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# ON THE STRESS ANALYSIS OF RAILS AND TIES


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16. Abstract


The report first reviews the methods presented in the literature for the stress analysis of the railroad track components and the results of a variety of validation tests. It was found that the equation $EIw^{iv} + kw = q$ yields deflections and bending stresses in the rails of longitudinal-tie and cross-tie tracks that agree sufficiently (for design purposes) with corresponding track test results, provided the coefficients that enter the analyses are properly chosen. This is followed by a review of discussion of the methods for determining the coefficients that enter these analyses. The report concludes with recommended analyses and test methods for the determination of stresses in the rail-tie structure.


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## PREFACE

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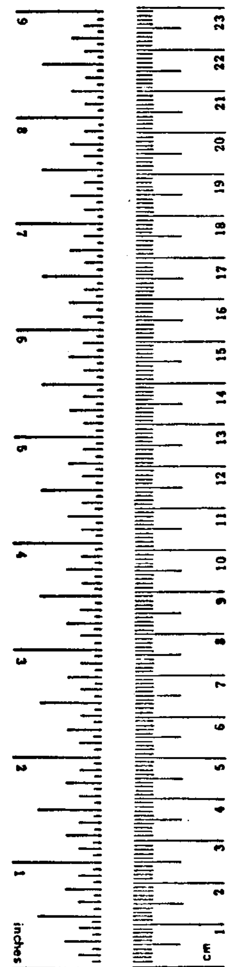
# METRIC CONVERSION FACTORS

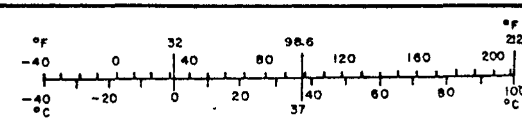
## Approximate Conversions to Metric Measures

| Symbol | When You Know | Multiply by | To Find | Symbol |
|---|---|---|---|---|
| | **LENGTH** | | | |
| in | inches | 2.5 | centimeters | cm |
| ft | feet | 30 | centimeters | cm |
| yd | yards | 0.9 | meters | m |
| mi | miles | 1.6 | kilometers | km |
| | **AREA** | | | |
| $in^2$ | square inches | 6.5 | square centimeters | $cm^2$ |
| $ft^2$ | square feet | 0.09 | square meters | $m^2$ |
| $yd^2$ | square yards | 0.8 | square meters | $m^2$ |
| $mi^2$ | square miles | 2.6 | square kilometers | $km^2$ |
| | acres | 0.4 | hectares | ha |
| | **MASS (weight)** | | | |
| oz | ounces | 28 | grams | g |
| lb | pounds | 0.45 | kilograms | kg |
| | short tons (2000 lb) | 0.9 | tonnes | t |
| | **VOLUME** | | | |
| tsp | teaspoons | 5 | milliliters | ml |
| Tbsp | tablespoons | 15 | milliliters | ml |
| fl oz | fluid ounces | 30 | milliliters | ml |
| c | cups | 0.24 | liters | l |
| pt | pints | 0.47 | liters | l |
| qt | quarts | 0.95 | liters | l |
| gal | gallons | 3.8 | liters | l |
| $ft^3$ | cubic feet | 0.03 | cubic meters | $m^3$ |
| $yd^3$ | cubic yards | 0.76 | cubic meters | $m^3$ |
| | **TEMPERATURE (exact)** | | | |
| °F | Fahrenheit temperature | 5/9 (after subtracting 32) | Celsius temperature | °C |




## Approximate Conversions from Metric Measures

| Symbol | When You Know | Multiply by | To Find | Symbol |
|---|---|---|---|---|
| | **LENGTH** | | | |
| mm | millimeters | 0.04 | inches | in |
| cm | centimeters | 0.4 | inches | in |
| m | meters | 3.3 | feet | ft |
| m | meters | 1.1 | yards | yd |
| km | kilometers | 0.6 | miles | mi |
| | **AREA** | | | |
| $cm^2$ | square centimeters | 0.16 | square inches | $in^2$ |
| $m^2$ | square meters | 1.2 | square yards | $yd^2$ |
| $km^2$ | square kilometers | 0.4 | square miles | $mi^2$ |
| ha | hectares (10,000 $m^2$) | 2.5 | acres | |
| | **MASS (weight)** | | | |
| g | grams | 0.035 | ounces | oz |
| kg | kilograms | 2.2 | pounds | lb |
| t | tonnes (1000 kg) | 1.1 | short tons | |
| | **VOLUME** | | | |
| ml | milliliters | 0.03 | fluid ounces | fl oz |
| l | liters | 2.1 | pints | pt |
| l | liters | 1.06 | quarts | qt |
| l | liters | 0.26 | gallons | gal |
| $m^3$ | cubic meters | 35 | cubic feet | $ft^3$ |
| $m^3$ | cubic meters | 1.3 | cubic yards | $yd^3$ |
| | **TEMPERATURE (exact)** | | | |
| °C | Celsius temperature | 9/5 (then add 32) | Fahrenheit temperature | °F |



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# 1. INTRODUCTION

After the introduction of metal rails, during the 19th century, two types of track were in use: the *longitudinal-tie track* and the *cross-tie track*. Whereas in the longitudinal-tie track the two metal rails are *continuously* supported by longitudinal ties, in the cross-tie track these rails are supported *discretely* by cross-ties which are spaced at a prescribed distance from each other [1].*

During the second half of the past century the longitudinal-tie track exhibited various deficiencies and its use diminished. As a consequence, in the past several decades the cross-tie track has become the dominant mode of track construction.

When the cross-tie track was introduced, the wheel loads were very small and the tie spacing relatively large. For example, around 1800, the tie spacing was about 1.8 meters [2]. As the wheel loads progressively increased the rail and tie cross-sections increased and the tie spacing decreased. According to E. Winkler [3], in 1875 the tie spacing on main lines was about 0.9 meters. A view of a typical track currently in use in the USA, with even smaller tie spacings, is shown in Fig. 1.

Although the development of the railroad track, up to the present, was mainly intuitive, based on the trial and error approach, since the second half of the 19th century railroad engineers have been attempting to analyze the stresses in the track components.

The purpose of this report is to critically review these analyses and the related test results, in order to establish which of the proposed methods are suitable for the analysis of tracks currently in use and the ones to be built in the future.

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*Number in brackets refer to references.

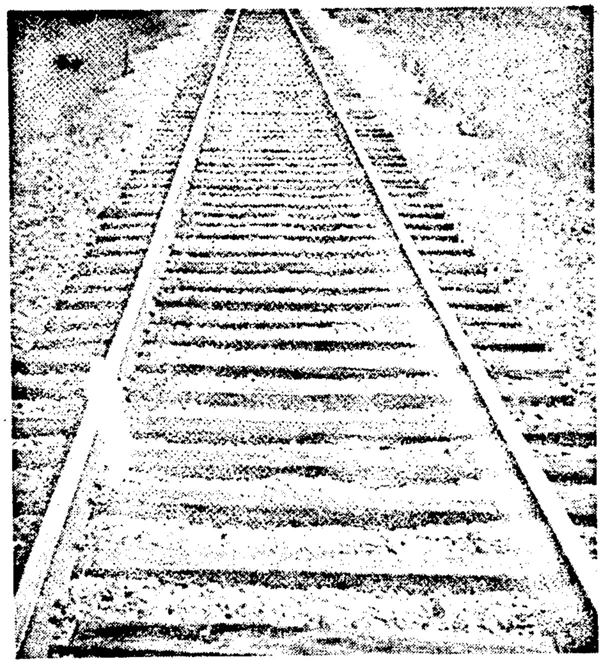


FIG. 1. A TYPICAL RAILROAD TRACK IN THE U.S.

