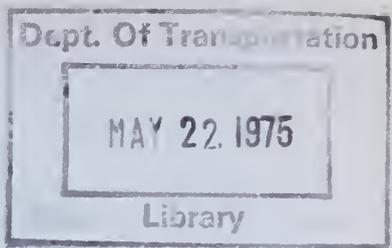


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MODULAR APPROACH TO STRUCTURAL SIMULATION  
FOR VEHICLE CRASHWORTHINESS PREDICTION

Pin Tong  
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FINAL REPORT

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## PREFACE

Computer simulation of structural deformation under vehicle crash conditions is an important tool for the prediction and design improvement of vehicle crashworthiness. As an alternative and aid to crash testing it also provides important cost benefits. There are various mathematical models, with varying degrees of sophistication, describing the dynamics behavior of structures. The present study is an attempt to unify the different approaches by using a modular concept. Such an approach can provide the flexibility to model different parts of a vehicle with different degrees of sophistication and computer efficiency, and, hopefully, it can help avoid the need for large, cumbersome software.



## CONTENTS

<u>Section</u>		<u>Page</u>
1	INTRODUCTION.....	1-1
2	MODULAR FORMULATION AND TIME INTEGRATION OF A DYNAMIC SYSTEM.....	2-1
3	ELEMENT MATRICES.....	3-1
	3.1 Beam Elements.....	3-1
	3.2 Spring Elements.....	3-13
	3.3 Rigid Body Elements.....	3-14
	3.4 Modal Element.....	3-16
4	CONCLUSIONS.....	4-1
	APPENDIX - EVALUATION OF $\underline{L}$ IN EQ. (3.1) AT TIME = 0.....	A-1
	REFERENCES.....	R-1



## LIST OF ILLUSTRATIONS

<u>Figure</u>		<u>Page</u>
1	Vehicle Schematic Sketch.....	2-1
2	Beam Orientation; Local and Global Axes.....	3-2
3	Force Resultants and Couples at Element Nodes.....	3-5
4	Schematic Diagram of the Stress Vector and the Incremental Plastic Strain Vector.....	3-8



## 1. INTRODUCTION

Computer simulation of structural deformation under vehicle crash conditions is an important tool for the prediction and design improvement of vehicle crashworthiness. As an alternative and aid to crash testing it also provides important cost benefits. Efficient computer models are needed to provide both economy and flexibility in all analytical simulation efforts. In particular, it is of value to be able to simulate a specific crash situation with a minimum of software. This should lead to simplicity of operation by engineers, and also aid in engineering judgement regarding prediction and design.

The Calspan-Shieh two dimensional frame program, the Battelle FMCCM lumped mass program, together with the 3-dimensional programs under development at Lockheed, the University of Michigan and TSC have been studied as to their feasibility and effectiveness for general usage by engineers.

The Battelle-FMCCM program<sup>1</sup> is essentially a large simplified spring-mass model which is restricted to unidirectional motion. This model handles collinear vehicle-to-vehicle as well as vehicle-to-barrier impact conditions. It can include four masses and up to 35 nonlinear resistances in the form of elasto-plastic springs, hydraulic energy absorber elements, and viscous dampers. The load deformation characteristics are obtained from theoretical and/or experimental data. The crash data must, of course, be interpreted carefully. This model provides the ability to predict overall vehicle response, and is useful in making parameter and design trade-off studies.

In regard to the frame models, the Calspan model<sup>2</sup> which represents the earliest development in this regard, is applicable to 2-dimensional frame structures. It is essentially a finite element model with straight-beam elements, lumped masses at nodes, and localized plastic hinges at pre-selected nodes. The program is operational, and at present, can be useful for simple front frame and bumper configurations.

The 3-D TSC/DOT model<sup>3</sup> presently being developed at the Transportation Systems Center can be regarded as a three-dimensional extension of the Calspan model, in the sense that the assumptions on ideal plastic hinges and lumped masses at nodes are the same as in Reference 2. The analysis and computer implementation of the TSC/DOT model is more heavily based on finite element techniques.<sup>4-6</sup> Another three-dimensional frame model, developed by Lockheed and referred to as KRASH<sup>7</sup> has been installed on TSC equipment. It is a 3-dimensional extension of the FMCCM model, consisting of mass connected straight "beam" elements. Forces and moments are determined from a linear elastic stiffness matrix which is adjusted for plasticity by multiplying by a stiffness reduction factor. The stiffness reduction factor concept is theoretically incorrect in three-dimensional problems, since the stress vector can move on the yield surface having one component increase and another component decrease in the same loading step. However, under some deformation mode, if the proper factor is used, it could produce a reasonable result. CRASH<sup>8</sup> is a more general three-dimensional frame model which uses plasticity theory. It has been used to model the vehicle to barrier impact of a Mustang. The data indicated that the simulation gave too stiff a result. The work at the University of Michigan falls in the general category of 3-dimensional frame models using the plastic hinge concept, and is discussed in Ref. 9. Reference 9 has also proposed a kind of modular concept to model major components of a vehicle.

Additional features which would produce more complete simulation should, for instance, include the ability to unite a rigid body element (such as the engine block) with a deformable structure, and the ability to allow an optimum mix of several finite elements (i.e., plate, beam and specialized spring elements) to represent vehicle components. Such features can be incorporated conveniently by means of a modular approach. Such an approach can provide both computer efficiency and detailed accuracy in a specific situation; hopefully, to avoid the need for large, cumbersome software.

The main purpose of this report is to describe a unified modular approach for vehicle structural simulation. Such

an approach will provide the flexibility to model different parts of a vehicle with different degrees of sophistication, depending on the type of information one is seeking. For example, in the frontal or oblique impact of a vehicle, detailed modelling of the front end may be necessary; however, the passenger compartment and the rear end can be modelled as a rigid body of finite size. In the case where more detailed information is required for the passenger compartment, such as differences in acceleration at several passenger locations, the elastic motion of the passenger compartment may have to be considered. In this case, a simple modal approximation can be used for the passenger compartment itself.

In the present study, an explicit scheme is used for numerical integration. A beam element which allows large three-dimensional rotations and plastic hinge action, a spring element which can only take tension or compression, a finite size rigid body element and an element using modal approximations are presented in detail in this report.



## 2. MODULAR FORMULATION AND TIME INTEGRATION OF A DYNAMIC SYSTEM

Consider a vehicle consisting of a finite number of components, such as bumpers, chassis, engine, passenger components, wheels, etc. A schematic sketch is shown in Figure 1. A component itself may consist of one or many individual elements. The behavior of a component (or an element) is expressed in terms of the generalized coordinates at a finite number of points of the component (or the element) in the form of component (or element) mass matrix  $\tilde{m}_i$ , damping matrix  $\tilde{c}_i$  and stiffness matrix  $\tilde{k}_i$ . Some or all of these points are connected to adjacent components and the discussion of the appropriate matrices will be given in Section 3.

