s(t) \left[\sin \theta_s(t) \pm \frac{v_T}{R}(s-t)\right] ds \qquad \text{(A.3-90)}$$

(2) Obtain the general solutions for the definite integrals bounded by limits s=t and s=(t+τ).

$$\int_{t}^{t+\tau} v_{sx}(s)\, ds = v_s(t) \left\{ \frac{(\pm)R}{v_T} \sin\left[\theta_s(t) \pm \frac{v_T}{R}(s-t)\right] \right\}_{s=t}^{s=t+\tau} \qquad \text{(A.3-91)}$$

$$\int_{t}^{t+\tau} v_{sy}(s)\, ds = v_s(t) \left\{ \frac{(\mp)R}{v_T} \cos\left[\theta_s(t) \pm \frac{v_T}{R}(s-t)\right] \right\}_{s=t}^{s=t+\tau} \qquad \text{(A.3-92)}$$

(3) Using $v_s(t) = v_T$ (Equation A.3-78), and also using Equation A.3-85, obtain the final solutions for the integrals

$$\int_{t}^{t+\tau} v_{sx}(s)\, ds \text{ and } \int_{t}^{t+\tau} v_{sy}(s)\, ds \text{ as:}$$

$$\int_{t}^{t+\tau} v_{sx}(s)\, ds = \pm R \left[\sin \theta_s(t+\tau) - \sin \theta_s(t)\right] \qquad \text{(A.3-93)}$$

$$\int_{t}^{t+\tau} v_{sy}(s)\, ds = \pm R \left[\cos \theta_s(t) - \cos \theta_s(t+\tau)\right] \qquad \text{(A.3-94)}$$

Note that in still water and air $\theta_s(s) = \theta(s)$, and that for mode 4, phase 2, $\theta_s(s)$ is defined by Equation A.3-79. Then, by substitution in Equations A.2-9 and A.2-10 of the expressions for

$\int_t^{t+\tau} v_{sx}(s)\,ds$ and $\int_t^{t+\tau} v_{sy}(s)\,ds$ (Equations A.3-93 and A.3-94) and

of the expression for $\theta(s)$ [$\theta_s(s)$ in Equation A.3-79], the position equations are obtained as follows:

$$\begin{aligned} p_x(t+\tau) = {} & p_x(t) \pm R\left[\sin\theta_s(t+\tau) - \sin\theta_s(t)\right] \\ & + \tau \cdot v_{tid} \cdot \cos\theta_{tid} \\ & + \cos\theta_{win} \cdot SC_8^{1/2}\,(0.0038)^{1/2}\,v_{win} \cdot \\ & \int_t^{t+\tau} \left|\sin\left[\theta_{win} - \theta(t) \pm \frac{v_T}{R}(s-t)\right]\right|^{1/2} ds \end{aligned} \qquad \text{(A.3-95)}$$

$$\begin{aligned} p_y(t+\tau) = {} & p_y(t) \pm R\left[\cos\theta_s(t) - \cos\theta_s(t+\tau)\right] \\ & + \tau \cdot v_{tid} \cdot \sin\theta_{tid} \\ & + \sin\theta_{win} \cdot SC_8^{1/2}\,(0.0038)^{1/2}\,v_{win} \\ & \int_t^{t+\tau} \left|\sin\left[\theta_{win} - \theta(t) \mp \frac{v_T}{R}(s-t)\right]\right|^{1/2} ds \end{aligned} \qquad \text{(A.3-96)}$$

A.3.4.3 Summary of Mode 4 Equations

Phase 1 (First ADV feet)

$$ADV = \frac{SC_{15} \cdot v_s^2(t_i)}{2\left[SC_{10} + SC_{16} \cdot v_s^2(t_i)\right]} \cdot \ln\left[\frac{v_s^2(t_i)}{v_T^2}\right]$$

$v_s(t+\tau) = v_s(t)$

$\theta_s(t+\tau) = \theta_s(t)$

$v_{sx}(t+\tau) = v_{sx}(t)$

$v_{sy}(t+\tau) = v_{sy}(t)$

$\theta(t+\tau) = \theta(t)$

$$p_x(t+\tau) = \tau\Big\{ v_{sx}(t) + v_{tid}\cos\theta_{tid} + \left[SC_8(0.0038)\cdot v_{win}^2\cdot|\sin(\theta_{win}-\theta(t))|\right]^{1/2}\cos\theta_{win}\Big\} + p_x(t)$$

$$p_y(t+\tau) = \tau\Big\{ v_{sy}(t) + v_{tid}\sin\theta_{tid} + \left[SC_8(0.0038)\cdot v_{win}^2\cdot|\sin(\theta_{win}-\theta(t))|\right]^{1/2}\sin\theta_{win} + p_y(t).$$

Phase 2 (After ADV feet)

$v_s(t+\tau) = v_s(t) = v_T$

$\theta_s(t+\tau) = \theta_s(t) \pm \frac{v_T}{R}\tau$

$v_{sx}(t+\tau) = v_s(t+\tau)\cos\theta_s(t+\tau)$

$v_{sy}(t+\tau) = v_s(t+\tau)\sin\theta_s(t+\tau)$

$\theta(t+\tau) = \theta_s(t+\tau)$

$$v_{sy}(t+\tau) = v_{sy}(t)$$

$$\theta(t+\tau) = \theta_s(t+\tau) = \theta(t)$$

$$p_x(t+\tau) = \tau\left\{v_{sx}(t) + v_{tid}\cos\theta_{tid} + \left[SC_8(0.0038).v_{win}^2\cdot\left|\sin(\theta_{win}-\theta(t))\right|\right]^{\frac{1}{2}}\cos\theta_{win}\right\} + p_x(t)$$

$$p_y(t+\tau) = \tau\left\{v_{sy}(t) + v_{tid}\sin\theta_{tid} + \left[SC_8(0.0038).v_{win}^2\cdot\left|\sin(\theta_{win}-\theta(t))\right|\right]^{\frac{1}{2}}\sin\theta_{win}\right\} + p_y(t).$$

B.4.2 Phase 2 (After ADV Feet)

$$v_s(t+\tau) = v_s(t) = v_T$$

$$\theta_s(t+\tau) = \theta_s(t) \pm \frac{v_T}{R}\tau$$

$$v_{sx}(t+\tau) = v_s(t+\tau)\cos\theta_s(t+\tau)$$

$$v_{sy}(t+\tau) = v_s(t+\tau)\sin\theta_s(t+\tau)$$

$$\theta(t+\tau) = \theta_s(t+\tau)$$

$$p_x(t+\tau) = p_x(t) \pm R\left[\sin\theta_s(t+\tau) - \sin\theta_s(t)\right] + \tau\ .\ v_{tid}\cdot\cos\theta_{tid} + \cos\theta_{win}\,SC_8^{\frac{1}{2}}\,(0.0038)^{\frac{1}{2}}v_{win} \cdot\int_t^{t+\tau}\left|\sin\left[\theta_{win}-\theta(t) \mp \frac{v_T}{R}(s-t)\right]\right|^{\frac{1}{2}}ds$$

$$
\begin{aligned}
p_y(t+\tau) = p_y(t) \pm R & \left[\cos \theta_s(t) - \cos \theta_s(t+\tau)\right] \\
& + \tau \cdot v_{tid} \cdot \sin \theta_{tid} \\
& + \sin \theta_{win} \; SC_8^{\frac{1}{2}} \; (0.0038)^{\frac{1}{2}} v_{win} \\
& \cdot \int_t^{t+\tau} \left| \sin \left[\theta_{win} - \theta(t) \mp \frac{v_T}{R} (s-t) \right] \right|^{\frac{1}{2}} ds
\end{aligned}
$$

Ship Characteristics (SC) and Constants

Symbol (SC or const)	Description/Defining Equation	Units/Const/Ratio
	Input into Program	
g	Gravitational constant	32.173 ft/sec^2
HUFA	Ship's human talent index	(Ratio)
NAVC	Navigational capability index	ft
NI	Navigational index as function of position	(Ratio)
RA	Ship's maximum rudder angle	deg
RDN	Conversion from degrees to radians	0.01745329252 rad /deg
ρ_F	Ship's propeller efficiency	(Forward ratio)
ρ_R	Ship's propeller efficiency	(Reverse ratio)
SC_1	Ship's length	ft
SC_2	Ship's width	ft
SC_3	Ship's mean draft	ft
SC_4	Ship's maximum velocity	Knots
SC_5	Ship's ahead horsepower	hp
SC_6	Ship's astern horsepower	hp
SC_7	Ship's weight	Tons
SC_8	Ship's wind sail	(Ratio)
SC_9	Ship's reverse time	sec

Ship Characteristics (SC) and Constants (Cont'd.)