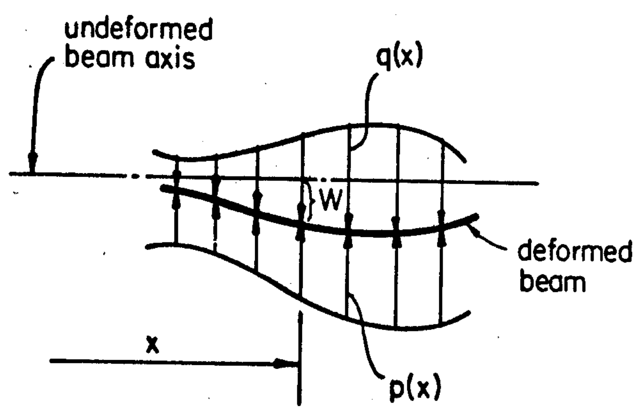



FIG. 2. EQUILIBRIUM POSITION OF DEFORMED BEAM

## 2. THE STRESS ANALYSIS OF THE LONGITUDINAL-TIE TRACK SUBJECTED TO VERTICAL LOADS

In 1867, E. Winkler [4] analyzed the stresses in the rails of a longitudinal-tie track by considering the rails as a continuously supported beam. The differential equation for the bending theory of an elastic beam

$$EI \frac{d^4w}{dx^4} + p(x) = q(x) \tag{1}$$

was established by this time. In this equation $w(x)$ is the vertical deflection at $x$, $EI$ is the flexural rigidity of the rail and tie, $q(x)$ is the distributed vertical load, and $p(x)$ is the continuous contact pressure between the ties and base, as shown in Fig. 2. For the base response Winkler proposed the relation

$$p(x) = k\, w(x) \tag{2}$$

where $k$ is the base parameter. This is the origin of the well-known Winkler foundation model. The resulting track equation

$$EI \frac{d^4w}{dx^4} + kw = q \tag{3}$$

is a fourth order ordinary differential equation. It represents the response of a beam which is attached to a spring base, as shown in Fig. 3.

In Refs. [3] and [4] Winkler presented a solution of equation (3) for the special case of a beam of infinite extent subjected to equidistant concentrated loads of the same intensity. In order to simplify the obtained results, Winkler also considered the case of increasing load spacing and obtained expressions which are the solution for a single concentrated load (It appears that Winkler did not realize that, since he refers to them as "approximate formulas" [3]).

In the analysis of the longitudinal-tie track, Winkler stipulated that

$$EI = E_r I_r + E_t I_t \tag{4}$$

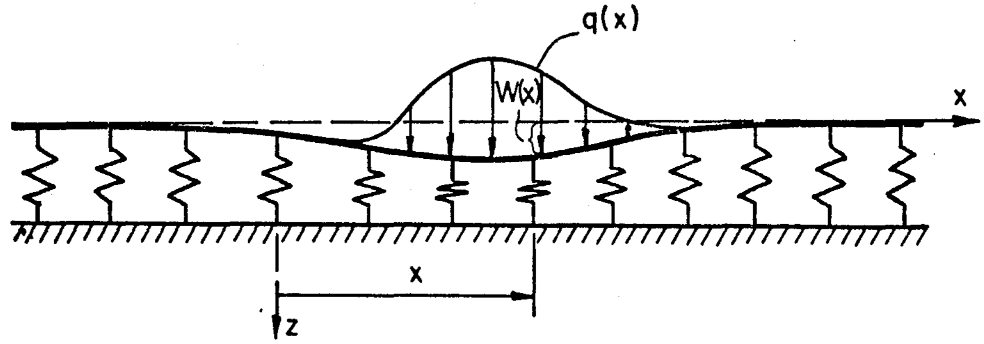



FIG. 3. CONTINUOUSLY SUPPORTED BEAM SUBJECTED TO A LOAD q (x)

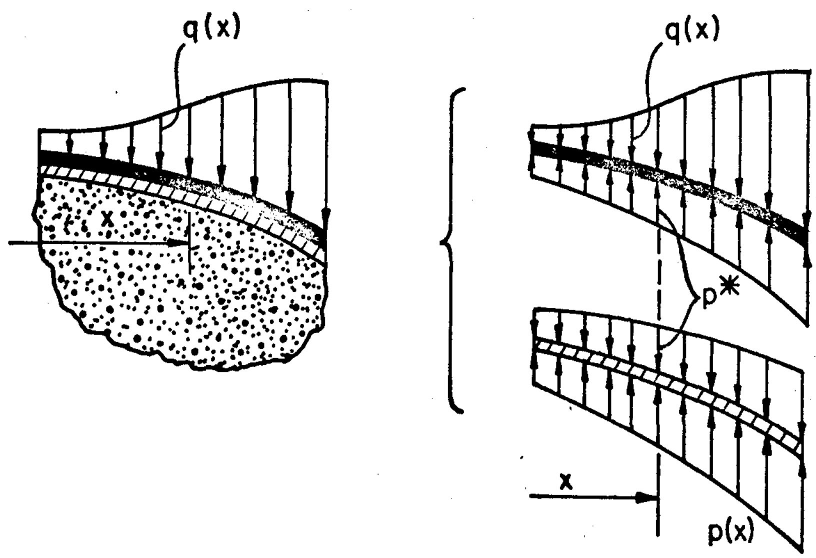



FIG. 4. TO THE DERIVATION OF THE BENDING RIGIDITY OF A LONGITUDINAL TIE-TRACK

where $E_rI_r$ and $E_tI_t$ are the flexural rigidities of the rail and tie with respect to their centroidal axes.

The above relation may be derived by assuming that, although the rail and longitudinal-tie press against each other, the friction forces in the contact area are negligible (The calculated maximum bending stresses and deflections will thus be larger than the actual ones). Noting that at each point x the vertical displacements of the rail and longitudinal tie are the same, namely

$$w_r(x) = w_t(x) = w(x) \tag{5}$$

and that at each x the contact pressures, $p^*(x)$, are equal but of opposite sign, as shown in Fig. 4, the differential equations for rail and tie may be written, respectively, as

$$\begin{aligned} E_rI_r\, w^{iv} &= q(x) - p^*(x) \\ E_tI_t\, w^{iv} &= p^*(x) - p(x) \end{aligned} \tag{6}$$

where $I_r$ and $I_t$ are the moments of inertia of rail and tie with respect to the respective centroidal axes. Adding these two equations, we obtain

$$(E_rI_r + E_tI_t)\, w^{iv} = q(x) - p(x) \tag{7}$$

Comparing eq. (7) with eq. (1) it may be concluded that for the longitudinal-tie track the EI to be used in eq. (1) or (3) is $\left(E_rI_r + E_tI_t\right)$, which agrees with eq. (4).

In 1882, J. W. Schwedler [5] in discussing bending stresses in the rails of a longitudinal-tie track, presented the solution of equation (3) for the case when the infinite beam is subjected to one concentrated force P

$$w(x) = \frac{P\beta}{2k}\,\eta(x) \tag{8}$$

and the corresponding expression for the bending moment

$$M(x) = -EI\,\frac{d^2w}{dx^2} = \frac{P}{4\beta}\,\mu(x) \tag{9}$$

where

$$\beta = \sqrt[4]{\frac{k}{4EI}}$$

and
$$\eta(x) = e^{-\beta x}\,[\cos\,(\beta x) + \sin\,(\beta x)] \tag{10}$$

$$\mu(x) = e^{-\beta x}\,[\cos\,(\beta x) - \sin\,(\beta x)]$$

Schwedler also used the above expressions as influence functions to determine the effect of several wheel loads. For example, according to this method, for the three wheel loads $P_1$, $P_2$, and $P_3$ shown in Fig. 5 the deflections and bending moments at point 0 are

$$w = \frac{\beta}{2k}\sum_{n=1}^{3} P_n\,\eta_n$$

and (11)

$$M = \frac{1}{4\beta}\sum_{n=1}^{3} P_n\,\mu_n$$

For the determination of *bending stresses* in the rail and the longitudinal-tie, note that the bending moment at x is

$$M(x) = M_r(x) + M_t(x) \tag{12}$$

where $M_r$ and $M_t$ are the corresponding bending moments in rail and tie and that because of eq. (5)