The component matrices can, in general, be functions of the generalized coordinates and/or their rates of change. At the common nodes of all different components one should use the same type of generalized coordinates. Then, all the component (or element)

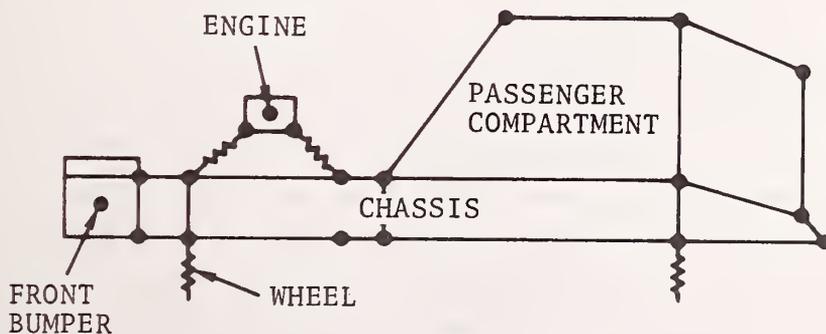


Figure 1. Vehicle Schematic Sketch

matrices can be assembled to yield the following system of equations of motion.

$$\underline{\underline{M}} \ddot{\underline{q}} + \underline{\underline{C}} \dot{\underline{q}} + \underline{\underline{K}} \underline{q} = \underline{F} \quad (2.1)$$

where  $\underline{\underline{M}}$ ,  $\underline{\underline{C}}$  and  $\underline{\underline{K}}$  are the assembled mass matrix, damping matrix, and stiffness matrix respectively;  $\underline{q}$  is the vector and unconstrained coordinates and  $\underline{F}$  is the loading vector due to external applied loads or prescribed constraints.

Equations (2.1) comprise a second order system. With given initial values of  $\underline{q}$  and  $\dot{\underline{q}}$  the time histories of  $\ddot{\underline{q}}$  can be obtained by numerical integration. There are many numerical integration schemes; implicit or explicit. To choose a particular scheme to perform numerical integration one will have to be concerned with the possibility of introducing artificial damping and numerical instability (i.e., introducing artificial energy to the system in the integration process). For a linear dynamic system, how these numerical problems are related to the integration stepsize is well understood. In most cases, the implicit schemes introduce artificial damping and are unconditionally stable, while the explicit schemes introduce energy if the stepsize is too large, and are only conditionally stable. However, these assertions have not been established and cannot be considered true for a nonlinear system. An unstable scheme can be made conditionally stable only. In general an implicit scheme will take much longer computer time for each integration step than that of an explicit scheme of comparable truncation error. This is because for a nonlinear system certain matrices must be redefined at each time step. In the implicit scheme, the updated system of equations would have to be resolved, while the explicit scheme only involves updating the right hand side of the system of equations. For large systems this can be very time consuming. Therefore, the explicit integration scheme is used in this study.

The simplest scheme is the central difference method, which allows us to write (2.1) in the form,

$$\begin{aligned} \tilde{M}_n (\Delta q_{n+1} - \Delta q_n) + \frac{\Delta t}{2} \tilde{C}_n (\Delta q_{n+1} + \Delta q_n) \\ + (\tilde{K}_n q_n - \tilde{F}_n) \Delta t^2 = 0 \end{aligned} \quad (2.2)$$

or

$$\begin{aligned} \left( \tilde{M}_n + \frac{\Delta t}{2} \tilde{C}_n \right) \Delta q_{n+1} = \left( \tilde{M}_n - \frac{\Delta t}{2} \tilde{C}_n \right) \Delta q_n \\ - (\tilde{K}_n q_n - \tilde{F}_n) \Delta t^2 \end{aligned} \quad (2.3)$$

where

$$q_n = q_{n+1} + \Delta q_n \quad (2.4)$$

and

$$\dot{q}_n = \frac{q_{n+1} - q_{n-1}}{\Delta t}; \quad \ddot{q}_n = \frac{q_{n+1} - 2q_n + q_{n-1}}{\Delta t^2}$$

The subscript n denotes the value of the quantity at time step n and  $\Delta t$  is the step size. From the given initial values  $q_0$  and  $\dot{q}_0$  we have

$$\Delta q_1 = \dot{q}_0 \Delta t + \frac{1}{2} (\Delta t)^2 a_0 \quad (2.5)$$

where  $a_0$  is the initial acceleration which is evaluated from (2.1)

$$\tilde{M} a_0 = - \left[ \tilde{C} \dot{q}_0 + \tilde{K}_0 q_0 - \tilde{F}_0 \right] \quad (2.6)$$

The subsequent  $q$ 's and  $\Delta q$ 's are evaluated from (2.3) and (2.4). For a problem where  $\tilde{M}$  and  $\tilde{C}$  are independent of time and  $q$  itself, or are block diagonal matrices,  $\Delta q$ , of each integration step can be determined very rapidly.

It is noted that in (2.3), one will only need the product of  $\tilde{K}_n$  and  $q_n$ . Thus, in actual computation, there is no need to assemble  $\tilde{K}_n$ . That is, at a given time step, if we let  $q^i$  and  $k_i$  be the generalized coordinates and the stiffness matrix of the  $i^{\text{th}}$  component (or element) respectively, then  $f^i = k_i q^i$  will be a

generalized force vector of the element due to the deformation  $\tilde{q}^i$ . In order to obtain  $\tilde{K} \tilde{q}$  at this time step, one only has to assemble  $\tilde{f}_i$  for all the elements. In this way, there is no multiplication of  $\tilde{K}$  and  $\tilde{q}$ . The actual multiplication is performed at the element level between  $\tilde{k}_i$  and  $\tilde{q}^i$ . The advantage of assembling  $\tilde{f}_i$  instead of  $\tilde{k}_i$  is the saving in core storage in the computer. Generally, larger core storage is required to store  $\tilde{K}$  (roughly number of degrees of freedom times the average semi-bandwidth). In some instances,  $\tilde{f}_i$  can be generated directly without even having to compute  $\tilde{k}_i$ . Detailed discussion of the computation of  $\tilde{f}_i$  will be given in the next section.