Symbol (SC or const)	Description/Defining Equation	Units/Const/Ratio
	Calculated in Program	
SC_{10}	$SC_{10} = 429\ SC_5/SC_4(1.6889)$	lbs
SC_{11}	$SC_{11} = SC_{15}\ (VMAX)^2/SC_{10}$	ft
VMAX	$VMAX = 1.6889\ {}_4$	ft/sec
SC_{12}	$SC_{12} = (1.56)\ SC_5/SC_6(VMAX)^2$	$(sec/ft)^2$
SC_{13}	$SC_{13} = \sqrt{SC_{12}}$	sec/ft
SC_{14}	$SC_{14} = SC_{15}/(0.95)\ SC_{10}$	sec^2/ft
SC_{15}	$SC_{15} = (1.05)\ (2240).\ SC_7/g$	$lbs/(ft/sec)^2$
SC_{16}	$SC_{16} = (0.016)SC_1 \cdot SC_3 \cdot \sin (RA \cdot RDN)$	$lbs/(ft/sec)^2$
SC_{17}	$SC_{17} = (VMAX)^2/(0.025)\ (RA)\ (RDN)$	ft^2/sec
SC_{18}	Ship's turning velocity variable	ft/sec
SC_{19}	Ship's turning velocity variable	Dimensionless
SC_{20}	Ship's turning velocity variable	$(ft/sec)^{-1}$

APPENDIX C

PROBABILITY OF COLLISIONS AND GROUNDINGS

The following discussion considers in detail an approach for using the vessel safety model to approximate the probability of a collision or of grounding for a ship as it passes through the restricted waters of a harbor. Simplifying assumptions made during the discussion are identified as they are presented in the development.

A general relationship is first developed between the probability of collisions and groundings on the whole track, and the probability of collisions and groundings on subintervals of the track. This relationship permits calculation of the probability of collision and grounding on the whole track iteratively, on the sole basis of information provided at discrete times by the computer program.

Once the iterative formula has been found for calculating the cumulative probability of collisions and groundings of the subintervals, a formula is derived which makes use of the available program data to compute these subinterval probabilities. This is done separately for probability of collision and probability of grounding. The algorithms to be used are discussed in detail, and full development of related equations is completed in Appendixes E through H.

C.1 REDUCTION FROM WHOLE INTERVAL TO SUBINTERVAL

The notation to be used in the discussion includes the following:

t_i = total time spent in the harbor by ship i

$NC_i(t_1,t_2)$ = number of collisions ship i is involved in between times t_1 and t_2

$NG_i(t_1,t_2)$ = number of groundings ship i experiences between times t_1 and t_2

$P[A]$ = probability that event A occurs

$P[A/B]$ = conditional probability of event A given event B, or the probability that event A occurs given that event B has occurred.

In addition to the above, designate the probability that ship i will be involved in a collision during its time in the harbor as $P[NC_i(0,T_i) > 0]$, and the probability that ship i will be involved in a grounding during its time in the harbor as $P[NG_i(0,T_i) > 0]$. These probabilities are expressed in a modified form by making use of the following basic theorem:

The probability that an event will happen in a sample space, P[A], is related to the probability that the event will not happen, P[NOT A], by the equation

$$P[A] = 1.0 - P[\text{NOT } A] \qquad (C.1\text{-}1)$$

Then

$$P\Big[NC_i(0,T_i) > 0\Big] = 1.0 - P\Big[NC_i(0,T_i) = 0)\Big] \qquad (C.1\text{-}2)$$

$$P\Big[NG_i(0,T_i) > 0\Big] = 1.0 - P\Big[NG_i(0,T_i) = 0)\Big] \qquad (C.1\text{-}3)$$

Let $(t_o, t_i, \ldots, t_{Ni})$ = partition of the interval $[0,T_i]$, so that $t_o = 0$ and $t_{Ni} = T_i$. Each t_j corresponds to the j^{th} step taken by the program. Therefore, the j^{th} time step covers the subinterval (t_{j-1}, t_j).

The formula for joint probability is used to determine the relationship, in the first instance, between $P[NC_i(0,T_i)=0]$ and $P[NC_i(t_{j-1},t_j)=0]$, and, in the second, between $P[NG_i(0,T_i)=0]$ and $P[NG_i(t_{j-1},t_j)=0]$.

$$P\Big[NC_i(t_o,t_{Ni}) = 0\Big] = \left(\prod_{j=2}^{N_i} \Big\{ P\Big[NC_i(t_{j-1},t_j) = 0/NC_i(t_o,t_{j-1}) = 0\Big]\Big\} \right) \cdot P\Big[NC_i(t_o,t_i) = 0\Big] \qquad (C.1\text{-}4)$$

$$P\left[NG_i(t_o,t_{Ni}) = 0\right] = \left(\prod_{j=2}^{N_i}\left\{P\left[NG_i(t_{j-1},t_j) = 0/NG_i(t_o,t_{j-1}) = 0\right]\right\}\right) \quad P\left[NG_i(t_o,t_1) = 0\right] \qquad \text{(C.1-5)}$$

Equations C.1-4 and C.1-5 satisfy the previously mentioned objective of obtaining iterative equations for the probabilities on the total interval in terms of the probabilities on the subintervals. These equations also permit attention to be focused on the following expressions:

$$P\left[NC_i(t_{j-1},t_j) = 0 \ / \ NC_i(t_o,t_{j-1}) = 0\right]$$

and

$$P\left[NG_i(t_{j-1},t_j) = 0 \ / \ NG_i(t_o,t_{j-1}) = 0\right]$$

The probabilities of collision and grounding are to be developed separately, beginning with the probability of collision. (The probability of grounding is discussed in Section C.6.)

C.2 PROBABILITY OF COLLISION ON SUBINTERVAL

Consider the probability

$$P\left[NC_i(t_{j-1},t_j) = 0 \ / \ NC_i(t,_o,t_{j-1}) = 0\right]$$

i.e., the probability that ship i experiences no collisions from time t_{j-1} to t_j, given that ship i has had no collisions before time t_{j-1}. In this discussion of probability calculation the following notation is introduced:

N_s = number of ships in the harbor

$\left[x_i(t_j),y_i(t_j)\right]$ = random variables denoting the true position of the center of ship i at time t_j

$d_{i,k}(t)$ = random variable denoting the true distance between ship i and ship k at time t_j

SL_i = length of ship i

$\delta_i(t_{j-1},t_j)$ = true distance which ship i moves from time t_{j-1}, t_j

$v_i(t_j)$ = true instantaneous velocity of ship i at time t_j

The random variable $d_{i,k}(t)$ is evaluated by the equation

$$d_{i,k}(t_j) = \left[\left(x_i(t_j) - x_k(t_j)\right)^2 + \left(y_i(t_j) - y_k(t_j)\right)^2\right]^{\frac{1}{2}} \quad \text{(C.2-1)}$$

The following assumptions are made pertinent to the further development of the calculation:

a. The true velocity of ship i is the same as the velocity generated by the vessel safety model for ship i, denoted by $v_i(t_j)$, for the j^{th} step.

b. Assume that

$$\delta_i(t_{j-1},t_j) \cong v_i(t_j)\ (t_j-t_{j-1}) \quad \text{(C.2-2)}$$

This implies that the velocity of ship i is constant on (t_{j-1},t_j).

Note: The above assumption is reasonable for small intervals. A more accurate assumption would consider velocities $v_i(t_j)$ and $v_i(t_{j-1})$ in the following equation:

$$\delta_i(t_{j-1},t_j) \cong \left[v_i(t_{j-1}) + v_i(t_j)\right]\ (t_j-t_{j-1})/2 \quad \text{(C.2-3)}$$

Although more accurate, this assumption involves more storage space and computer time requirements.

c. Assume that for all values of $k = 1,2,\ldots,i-1,i+1,\ldots,N_s$

$$P\left[NC_i(t_{j-1},t_j) = 0 \;/\; NC_i(t_o,t_{j-1}) = 0\right] = P\left[d_{i,k}(t_j) > \left[(SL_i+SL_k) + \delta_i(t_{j-1},t_j) + \delta_k(t_{j-1},t_j)\right] \;/NC_i(0,t_{j-1}) = 0\right] \qquad (C.2\text{-}4)$$

This should contain the true probability of collision for the subinterval (t_{j-1},t_j). (A complete development of equation C.2-4 is presented in Appendix E.)

d. Assume that the distances, $d_{i,k}(t_j)$ and $d_{i,\ell}(t_j)$ $(k \neq \ell)$ are mutually independent for each k and ℓ, given that there has been no previous collision for ship i. This assumption permits use of the following equation.

$$P\left[NC_i(t_{j-1},t_j) = 0 \;/\; NC_i(t_o,t_{j-1}) = 0\right] =$$

$$\prod_{\substack{k=1\\k\neq i}}^{N_s} P\left\{d_{i,k}(t_j) > \left[\frac{1}{2}\,(SL_i+SL_k) + \delta_i(t_{j-1},t_j) + \delta_k(t_{j-1},t_j)\right] \;/\; NC_i(t_o,t_{j-1}) = 0\right\}. \qquad (C.2\text{-}5)$$

Equation C.2-5 indicates that only two ships at a time need be considered in the development of the probability of collision on a subinterval. This greatly simplifies the algorithm. The assumption is expected to be true in almost all cases of interest. However, it will not be true if any of the following situations exists:

(1) Ship k is towing ship ℓ (or vice versa).

(2) Ship k is attempting to intercept or leave ship ℓ (e.g., a pilot boat or tug joining or leaving a tanker) (or vice versa).

(3) Ship k is maneuvering to avoid ship ℓ (or vice versa).

In these cases more than two ships at a time would have to be considered. This much more difficult analysis is beyond the scope of this report.