$$M_r(x) = -E_rI_rw''(x) \tag{13}$$

$$M_t(x) = -E_tI_tw''(x) \tag{14}$$

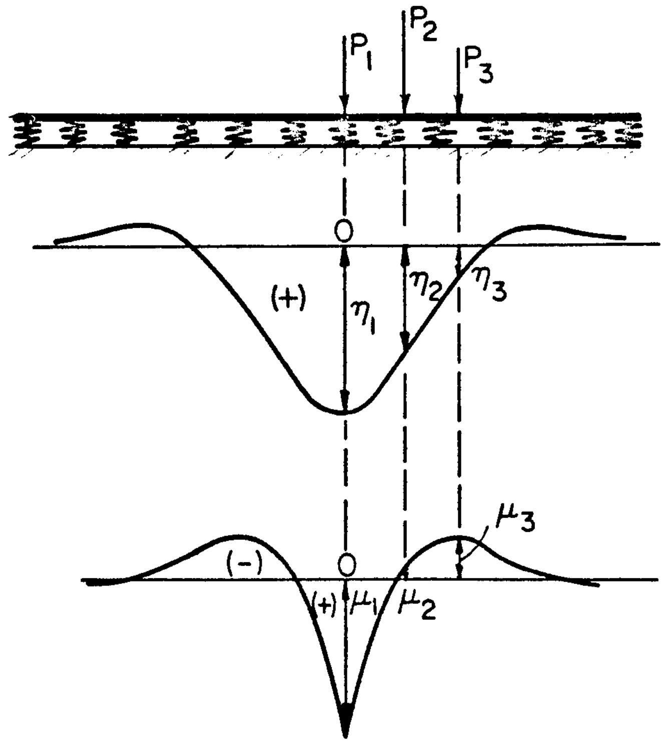



FIG. 5. TO INFLUENCE LINE METHOD

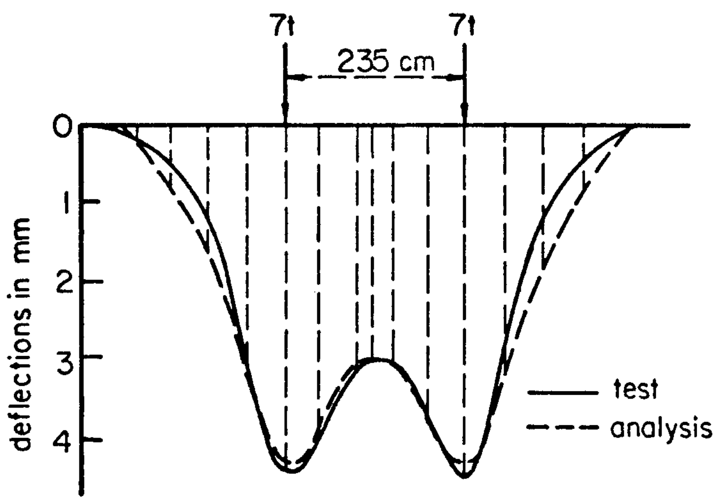



FIG. 6. COMPARISON OF DEFLECTIONS FOR LONGITUDINAL-TIE TRACK

and hence

$$\frac{M_r}{E_r I_r} = \frac{M_t}{E_t I_t} \tag{15}$$

Elimination of $M_t$ from eq. (12), by using eq. (15), yields

$$M_r = \frac{E_r I_r}{EI} M \tag{16}$$

and similarly

$$M_t = \frac{E_t I_t}{EI} M \tag{17}$$

The largest normal bending stresses in the rails and ties are then obtained from the well known stress formulas

$$(\sigma_r)_{max} = \frac{M_r c_r}{I_r} = \frac{E_r M c_r}{EI} \tag{18}$$

$$(\sigma_t)_{max} = \frac{M_t c_t}{I_t} = \frac{E_t M c_t}{EI} \tag{19}$$

where EI is given in (4).

In 1888, H. Zimmermann [6] published a book which contained solutions of eq. (3) for many special cases of interest for the analysis of a railroad track. Zimmermann like Schwedler, utilized the obtained solutions to analyze the longitudinal-tie track as well as the ties of the cross-tie track. Of interest is the presented comparison of the deflection curves for a longitudinal-tie track caused by two loads of seven tons each, obtained analytically and from a test, which are reproduced in Fig. 6. The close agreement between the measured and calculated ordinates pointed to the conclusion that the linear bending theory for a beam on a linear Winkler base was sufficient for the analysis of the longitudinal-tie track.

## 3. THE STRESS ANALYSIS OF THE CROSS-TIE TRACK SUBJECTED TO VERTICAL LOADS

The development of the analyses of rails for a cross-tie track was more involved. It started by considering the rail as a beam resting on *discrete* rigid, then elastic supports, and then as a *continuously* supported beam.

In 1875, E. Winkler [3] presented an analysis of bending stresses in the rails, by considering each rail as an infinitely long elastic beam which rests on an infinite number of discrete rigid supports, as shown in Fig. 7(I). For the shown load distribution he found that the largest possible bending moment is

$$M = 0.1888 \text{ Pa} \tag{20}$$

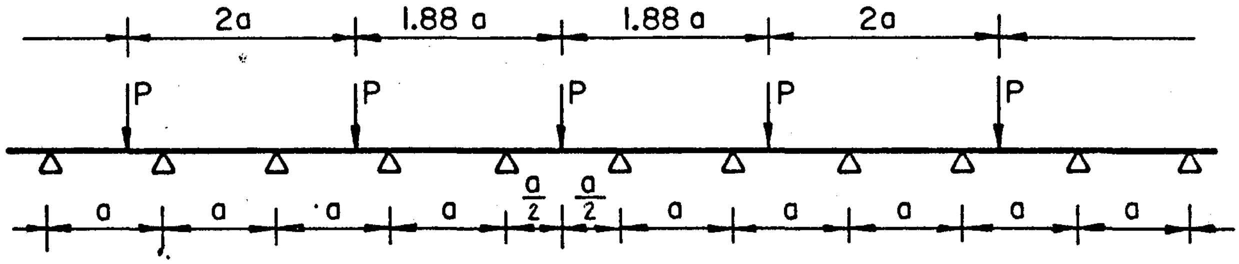
Realizing the shortcoming of the Winkler assumption of rigid supports, Zimmermann [6] presented a bending stress determination considering the rail as a finite elastic beam on four discrete *elastic* supports (in order to simplify the analysis), as shown in Fig. 7(II). The obtained expression for the largest bending moment, which takes place under the load P, was given as

$$M = \frac{8\gamma + 7}{4\gamma + 10} \frac{Pa}{4} \tag{21}$$

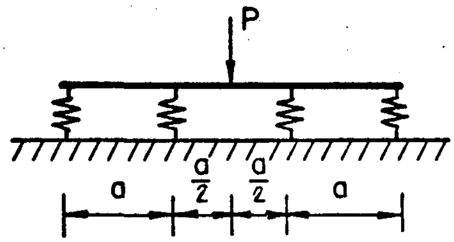
where $\gamma$ is the parameter of the discrete elastic support.

Schwedler [5] proposed to analyze the rail by considering it as a beam over eight elastic supports subjected to one concentrated force. For the largest moment Schwedler obtained a similar expression to the one shown in eq. (21).

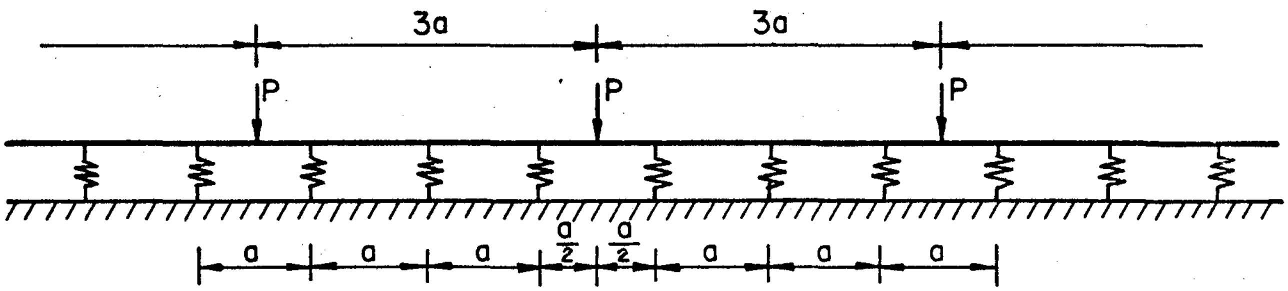
F. Engesser [7] analyzed the rail by considering it as an infinite beam on equidistant elastic supports subjected to a



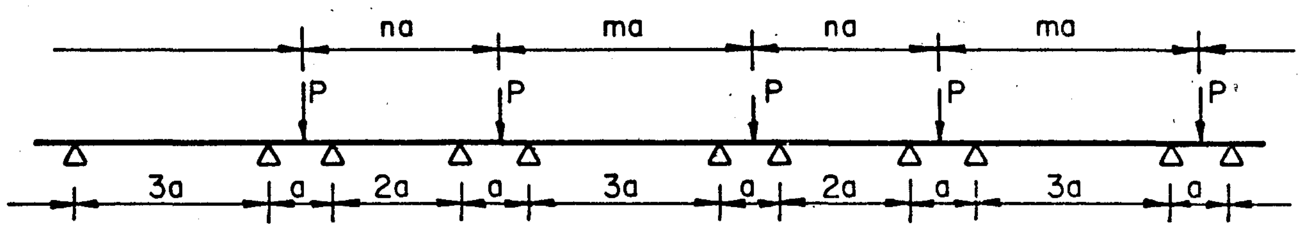

(I) Winkler model




(II) Zimmermann model




(III) Engesser model




(IV) VMEV model

FIG. 7. PROPOSED ANALYTICAL MODELS FOR THE DETERMINATION OF LARGEST BENDING MOMENT IN RAILS

periodic arrangement of forces, as shown in Fig. 7(III). For the largest moment, Engesser obtained the expression

