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ANALYTICAL FINITE ELEMENT SIMULATION MODEL FOR STRUCTURAL CRASHWORTHINESS PREDICTION

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16. Abstract

The analytical development and appropriate derivations are presented for a simulation model of vehicle crashworthiness prediction. Incremental equations governing the nonlinear elasto-plastic dynamic response of three-dimensional frame structures are derived, where the associated stiffness and compatibility matrices also incorporate large geometry changes. A discussion of yield criteria is given, together with bound type estimates for thin walled cross section beams. The Newmark beta method is then used to solve the equations of motion, and is oriented toward the particular incremental equations typical of the present application.

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PREFACE

The work described here was performed under the program for analytical crashworthiness prediction at the Transportation Systems Center sponsored by the Research Institute of the National Highway Traffic Administration of the U.S. Department of Transportation. This program is intended to provide engineering data that can be applied to establishing regulations relating to vehicle collision performance to improve motor vehicle safety. The principal author, Dr. J. Rossettos, Associate Professor of Mechanical Engineering at Northeastern University, held a temporary appointment as a staff consultant to the Transportation Systems Center during the summer of 1973.

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LIST OF SYMBOLS

$\tilde{C}$	Compatibility matrix for frame structure
$\tilde{C}_i$	Compatibility matrix for beam element i
$\tilde{g}_i$	Matrix relating increments in internal forces to deformation increments as defined by Equation (2.1)
$\tilde{K}$	Stiffness matrix of overall frame structure
$\tilde{K}_r$	Reduced stiffness matrix as defined by Equation (5.7)
$\tilde{\ell}$	Vector in the direction of line connecting the ends of a beam element and defined in Equation (3.2)
$\tilde{P},\tilde{Q}$	Loading vectors on the right hand side of the equations of motion and defined in conjunction with Equation (2.27)
$\tilde{T},\tilde{S}$	Transformation matrices of direction cosines associated with end faces of beam elements and defined by Equations (3.7) and (3.21) respectively
X_k,Y_k,Z_k	Global coordinates of node k
$\Delta M_y,\Delta M_z$	Incremental bending moments about the local y and z respectively
ΔN	Increment in axial force
ΔT	Increment in torque
$\Delta\tilde{R}_i$	Increment in vector of deformation quantities in a beam element
$\Delta\tilde{S}_i$	Increment in vector of internal forces in a beam element
$\Delta X_k,\Delta Y_k,\Delta Z_k$	Incremental changes in location of node k as referred to the global axes (X,Y,Z)
$\Delta\delta$	Increment in beam element elongation
$\Delta\theta_y,\Delta\theta_z$	Increments in angular deformations about the local y and z axes respectively
$\Delta\phi_x,\Delta\phi_y,\Delta\phi_z$	Increments in rotations of beam element end face with respect to the global axes
$\Delta\psi$	Increment in twist angle between the end faces of a beam element

LIST OF SYMBOLS (CONTINUED)

$\underset{\sim}{\xi}$	Vector of increments in displacements as defined by Equation (5.6).
θ	Angle between the vector normal to a beam element end face and the $\underset{\sim}{\ell}$ vector
$\underset{\sim}{\Omega}$	Rotation matrix associated with an end face of a beam element, as defined by Equation (3.15a)

1. INTRODUCTION

Structural crashworthiness plays an important role in the NHTSA mission. There are definite requirements for analytic models which can serve as interpolation tools in conjunction with crash testing. These simulation models can also serve as design feasibility and evaluation tools.

The Transportation Systems Center has been asked by NHTSA to provide support in the formulation and implementation of such analytic models and computer programs in order to predict vehicle crashworthiness. This support will include the acquisition, implementation and extension of existing computer codes.