C.3 SUMMARY OF RESULTS AND DEVELOPMENT OF $P[NC_i(0,T_i) > 0]$

In this section we briefly summarize the results of the previous development. The probability that ship i will be involved in a collision during its time in the harbor is expressed by

$$P\left[NC_i(0,T_i) > 0\right] = 1.0 - P\left[NC_i(0,T_i) = 0\right]. \tag{C.1-2}$$

With

$$P\left[NC_i(t_o,t_{Ni}) = 0\right] = \prod_{j=2}^{N}\left\{P\left[NC_i(t_{j-1},t_j) = 0 \,/\, NC_i(t_o,t_{j-1}) = 0\right]\right\} . \; P\left[NC_i(t_o,t_1) = 0\right], \tag{C.1-4}$$

the following equation applies.

$$P\left[NC_i(0,T_i) > 0\right] = 1.0 - \prod_{j=2}^{N}\left\{P\left[NC_i(t_{j-1},t_j) = 0 \,/\, NC_i(t_o,t_{j-1}) = 0\right]\right\} . \; P\left[NC_i(t_o,t_1) = 0\right]. \tag{C.3-1}$$

Then, together with assumptions a through d of Section C.2, Equation C.2-5 leads to the following conclusion:

$$P\left[NC_i(0,T_i) > 0\right] = 1.0 - \prod_{j=2}^{N_i} \prod_{\substack{k=1 \\ k\neq i}}^{N_s} P\left\{d_{i,k}(t_j) > \left[\frac{1}{2}(SL_i+SL_k) + \delta_i(t_{j-1},t_j) + \delta_k(t_{j-1},t_j)\right] \,/\, NC_i(t_o,t_{j-1}) = 0\right\}, \tag{C.3-2}$$

where the summation is over ships k=1 through N_s (not including ship i) and over j to yield all times from t_o to t_{Ni}. Therefore, the problem has been reduced to finding the probability of a collision between two ships on a subinterval of time.

C.4 PROBABILISTIC INTERPRETATION OF MODEL OUTPUTS

With the development of Equation C.3-2 above, the necessary preliminary analysis required to produce a probability of collision measure with the vessel safety model is complete, and can be integrated into the model. This requires probabilistic interpretations of the model outputs.

Assume a ship's cartesian coordinates of position in the waterway are the jointly gaussian random variables $x_i(t)$ and $y_i(t)$. In contrast to these random variables, the vessel safety model determines the coordinates of ship i to be $\left[\hat{x}_i(t_j), \hat{y}_i(t_j)\right]$ at time step t_j. These are the values predicted by the model on the basis of the given equations of motion and on the acceleration, deceleration, and turning capabilities of the ship.

Relate $\left[\hat{x}_i(t_j), \hat{y}_i(t_j)\right]$ to $\left[x_i(t_j), y_i(t_j)\right]$ by assuming that the expected values of the x and y coordinates of ship i at time t_j are equal to the values as propagated by the model:

$$E\left[x_i(t_j)\right] = \hat{x}(t_j) \tag{C.4-1}$$

and

$$E\left[y_i(t_j)\right] = \hat{y}(t_j) \tag{C.4-2}$$

In operation the vessels make position errors, which are defined below.

$$e_{xi}(t_j) = x_i(t_j) - \hat{x}_i(t_j), \tag{C.4-3}$$

and

$$e_{yi}(t_j) = y_i(t_j) - \hat{y}_i(t_j). \tag{C.4-4}$$

The following independence assumptions are also made:

a. $e_{xi}(t_j)$ and $e_{xi}(t_k)$ are independent if $j \neq k$,

b. $e_{yi}(t_j)$ and $e_{yi}(t_k)$ are independent if $j \neq k$,

c. $e_{xi}(t_j)$ and $e_{xk}(t_j)$ are independent if $i \neq k$, and

d. $e_{yi}(t_j)$ and $e_{yk}(t_j)$ are independent if $i \neq k$.

This completes the major probabilistic assumptions required to perform the probability of collision calculations.

C.5 PROBABILITY OF COLLISION CALCULATIONS

It is now possible to calculate the probability:

$$P\left\{ d_{i,k}^2(t_j) > \left[\frac{1}{2}(SL_i+SL_k) + \delta_i(t_{j-1},t_j) + \delta_k(t_{j-1},t_j)\right]^2 / NC_i(t_o,t_{j-1}) = \right\} 0 \quad . \qquad (C.5\text{-}1)$$

For simplification of future operations, the following notations are first described, and then defined by their associated equations C.5-2, C.5-3, C.5-4, and C.5-5.

$\Delta_{i,k}(t_{j-1},t_j)$ = distance between ships i and k at time t_j, made up of one-half the lengths of ships i and k and the distance moved by ships i and k during time interval (t_j-t_{j-1}).

$$\Delta_{i,k}(t_{j-1},t_j) = \frac{1}{2}(SL_i+SL_k) + \delta_i(t_{j-1},t_j) + \delta_k(t_{j-1},t_j) \qquad (C.5\text{-}2)$$

$\left.\begin{matrix}\sigma_{xi}(t_j)\\ \sigma_{yi}(t_j)\end{matrix}\right\}$ = x,y variances from expected values at time t_j, defined in equations C.5-3 and C.5-4.

$P_i(t_j)$ = correlation function, defined in equation C.5-5.

$$\sigma_{xi}^2(t_j) = E\left\{\left[x_i(t_j) - \hat{x}_i(t_j)\right]^2\right\} \qquad (C.5\text{-}3)$$

$$\sigma_{yi}^2(t_j) = E\left[\left(y_i(t_j) - \hat{y}_i(t_j)\right)^2\right] \qquad (C.5\text{-}4)$$

$$\rho_i(t_j) = \frac{E\left\{\left[\left(x_i(t_j) - \hat{x}_i(t_j)\right)\left(\left(y_i(t_j) - \hat{y}_i(t_j)\right)\right]\right\}}{\sigma_{xi}(t_j) \cdot \sigma_{yi}(t_j)} \qquad (C.5\text{-}5)$$

where ρ is the cross correlation term between the x-and y- coordinates of ship i.

The following equation makes use of the above notation.

$$P\left[NC_i(0,T_i) > 0\right] = 1.0 - \prod_{j=1}^{N_i} \prod_{\substack{k=1 \\ k\neq i}}^{N_S} P\left[d_{i,k}^{2}(t_j) > \Delta_{i,k}^{2}(t_{j-1},t_j) \;/\; NC_i(0,t_{j-1}) = 0\right] \tag{C.5-6}$$

The analysis continues with a simplification of the expression

$$P\left[d_{i,k}^{2}(t_j) > \Delta_{i,k}^{2}(t_{j-1},t_j) \;/\; NC_i(0,t_{j-1}) = 0\right]$$

From equation C.2-1,

$$d_{i,k}^{2}(t_j) = \left[x_i(t_j) - x_k(t_j)\right]^2 + \left[y_i(t_j) - y_k(t_j)\right]^2. \tag{C.5-7}$$

Since, by the jointly gaussian assumption, $x_i(t_j)$, $x_k(t_j)$, $y_i(t_j)$ and $y_k(t_j)$ are gaussian random variables, their differences are also gaussian random variables. These differences are expressed as

$$\Delta x_{i,k}(t_j) = x_i(t_j) - x_k(t_j) \tag{C.5-8}$$

$$\Delta y_{i,k}(t_j) = y_i(t_j) - y_k(t_j) \tag{C.5-9}$$

when $\Delta x_{i,k}(t_j)$ and $\Delta y_{i,k}(t_j)$ are jointly gaussian random variables.

Let

$$\hat{\Delta x}_{i,k}(t_j) = E\left[\Delta x_{i,k}(t_j)\right] \tag{C.5-10}$$

$$\hat{\Delta y}_{i,k}(t_j) = E\left[\Delta y_{i,k}(t_j)\right] \tag{C.5-11}$$

Also, let

$$\sigma_{\Delta x_{i,k}}^{2}(t_j) = E\left\{\left[\Delta x_{i,k}(t_j) - \hat{\Delta x}_{i,k}(t_j)\right]^2\right\} \tag{C.5-12}$$

$$\sigma_{\Delta y_{i,k}}^{2}(t_j) = E\left\{\left[\Delta y_{i,k}(t_j) - \hat{\Delta y}_{i,k}(t_j)\right]^2\right\} \tag{C.5-13}$$

and

$$\rho_{\Delta_{i,k}}(t_j) = \frac{E\left\{\left[\Delta x_{i,k}(t_j) - \Delta\hat{x}_{i,k}(t_j)\right]\left[\Delta y_{i,k}(t_j) - \Delta\hat{y}_{i,k}(t_j)\right]\right\}}{\sigma_{\Delta x_{i,k}}(t_j)\ \sigma_{\Delta y_{i,k}}(t_j)} \qquad \text{(C.5-14)}$$

In Appendix F the following equations relating $\sigma_{\Delta x^2_{i,k}}(t_j)$, $\sigma_{\Delta y^2_{i,k}}(t_j)$, and $\rho_{\Delta_{i,k}}(t_j)$ to the original quantities are derived.

$$\sigma_{\Delta x_{i,k}}^{2}(t_j) = \sigma_{x_i}^{2}(t_j) + \sigma_{x_k}^{2}(t_j) \qquad \text{(C.5-15)}$$

$$\sigma_{\Delta y_{i,k}}^{2}(t_j) = \sigma_{y_i}^{2}(t_j) + \sigma_{y_k}^{2}(t_j) \qquad \text{(C.5-16)}$$

$$\rho_{\Delta_{i,k}}(t_j) = \frac{\rho_i(t_j)\sigma x_i(t_j)\sigma y_i(t_j) + \rho_k(t_j)\sigma x_k(t_j)\sigma y_k(t_j)}{\sigma_{\Delta x_{i,k}}(t_j) \cdot \sigma_{\Delta y_{i,k}}(t_j)} \qquad \text{(C.5-17)}$$