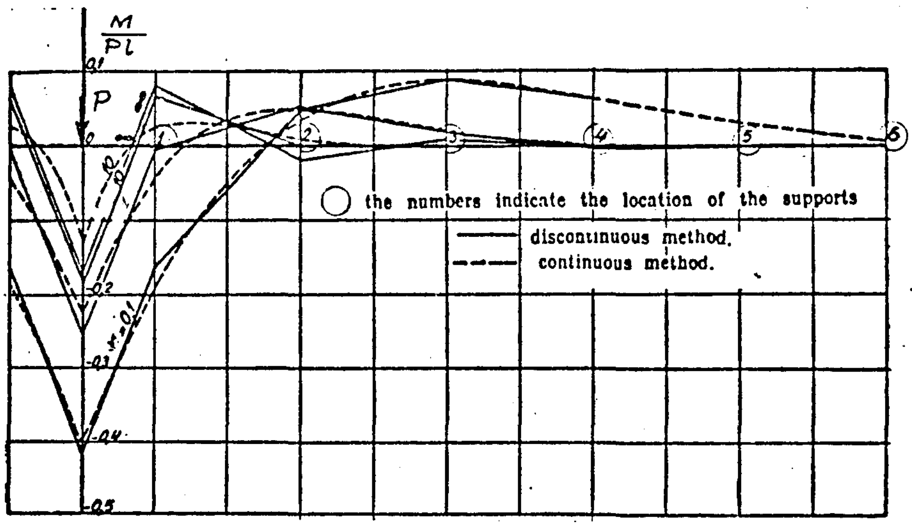
$$M = \frac{19\gamma + 4}{3\gamma + 1} \frac{Pa}{24} \tag{22}$$

A similar approach was also utilized by a number of other investigators (many of these papers appeared in the journal *Organ für die Fortschritte des Eisenbahnwesens*). These and related results are discussed by R. Hanker in Section B.V.3 of reference [8].

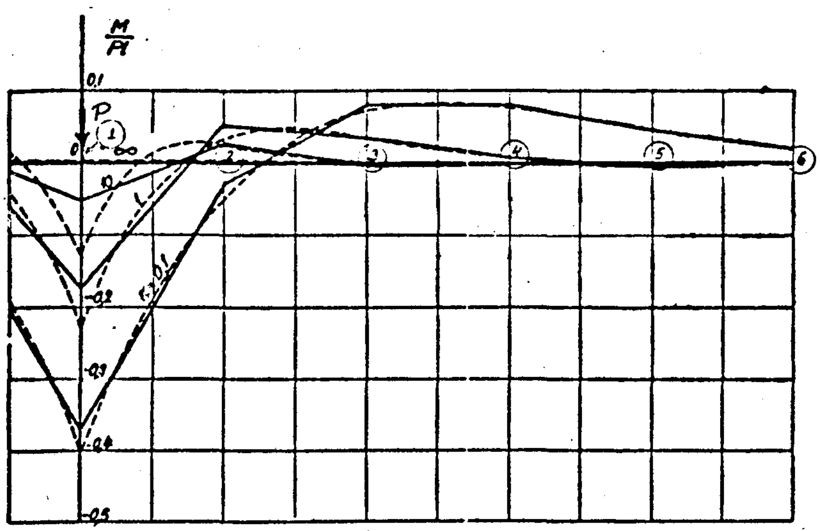
An analysis of a beam on many discrete supports was at the time rather cumbersome, since it involves the solution of many simultaneous algebraic equations. It was therefore natural that attempts were made to analyze the bending stresses in the rails by assuming that also for a *cross-tie track* the rails respond like a *continuously supported beam.* Early investigators who adopted this approach are:A. Flamache [9] in 1904, S. Timoshenko [10] in 1915, and the ASCE-AREA Special Committee on Stresses in the Railroad Track [11] in 1917. The tendency of steadily increasing wheel loads, which was countered by a steady decrease of the cross-tie spacings, enhanced the justification of the "continuity" assumption.

The use of the "continuity" assumption, in conjunction with eq. (2), for the *cross-tie track* prompted a number of studies to determine whether this assumption is justified for the track parameters in use.

One approach was to analyze the track as a beam on *discrete* elastic supports, then as a beam on a *continuous* Winkler base, and then compare the obtained results. Such a comparison was performed, for example, by G. S. Gough [12], E. Czitary [13], A. Wasiutynski [14] and A. D. dePater [15].




(a) Bending moments along the beam; load applied between two supports




(b) Bending moments along the beam; load applied over support

FIG. 8. COMPARISON OF BENDING MOMENTS [15]

Graphs from Ref. [15] which compare the bending moment distributions, are reproduced in Fig. 8. Note the good agreement of the shown results. A more recent comparative study, with error estimate, was presented by H. Luber [16]. Related results were published by C. B. Biezeno [17], C. Popp [18-19], G. Hutter [20], and J. P. Ellington [21]. Gough [12] and Ellington [21] also studied the effect of a missing tie. According to [21] when the base is relatively soft, for a concentrated load over the missing tie, the increase in the largest possible bending moment at the missing tie is about 30%. The increase of the largest possible bending moment for a relatively rigid base is over 100%.

Another approach was to compare the results based on *equation (3)* with corresponding test results obtained using an *actual track*. For examples of this approach refer to the studies by the ASCE-AREA Special Committee on Stresses in Railroad Track [11] [22]. One of the comparisons from Reference [11] is reproduced in Fig. 9. Note that also in this study the results for the bending moments show good agreement.

Because of the agreement found in such comparative studies, and the absence of a better (and simple) analytical approach, the validity of the "continuity" assumption, in conjunction with the Winkler hypothesis (2) was accepted by a number of railroads as a basis for the analysis also of *cross-tie tracks* [23-24].

The acceptance of this method was not universal, however, and many railroads had their own methods of track analysis. To demonstrate this point let us consider the corresponding developments at the railroads of central Europe.

Since World War I an attempt was made by the central European railroads (Verein Mitteleuropäischer Eisenbahnverwaltungen,VMEV)

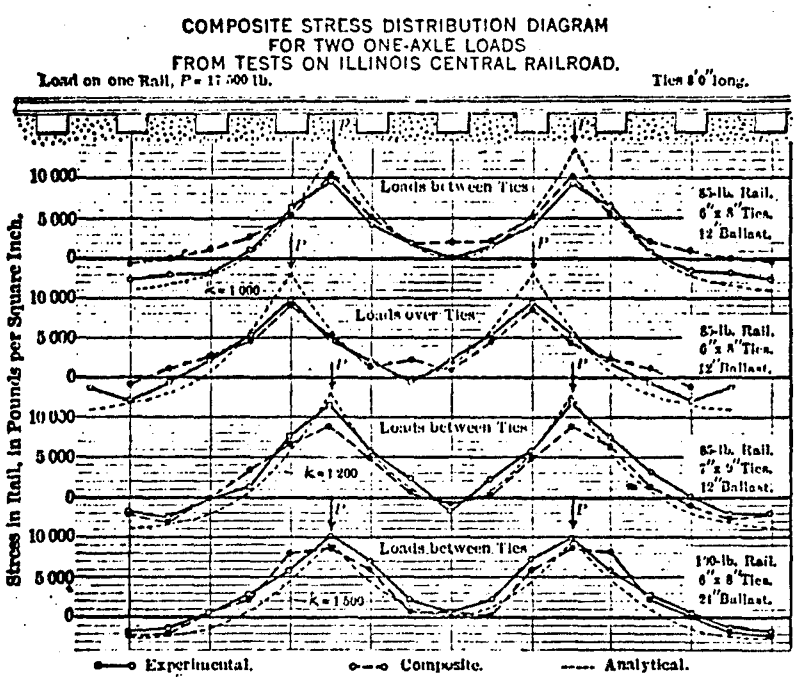



FIG. 9. COMPARISON OF STRESSES FOR CROSS-TIE TRACK [11]

to standardize the analyses of the track, by comparing the available analyses with test results. As part of this effort, for several years stresses were measured in the rails of the German and Dutch tracks which were caused by a train of a prescribed composition [25]. The obtained results were then compared with the calculated stresses based on the bending moment formulas of Winkler, Zimmermann, and van Dijk (who like Zimmermann used discrete elastic supports, but took into consideration the effect of adjoining wheel loads). On the basis of this comparison it was concluded that the axle-load spacing is an important parameter,that the use of discrete elastic supports led to too high stresses, and that the analysis based on rigid discrete supports yielded stress values which in the mean agreed with the measured ones.