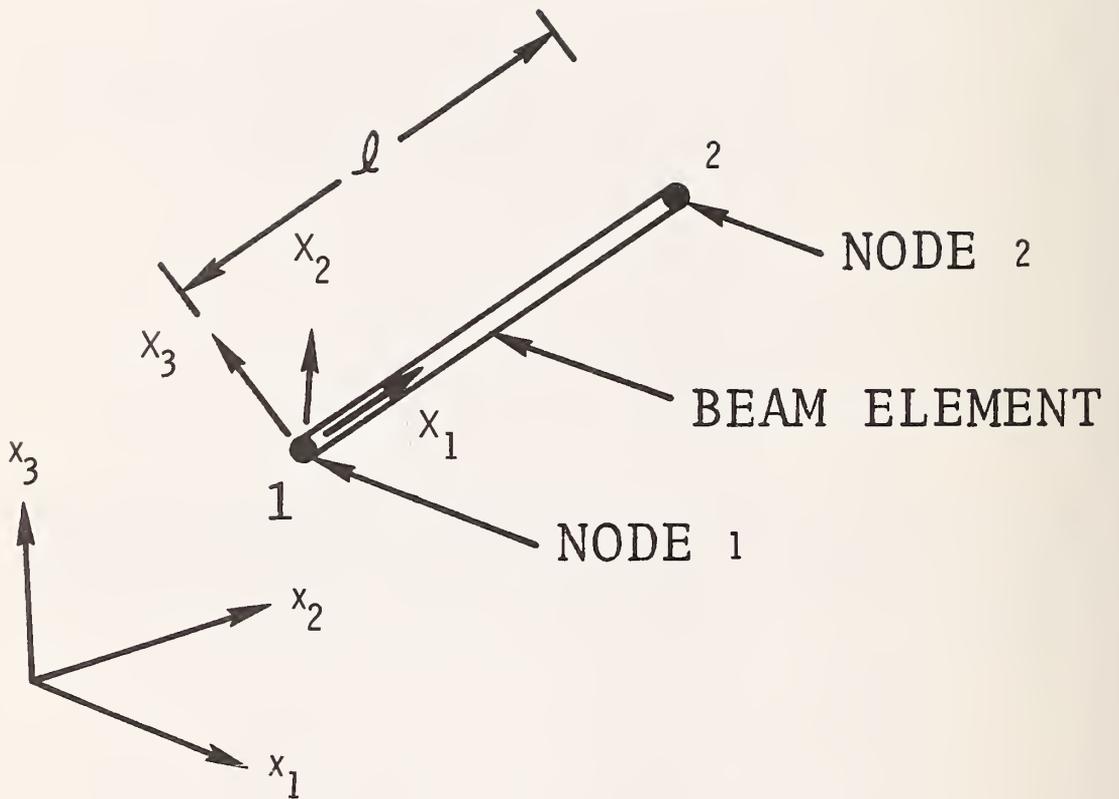


Figure 2. Beam Orientation; Local and Global Axes

in which

$$\begin{aligned}
 a &= \frac{N}{\ell} - \frac{2b}{\ell} \\
 b &= (\cosh \beta - 1)/D \\
 c &= \left( \cosh \beta - \frac{1}{\beta} \sinh \beta \right) / D \\
 d &= \left( \frac{\sinh \beta}{\beta} - 1 \right) / D
 \end{aligned} \tag{3.3}$$

and where

$$\begin{aligned}
 D &= \frac{\ell^2}{EI_2} \frac{\sinh \beta}{\beta} \left( \frac{\sinh \beta}{\beta} - 1 \right) - (\cosh \beta - 1)^2 / N \\
 \beta &= \sqrt{\frac{N\ell^2}{EI_2}}
 \end{aligned}$$

and similarly for  $a'$ ,  $b'$ ,  $c'$  and  $d'$  with  $I_2$  being replaced by  $I_3$  in the values of  $\beta$  and  $D$  given in (3.3). The generalized coordinates associated with the element stiffness matrix given by (3.2) are given in incremental form as

$$\Delta \underline{q}^T = \left\{ \Delta \underline{u}_{\sim 1}^T, \Delta \underline{\theta}_{\sim 1}^T, \Delta \underline{u}_{\sim 2}^T, \Delta \underline{\theta}_{\sim 2}^T \right\} = \left\{ \Delta q_1, \Delta q_2, \Delta q_3, \dots, \Delta q_{12} \right\} \tag{3.4}$$

which are the three displacements  $\Delta \underline{u}_{\sim 1}$  followed by the three rotations  $\Delta \underline{\theta}_{\sim 1}$  of node 1 in the local directions shown in Figure 2; then followed by the corresponding quantities of node 2. The element stiffness matrix given in 3.2 is an exact representation of the beam equation given below. The matrix 3.2 yields values which are equal to those of the closed form solutions at the nodes. It is derived by using the homogeneous solution of the straight beam equation with initial stress as the interpolation function (Ref. 10). In our case the interpolation function is

$$w = C_1 \cosh \sqrt{\frac{N}{EI}} s + C_2 \sinh \sqrt{\frac{N}{EI}} s + C_3 s + C_4$$

which is a solution to

$$EI \frac{d^4 w}{ds^4} + N \frac{d^2 w}{ds^2} = 0$$

and  $N$  is the initial stress.

For small increments of elastic deformation, the incremental generalized forces of the element, in correspondence to the generalized coordinates in (3.4), are

$$\Delta \tilde{f} = \left\{ (\Delta \tilde{F})_1^T, (\Delta \tilde{M})_1^T, (\Delta \tilde{F})_2^T, (\Delta \tilde{M})_2^T \right\} = \tilde{k} \Delta \tilde{q} \quad (3.5)$$

where

$$(\Delta \tilde{F})_i = \left\{ \begin{array}{c} \Delta N \\ \Delta Q_2 \\ \Delta Q_3 \end{array} \right\}_i \quad (\Delta \tilde{M})_i = \left\{ \begin{array}{c} \Delta M_1 \\ \Delta M_2 \\ \Delta M_3 \end{array} \right\}_i$$

and  $(\Delta \tilde{F})_i$  is the force resultant vector at node  $i$  while  $(\Delta \tilde{M})_i$  is the moment resultant vector at node  $i$ , as shown in Figure 3. Note that  $\Delta \tilde{F}_1 = -\Delta \tilde{F}_2$  for equilibrium.

For increments involving plastic deformation,  $\Delta \tilde{f}$  must be computed differently. The detailed procedure will depend upon the type of plasticity theory used. In the present report, we shall only use the incremental theory of the plastic hinge concept. The yielding surface  $\Phi$ , is assumed to be expressible in the form

$$\Phi (M_i, N, K) = 0 \quad (3.6)$$

where  $K$  is a hardening parameter which depends on the strain history,  $M_i$  ( $i = 1, 2, 3$ ) and  $N$  are moments and stress resultant respectively. If  $\Phi < 0$  or  $\Phi = 0$  and the first variation

$$\Delta \Phi = \frac{\partial \Phi}{\partial M_i} \Delta M_i + \frac{\partial \Phi}{\partial N} \Delta N + \frac{\partial \Phi}{\partial K} \Delta K < 0$$

there will be only elastic deformation. When  $\Phi = 0$ ,  $\Delta \Phi \geq 0$  plastic deformation occurs. However,  $\Delta \Phi > 0$  is not permitted. We must determine the portion of total deformation that is plastic to