Using equations C.5-7 and C.5-8, the following relationship holds for $d_{i,k}^{2}(t_j)$.

$$d_{i,k}^{2}(t_j) = \Delta x_{i,k}^{2}(t_j) + \Delta y_{i,k}^{2}(t_j). \qquad \text{(C.5-18)}$$

Since $\Delta x_{i,k}(t_j)$, and $\Delta y_{i,k}(t_j)$ are both gaussian random variables, $d_{i,k}^{2}(t_j)$ must be some generalized chi-squared random variable. If the distribution for $d_{i,k}(t_j)$ could be derived, then the expression

$$P\left[d_{i,k}^{2}(t_j) > \Delta_{i,k}^{2}(t_{j-1},t_j) \;/\; NC_i(0,t_{j-1}) = 0\right].$$

could be easily evaluated. However, since no convenient expression has been found for the distribution of $d_{i,k}^2(t_j)$, the following approach has been adopted.*

Consider the following equation:

$$P\left[d_{i,k}^2(t_j) > \Delta_{i,k}^2(t_{j-1},t_j) \;/\; NC_i(0,t_{j-1}) = 0\right] = 1.0 - \int_{-\Delta_{i,k}(t_{j-1},t_j)}^{+\Delta_{i,k}(t_{j-1},t_j)} \int_{-\sqrt{\Delta_{i,k}^2(t_{j-1},t_j) - y^2}}^{+\sqrt{\Delta_{i,k}^2(t_{j-1},t_j) - y^2}} p_\Delta(x,y)\,dx\,dy \qquad (C.5\text{-}19)$$

where

$$p_\Delta(x,y) = \frac{1}{2\pi\,\sigma_{\Delta x_{i,k}}(t_j)\,\sigma_{\Delta y_{i,k}}(t_j)\sqrt{1-\rho_{\Delta_{i,k}}^2(t_j)}} \cdot \exp\left\{-\frac{1}{2\left(1-\rho_{\Delta_{i,k}}^2(t_j)\right)}\left[\frac{\left(x-\Delta\hat{x}_{i,k}(t_j)\right)^2}{\sigma_{\Delta x_{i,k}}^2(t_j)} + \frac{\left(y-\Delta\hat{y}_{i,k}(t_j)\right)^2}{\sigma_{\Delta y_{i,k}}^2(t_j)} - 2\rho_{\Delta_{i,k}}(t_j)\frac{\left(x-\Delta\hat{x}_{i,k}(t_j)\right)\left(y-\Delta\hat{y}_{i,k}(t_j)\right)}{\sigma_{\Delta x_{i,k}}(t_j)\,\sigma_{\Delta y_{i,k}}(t_j)}\right]\right\} \qquad (C.5\text{-}20)$$

This expression is simplified to eliminate the requirement for performing double integration on the computer. The simplification is achieved by use of the rotational properties of jointly gaussian random variables.

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*Since this writing, it has been found that the distribution is related to a Ricean distribution (in one variable) and, generally, may be expressed as a non-central chi-squared distribution.

J.M. Wozencraft and I.M. Jacobs", Principles of Communication Engineering," John Wiley and Sons, May, 1965, Page 362.

J.P. Clark, S. Karp, Proceedings of the IEEE, Volume 58, No. 12, December 1970, Pages 1964-1965.

D. Middleton, "Introduction to Statistical Communication Theory," McGraw-Hill Book Company, New York, 1960, page 414, Equation 9.50, and chapter 17.

Although use of the fact that the distribution of $d_{i,k}^2(t_j)$ is a Ricean distribution would provide greater accuracy in the resulting derivation, the present, less elegant approach also yields quite accurate results in the sense that the method well approximates the function (though time consuming).

Perform the following transformation:

$$u = x \cos \alpha - y \sin \alpha \tag{C.5-21}$$

$$u = y \cos \alpha - x \sin \alpha \tag{C.5-22}$$

where

$$\alpha = \frac{1}{2} \tan^{-1} \left[\frac{2\rho_{\Delta_{i,k}}(t_j)\sigma_{\Delta x_{i,k}}(t_j)\, \sigma_{\Delta y_{i,k}}(t_j)}{\sigma_{\Delta y_{i,k}}{}^2(t_j) - \sigma_{\Delta x_{i,k}}{}^2(t_j)} \right]. \tag{C.5-23}$$

Then this expression simplifies to

$$P\left[d_{i,k}^2(t_j) > \Delta_{i,k}^2(t_{j-1},t_j) \;/\; NC_i(0,t_{j-1}) = 0 \right] = 1.0 -$$

$$\int_{-\Delta_{i,k}(t_{j-1},t_j)}^{+\Delta_{i,k}(t_{j-1},t_j)} \int_{-\sqrt{\Delta_{i,k}^2(t_{j-1},t_j) - v^2}}^{+\sqrt{\Delta_{i,k}(t_{j-1},t_j) - v^2}} P_{uv}(u,v)\,du\,dv \tag{C.5-24}$$

where $P_{uv}(u,v)$ is defined by

$$P_{uv}(u,v) = \frac{1}{2\pi\sigma_u\sigma_v} \exp\left\{ -\frac{1}{2}\left[\frac{(u-\hat{u})^2}{\sigma_u{}^2} + \frac{(v-\hat{v})^2}{\sigma_v{}^2} \right] \right\}, \tag{C.5-25}$$

and σ_u^2, σ_v^2, $\hat{u}$, and $\hat{v}$ are defined in terms of $\sigma_{\Delta x_{i-k}}{}^2(t_j)$, $\sigma_{\Delta y_{i,k}}{}^2(t_j)$, $\Delta\hat{x}_{i,k}(t_j)$, $\Delta\hat{y}_{i,k}(t_j)$, and $\rho_{\Delta_{i,k}}(t_j)$. (See Appendix G.)

Further simplification is accomplished by approximating the circle of radius $\Delta_{i,k}(t_{j-1},t_j)$ to a square with sides $2\Delta_{i,k}(t_{j-1},t_j)$. With this approximation the equation becomes

$$P\left[d_{i,k}^2(t_j) > \Delta_{i,k}^2(t_{j-1},t_j) \;/\; NC_i(0,t_{j-1}) = 0 \right] =$$

$$\int_{-\Delta_{i,k}(t_{j-1},t_j)}^{+\Delta_{i,k}(t_{j-1}t_j)} P_u(u)\,du \cdot \int_{-\Delta_{i,k}(t_{j-1},t_j)}^{+\Delta_{i,k}(t_{j-1},t_j)} P_v(v)\,dv \tag{C.5-26}$$

where

$$P_u(u) = \frac{1}{\sqrt{2\pi}\ \sigma_u} \exp\left[- \frac{(u-\hat{u})^2}{2\sigma_u^2} \right] \qquad \text{(C.5-27)}$$

and

$$P_v(v) = \frac{1}{\sqrt{2\pi}\ \sigma_v} \exp\left[- \frac{(v-\hat{v})^2}{2\sigma_v^2} \right] \qquad \text{(C.5-28)}$$

(Full details for approximation of equation C.5-24 are given in Appendix G.) Equation C.5-26 is the simplified expression for the probability of collision for a ship as it proceeds through the waterway with the vessel safety model.

C.6 PROBABILITY OF GROUNDING ON SUBINTERVALS

A general relationship was previously developed for the probability of grounding for ship i on its whole track to the conditional probability on the subinterval (t_{j-1}, t_j), given that ship i has had no previous groundings, i.e.,

$$P\left[NG_i(t_o, t_{Ni}) = 0\right] = \prod_{j=1}^{N_i} P\left[NG_i(t_{j-1}, t_j) = 0 \,/\, NG_i(t_o, t_{j-1}) = 0\right] . \qquad \text{(C.6-1)}$$

The following discussion will center on consideration of the right side expression of equation C.6-1:

$$P\left[NG_i(t_{j-1}, t_j) = 0 \,/\, NG_i(t_o, t_{j-1}) = 0\right] .$$

The following concepts supplement the notation on ship grounding already introduced in Section C.1.

Relative ship safety in harbor waters for depths at particular points is indicated in the notation below.

H_A^i = [(x,y): the water at point (x,y) in the harbor is deep enough for ship i] (Safe sailing depth)

$H_A^{ic} = [(x,y): \; (x,y) \notin H_A^i\,]$ (The water is not deep enough for ship i.)

Assume that the no-grounding event $\left[NG_i(t_{j-1},t_j) = 0\right]$ is equivalent to the event $\left[x_i(t_j),y_i(t_j)\varepsilon H_A^i\right]$, so that the probability of no grounding is expressed as follows:

$$P\left[NG_i(t_{j-1},t_j) = 0 \;/\; NG_i(0,t_{j-1}) = 0\right] = P\left[x_i(t_j),y_i(t_j)\,\varepsilon H_A^i \;/\; NG_i(0,t_{j-1}) = 0\right] \quad \text{(C.6-2)}$$

Since the only available information is on the position of the ships at time t_j, the probability expressed in equation C.6-2 must be approximated on the basis of this available information. The approximation is accomplished after first introducing an additional parameter:

$SH_A^i(\Delta) = [(x,y): \; (x,y)\varepsilon H_A^i$, and the distance from (x,y) to the boundary of H_A^i is at least $\Delta]$

$SH_A^i(\Delta)$ denotes that portion of the harbor in which it is certain that ship i will not run aground unless it moves a distance of more than Δ.