Based on these conclusions, the Technical Committee of VMEV, at its meeting held September 16-18, 1930 in Münster Germany, recommended an analysis of the rails, developed by the Dutch railroads, which is based on a weightless beam which rests on rigid discrete supports with possibility of lift-off and is loaded by a periodic load distribution, as shown in Fig. 7(IV). The recommended expression for the bending moments under a load, which is always located in midspan, is

$$M = \frac{12mn - 7(m + n) + 4}{16\,[3mn - (m + n)]}\,Pa \tag{23}$$

where m, n are wheel set separation parameters shown in Fig. 7 (IV). For additional details of the recommendations the reader is referred to Ref. [25] p. 120. For a discussion of this method of analysis refer to Hanker [8] p. 39.

As part of the above effort, preliminary tests were also conducted for the determination of the base parameter k which enters eq. (3), in order to determine whether eq. (3) is suitable for the analysis of the cross-tie track, and thus whether the "standard" track analysis of VMEV could be based on eq. (3). Since no conclusive results were obtained, the Technical Committee recommended (also in 1930) that member railroads conduct tests to determine k using a standard test.

The recommended test consisted of loading vertically one tie, which was separated from the rails by removal of the fasteners, and then by recording its vertical displacement due to the load. Two "point" loads were generated by a loaded freight car (about 16 tons) which was equipped with two hydraulic pumps mounted between the two wheel sets. The pumps, when activated, pressed against the tie, lifting up the car; thus exerting about 16 tons on the tie ([25] p. 121).

A total of 385 tests were conducted on the lines of the German, Dutch, and Swiss railroads. The Technical Committee of VMEV could not detect definite effects of the various types of ballast, of the condition of the ballast, or of the type of the ties on the vertical tie displacements. It did notice a strong effect on the tie response by the type of sub-base, but could not establish a tendency based on sub-base properties. On the basis of these findings, the Technical Committee of VMEV, at its meeting in Stockholm May 28-30, 1935, passed a resolution to recommend to its member railroads *not* to use eq. (3) for the analysis of the railroad track [25].

It appears that the main problem with the above study (and the VMEV resolution) was that the test set-up used to obtain the base parameter k, which loads only one tie, is conceptually incorrect.

The first shortcoming is that the base parameter k depends on the size of the loading area [27] [28]. Thus, the loading with one tie does not yield the same coefficient k as the loading by a row of closely spaced ties encountered in an actual track. The conceptual difficulties encountered by Driessen ([25] p. 125), who observed that the adjacent ties when separated from the rails although unloaded also displaced vertically, are closely connected to this phenomenon.

The second shortcoming is that the material properties of the ballast and sub-soil, because of their granular character, vary locally. Thus the loading of only one tie, at different locations along the track, will necessarily show a wide scatter in the obtained data. This is very apparent from the test data presented by Driessen ([25] p.123).

For a critical discussion of some of the arguments presented by Driessen [25],who favored the VMEV decision, refer to R. Hanker ([29] Section IV). It should be noted that the VMEV decision was made in spite of the findings of the extensive study by the ASCE-AREA Committee [11] [22] discussed previously and the opinion of many central European track experts [30-37], who favored the use of eq. (3) for the analysis of rail stresses.

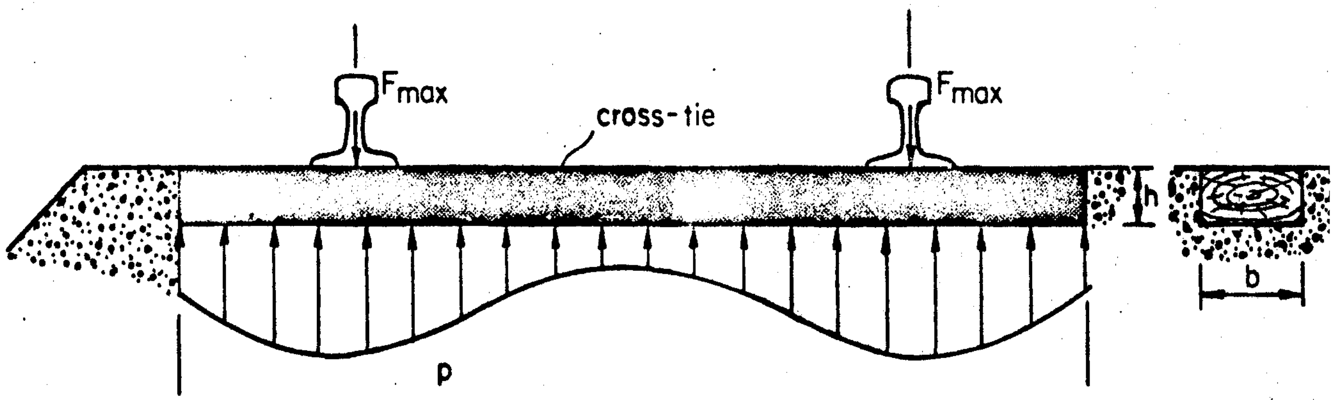
In 1937, A. Wasiutynski [14] published the results of an extensive experimental program performed on a main line track by subjecting it to moving locomotives and by comparing the obtained results with those based on eq. (3). The presented graphs show general agreement between the measured and calculated deflections and bending moments for the rails; thus confirming the findings of the ASCE-AREA Committee [11] [22] that eq. (3) is suitable for the rail analysis of the cross-tie track.

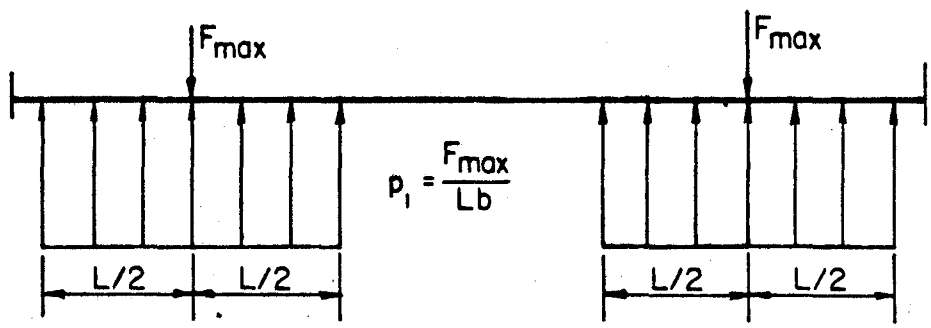
In the course of the following decades the use of the analysis based on eq. (3) found general acceptance, as evident from the writings by S. Timoshenko and B.F. Langer [38], C. W. Clarke [39], G. Sauvage [40], J. Eisenmann [41], H. C. Meacham, R. H. Prause and J. D. Waddell [42] and the Association of American Railroads [43] as well as from the books by W. W. Hay [44], M. Shrinivasan [45], R. Hanker [8], A. Schoen [46], G. Schramm [47], G. M. Shakhunyants [48], M. A. Frishman and co-authors [49], and V. V. Basilov and M. A. Chernyshev [50]. However, as pointed out by Schoen ([46] p. 258), the simple formulas (20) and (21), in spite of their known deficiencies, are still being used by a number of railroads for the determination of the largest bending moment in the rails of a cross-tie track.

A shortcoming of track analyses based entirely on eq. (3) was suggested by the observation that, for example, in front of a locomotive over a certain interval the track lifts off the ballast. Because of the separation of the rail-tie frame from the ballast, in this domain eq. (3) is not valid, since $k = 0$. Problems of this type were recently solved by Y. Weitsman [51].

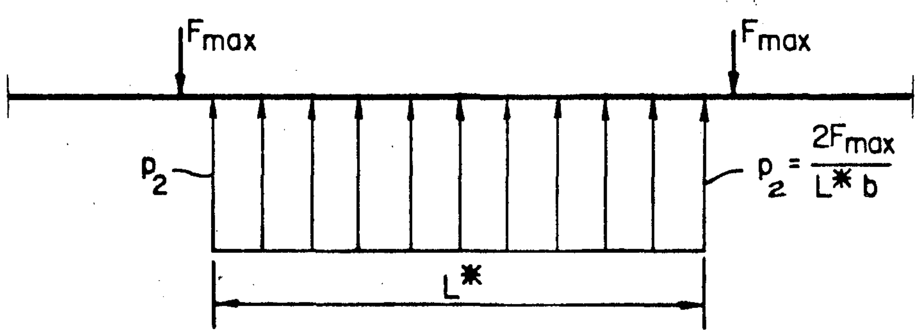
Once the "continuity" of rail support is adopted, the *determination of the force the rail exerts on a tie* is very simple; it is the contact pressure integrated from half span to half span; or approximately, the pressure ordinate at the tie multiplied by the center to center tie spacing. The determined largest force, $F_{max}$, that each rail could exert on a cross-tie caused by the anticipated wheel loads of a moving train, are then used for the stress analysis of the cross-tie, as shown in Fig. 10.