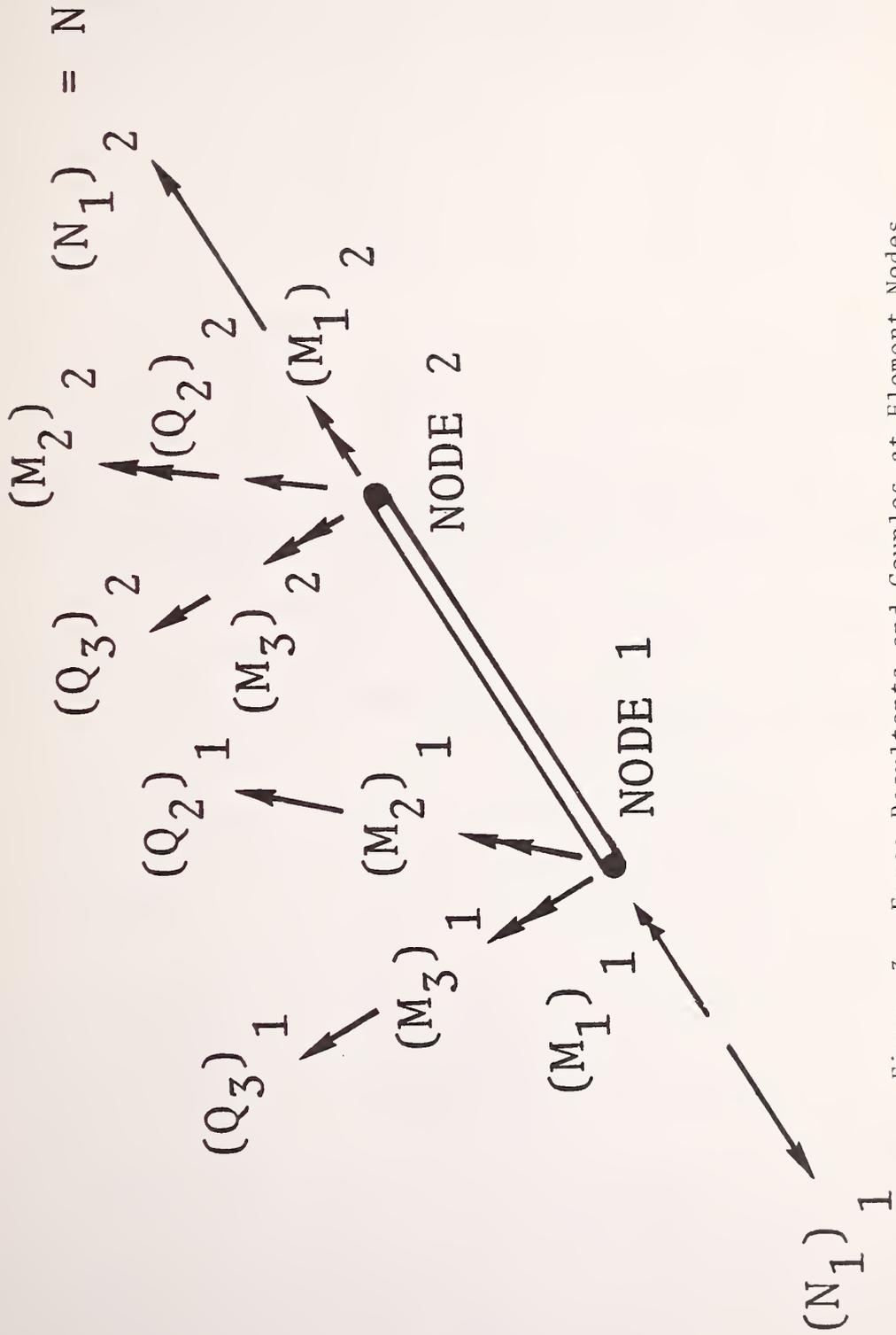


Figure 3. Force Resultants and Couples at Element Nodes

insure  $\Delta\phi = 0$ . Let  $\Delta\tilde{\gamma}$  denote the total incremental strain due to the incremental deformation associated with  $\Delta q$  so that

$$\Delta\tilde{\gamma} = \begin{Bmatrix} \Delta e_1 \\ \Delta K_1 \\ \Delta K_2 \\ \Delta K_3 \end{Bmatrix} \quad (3.7)$$

where  $\Delta e_1$ , is due to stretching,  $\Delta K_1$ , is due to torsion and  $\Delta K_2$  and  $\Delta K_3$  are due to bending which can be expressed in terms of incremental displacements. For example,  $\Delta K_3$  is the bending strain about the  $x_3$ -axis, which is equal to  $EI_3 d^2(\Delta u_2)/dx_1^2$  where  $\Delta u_2$  is the incremental displacement in the  $x_2$  direction. The total incremental strain can be separated into a part due to elastic deformation and a part due to plastic deformation, namely

$$\Delta\tilde{\gamma} = \Delta\tilde{\gamma}^e + \Delta\tilde{\gamma}^p$$

The incremental stresses will be computed from  $\Delta\tilde{\gamma}^e$ , the elastic incremental strain as follows,

$$\Delta\tilde{\sigma} = \begin{Bmatrix} \Delta N \\ \Delta M_1 \\ \Delta M_2 \\ \Delta M_3 \end{Bmatrix} = \tilde{E} \Delta\tilde{\gamma}^e \quad (3.8)$$

or

$$\Delta\tilde{\sigma} = \tilde{E} \Delta\tilde{\gamma} - \Delta\tilde{\gamma}^p = \Delta\tilde{\sigma}' - \tilde{E} \Delta\tilde{\gamma}^p \quad (3.9)$$

where

$$\Delta\tilde{\sigma}' = \tilde{E}\Delta\tilde{\gamma} = \begin{bmatrix} AE & 0 & 0 & 0 \\ 0 & GJ & 0 & 0 \\ 0 & 0 & EI_3 & 0 \\ 0 & 0 & 0 & EI_2 \end{bmatrix} \Delta\tilde{\gamma} \quad (3.10)$$

in which  $\Delta\tilde{\sigma}$  is a function of distance along the beam element. When evaluated at an end point,  $\Delta\tilde{\sigma}'$  will yield exact values by virtue of (3.2), and contribute to corresponding components in (3.5). ( $\Delta\tilde{\sigma}'$  is equal to the incremental stresses  $\Delta\tilde{\sigma}$ , if this increment due to  $\Delta q$  is elastic.) We shall evaluate  $\Delta\tilde{\gamma}^P$ , the incremental strain vector based on incremental plasticity theory. According to the flow rule,  $\Delta\tilde{\gamma}^P$  is normal to the yielding surface given by (3.6), i.e., (a schematic diagram is given in Figure 4).

$$\Delta\tilde{\gamma}^P = \lambda \begin{Bmatrix} \partial\phi/\partial N \\ \partial\phi/\partial M_1 \\ \partial\phi/\partial M_2 \\ \partial\phi/\partial M_3 \end{Bmatrix} \quad (3.11)$$

evaluated on the yield surface

where  $\lambda$  is a proportionality factor, which is to be determined so that the total cumulative stress vector,  $\tilde{\sigma}$ , lies on the yield surface. The total cumulative stress vector can be written as the sum of  $\tilde{\sigma}_0$ , the stress vector just before the present increment and  $\Delta\tilde{\sigma}$ , i.e.,

$$\tilde{\sigma} = \begin{Bmatrix} N \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \tilde{\sigma}_0 + \Delta\tilde{\sigma} = \tilde{\sigma}_0 + \Delta\tilde{\sigma}' - \lambda E \begin{Bmatrix} \partial\phi/\partial N \\ \partial\phi/\partial M_1 \\ \partial\phi/\partial M_2 \\ \partial\phi/\partial M_3 \end{Bmatrix} \quad (3.12)$$

on yielding surface

When the expressions given by (3.12) are substituted into (3.6) an equation is obtained which allows determination of  $\lambda$ .