The approximation to be obtained involves the parameter $\Delta_i(t_j)$, which is defined as follows:

$$\Delta_i(t_j) = \frac{1}{2}\,SL + v_i(t_j)(t_j - t_{j-1}) \quad \text{(C.6-3)}$$

This parameter is contained in the probability equation below.

$$P\left[NG_i(t_{j-1},t_j) = 0 \;/\; NG_i(0,t_{j-1}) = 0\right] = P\left\{\left[x_i(t_j),y_i(t_j)\right] \varepsilon SH_A^i\left[\Delta_i(t_j)\right] \;/\; NG_i(0,t_{j-1}) = 0\right\} \quad \text{(C.6-4)}$$

where $SH_A^i[\Delta_i(t_j)]$ is a safe area in the harbor (water deep enough for ship i) for a distance $\Delta_i(t_j)$ equal to one-half the ship's length plus the distance the ship moves during the time interval $(t_j - t_{j-1})$.

The same assumptions about the model outputs make it possible to write the following equation for the probability density function.

$$P_{x_i(t_j),y_i(t_j)}[x,y] = \frac{1}{2\pi\, \sigma_{x_i}(t_j)\, \sigma_{y_i}(t_j) \sqrt{1-\rho_i^2(t_j)}} \cdot$$

$$\exp\left\{ -\frac{1}{2\left[1-\rho_i^2(t_j)\right]} \left[\frac{\left(x-\hat{x}_i(t_j)\right)^2}{\sigma_{x_i}^2(t_j)} + \frac{\left(y-\hat{y}_i(t_j)\right)^2}{\sigma_{y_i}^2(t_j)} \right.\right.$$

$$\left.\left. \frac{-2\cdot\rho_i(t_j)\left(x-\hat{x}_i(t_j)\right)\left(y-\hat{y}_i(t_j)\right)}{\sigma_{x_i}(t_j)\, \sigma_{y_i}(t_j)} \right] \right\} \qquad \text{(C.6-5)}$$

where $P_{x(t_j),\, y_i(t_j)}[x,y]$ is the joint probability density of $x_i(t_j)$ and $y_i(t_j)$, previously defined in Section C.2.

Using this probability density function, one may evaluate the probability of not grounding during the jth subinterval as follows:

$$P\left[NG_i\ (t_{j-1},t_j) = 0 \ / \ NG_i(0,t_{j-1}) = 0\right] =$$

$$\iint_{SH_A^i\left[\Delta_i(t_j)\right]} P_{x\,(t_j),\ y_i(t_j)}[x,y]\ dx\ dy \qquad \text{(C.6-6)}$$

Note: Certain factors not explicitly stated are implicit in equation C.6-6.

a. There are several ways to represent analytically the safe harbor area $SH_A^i[\Delta_i(t_j)]$ for ship i at time t_j.

b. When one of these ways has been chosen, the double integral in equation C.6-6 must be approximated. Appendix H provides one approach for the solution of Equation C.6-6.

c. The approach chosen and all other possible approaches have two disadvantages in common:

(1) They require large computer times for moderately complex harbors.

(2) They require large amounts of storage for moderately complex harbors.

Equation C.6-6 (including the implicit factors referred to above) provides a means for computing an approximation to the probability of grounding for a ship as it proceeds through the waterway with the vessel safety model. Suggestions for future mathematical model development are provided in Appendix I.

APPENDIX D

RELATING PROBABILITY OF WHOLE INTERVAL TO SUBINTERVALS

This appendix concerns itself with calculation of $P[NC_i(0,T_i)=0]$ and $P[NG_i(0,T_i)=0]$ in terms of the events on the subintervals (t_o,t_1), $(t_1,t_2),\ldots,(t_{N_i-1}, t_{N_i})$.

The definition of the joint probability of interdependent events is:

$$P[A\cdot B] = P[A/B]\cdot P[B] \tag{D-1}$$

If events A and B are defined by

$$A = \left[NC_i(t_{N-1},t_N) = 0\right] \tag{D-2}$$

and

$$B = \left[NC_i(0,t_{N-1}) = 0\right] \tag{D-3}$$

then

$$P\left[NC_i(0,T_i) = 0\right] = P\left[NC_i(t_{N-1},t_N) = 0 \;/\; NC_i(0,t_{N-1}) = 0\right] \cdot P\left[NC_i(0,t_{N-1}) = 0\right] \quad . \tag{D-4}$$

Applying this reasoning recursively yields,

$$P\left[NC_i(0,T_i) = 0\right] = \left\{\prod_{j=2}^{N_i} \left(P\left[NC_i(t_{j-1},t_j) = 0 \;/\; NC_i(t_o,t_{j-1}) = 0\right]\right)\right\}\cdot P\left[NC_i(t_o,t_1) = 0\right] \tag{D-5}$$

Using the same development for $NG_i(0,T_i)$ gives

$$P\left[NG_i(0,T_i) = 0\right] = \left\{ \prod_{j=2}^{N_i} \left(P\left[NG_i(t_{j-1},t_j) = 0 \;/\; NG_i(t_o,t_{j-1}) = 0\right]\right)\right\} \cdot P\left[NG_i(t_o,t_1) = 0\right] . \tag{D-6}$$

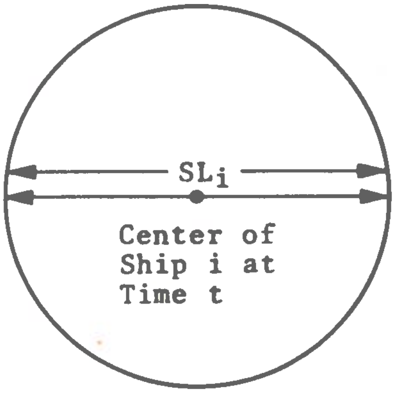
These equations corroborate equations C.1-4 and C.1-5, which were stated without proof.

APPENDIX E
DEVELOPMENT OF EQUATION C.2-4

This development involves specific definitions of factors relating to the probability of collision in a subinterval. These definitions are given for the topics below.

E.1 SPACE OCCUPIED BY A SHIP

This space is enclosed by a circle (Figure E-1), of which the center is at the midpoint of the ship and the radius is one-half the ship's length.



NOTE: For time steps of a minute or more, or for ship velocities greater than ten knots, δ_i is large relative to SL_i.

Figure E-1. Area Occupied By Ship i at Time t

E.2 COLLISION BETWEEN SHIPS i AND k

a. A collision takes place if the circle approximating the space occupied by ship i and the circle approximating the space occupied by ship k overlap at any time. (This definition involves a degree of reservation, since ships traveling in parallel paths can pass side by side without colliding, even though the circles representing the spaces occupied by the respective ships overlap. At high speeds, however, such a maneuver would be considered dangerous.)

b. The formal expression for a collision to take place between ship i and k is given by equation E-1 for time t in the interval (t_{j-1},t_j).

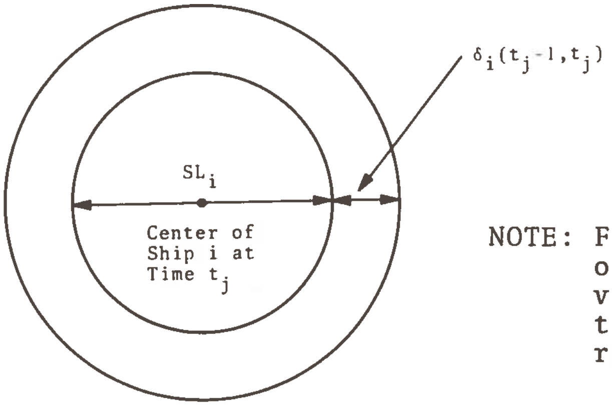
$$\left\{\left[x_i(t) - x_k(t)\right]^2 + \left[y_i(t) - y_k(t)\right]^2\right\}^{\frac{1}{2}} \leq \frac{1}{2}\,(SL_i + SL_k) \qquad \text{(E-1)}$$

Conversely, no collision will take place between ships i and k in the interval (t_{j-1},t_j) if

$$\left\{\left[x_i(t) - x_k(t)\right]^2 + \left[\ _i(t) - y_k(t)\right]^2\right\}^{\frac{1}{2}} > \frac{1}{2}\,(SL_i + SL_k) \qquad \text{(E-2)}$$

for all t in the interval (t_{j-1},t_j).

c. The next step in the development relates the conditions above to the ship positions at times t_{j-1} and t_j. In this case, the concept of space occupied by a ship is extended to account for the distance moved during an interval. The extended definition now states that the space occupied by a ship at time t is a circle with radius equal to one-half the ship's length plus the ship's velocity times the length of the time interval, indicated graphically in Figure E-2 as $[\frac{1}{2}\,SL_i + \delta_i(t_{j-1},t_j)]$.



NOTE: For time steps of a minute or more, or for ship velocities greater than ten knots, δ_i, is large relative to SL_i.