A *cross-tie analysis* based on eq. (3) was presented by Zimmermann [6] in 1888. A major shortcoming of this analysis is the assumption




(I) Assumed pressure distribution for determination of $M_{max}$ at rail seat




(II) Assumed pressure distribution for determination of $M_{max}$ at midspan

FIG. 10. TO THE STRESS ANALYSIS OF CROSS-TIES

that the tie rests on a uniform linear Winkler base, which is not the case in the field. In order to prevent "end bound" or "center bound" ties, the ballast is usually tamped under each rail seat. Thus, the resulting contact pressure distribution is often as shown in Fig. 10 (For actual test results refer to [11], Second Progress Report, 1920). Because of the continuously varying contact pressure distribution, caused by changing ballast properties due to the moving trains and environmental factors, the design analysis of ties is often based on the simplifying assumption that the contact pressure distribution is uniform and extends over a distance L or L*, as indicated in Fig. 10 (Distributions I and II). The values L and L* are based on experience ([52] p. 285, [39] p. 159, [42] I p. 52). Because of the uncertainty in the tie support conditions during the tie service life, this method, although very simple, yields an upper bound on the expected tie bending stresses and thus seems sufficient for tie design purposes.

# 4. DETERMINATION OF THE PARAMETERS IN EQ. (3) FOR THE CROSS-TIE TRACK

The utilization of eq. (3), for the stress analysis of the rails and the determination of the forces the rails exert on the cross-ties, requires the knowledge of three entites : the load parameter q, the bending rigidity EI, and the base parameter k.

## 4.1 THE LOAD PARAMETER q

The load parameter is determined from the geometry and the axle loads of the locomotives and cars to be used on the track under consideration. Thus, once the anticipated rolling stock and admissible train speeds are established, the parameter q, which enters eq. (3), is known.

Since q usually consists of a large number of concentrated forces (wheel loads), the resulting deflections and bending moments are determined using the influence function method, as indicated in eq. (11).

## 4.2 THE BENDING RIGIDITY EI IN THE VERTICAL PLANE

The bending rigidity of the rail-tie structure in the vertical plane is usually assumed to be the product of E for rail steel multiplied by the 2 moments of inertia of each rail with respect to the horizontal axis which passes through their centroids.

The determination of EI was the subject of a major controversy in the early 1930's. Nemcsek [30, 35] and Janicsek [32, 36] were of the opinion that the cross-ties contribute to the rigidity of the track and added the I of the ties divided by the tie spacing to the I of the rails, whereas Saller [31, 34] and Hanker [37] argued that the effect of the ties is negligible. This controversy ended without agreement among the authors [53].

The inclusion of the I of the cross-ties, as done by Nemcsek and Janicsek, is definitely not correct. However, the close spacing of the ties and the rigidity of some of the fasteners currently in use, may contribute to an increase of the "effective" bending rigidity of the track, which in turn may have an effect on $w(x)$. To illustrate this phenomenon, consider the strongly exaggerated situation, shown in Fig. 11, in which a beam is periodically "rigidized". The effect of the rigid parts on the global response of the beam is obvious: It leads to smaller deflections and an increased "effective" rigidity.

Because of the close tie spacing currently in use on main lines, it may be advisable to measure the rail deflections and stresses in an actual track, in order to establish whether the "rigidization" of the rails in the fasteners noticeably affects them. As part of this test program one could also study the effect of the tie resistance to rotation about their long axes on the track response, as discussed by Hanker [29] and Kerr [1]. These rotational resistances do occur, but are not taken into consideration in eq. (3). As shown by Kerr [1], the corresponding differential equation is

$$EI \frac{d^4w}{dx^4} - \rho \frac{d^2w}{dx^2} + kw = q \qquad (3')$$

where $\rho$ is the rotational proportionality coefficient.

## 4.3 THE BASE PARAMETER

The Winkler assumption for the base response

$$p(x) = k\,w(x) \qquad (2)$$

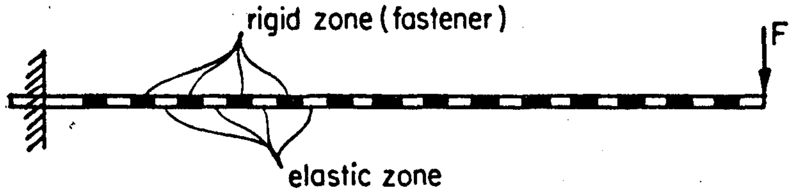



FIG. 11. IDEALIZED BEAM MODEL TO DEMONSTRATE THE EFFECT OF THE FASTENERS ON THE BENDING RIGIDITY OF THE RAILS

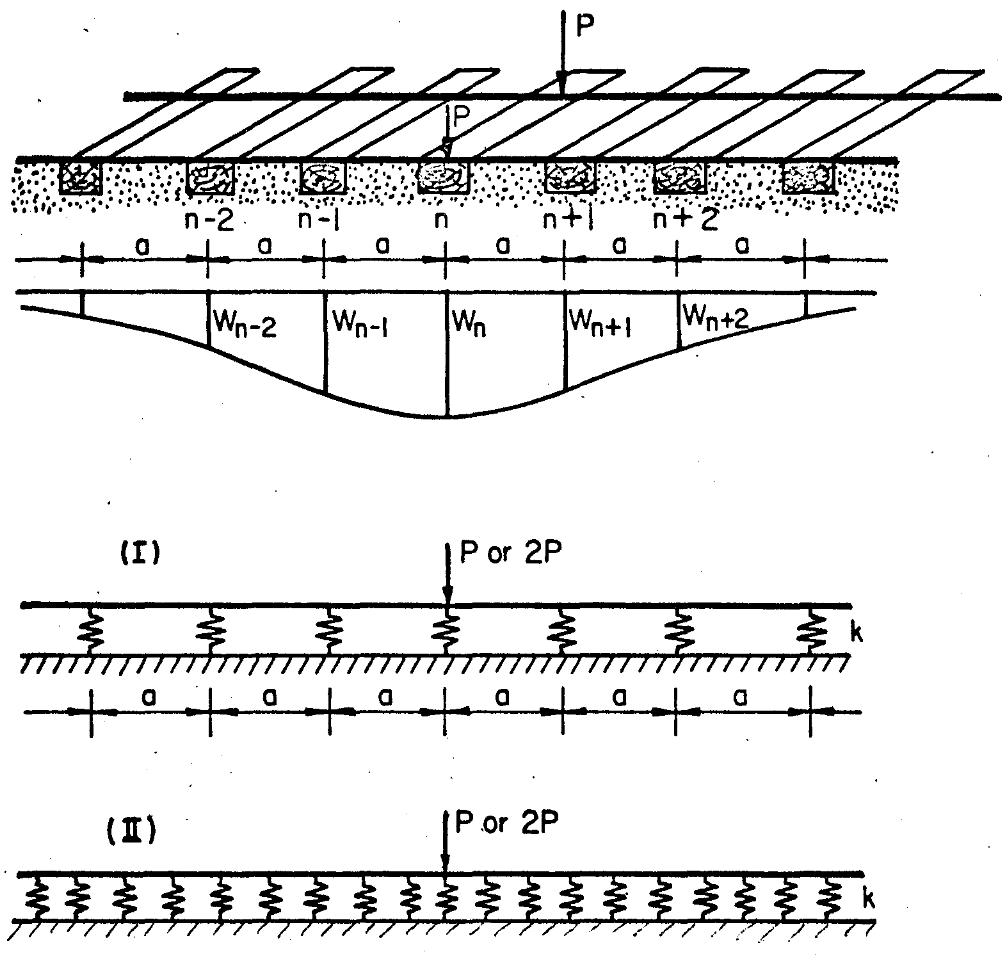



FIG. 12. TO THE DERIVATION OF EXPRESSIONS FOR THE DETERMINATION OF THE BASE PARAMETERS

is an approximation. A major shortcoming of this representation is the absence of shear interactions between the vertical base elements. These questions were discussed by A. D. Kerr [54] in 1964. Refer also to [27] and [28].