A simple example of a possible yield criterion is

$$\phi = a_4 N^2 + a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2 - K_0 = 0 \quad (3.13)$$

where the a's are constants which depend on the geometry of the cross-section of the beam, and  $K_0$  is another constant related to the square of the yield stress. In the case of elastic perfectly plastic material,  $K_0$  is independent of strain history. To

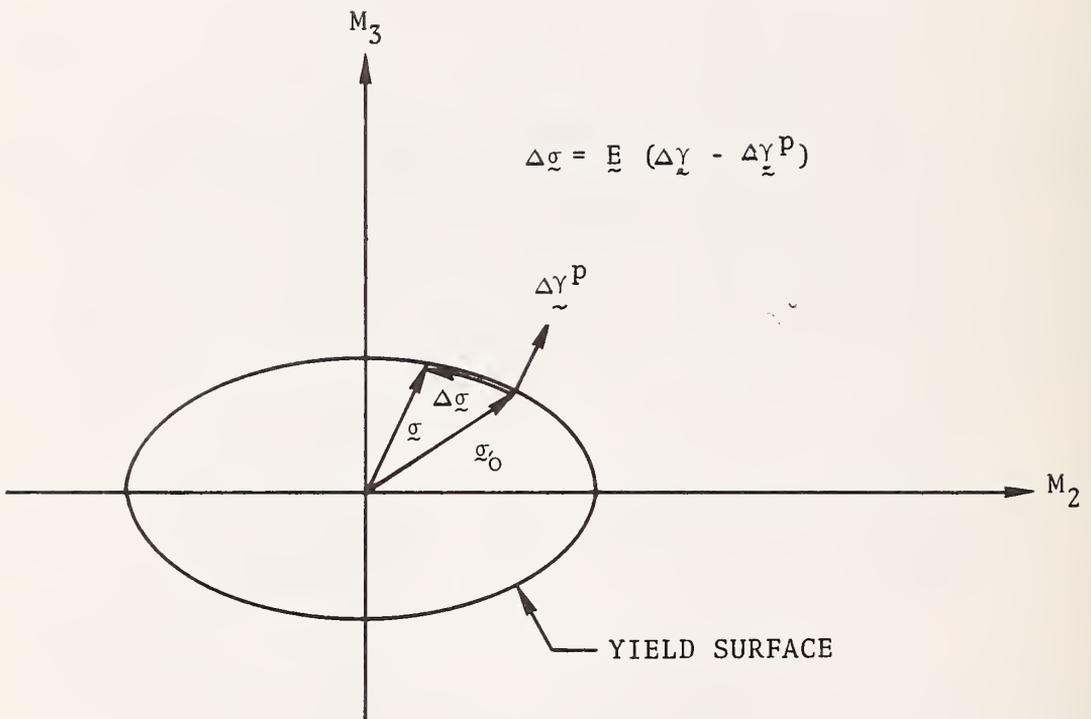


Figure 4. Schematic Diagram of the Stress Vector and the Incremental Plastic Strain Vector

determine  $\lambda$  in this case, we suppose that  $\underline{\sigma}_0$  lies within the yield surface given by  $\phi = 0$  (Eq. 3.13), while  $\underline{\sigma}_0 + \Delta\underline{\sigma}'$  lies outside. The next step is to determine the values of the M's and N on the yield surface. To do this let

$$\underline{\sigma}'' = \underline{\sigma}_0 + \lambda' \Delta\underline{\sigma}' \quad (3.14)$$

and determine the factor  $\lambda'$  so that  $\underline{\sigma}''$  just falls on the yield surface (i.e.,  $\lambda' \Delta\underline{\sigma}'$  is an elastic increment). Substituting (3.14) into (3.13) get

$$A(\lambda')^2 + 2B\lambda' + C = 0 \quad (3.15)$$

where

$$A = a_4 (\Delta N')^2 + \sum_{i=1}^3 a_i (\Delta M_i')^2$$

$$B = - \left[ a_4 N_0 \Delta N' + \sum_{i=1}^3 a_i (M_i)_0 \Delta M_i' \right] \quad (3.16)$$

$$C = a_4 N_0^2 + \sum_{i=1}^3 a_i (M_i)_0^2 - K_0$$

Then

$$\lambda' = \frac{C}{B \left( 1 + \sqrt{1 - AC/B^2} \right)} \quad (3.17)$$

With  $\lambda'$  defined by (3.17),  $\underline{\sigma}''$  will be on the yield surface., whose components are  $N_1''$ ,  $M_1''$ ,  $M_2''$ , and  $M_3''$ . Now in order to compute  $\lambda$  given in (3.11) the following partial derivatives are needed

$$\left\{ \begin{array}{l} \partial\Phi/\partial N \\ \partial\Phi/\partial M_1 \\ \partial\Phi/\partial M_2 \\ \partial\Phi/\partial M_3 \end{array} \right\} = \left\{ \begin{array}{l} 2a_4 N'' \\ 2a_1 M_1'' \\ 2a_2 M_2'' \\ 2a_3 M_3'' \end{array} \right\} \quad (3.18)$$

On substituting (3.12) (also using (3.18)) into (3.13), an equation of the same form as that of (3.15) is obtained to determine  $\lambda$ . Once  $\lambda$  is determined,  $\Delta\sigma$  within the element can be evaluated from (3.9) and (3.11).

In the present paper we are restricting our consideration to the plastic hinge concept with hinges limited to the end nodes only. In the finite element theory of beams,  $N$  and  $M_1$  are constant throughout the element while  $M_2$  and  $M_3$ , in general, vary. In order that internal equilibrium always be maintained (i.e.,  $N$  and  $M_1$  remain constant throughout the element even if yielding occurs at both ends), the yield surface at node 2 must be modified to the form given by Eq. (3.19) if node 1 has already yielded.

$$\Phi = a_2 M_2^2 + a_3 M_3^2 - K = 0 \quad (3.19)$$

where the constant  $K$  is given by

$$K = K_0 - a_4 N^2 - a_1 M_1^2$$

The stress resultant,  $N$ , and the twisting moment  $M_1$  are the values at node 1, and fall on the yield surface at node 1. The same remarks hold when node 2 yields first, in which case node 1 and node 2 interchange roles.

Now, when a hinge appears at a node, all plastic deformation is assumed to be concentrated at that node, and the rest of the elements still deforms elastically (i.e., the matrix in (3.2) still applies just outside the plastic zone). For the node which becomes a plastic hinge the moment increments and the stress resultant increments at such a hinge are determined according to Eqs. (3.8)-(3.12); then one can determine the incremental displacements and

rotations at the hinged node that are associated with such incremental moments and forces. For example, if node 1 yields and node 2 does not we can solve for  $\Delta\theta_1$  at node one by making use of the three equations implied by the fourth through sixth rows of (3.2). The following expressions are easily derived

$$\begin{aligned} (\Delta\theta_1)_1 &= \lambda (\Delta M_1)_1 / GJ + (\Delta\theta_1)_2 \\ (\Delta\theta_2)_1 &= \frac{1}{c} \left\{ (\Delta M_2)_1 + b \left[ (\Delta u_3)_2 - (\Delta u_3)_1 \right] + d (\Delta\theta_2)_2 \right\} \\ (\Delta\theta_3)_1 &= \frac{1}{c'} \left\{ (\Delta M_3)_1 + b' \left[ (\Delta u_2)_2 - (\Delta u_2)_1 \right] + d' (\Delta\theta_3)_2 \right\} \end{aligned} \quad (3.20)$$

where the subscript of a parenthesis denotes the node while the subscript of a quantity denotes direction or sense. The  $\Delta M$ 's given in (3.20) are the incremental moments at node 1, which have been calculated using the flow rule as discussed in relation to Eqs. (3.11) through (3.18), so that plastic deformation has already been accounted for. Subsequently, the  $\Delta M$ 's and  $\Delta Q$ 's at node 2 should be recomputed utilizing the equations implied by the eight through twelfth rows of (3.2), where values of the rotations  $\Delta\theta$ , as computed in (3.20) for node 1 are used. It is clear, then, that the incremental changes in quantities at node 1 due to plastic hinge action will affect the quantities at node 2.