Figure E-2. Area Occupied by Ship i in the Interval From Time t_{j-1} to t_j

This exceeds the space required by a ship during a time interval; however, if the time steps are small (on the order of several seconds) the approximation will be reasonable, although the computer run time, if many ships are involved, may be excessive. For larger time steps, a directional capability to the space occupied by a ship must be developed to reduce the number of false alarms generated by this circle-of-influence collision probability calculation. This directional capability is not included in the present collision probability calculations.

d. The definition of a collision can now be modified to cover the above considerations. It becomes convenient to specify (for the probability of collision) that ships i and k will not collide between times t_{j-1} and t_j if

$$\left[\left(x_i(t_j) - x_k(t_j)\right)^2 + \left(y_i(t_j) - y_k(t_j)\right)^2\right]^{\frac{1}{2}} >$$
$$\left\{\left[\frac{1}{2}\,(SL_i+SL_k) + \left(v_i(t_j) + v_k(t_j)\right)\,(t_j-t_{j-1})\right]\right\} . \qquad \text{(E-3)}$$

This may be extended to include any number of ships, giving rise to the desired equation:

$$P\left[NC_i(t_{j-1},t_j) = 0 \,/\, NC_i(t_o,t_{j-1}) = 0\right] = P\left[d_{i,k}(t_j) >\right.$$
$$\left[\frac{1}{2}\,(SL_i+SL_k) + \delta_i(t_{j-1},t_j) + \delta_k(t_{j-1},t_j)\right]$$
$$\left. / \; NC_i(t_o,t_{j-1}) = 0\right] \qquad \text{(E-4)}$$

for all values of k=1, 2, ... , i-1, i+1, ... , N_s.

APPENDIX F

DERIVATION OF EQUATIONS C.5-15, C.5-16, AND C.5-17

In this appendix relations are developed among $\sigma_{\Delta x_{i,k}}(t_j)$, $\sigma_{\Delta y_{i,k}}(t_j)$, $\rho_{\Delta_{i,k}}(t_j)$; $\sigma_{x_i}^2(t_j)$, $\sigma_{y_i}^2(t_j)$, $\rho_i(t_j)$; and $\sigma_{x_k}^2(t_j)$, $\sigma_{y_k}^2(t_j)$, $\rho_k(t_j)$.

$$\begin{aligned}
\sigma_{\Delta x_{i,k}}^2(t_j) &= E\left\{\left[\Delta x_{i,k}(t_j) - \Delta x_{i,k}(t_j)\right]^2\right\} \\
&= E\left\{\left[\Delta x_{i,k}(t_j)\right]^2\right\} - \Delta\hat{x}_{i,k}^2(t_j) \\
&= E\left\{\left[x_i(t_j) - x_k(t_j)\right]^2\right\} - \left[\hat{x}_i(t_j) - \hat{x}_k(t_j)\right]^2 \\
&= E\left[x_i^2(t_j)\right] + E\left[x_k^2(t_j)\right] \\
&\quad - \hat{x}_i^2(t_j) - \hat{x}_k^2(t_j)
\end{aligned} \tag{F-1}$$

$$\sigma_{\Delta x_{i,k}}^2(t_j) = \sigma_{x_i}^2(t_j) + \sigma_{x_k}^2(t_j) \tag{F-2}$$

By an approach similar to the above,

$$\sigma_{\Delta y_{i,k}}^2(t_j) = \sigma_{y_i}^2(t_j) + \sigma_{y_k}^2(t_j) \ . \tag{F-3}$$

Consider the following steps for $\rho_{\Delta_{i,k}}(t_j)$.

$$\rho_{\Delta_{i,k}}(t_j) = \frac{E\left\{\left[\Delta x_{i,k}(t_j) - \Delta\hat{x}_{i,k}(t_j)\right]\left[\Delta y_{i,k}(t_j) - \Delta\hat{y}_{i,k}(t_j)\right]\right\}}{\sigma_{\Delta x_{i,k}}(t_j)\ \sigma_{\Delta y_{i,k}}(t_j)}$$

$$= \frac{E\left\{\left[\Delta x_{i,k}(t_j)\Delta y_{i,k}(t_j)\right]\right\} - \Delta\hat{x}_{i,k}(t_j)\Delta\hat{y}_{i,k}(t_j)}{\sigma_{\Delta x_{i,k}}(t_j)\,\sigma_{\Delta y_{i,k}}(t_j)} . \qquad (F-4)$$

$$\rho_{\Delta_{i,k}}(t_j) = E\left[\left(x_i(t_j) - x_k(t_j)\right)\left(y_i(t_j) - y_k(t_j)\right)\right]$$

$$- \frac{\left[\hat{x}_i(t_j) - \hat{x}_k(t_j)\right]\left[\hat{y}_i(t_j) - \hat{y}_k(t_j)\right]}{\sigma_{\Delta x_{i,k}}(t_j)\cdot\sigma_{\Delta y_{i,k}}(t_j)}$$

$$= E\left[x_i(t_j)\;y_i(t_j)\right] - \hat{x}_i(t_j)\cdot\hat{y}_i(t_j)$$

$$+ \frac{E\left[x_k(t_j)\;y_k(t_j)\right] - \hat{x}_k(t_j)\;\hat{y}_k(t_j)}{\sigma_{\Delta x_{i,k}}(t_j)\cdot\sigma_{\Delta y_{i,k}}(t_j)} \qquad (F-5)$$

$$\rho_{\Delta_{i,k}}(t_j) = \frac{\rho_i(t_j)\;\sigma_{x_i}(t_j)\;\sigma_{y_i}(t_j) - \rho_k(t_j)\;\sigma_{x_k}(t_j)\;\sigma_{y_k}(t_j)}{\sigma_{\Delta x_{i,k}}(t_j)\cdot\sigma_{\Delta y_{i,k}}(t_j)} \qquad (F-6)$$

The desired relationship has now been obtained. The variances $\sigma_{\Delta x_{i,k}}$ and $\sigma_{\Delta y_{i,k}}$ associated with x and y distances and the cross correlation function for distances $\Delta x_{i,k}$ and $\Delta y_{i,k}$, namely $\rho_{\Delta_{i,k}}$, have been expressed in terms of the variances σ_{x_i}, σ_{y_i} of ship's x and y coordinates and the cooridnate cross correlation function ρ_i.

APPENDIX G
APPROXIMATION OF EQUATION C.5-24

In this appendix, computational methods are considered for approximating the following equation:

$$P\left[d_{i,k}^{2}(t_j) > \Delta_{i,k}^{2}(t_{j-1},t_j) \;/\; NC_i(0,t_{j-1}) = 0\right] = 1.0$$

$$- \int_{-\Delta_{i,k}(t_{j-1},t_j)}^{+\Delta_{i,k}(t_{j-1},t_j)} \int_{-\sqrt{\Delta_{i,k}(t_{j-1},t_j) - y^2}}^{+\sqrt{\Delta_{i,k}(t_{j-1},t_j) - y^2}} P_\Delta(x,y)\, dx\, dy, \qquad \text{(G-1)}$$

where

$$P_\Delta(x,y) = \frac{1}{2\pi\, \sigma_{\Delta x_{i,k}}(t_j)\, \sigma_{\Delta y_{i,k}}(t_j) \sqrt{1-\rho_{\Delta_{i,k}}^{2}(t_j)}}$$

$$\cdot \exp\left\{-\frac{1}{2\left(1-\rho_{\Delta_{i,k}}^{2}(t_j)\right)}\left[\frac{\left(x-\Delta\hat{x}_{i,k}(t_j)\right)^2}{\sigma_{\Delta x_{i,k}}^{2}(t_j)} + \frac{\left(y-\Delta\hat{y}_{i,k}(t_j)\right)^2}{\sigma_{\Delta y_{i,k}}^{2}(t_j)} - \frac{2\rho_{\Delta_{i,k}}(t_j)\left(x-\Delta\hat{x}_{i,k}(t_j)\right)\left(y-\Delta\hat{y}_{i,k}(t_j)\right)}{\sigma_{\Delta x_{i,k}}(t_j)\, \sigma_{\Delta y_{i,k}}(t_j)}\right]\right\} \qquad \text{(G-2)}$$

To simplify the algebraic manipulations, make the following definitions:

$$\hat{x} = \Delta\hat{x}_{i,k}(t_j) \qquad \text{(G-3)}$$

$$\hat{y} = \Delta\hat{y}_{i,k}(t_j) \qquad \text{(G-4)}$$

$$\sigma_x^2 = \sigma_{\Delta i,k}^{2}(t_j) \qquad \text{(G-5)}$$

$$\sigma_y^2 = \sigma_{\Delta y_{i,k}}^{2}(t_j) \qquad \text{(G-6)}$$

$$\rho_1 = \rho_{\Delta_{i,k}}(t_j) \qquad \text{(G-7)}$$

$$\Delta = \Delta_{i,k}(t_{j-1}, t_j) \tag{G-8}$$

Rewrite equations (G-1) and (G-2) with the new notation to obtain:

$$P\left[d_{i,k}^{2}(t_j) > \Delta_{i,k}^{2}(t_{j-1}, t_j) \;/\; NC_i(0, t_{j-1}) = 0\right] = 1.0$$

$$- \int_{-\Delta}^{\Delta} \int_{-\sqrt{\Delta^2 - y^2}}^{\sqrt{\Delta^2 - y^2}} P_\Delta(x,y)\; dx\; dy, \tag{G-9}$$

where

$$P_\Delta(x,y) = \frac{1}{2\pi\; \sigma_x\; \sigma_y\; \sqrt{1-\rho^2}}$$

$$\cdot \exp\left\{- \frac{1}{2(1-\rho_1^2)}\left[\frac{(x-\hat{x})^2}{\sigma_x^2} + \frac{(y-\hat{y})^2}{\sigma_y^2} - \frac{2\rho_1\; (x-\hat{x})(y-\hat{y})}{\sigma_x\; \sigma_y}\right]\right\}. \tag{G-10}$$