Because of the simplifying assumptions implicit in relation (2), the k coefficient which enters eq. (3) is not a true constant, but depends on the size of the loading area. As pointed out previously, this was one of the reasons why the *loading of only one tie*, as suggested and practiced by VMEV, did not yield meaningful results. In this connection note also the more recent test results reported by F. Birmann [55], which were obtained utilizing this approach. According to H. Nagel [56], this test method is being used by the DB for the determination of the efficiency of track compaction.

In view of the approximate nature of eq. (2) and of the governing eq. (3), the method for the determination of k should be such that the analytically obtained quantities (like rail deflections and/or the stress distribution in the rails) should represent the corresponding actual quantities as closely as possible. To achieve this objective, first of all a test for the determination of k should involve a relatively long section of track subjected to vertical loads, similar to the actual situation in the field. In the following, three methods for the determination of k are discussed which utilize the entire track and in which the rails are loaded vertically, as shown in Fig. 12. The difference between these methods is the way the k-value is computed from the obtained test data.

One method for the determination of k was used by the AREA-

ASCE Special Committee ([11] First Progress Report, 1918) and by Wasiutynski [14]. Although the agreement found between analytical and test results for rail deflections and stresses appears satisfactory, questions may be raised regarding the validity of the used method for calculating k from the test data.

To demonstrate this point, let us derive the formulas used in [11] and [14] for the determination of k, and thus establish the assumptions they are based on. For this purpose consider the track subjected to two loads P, as shown in Fig. 12. The corresponding averaged deflection of the two rails over each tie caused by 2P, is denoted by $w_m$. To avoid future misunderstandings, in the following the foundation modulus for the two rails is denoted by k and the one for only one rail by $k_r$.

Consider the two rails as a beam which rests on *discrete* linearly elastic springs with spacing a as shown in Fig. 12(I). Assuming that $\kappa$ is the spring constant for the two rails and $\kappa_r$ for one rail, it follows from vertical equilibrium that

$$2P = \kappa \sum_{n=-\infty}^{\infty} w_n \qquad \text{or} \qquad P = \kappa_r \sum_{n=-\infty}^{\infty} w_n \tag{24}$$

Thus, the spring constants, which are assumed not to vary along the track, are

$$\kappa = \frac{2P}{\sum_{n=-\infty}^{\infty} w_n} \qquad \text{and} \qquad \kappa_r = \frac{P}{\sum_{n=-\infty}^{\infty} w_n} \tag{25}$$

where $w_n$ are the measured (averaged) rail deflections at each tie and

$$\kappa_r = \kappa/2 \tag{25'}$$

Next, consider the two rails as a beam which rests on a *continuous* Winkler base, as shown in Fig. 12(II). Then vertical equilibrium yields

$$2P = \int_{-\infty}^{\infty} p(x)dx = k\int_{-\infty}^{\infty} w(x)dx \qquad \text{or} \qquad P = k_r\int_{-\infty}^{\infty} w(x)dx \tag{26}$$

Thus

$$k = \frac{2P}{\int_{-\infty}^{\infty} w(x)dx} \qquad \text{and} \qquad k_r = \frac{P}{\int_{-\infty}^{\infty} w(x)dx} \tag{27}$$

and

$$k_r = k/2 \tag{27'}$$

In order to find the dependence between $\kappa$ and k, or $\kappa_r$ and $k_r$, we equate the corresponding right hand sides of (24) and (26) since, for a given test, the P values are the same. This results in

$$k\int_{-\infty}^{\infty} w(x)dx = \kappa\sum_{n=-\infty}^{\infty} w_n \qquad \text{and} \qquad k_r\int_{-\infty}^{\infty} w(x)dx = \kappa_r\sum_{n=-\infty}^{\infty} w_n \tag{28}$$

or rewritten, noting that the tie spacing **a** is constant,

$$k\int_{-\infty}^{\infty} w(x)dx = \frac{\kappa}{a}\sum_{n=-\infty}^{\infty} w_n a \qquad \text{and} \qquad k_r\int_{-\infty}^{\infty} w(x)dx = \frac{\kappa_r}{a}\sum_{n=-\infty}^{\infty} w_n a \tag{29}$$

The integral in eq. (29) is the area formed by the deflection curve. When the tie spacing is so small that

$$\int_{-\infty}^{\infty} w(x)dx \cong \sum_{n=-\infty}^{\infty} w_n a \tag{30}$$

then the equations in (29) reduce to

$$k = \frac{\kappa}{a} \qquad \text{and} \qquad k_r = \frac{\kappa_r}{a} \tag{31}$$

This is the relationship introduced by Timoshenko [10], also presented by M. Hetényi [57] p. 27, whose derivation seems to be missing in the literature (Hanker [37] p. 93; Saller [53] p. 97).

Eliminating $\kappa$ and $\kappa_r$ from (31), using (25), yields

$$k = \frac{2P}{\sum_{n=-\infty}^{\infty} w_n a} \quad \text{and} \quad k_r = \frac{P}{\sum_{n=-\infty}^{\infty} w_n a} \tag{32}$$

which are equivalent to the equations in (27), when eq. (30) is valid.

According to the above procedure two loads P are placed on the track, as shown in Fig. 12, and the deflections w of the rails over each tie are measured. Then the $\kappa$ or $\kappa_r$ value is obtained using (25) and the corresponding k or $k_r$ value using (32). Note that for tests which use more wheel load, the above derivations are valid by replacing P with the sum of the used loads.

Determining the rail foundation modulus in this manner, the ASCE-AREA Special Committee [11]* and Wasiutynski [14] found good agreement between the measured rail deflections and rail stresses and the corresponding values based on eq. (3). The result of one such comparison is shown in Fig. 9.

Nevertheless a question may be raised regarding the general validity of eq. (25) and eq. (32) from a conceptual viewpoint. For this purpose, consider a long straight track which, when subjected to loads 2P, deflects as shown in Fig. 12. According to (32), the corresponding k-value is

$$k = \frac{P}{A_1} \tag{33}$$

*) According to [11], 1st Progress Report, Section 47, the expression for the determination of k is the same as the one shown in (32), except that the denominator is multiplied by the number of ties in the track test section. In view of the above derivations, this is incorrect.

where $A_1$ is the area formed by the straight and deflected rail axes. Next, imagine the rails replaced by much lighter (or much heavier rails), without disturbing the base, and then subjected again to 2P. It is easy to realize that the area formed by the straight and the new deflection curve, $A_2$, will generally not be equal to $A_1$. Thus, the calculated k-value will not be the same, although the ties, ballast, and sub-base (whose properties k represents) are identical for both cases. It appears that for eq. (32) to yield reasonable k-values, in the test the contact shape and the EI of the structure (here the rail-tie system) should be as close as possible to the one to be analyzed.

The other two methods for the determination of k or $k_r$ also utilize the test set up shown in Fig. 12, but determine the value by comparing a measured deflection or a strain with the corresponding value based on eq. (3). This is a proper approach, since *the criterion for determination of k should be that the analytical results represent the actual ones as closely as possible*. These two methods are described in the following.

In one method [38] the deflections of both rails under the loads 2P are measured. The average of these two deflections, $w_m$, is substituted in eq. (8). Placing the origin of x at the wheel load, thus x = 0 at P, it follows that

$$w_m = \frac{2P\sqrt[4]{\dfrac{k}{4E(2I_r)}}}{2k} \qquad \text{or} \qquad w_m = \frac{P\sqrt[4]{\dfrac{k_r}{4EI_r}}}{2k_r} \tag{34}$$

where E is Young's modulus for rail steel and $I_r$ is the moment of inertia of a rail with respect to the horizontal axis which passes through its centroid. Comparing the equations in (34) it follows that

$k = 2k_r$, in agreement with eq. (27'). Solving each equation in (34) for k, the only unknown, we obtain

$$k = \frac{1}{2}\sqrt[3]{\frac{P^4}{EI_r\, w_m^4}} \qquad \text{and} \qquad k_r = \frac{1}{4}\sqrt[3]{\frac{P^4}{EI_r\, w_m^4}} \tag{35}$$

The other method for the determination of k is based on the measurement of the strain at the bottom of each rail under the load, by means of strain gauges. The average value of the two measurements, $\varepsilon_m$, multiplied by E for rail steel yields the corresponding stress, $\sigma_m = E\varepsilon_m$. Noting that the bending moment M(0) is given in eq. (9), and that the bending stress at the bottom is $\sigma = M(0)z_o/I$, it follows that

$$\sigma_m = \frac{2P}{4\sqrt[4]{\frac{k}{4E(2I_r)}}}\,\frac{z_o}{(2I_r)} \qquad \text{or} \qquad \sigma_m = \frac{P}{4\sqrt[4]{\frac{k_r}{4EI_r}}}\,\frac{z_o}{I_r} \tag{36}$$

where $z_o$ is the vertical distance between the centroid of the rail and its bottom surface. Solving for k, the only unknown, we obtain

$$k = \frac{P^4 z_o^4}{32(EI_r)^3\varepsilon_m^4} \qquad \text{and} \qquad k_r = \frac{P^4 z_o^4}{64(EI_r)^3\varepsilon_m^4} \tag{37}$$

Note that when eq. (35) is used, k is determined by equating the deflection obtained analytically with the measured value at only one point, namely at P. When eq. (37) is used, k is determined by equating the analytical expression and the measured value of the bending strain also at only one point. Thus, in both approaches k is determined by equating only one analytical and test quantity at one point. In view of the good agreement between the deflection curves and bending moments, based on eq. (3) and the corresponding test data of actual tracks shown in [11] and [14], the determination of k by equating *only one analytical and test quantity at one point* may be sufficient.