The total cumulative forces and moments are the sum of those existing at the last time step and those of the present increment. In these computations the forces and the moments are expressed in the local beam coordinate system. They must be transformed to global coordinates before they can be assembled properly. First, the matrix  $\tilde{L}$  which relates the global coordinates to local coordinates, as defined in Appendix A, must be updated using the results in Appendix A, namely

$$\tilde{L}_i = (L_i)_0 \begin{bmatrix} 1 & \Delta\phi_3 & -\Delta\phi_2 \\ -\Delta\phi_3 & 1 & \Delta\phi_1 \\ \Delta\phi_2 & -\Delta\phi_1 & 1 \end{bmatrix}_i \quad (i = 1, 2) \quad (3.21)$$

where  $(L_{\tilde{i}})_i$  is the value of  $L_{\tilde{i}}$  at the previous step, and the subscript  $i$  denotes node 1 or 2. The  $\Delta\phi$ 's are rotation increments with respect to global axes. (Note that the vector  $\Delta\phi_i$  as defined previously is given by  $\Delta\theta_{\tilde{i}} = (\Delta\theta_1)_i (\underline{e}_1)_i + (\Delta\theta_2)_i (\underline{e}_2)_i + (\Delta\theta_3)_i (\underline{e}_3)_i = (\Delta\phi_1)_i \underline{i} + (\Delta\phi_2)_i \underline{j} + (\Delta\phi_3)_i \underline{k}$  where the local base vector  $\underline{e}_1$ ,  $\underline{e}_2$  and  $\underline{e}_3$  are defined in Appendix A.)

If we let  $\underline{\tilde{f}}$  denote the vector of cumulative values of the forces and moments for an element in the local coordinate system, then the corresponding values in the global coordinate system will be

$$\begin{bmatrix} L_{\tilde{1}} & 0 & 0 & 0 \\ 0 & L_{\tilde{1}} & 0 & 0 \\ 0 & 0 & L_{\tilde{2}} & 0 \\ 0 & 0 & 0 & L_{\tilde{2}} \end{bmatrix}^T \underline{\tilde{f}} \quad (3.22)$$

where  $L_{\tilde{1}}$  and  $L_{\tilde{2}}$  are the matrices defined by Eq. (3.1).

This section will now be completed with a discussion of the mass matrix. It is clear that a consistent element mass matrix can be derived in a straightforward manner by means of well known methods in finite element theory (Refs. 4-6). However, based on considerations involving questions of numerical instability (Ref. 2), we shall use a lumped element mass matrix herein, given by

$$\underline{\tilde{m}} = \begin{bmatrix} m_{\tilde{1}} & 0 & 0 & 0 \\ 0 & m_{\tilde{2}} & 0 & 0 \\ 0 & 0 & m_{\tilde{1}} & 0 \\ 0 & 0 & 0 & m_{\tilde{2}} \end{bmatrix} \quad (3.23)$$

where

$$\tilde{m}_1 = \rho \frac{\ell A}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \tilde{m}_2 = \begin{bmatrix} \frac{\rho \ell A}{2} & 0 & 0 \\ 0 & \rho \frac{\ell^3 A}{12} & 0 \\ 0 & 0 & \rho \frac{\ell^3 A}{12} \end{bmatrix} \quad (3.24)$$

where  $\rho$ ,  $\ell$  and  $A$  are the density, length and cross-sectional area, respectively, of the uniform element. The above element mass matrix is associated with the local coordinate system, where the  $\tilde{m}_2$  matrix which is concerned with rotations is based on local principal axes. When transformed to global coordinates,

$$m = \begin{bmatrix} \tilde{m}_1 & 0 & 0 & 0 \\ 0 & L_1^T \tilde{m}_2 L_1 & 0 & 0 \\ 0 & 0 & \tilde{m}_1 & 0 \\ 0 & 0 & 0 & L_2^T \tilde{m}_2 L_2 \end{bmatrix} \quad (3.25)$$

the part of the mass matrix due to rotary inertia is a function of time, (since the matrices  $L_1$  and  $L_2$  are functions of time). However, it is felt that the rotary inertia of the beam type structural elements will not affect greatly the dynamics of the system. In the numerical integration we shall treat the element mass matrix as a constant in the form of (3.25), after it is updated for each time step. This approach is correct for the translational inertia portion depicted by  $\tilde{m}_1$ . However, for the rotary inertia portion, depicted by  $L_1^T \tilde{m}_2 L_1$  and  $L_2^T \tilde{m}_2 L_2$  the changes from constant values are of the order  $\dot{\theta}^2/\ddot{\theta}$  with respect to unity, which is deemed small in crash situations.

### 3.2 SPRING ELEMENTS

Spring elements can be regarded as special cases of the beam element with  $I_2 = I_3 = 0$ , and can be handled as such computationally. There are, however, certain special springs which arise naturally in vehicle crashworthiness studies, and so are worthy of

discussion here. In what follows some computational features of such springs will be developed.

(a) Ground spring - This is used to simulate a wheel supporting the vehicle on the ground. The force transmitted through a wheel is always assumed to be in the vertical direction and is either zero or in compression. Let us assume that the point where the wheel touches the ground is node 1 and the point where the wheel supports the vehicle is node 2. In local coordinates, with the  $X_1$  -axis pointing from node 1 to node 2, and considering the order of components in the force vector given in (3.5), all the components in  $\tilde{f}$  will be zero in this case except  $f_1$  and  $f_7$  which are

$$f_1 = \max \left[ 0, (f_1)_0 + k (\Delta q_1 - \Delta q_7) \right] = -f_7 \quad (3.26)$$

where  $(f_1)_0$  is the value of  $f_1$  at the previous time step,  $\Delta q_1$  and  $\Delta q_7$  are defined in (3.4) and  $k$  is a spring constant to simulate the wheel. It is usual to lump the mass of the wheel at node 2.

(b) Tension spring - This is used to simulate a cable restraint system such as a seat belt. Again using local coordinates the only non-zero force components are  $f_1$  and  $f_7$  which in this case are given by

$$f_1 = \min \left[ 0, (f_1)_0 + k (\Delta q_1 - \Delta q_7) \right] = -f_7 \quad (3.27)$$

with similar definitions for quantities as in (3.26).

### 3.3 RIGID BODY ELEMENTS

We may simulate the engine block, or a portion of the vehicle which does not involve significant deformation in a crash by a rigid body which is connected at a finite number of points to the deformable part of the vehicle. This rigid body will be referred to as a rigid body element.

The motion of a rigid body can be computed as follows. Consider a rigid body of mass  $m$ , and moments of inertia  $I_1$ ,  $I_2$  and  $I_3$

with respect to its principal axes. Let  $\underline{u}$  and  $\underline{\phi}$  denote the linear and angular displacement vectors, respectively, of its center of gravity with respect to global axes. Then the equations of motion can be written as

$$m \ddot{\underline{u}} = \underline{F} \quad (3.28)$$

$$\frac{d}{dt} \left\{ \underline{L}^T \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \underline{L} \dot{\underline{\phi}} \right\} = \underline{M} \quad (3.29)$$

where, as shown in Appendix A

$$\dot{\underline{L}} = \underline{L} \begin{bmatrix} 0 & \dot{\phi}_3 & -\dot{\phi}_2 \\ -\dot{\phi}_3 & 0 & \dot{\phi}_1 \\ \dot{\phi}_2 & -\dot{\phi}_1 & 0 \end{bmatrix} \quad (3.30)$$

and where  $\underline{F}$  and  $\underline{M}$  are the total force and the total moment on the rigid body element;  $\underline{L}$  is the matrix relating the global cartesian coordinates to the principal axes, and is defined similar to that in Eq. (3.1).