Some promising analytic tools are the Calspan-Shieh two-dimensional frame analysis program,[1] the Battelle-FMCCM lumped mass program,[2] and the Lockeed three-dimensional KRASH program.[3]

In recent years various simulation programs have been developed to model the dynamic structural response under vehicle impact conditions. The models vary from the very simple which can give only average features of the overall response, to the rather complex, where greater detail in the response can be provided. The simplest form of analytic simulation to date has been embodied in simplified spring-mass models with 2-3 lumped masses and less than ten degrees of freedom, while generalized resistances are made to represent gross vehicle structural properties. An example of such models is given in Reference 4. Correlation with tests depends heavily on making a judicious choice for the parameters which measure the generalized resistances. Therefore, their use as predictive tools is limited although they can be used to establish general behavior. A good example of the proper use of such models is given in Reference 5.

The next step beyond the simplified spring-mass model exists in the BCL simulation program.[2] Four masses are represented together with 35 individual nonlinear resistances. There is a restriction to unidirectional motion. Judgment in the selection of mass and resistance parameters is required, especially in interpreting the crush data to be used for the six different types of force-deformation curves which are available. Of course, this offers a degree of flexibility to the user.

There are certain simulation programs which are labelled as "hybrid" models because they require as necessary input, experimental crush data.[6] Correlation of the static deformation mode with the dynamic mode is at present a very difficult problem, so that extrapolation to other environments is not assured, and again experience and judgment are important for any reasonable prediction capability.

The next step in complexity involves the frame models[1,2,3,7] which are comprised of a large number of beam elements and lumped masses. Increasing the number of degrees of freedom would, of course provide increased ability for the evaluation of detailed structural response of components in vehicle impacts. The simplest of the frame models is the Calspan-Shieh program,[1] a two-dimensional model which provides for elasto-plastic response by the use of plastic hinges. The plastic hinge idea is, of course, a simplified approach to yielding of a beam cross section since details of the stress distribution over the cross section are not taken into account. Correlation with limited tests has been shown to be satisfactory. Another frame model, KRASH program,[3] was originally developed for aircraft structure, and in principle, it can be made to apply to vehicle impact. It can be regarded as an extension of the BCL model, consisting of lumped masses connected by straight beam elements, where each mass has three translational and three rotational degrees of freedom. The codes can include energy absorber devices and seat collapse mechanisms. The large deformation characteristics are treated by piecewise linearization, whereby the linear stiffness matrix is adjusted for plasticity at each time step by multiplying by a stiffness reduction factor. This factor is determined from static crush data, so that again experimental difficulties similar to the Kamal model[6] exist, since static and dynamic mode behavior is not easily correlated.

A more general element frame model is the three-dimensional CRASH program,[7] which contains additional features not found in the last two models. No prior assumption on the locations of plastic hinges is required. Moments and forces at the nodes are computed by numerical integration of the stress distribution over the cross section. The actual stress-strain behavior of the material may be used directly but the stress state must be monitored at various locations accross the cross section. It is clear that the additional effort and computer time required for this program, still does not allow prediction of detailed vehicle response much beyond the capability of the previous two models. This is because the frame concept cannot really model an entire vehicle body, and also since local deformation of the cross section and joint inefficiency are not accounted for.

TSC and the University of Michigan under NHTSA contract conclude that there is a need for a hybrid finite element program, which would, for instance, incorporate shell, frame, lumped parameter and finite difference models. These models would in fact form modules for the overall program, where each module can be regarded as a "super-element". It should be pointed out that the use of various simulation programs will be dictated by the particular impact situation to be modelled. For instance, in some low speed impact situations where the bumpers alone may be involved, the simple BCL model, or in combination with the Calspan-Shieh model may be sufficient to handle the significant features. In any case, it is clear that the three-dimensional frame structure will form an important module, and it will be the main concern of the present report. Figure 1-1 shows a typical vehicle where some of the module representations are indicated. A spring suspended large mass which may be the engine is shown, together with supporting frame and plate and shell portions.

TSC has become familiar with the Calspan model and has simplified the input to the two-dimensional program, so that it can be run by personnel with little knowledge of the development details. The present report summarizes the analytic requirements for extension of the Calspan two-dimensional frame to a three-dimensional frame. A future document will discuss the programming requirements for implementing the analytical development. Program modifications made by TSC to expedite usage by engineers will also be described.