From the above discussion it follows that the value of k or $k_r$ could also be determined from a least square (or any other suitable) fit of the measured and calculated deflections, or stresses, or other quantities of special interest which can be easily measured.

It appears, that the method for the determination of k which uses relations (35) or (37) or both, because of its simplicity and non-destructive character, may be the most suitable one also for the determination of the efficiency of track compaction machinery, if the value of k is a suitable indicator.

In connection with the above discussion of the loads, bending rigidities, and rail foundation moduli, note that when both rails are considered as a *track-beam* then eq. (3) becomes

$$2EI_r \frac{d^4w}{dx^4} + k\,w = q_{track} \tag{38}$$

whereas when only one rail is considered then the corresponding differential equation is

$$EI_r \frac{d^4w}{dx^4} + k_r w = q_{rail} \tag{39}$$

where $q_{track} = 2\,q_{rail}$ .

# 5. THE STRESS ANALYSIS OF RAILS SUBJECTED TO NON-CENTRAL AND LATERAL LOADS

The vertical force of an actual railroad wheel does not act on the rail centrally. Furthermore a wheel of a moving train exerts on the rail also lateral forces. Corresponding stress analyses and test results were presented by S. Timoshenko [58] and by S. Timoshenko and B. F. Langer [38]. A more recent discussion of these stresses is contained in a paper by J. Eisenmann [59] and in an ORE Report [60].

It may be of interest to note, however, that to date many railroads do not require such analyses. For example, according to Schoen [46], Vol. I, p. 250, the effect of the lateral forces is taken into consideration by choosing a low value for $\sigma_{all}$ in the bending analysis for vertical wheel loads. Schramm ([47], p. 58) used a similar approach, by estimating these additional stresses using test data as a guide. Hay [44] describes a similar procedure (p. 417), as do Frishman and co-authors [49].

## 6. THE WHEEL-RAIL CONTACT STRESSES

In the wheel-rail contact region, where as much as 16 tons are transmitted from the wheel to the rail over a very small area, the stresses deviate considerably from those obtained from the bending theory of beams discussed above. The occurrence of "shelling" rail failures prompted many analytical and experimental studies of this problem. For an early study refer to N. M. Belyaev [61]. For more recent results refer to N. Gössl [62], J. Eisenmann [63], and G. C. Martin and W. W. Hay [64]. For an extensive up-to-date survey of analyses and tests dealing with rail-wheel contact stresses refer to B. Paul [65].

These studies are of utmost importance, in view of the continuously increasing car loads and the resulting increase of rail failures. If it can be proven to the railroad community that

(1) the *condition* of a main line track for the usual train speeds has a relatively small effect on the contact stresses, and

(2) that these increasing stresses are directly related to the increasing number of rail failures

then, unless a major metallurgical breakthrough is achieved and realizing the practical limitations on the wheel size, this may lead to a restriction of the axle loads. This in turn may induce the car builders to increase the number of axles per car or to limit the weight of cars and locomotives.

## 7. EFFECT OF TRAIN SPEED ON TRACK STRESSES

According to test results (for example [14]), the forces a moving train exerts on the rails of a well maintained track are very little affected for train speeds of up to about 50 km/h (30 mph). With increasing train speeds, however, the forces increase noticeably. A number of analytical papers were published in this area. However, to date this problem because of its complexity (since it involves the dynamic interaction between the train and track), is not yet solved. A review of the published tests and analytical results would require a separate study.

It should be pointed out, however, that many railroads when analyzing the track components for design purposes, take into account the effect of the train speed by multiplying the static forces by a "speed coefficient". For example, according to Schramm([47], p. 49), the bending moment due to dynamic loads is

$$M_{dyn} = \alpha M_{static} \tag{40}$$

where $\alpha$ is the train speed coefficient

$$\alpha = 1 + 4.5 \times 10^{-5}v^2 - 1.5 \times 10^{-7}v^3 \qquad \text{for } v < 170 \text{ km/h} \tag{41}$$

In the above equation v (in km/h) is the largest admissible train speed. According to the above formula

| v km/h (mph) | 0 | 30 (18.7) | 70 (43.5) | 140 (87.0) |
|---|---|---|---|---|
| $\alpha$ | 1.00 | 1.04 | 1.17 | 1.47 |

For a discussion of other speed coefficient formulas refer to Clarke ([39], Part 7).

## 8. CONCLUSIONS AND RECOMMENDATIONS

The above study leads to the conclusion that, to date, eq. (38) or (39) is the most suitable one (also because it is very simple) for the determination of bending stresses in the rails and for the determination of the forces the rails exert on the cross-ties due to vertical loads. The extensive test results presented by the ASCE-AREA Special Committee [11] and by Wasuitynski [14] prove that the obtained accuracy is sufficient for many design purposes. The coefficients which enter into (38) and (39) are defined in the previous sections.

For the determination of the k or $k_r$ value the actual track should be used by loading it with a *slowly moving* (about 10 km/h) locomotive or loaded car with known axle loads. The speed of the moving vehicles should be high enough to avoid the occurrence of non-elastic deformations and low enough not to create noticeable inertia effects in the track, effects not represented in eq. (3). The calculation of k from the obtained test data should be based on a comparison of the test data and the corresponding results based on eq. (3) of the type shown in (35) and (37).

In conclusion it should be noted that the stress analysis of a track, as presently prescribed by many railroads, usually involves the determination of the following quantities for an anticipated load environment:

- largest bending stress $\sigma^{(1)}_{max}$ in a rail (due to wheel loads and temperature change),
- largest bending stress $\sigma^{(2)}_{max}$ in a cross-tie,

- largest contact pressure $\sigma_{max}^{(3)}$ between the rail and the cross-tie or when tie plates are used between the tie plates and cross-tie,
- largest contact pressure $\sigma_{max}^{(4)}$ between the cross-tie and the ballast, and
- largest contact pressure $\sigma_{max}^{(5)}$ between the ballast and subsoil.

The obtained values $\sigma_{max}$ are then compared with the corresponding allowable stresses $\sigma_{all}$, which are usually prescribed in the codes of the various railroads. The design criteria used are

$$\sigma_{max}^{(n)} < \sigma_{all}^{(n)} \qquad n = 1, 2, \ldots, 5.$$

The above analyses have to be conducted taking into consideration the anticipated wear of the rail heads, because rail wear decreases the bending rigidity of the rail. Additionally, it has to be noted that in a new or newly renovated track, the ballast properties may differ substantially from those of a track compacted by traffic, and hence for the same loads the track response may differ. The same applies to the base properties during the winter and summer periods, because the track is exposed to the seasonal weather conditions and field tests show that the ballast and subsoil characterisitics are affected by it.

The above is a simple method of track analysis which for standard track designs yields reasonable results. This is to be expected because for standard tracks the design and the various elements are similar (by definition) and the stipulation of a $\sigma_{all}$ value, for each case n, is based on closely related test results. However, with increasing wheel loads and train speeds more sophisticated analyses may be needed for the analysis of the existing tracks as well as for the planned new ones.

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## APPENDIX:

## REPORT OF INVENTIONS

After a review of the work performed under this phase of the contract, it was determined that no technical innovation, discovery, or invention has been made. The purpose of this project was to write a survey of the bending stress analyses of rails and ties and a discussion of the related tests to be conducted.