To take advantage of the rotational properties of gaussian random variables, consider the following change of variables.

$$\begin{aligned} u &= x \cos(\alpha) - y \sin(\alpha) \\ v &= y \cos(\alpha) + x \sin(\alpha) \quad . \end{aligned} \tag{G-11}$$

The corresponding inverse transformation may be written as

$$\begin{aligned} x &= u \cos(\alpha) + v \sin(\alpha) \\ y &= v \cos(\alpha) + u \sin(\alpha) \quad . \end{aligned} \tag{G-12}$$

Since it is desired to integrate over an area such that

$$x^2 + y^2 < \Delta^2,$$

this means that, in terms of the new variables, integration is to be performed over an area such that

$$u^2 + v^2 < \Delta^2 \quad .$$

Rewriting equations (G-9) and (G-10) in terms of the new variables provides the following relationship:

$$P\left[d_{i,k}^{2}(t_j) > \Delta_{i,k}^{2}(t_{j-1},t_j) \;/\; NC_i(0,t_{j-1}) = 0\right] = 1.0$$

$$- \int_{-\Delta}^{\Delta} \int_{-\sqrt{\Delta^2-v^2}}^{\sqrt{\Delta^2-v^2}} P_{u,v}\,(u,v)\;du\;dv \qquad \text{(G-13)}$$

where

$$P_{uv}(u,v) = P_{\Delta}\big(u\cos(\alpha) + v\sin(\alpha),\; v\cos(\alpha) - u\sin(\alpha)\big) \qquad \text{(G-14)}$$

If the following variable assignments are made:

$$\sigma_u^2 = (1-\rho_1^2)\left[\frac{\cos^2(\alpha)}{\sigma_x^2} + \frac{2\rho_1\,\sin(\alpha)\cos(\alpha)}{\sigma_x\;\sigma_y} + \frac{\sin^2(\alpha)}{\sigma_y^2}\right]^{-1} \qquad \text{(G-15)}$$

$$\sigma_v^2 = (1-\rho_1^2)\left[\frac{\cos^2(\alpha)}{\sigma_y^2} + \frac{2\rho_1\,\sin(\alpha)\cos(\alpha)}{\sigma_x\;\sigma_y} + \frac{\sin^2(\alpha)}{\sigma_x^2}\right]^{-1} \qquad \text{(G-16)}$$

$$r = \sin\,(2\alpha)\left[\frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2}\right] - \frac{2\rho_1}{\sigma_x\sigma_y}\left[\cos(2\alpha)\right], \qquad \text{(G-17)}$$

then equation G-18 is obtained:

$$P_{uv}(v,u) = \frac{1}{2\pi\,\sigma_x\sigma_y\sqrt{1-\rho_1^2}} \cdot \exp\left\{-\frac{1}{2}\left[\frac{(u-\hat{u})^2}{\sigma_u^2} + \frac{(v-\hat{v})^2}{\sigma_v^2} - r\,(u-\hat{u})(v-\hat{v})\right]\right\}$$

$$\text{(G-18)}$$

The equation will be easier to integrate if r = 0. From the definition of r, if r = 0 (so that the cross correlation term in the exponential of equation G-18 is zero), then α^* must be defined as follows

$$\alpha^* = \frac{1}{2}\tan^{-1}\left[\frac{2\rho_1\;\sigma_x\sigma_x}{\sigma_y^2-\sigma_x^2}\right] \qquad \text{(G-19)}$$

Using the value α^* for α,

$$P\left[d_{i,k}^{2}(t_j) > \Delta_{i,k}^{2}(t_{j-1},t_j) \;/\; NC_i(0,t_{j-1}) = 0\right] = 1.0$$

$$- \int_{-\Delta}^{\Delta} \int_{-\sqrt{\Delta^2-v^2}}^{+\sqrt{\Delta^2-v^2}} P_u(u)\cdot P_v(v)\; du\; dv \qquad \text{(G-20)}$$

where

$$P_u(u) = \frac{1}{\sqrt{2\pi\,\sigma_u^2}} \exp\left[-\,\frac{(u-\hat{u})}{2\sigma_u^2}\right] \qquad \text{(G-21)}$$

$$P_v(v) = \frac{1}{\sqrt{2\pi\,\sigma_v^2}} \exp\left[-\,\frac{(v-\hat{v})}{2\sigma_v^2}\right] \qquad \text{(G-22)}$$

Finally, it is possible to overbound this integral by the following integral:

$$P\left\{d_{i,k}^{2}(t_j) > \Delta_{i,k}^{2}(t_{j-1},t_j) \;/\; NC_i(0,t_{j-1}) = 0\right\} \cong 1.0$$

$$-\left(\int_{-\Delta}^{+\Delta} P_u(u)\; du\right)\left(\int_{-\Delta}^{+\Delta} P_v(v)\; dv\right) \;. \qquad \text{(G-23)}$$

Then the integral may be computed by calls to a standard program to compute the gaussian distribution. This is the method used to evaluate the integral in equation (G-1).

APPENDIX H

APPROXIMATION OF EQUATION C.6-6

This appendix provides a method of approximating the following integral:

$$P\left[NG_i(t_{j-1},t_j) = 0 \;/\; NG_i(0,t_{j-1}) = 0\right] =$$

$$\iint\limits_{SH_A^i\left[\Delta_i(t_j)\right]} p_{X(t_j)Y(t_j)}(x,y)\;dx\;dy \qquad \text{(H-1)}$$

which is the probability that ship i will not ground in the jth time interval. $SH_A[\Delta_i(t_j)]$ is a safe area in the harbor providing water is deep enough for ship i for a distance equal to one-half the ship's length plus the distance the ship moves during the jth time interval, and

$$p_{X(t_j)Y(t_j)}(x,y) = \frac{1}{2\pi\,\sigma_{x_i}(t_j)\,\sigma_{y_i}(t_j)\sqrt{1-\rho_i^2(t_j)}}$$

$$\exp\left\{-\frac{1}{2\left(1-\rho_i^2(t_j)\right)}\left[\frac{\left(x-\hat{x}_i(t_j)\right)^2}{\sigma_{x_i}^{2}(t_j)} + \frac{\left(y-\hat{y}_i(t_j)\right)^2}{\sigma_{y_i}^{2}(t_j)}\right.\right.$$

$$\left.\left.-\frac{2\rho_i(t_j)\left(x-\hat{x}_i(t_j)\right)\left(y-\hat{y}_i(t_j)\right)}{\sigma_{x_i}(t_j)\,\sigma_{y_i}(t_j)}\right]\right\} \qquad \text{(H-2)}$$

To evaluate this integral, it is necessary to express the boundary of $SH_A^i[\Delta_i(t_j)]$ by some analytic expression. One possibility is to use a series of straight-line segments. Assume that the safe harbor area is closed, and that the straight-line segments close the harbor in a clockwise manner. Islands will be accounted for by integrating over them separately and subtracting their integrals from the whole harbor area.

To simplify the discussion, consider the following integral. The transformation back to the original problem should be obvious. The new problem is to evaluate the following integral:

$$I = \iint_A P_{XY}(x,y)\ dx\ dy, \tag{H-3}$$

where

$$P_{XY}(x,y) = \frac{1}{2\pi\ \sigma_x \sigma_y \sqrt{1-\rho^2}}$$

$$\cdot \exp\left\{-\ \frac{1}{2\sqrt{1-\rho^2}} \left[\frac{(x-\hat{x})^2}{\sigma_x^2} + \frac{(y-\hat{y})^2}{\sigma_y^2} - \frac{2\rho(x-\hat{x})(y-\hat{y})}{\sigma_x\ \sigma_y}\right]\right\}, \tag{H-4}$$

and where A is described by a closed set of straight line segments connected in a clockwise manner to surround A. Let

(x_j, y_j) = the end point of the j<u>th</u> line enclosing A

$(j = 1, 2, \cdots N)$

N = the total number of lines enclosing A.

Following the approach used in Appendix G, define (u,v) by the following relationships:

$$\begin{aligned} u &= x \cos(\alpha^*) - y \sin(\alpha^*) \\ v &= y \cos(\alpha^*) + x \sin(\alpha^*) \end{aligned} \tag{H-5}$$

and $(\hat{u}, \hat{v})$ by

$$\begin{aligned} \hat{u} &= \hat{x} \cos(\alpha^*) - \hat{y} \sin(\alpha^*) \\ \hat{v} &= \hat{y} \cos(\alpha^*) + \hat{x} \sin(\alpha^*) \end{aligned} \tag{H-6}$$

where

$$\alpha^* = \frac{1}{2} \tan^{-1}\left[\frac{2\rho\ \sigma_x \sigma_y}{\sigma_y^2 - \sigma_x^2}\right]\ . \tag{H-7}$$

Also, let

$$u_i = x_i \cos(\alpha^*) - y \sin(\alpha^*),\ i = 1,2,\cdots,N$$
$$v_i = y_i \cos(\alpha^*) + x_i \sin(\alpha^*),\ i = 1,2,\cdots,N \qquad \text{(H-8)}$$

and

$$\sigma_u^2 = (1-\rho^2)\left[\frac{\cos^2(\alpha^*)}{\sigma_x^2} + \frac{2\rho \sin(\alpha^*) \cos(\alpha^*)}{\sigma_x \sigma_y} + \frac{\sin^2 (\alpha^*)}{\sigma_y^2}\right] \qquad \text{(H-9)}$$

$$\sigma_v^2 = (1-\rho^2)\left[\frac{\cos^2(\alpha^*)}{\sigma_y^2} - \frac{2\rho \sin(\alpha^*) \cos(\alpha^*)}{\sigma_x \sigma_y} + \frac{\sin^2(\alpha^*)}{\sigma_x^2}\right] \qquad \text{(H-10)}$$

Using these new variables, write

$$I = \iint_A P_{UV} (u,v)\, du\, dv, \qquad \text{(H-11)}$$

where

$$P_{UV}(u,v) = \frac{1}{2\pi\, \sigma_u \sigma_v} \cdot \exp\left\{ - \frac{(u-\hat{u})^2}{2\sigma_u^2} - \frac{(v-\hat{v})^2}{2\sigma_v^2} \right\}, \qquad \text{(H-12)}$$

and A is enclosed by a series of straight line segments connecting (u_i,v_i) i = 1,2,$\cdots$$N_s$., in a clockwise direction.