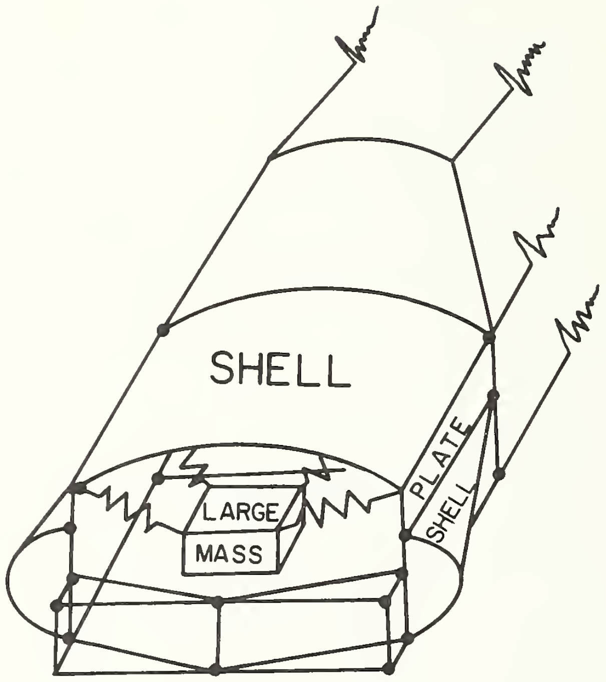

Figure 1-1. Hybrid Vehicle Model

2. ANALYSIS OF THE THREE-DIMENSIONAL FRAME

The three-dimensional frame is a necessary module, since it is possible with it to simulate at least the significant features of the large dynamic plastic deformation and geometry changes of the vehicle. This is done by means of an appropriate breakdown into straight beam elements which are connected at nodes, with inertial effects treated by means of lumped point or rigid body masses at the nodes.

With a sufficient number of beam elements and nodes, one should be able to obtain in adequate detail, displacement and acceleration time histories of nodal positions which represent relatively important points in the vehicle structure, including of course the three-dimensional motions of the occupant compartments. By combining this information with up-to-date biomechanics data, one can evolve an estimate of crashworthiness in a particular environment.

By use of the frame model, the basic overall structural dynamic response phenomenon can be identified. Therefore with some experience and careful interpretation, it can also be used in a very important way to establish the usefulness of any selected experimental parameters in taking and using test data, so that costs of tests are minimized.

In order to define the various quantities to be used in the analysis a hypothetical frame is shown in Figure 2.1. It should be clear that in this figure not all nodes and elements are necessarily shown. This is to avoid cluttering the figure, since its purpose is mainly to define nomenclature.

With reference to Figure 2-1 the following definitions are used:

<u>Nodes</u> - are fictitious bodies to which two or more beam elements may be connected. The end of each element connected to a particular node takes on the displacements and rotations of that particular node. Plastic hinges and