If there are  $N_r$  nodes in common between the deformable structure and the rigid body, the total force and moment ( $\underline{F}$  and  $\underline{M}$ ) are obtained by summing over all  $N_r$  nodes the forces,  $\underline{f}_i$  and torques  $\underline{T}_i$  which act on the rigid body at the  $i$ th node, so that,

$$\underline{F} = \sum_{i=1}^{N_r} \underline{f}_i \quad (3.31)$$

$$\underline{M} = \sum_{i=1}^{N_r} (\underline{T}_i + \underline{r}_i \times \underline{f}_i)$$

where  $\underline{r}_i$  is the position vector measured from the center of gravity to node  $i$ .

The central difference explicit scheme used earlier will also be used here to integrate (3.29) and (3.30), namely,

$$\Delta \underline{u}_{\sim n+1} = \Delta \underline{u}_{\sim n} + \frac{(\Delta t)^2}{m} \underline{F}_{\sim n} \quad (3.32a)$$

$$\underline{L}_{\sim n}^T \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \underline{L}_{\sim n} \Delta \underline{\phi}_{\sim n+1} = \underline{L}_{\sim n-1}^T \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \underline{L}_{\sim n-1} \Delta \underline{\phi}_{\sim n} + (\Delta t)^2 \underline{M}_{\sim n} \quad (3.32b)$$

$$\underline{L}_{\sim n+1} = \underline{L}_{\sim n} \begin{bmatrix} 1 & \Delta \phi_3 & -\Delta \phi_2 \\ -\Delta \phi_3 & 1 & \Delta \phi_1 \\ \Delta \phi_2 & -\Delta \phi_1 & 1 \end{bmatrix}_{n+1} \quad (3.32c)$$

Where the subscript n denotes the value of the quantity at the nth step, and  $\Delta \underline{u}_{\sim n+1} = \underline{u}_{\sim n+1} - \underline{u}_{\sim n}$ . Now, the incremental rotations at all the  $N_r$  nodes on the rigid body are also  $\Delta \underline{\phi}_{\sim n+1}$  and so the incremental displacements at node i are given by,

$$(\Delta \underline{u}_{\sim n+1})_i = \Delta \underline{u}_{\sim n+1} + \Delta \underline{\phi}_{\sim n+1} \times \underline{r}_i \quad (3.33)$$

in which  $\underline{r}$  is the same as that in (3.31).

### 3.4 MODAL ELEMENT

During a crash, a component or a part of the vehicle may experience only a small elastic deformation. In such a case one could probably approximate this component as a rigid body. If, however, for one reason or another more detailed response information is desired within this component, it is possible to obtain adequate engineering accuracy by approximating the response using a few natural modes.

In what follows, the procedure of deriving the component matrix using natural modes is briefly described. Let the mass and

stiffness matrix of a given component be denoted by  $\underline{m}_c$  and  $\underline{k}_c$ , respectively. The generalized coordinates associated with these matrices will be denoted by  $\underline{q}_c$ . Now, these matrices may be determined either by detailed finite element modelling of the component itself or by other well established methods. For complex components the size of such matrices can be very large. This, of course, will depend on how extensive the original modelling is. In such cases, to reduce  $\underline{m}_c$  and  $\underline{k}_c$  to what will be referred to as a modal form, we may either use assumed modes to approximate the response, or more rationally, derive the modes by solving the natural vibration eigenvalue problem associated with  $\underline{m}_c$  and  $\underline{k}_c$ . The number of modes required, will, of course, depend on the degree of accuracy desired.

Consider the eigenvalue problem

$$\lambda \underline{m}_c \underline{x} = \underline{k}_c \underline{x} \quad (3.34)$$

We can easily determine the first few, say  $r$  eigenvalues,  $\lambda_j$  and eigenvectors,  $\underline{x}_j$  ( $j = 1, 2, \dots, r$ ) [Refs. 3]. The number  $r$  should be sufficiently large or at least equal to the number of rigid body degrees of motion. The generalized coordinate vector,  $\underline{q}_c$ , is then approximated by a combination of the eigenvectors or modes, namely

$$\underline{q}_c = \underline{X} \underline{a} \quad (3.35)$$

in which

$$\underline{X} = \{ \underline{x}_1, \underline{x}_2, \dots, \underline{x}_r \}$$

The quantity  $\underline{a}$  can be called the vector of modal amplitudes [when using  $\underline{a}$  as generalized coordinates, the corresponding mass and stiffness matrices can be made diagonal].

To insure compatibility between the component and the rest of the vehicle, it is not convenient to use  $\underline{a}$  directly. We shall partition  $\underline{q}_c$ ,  $\underline{X}$  and  $\underline{a}$  such that

$$\underline{q}_c \begin{Bmatrix} \underline{q}_\alpha \\ \underline{q}_\beta \end{Bmatrix} = \begin{bmatrix} X_{\alpha\alpha} & X_{\alpha\gamma} \\ X_{\beta\alpha} & X_{\beta\gamma} \end{bmatrix} \begin{Bmatrix} \underline{a}_{\sim\alpha} \\ \underline{a}_{\sim\gamma} \end{Bmatrix} \quad (3.36)$$

where the components of  $\underline{q}_\alpha$  and  $\underline{q}_\beta$  are the generalized coordinates associated with nodes connected and not connected respectively to the rest of the vehicle. By expressing  $\underline{a}_\alpha$  in terms of  $\underline{q}_\alpha$  we have

$$\underline{a} = \begin{Bmatrix} \underline{a}_{\sim\alpha} \\ \underline{a}_{\sim\gamma} \end{Bmatrix} \begin{bmatrix} X_{\alpha\alpha}^{-1} & -X_{\alpha\alpha}^{-1} X_{\alpha\gamma} \\ 0 & I \end{bmatrix} \begin{Bmatrix} \underline{q}_\alpha \\ \underline{a}_{\sim\gamma} \end{Bmatrix} = \underline{A} \begin{Bmatrix} \underline{a}_{\sim\alpha} \\ \underline{a}_{\sim\gamma} \end{Bmatrix} \quad (3.37)$$

(note that  $X$  may have to be reordered to assure that  $X_{\alpha\alpha}$  is nonsingular). The vector  $\underline{q}_\alpha$  and  $\underline{a}_{\sim\gamma}$  can be used as the new generalized coordinates. Accordingly, the associated reduced mass and stiffness matrices are then given by

$$\underline{m}_r = \underline{A}^T \underline{A} \quad (3.38)$$

$$\underline{k}_r = \underline{A}^T \underline{D} \underline{A}$$

where  $\underline{D}$  is a diagonal matrix of order  $r$  with the elements on the diagonal being the first  $r$  eigenvalues  $\lambda_j$  ( $j = 1, 2, \dots, r$ ) of (3.34). It can be seen that the order of  $\underline{m}_r$  and  $\underline{k}_r$  can be much less than that of  $\underline{m}_c$  and  $\underline{k}_c$ ; therefore, the total number of degrees of freedom for the final system of equations is reduced.