Scaling and converting to polar coordinates make it possible to partially integrate this equation. That is, if

$$r^2 = \frac{(u-\hat{u})^2}{\sigma_u^2} + \frac{(v-\hat{v})^2}{\sigma_v^2} \qquad \text{(H-13)}$$

and if the end points of the line segments are scaled as follows, then it is possible to partially integrate the expression. To scale the endpoints of the line segments, set

$$\left.\begin{aligned}\tilde{u}_i &= u_i/\sigma_u \\ \tilde{v}_i &= v_i/\sigma_v\end{aligned}\right\} \quad i = 1,2,\cdots,N, \tag{H-14}$$

Equation H-15 can now be written,

$$I = \int_{\theta_i}^{\theta_f} \int_{r_i(\theta)}^{r_f(\theta)} \frac{r(\theta)}{2\pi} \exp\left[-\frac{r^2(\theta)}{2}\right] dr\, d\theta \quad ,$$

$$I = \frac{1}{2\pi} \int_{\theta_i}^{\theta_f} \left\{\exp\left[-r_i^2(\theta)\right] - \exp\left[-r_f^2(\theta)\right]\right\} d\theta \tag{H-15}$$

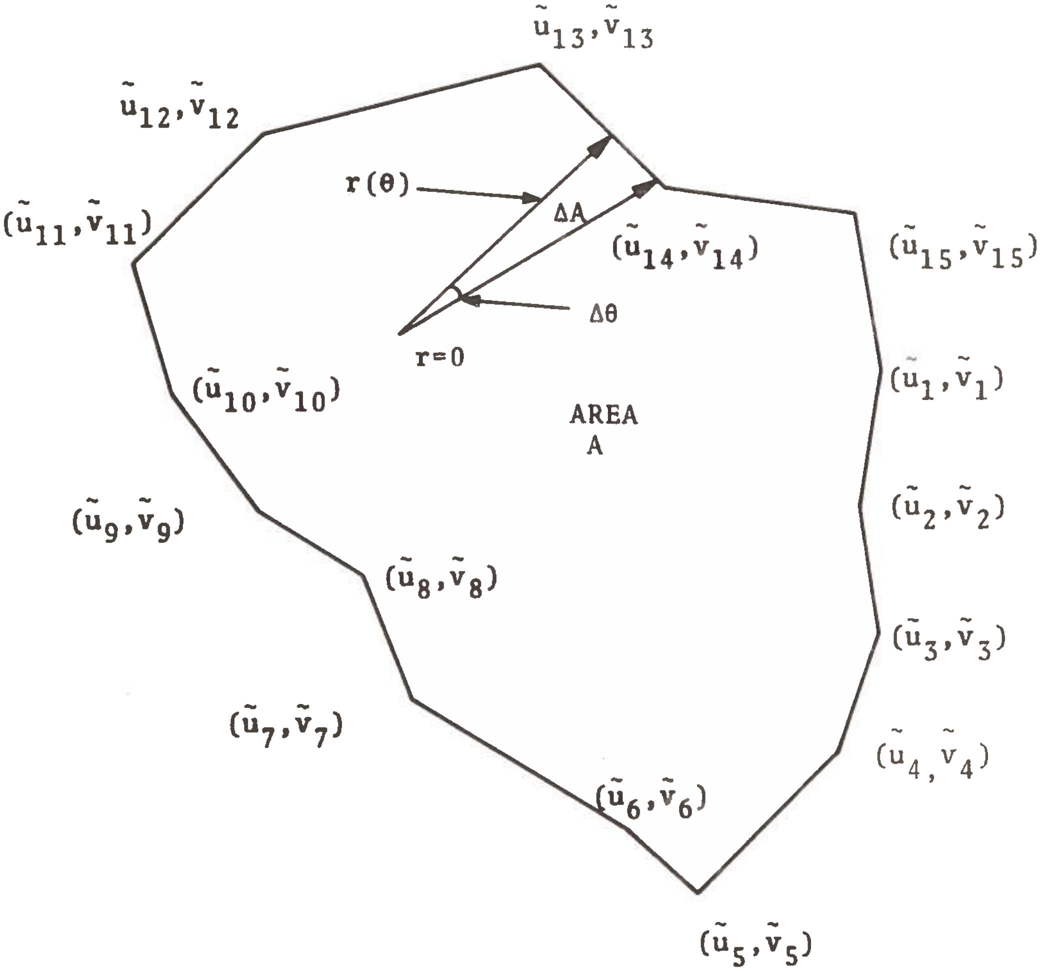
At this point, consider the solution for two different cases.

a. Case I: $(\hat{x},\hat{y}) \varepsilon A$

If $(\hat{x},\hat{y})$ are in A, it implies that the point $r = 0$ is also in A. This means that the limits of the integral must be $\theta_i = 0$, $\theta_f = 2\pi$, $r_i = 0$, and the value of $r_f(\theta)$ must be determined from the straight lines connecting the points $[(\tilde{u}_i,\tilde{v}_i), i = 1,2,\cdots,N]$. Figure H-1 provides an insight into how the integral is approximated.

b. Case II. $(\hat{x},\hat{y}) \not\varepsilon A$

If the point $(\hat{x},\hat{y})$ is not inside A, then the integral is evaluated in the same way, except that the limits are different We no longer go in a complete circle. However, we still go around the harbor. We add the contribution if θ increases and substacts it if θ decreases. The following diagram (Fig. H-2) will help explain the process. The integral may be approximated by going around the harbor enclosed by the straight lines and adding the contributions from each sector.

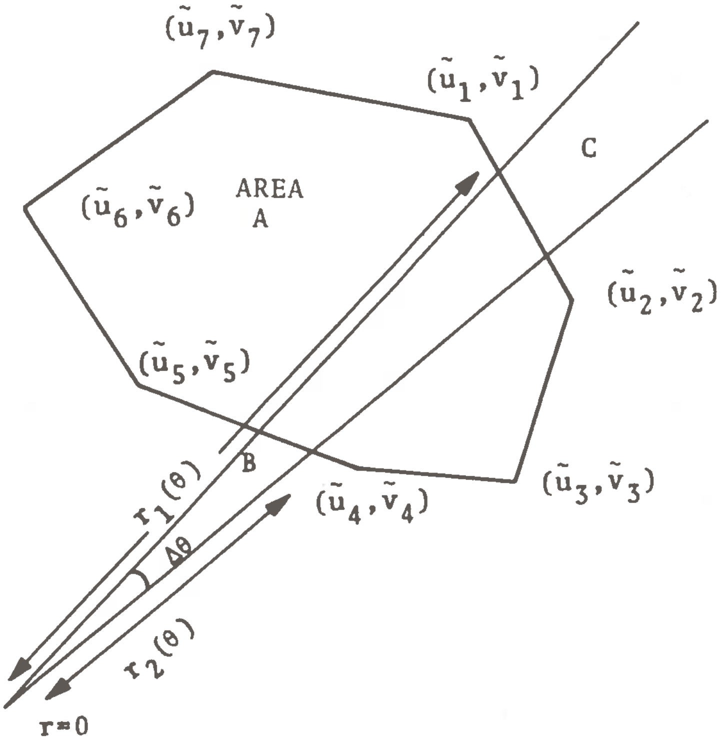


NOTE:

The area of sector ΔA contributes the following to the integral:

$$\frac{\Delta\theta}{2\pi}\left\{1 - \exp -\left[\frac{r^2(\theta)}{2}\right]\right\}$$

Figure H-1. Schematic for Integral Evaluation



NOTES:

1. The contribution of sector C is

$$\frac{\Delta\theta}{2\pi}\left\{1 - \exp\left[\frac{r_1^2(0)}{2}\right]\right\}$$

2. The contribution of sector B is

$$-\frac{\Delta\theta}{2\pi}\left\{1 - \exp\left[-\frac{r_2^2(0)}{2}\right]\right\}$$

Figure H-2. Schematic for Integral Evaluation When $(\hat{x},\hat{y})$ is Outside Area

Thus, by again going around the harbor boundary and adding or subtracting the contributions of each section, the value of the integral can be approximated.

As mentioned above, this procedure may be used for any island and the whole harbor as required. Unfortunately, this approach seems to require both a great deal of time and storage for each step.