#### 4. CONCLUSIONS

The basic concept of a modular approach and the procedure of numerical integration for crashworthiness prediction has been developed in the present report. In addition, several elements have been examined which are useful for this purpose. One of the important advantages of using the modular approach in vehicle structure simulation is that it provides great flexibility in modelling specific dynamic behavior economically. Depending on the degree of accuracy desired, one may use simplified spring-mass, or detailed finite element modelling for part or the entire structure.

A computer program based on the modular concept has been developed at TSC. Many examples are being run, and extensive numerical results will be reported in a separate report.



## APPENDIX

### EVALUATION OF $\underline{L}$ IN EQ. (3.1) AT TIME = 0

The matrix  $\underline{L}$  relates the global coordinates to local coordinates. It can be defined either in terms of Euler angles or in terms of the position vectors of three noncollinear points in space. The former approach requires less information, while the latter is easier to visualize. In what follows,  $\underline{L}$  shall be defined using the latter approach. Let  $(x_{i1}, x_{i2}, x_{i3})$  be the coordinates of point  $i$ , ( $i = 1,2,3$ ). The local coordinate frame will have the first base vector  $\underline{e}_1$ , pointing from point 1 to point 2; the second base vector,  $\underline{e}_2$ , lies in the plane spanned by the three points and is normal to the first base vector. The third base vector,  $\underline{e}_3$ , is normal to base vectors  $\underline{e}_1$  and  $\underline{e}_2$ , and satisfies the right hand rule. Therefore, from these definitions, we have

$$\underline{e}_1 = \frac{1}{\ell_1} \begin{Bmatrix} x_{21} - x_{11} \\ x_{22} - x_{12} \\ x_{23} - x_{13} \end{Bmatrix} \quad (\text{A.1})$$

$$\underline{E}_2 = \frac{1}{\ell_2} \begin{Bmatrix} x_{31} - x_{11} \\ x_{32} - x_{12} \\ x_{33} - x_{13} \end{Bmatrix} \quad (\text{A.2})$$

$$\underline{e}_2 = (\underline{e}_1 \times \underline{E}_2) \times \underline{e}_1 = \underline{E}_2 - (\underline{E}_2 \cdot \underline{e}_1) \underline{e}_1 \quad (\text{A.3})$$

$$\underline{e}_3 = \underline{e}_1 \times \underline{e}_2 \quad (\text{A.4})$$

in which

$$\ell_1 = \left[ (x_{21} - x_{11})^2 + (x_{22} - x_{12})^2 + (x_{23} - x_{13})^2 \right]^{1/2}$$

and

$$l_2 = \left[ (x_{31} - x_{11})^2 + (x_{32} - x_{12})^2 + (x_{33} - x_{13})^2 \right]^{1/2}$$

Then

$$\underline{\tilde{L}} = \begin{bmatrix} \underline{e}_1^T \\ \underline{e}_2^T \\ \underline{e}_3^T \end{bmatrix} \quad (A.5)$$

i.e., the  $i$ th row of  $\underline{\tilde{L}}$  is the  $i$ th base vector of the local frame having its components expressed in terms of the global coordinates.

For a beam element, the three noncollinear points are taken as the two end nodes which are chosen as the first two points; a third point is a point in space to indicate the direction of one of the principal axes of bending.

Evaluation of  $\Delta \underline{\tilde{L}}$ , and the updating of  $\underline{\tilde{L}}$

The matrix  $\underline{\tilde{L}}$  is

$$\underline{\tilde{L}} = \begin{bmatrix} \underline{e}_1^T \\ \underline{e}_2^T \\ \underline{e}_3^T \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \quad (A.6)$$

To determine  $\dot{\underline{\tilde{L}}}$ , find  $\dot{\underline{e}}_i$  ( $i = 1, 2, 3$ ) where the base vectors  $\underline{e}_i$  are defined in the discussion leading to (A.5). The following relations yield the desired results.

$$\dot{\underline{e}}_i = \underline{\omega} \times \underline{e}_i$$

where

$$\underline{\omega} = \omega_1 \underline{i} + \omega_2 \underline{j} + \omega_3 \underline{k} \quad (A.7)$$

Then,

$$\dot{\tilde{e}}_i = \begin{bmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ L_{i1} & L_{i2} & L_{i3} \end{bmatrix} = \begin{pmatrix} \omega_2 L_{i3} - \omega_3 L_{i2} \\ \omega_3 L_{i1} - \omega_1 L_{i3} \\ \omega_1 L_{i2} - \omega_2 L_{i1} \end{pmatrix} \tilde{i} + \begin{pmatrix} \omega_3 L_{i1} - \omega_1 L_{i3} \\ \omega_1 L_{i2} - \omega_2 L_{i1} \\ \omega_2 L_{i3} - \omega_3 L_{i2} \end{pmatrix} \tilde{j} + \begin{pmatrix} \omega_1 L_{i2} - \omega_2 L_{i1} \\ \omega_2 L_{i3} - \omega_3 L_{i2} \\ \omega_3 L_{i1} - \omega_1 L_{i3} \end{pmatrix} \tilde{k}$$

So that

$$\dot{\tilde{L}} = \begin{bmatrix} \omega_2 L_{13} - \omega_3 L_{12} & \omega_3 L_{11} - \omega_1 L_{13} & \omega_1 L_{12} - \omega_2 L_{11} \\ \omega_2 L_{23} - \omega_3 L_{22} & \omega_1 L_{21} - \omega_1 L_{23} & \omega_1 L_{22} - \omega_2 L_{22} \\ \omega_2 L_{33} - \omega_3 L_{32} & \omega_3 L_{31} - \omega_1 L_{33} & \omega_1 L_{32} - \omega_2 L_{31} \end{bmatrix}$$

or

$$\dot{\tilde{L}} = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} = \tilde{L} \begin{bmatrix} 0 & \dot{\phi}_3 & -\dot{\phi}_2 \\ -\dot{\phi}_3 & 0 & \dot{\phi}_1 \\ \dot{\phi}_2 & -\dot{\phi}_1 & 0 \end{bmatrix}$$

(A.8)

where

$$\dot{\phi}_i = \omega_i$$

Equation (A.8) implies

$$\Delta \tilde{L} = \tilde{L} \begin{bmatrix} 0 & \Delta \phi_3 & -\Delta \phi_2 \\ -\Delta \phi_3 & 0 & \Delta \phi_1 \\ \Delta \phi_2 & -\Delta \phi_1 & 0 \end{bmatrix}$$

Then  $\tilde{L}$  at the  $n+1$  time step is given in terms of its value,  $\tilde{L}_n$ , at the  $n$ th time step as

$$\tilde{L}_{n+1} = \tilde{L}_n + \Delta \tilde{L}_n = \tilde{L}_n + \tilde{L}_n \begin{bmatrix} 0 & \Delta\phi_3 & -\Delta\phi_2 \\ -\Delta\phi_3 & 0 & \Delta\phi_1 \\ \Delta\phi_2 & -\Delta\phi_1 & 0 \end{bmatrix}$$

or

$$\tilde{L}_{n+1} = \tilde{L}_n \begin{bmatrix} 1 & \Delta\phi_3 & -\Delta\phi_2 \\ -\Delta\phi_3 & 1 & \Delta\phi_1 \\ \Delta\phi_2 & -\Delta\phi_1 & 1 \end{bmatrix} \quad (\text{A.9})$$

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