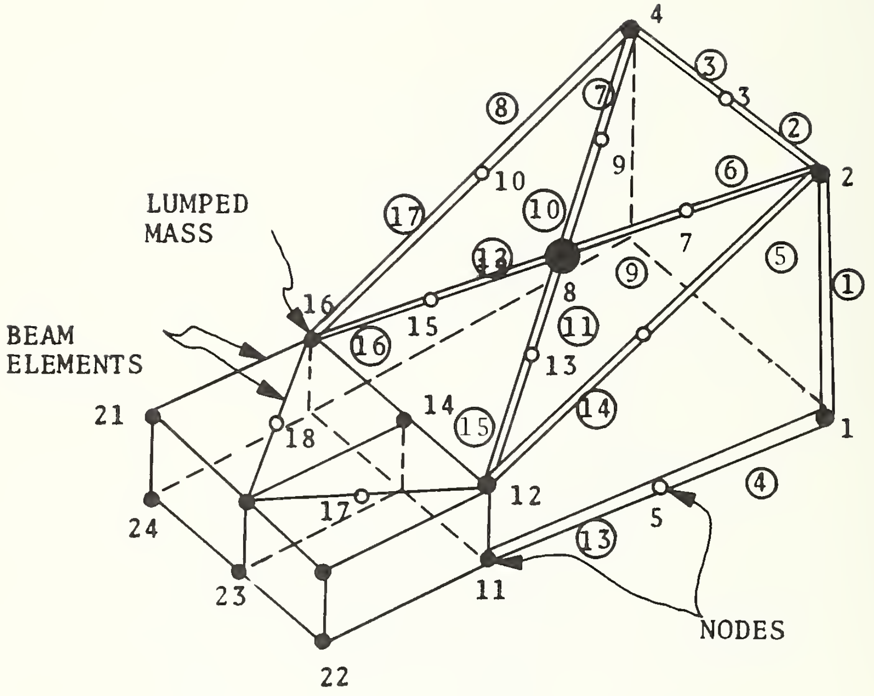

Figure 2-1. Frame Geometry and Notation

external forces may be concentrated at nodes. Lumped masses may or may not be assigned to a node. For instance, the locations numbered 1, 2, 3, ... 24, are nodes. The solid circles have assigned lumped masses while the hollow circles do not. In a three-dimensional analysis there are six degrees of freedom per node, so that for n nodes a system of 6 n equations would need to be solved.

Beam elements - are uniform straight members connecting two nodes. In Figure 2-1 beam elements are denoted by the circled numbers (i.e., 1, 2, 3 .....).

Beam members - are actual beams which make up the frame. They differ from beam elements in that two or more nodes can occur on a beam member. For instance, in Figure 2-1, the lines between nodes 2 and 12 and nodes 12 and 16 are beam members.

The physical assumptions of our model are essentially those adopted in the CALSPAN model of Reference 1. The implications of these assumptions are as follows. All plastic deformation is to occur at the nodes of our beam element. The location of hinges must be chosen a priori, and this gives the lengths of our elements. The plastic hinge is operative when appropriate stress resultants and bending moments lie on a given yield surface for the cross section. Perfectly plastic behavior is assumed so that material strain hardening is neglected. Also, the stress resultants at the cross section at initial yield are not significantly different from those at the fully plastic section. Finally, the frame structure may undergo large rigid body translations and rotations. These assumptions are felt to be reasonable for mild steel, thin members, and vehicle type frame loadings.

In the analysis, the matrix displacement method of frame analysis is to be used, and the dynamic problem is reduced to a nonlinear initial value problem, which is governed by simultaneous second order differential equations in time. The solution is carried out incrementally, and the coefficients of the 6 n dependent variables (n = number of nodes in the frame) are updated at each time increment to take into account the possible initiation of either a new plastic loading or unloading condition at some of the nodes. This means that the stiffness matrix, which will be defined shortly, is to be modified at each time step.

In the next few paragraphs, we will define the coefficients which enter into the equations of motion and the dependent variables whose solution is sought. Since the basic building block of our model is the beam element let us first describe the forces acting on it at the nodes, and the resulting deformations and displacements which it experiences.

If we imbed a local coordinate system at each end face (or node) of the element, we can describe large rotations (rigid body and deformation type) of the beam elements and their nodes by studying how such local coordinate systems translate and rotate with respect to a fixed (global) system. In regard to Figure 2-2(a) line 1 joins the beam element end points (or nodes) at faces "a" and "b". The global coordinates of nodes a and b are X_1, Y_1, Z_1 and X_2, Y_2, Z_2, respectively. The quantities $\underset{\sim}{N}$ and $\underset{\sim}{M}$ are stress resultant and moment vectors. Note that from here on a wavy symbol under a letter will denote a vector or a matrix, and will be clear from the context. In Figure 2-2 (b) $\underset{\sim}{1}_a$ is a unit vector along line 1, and $\underset{\sim}{x}_a$, $\underset{\sim}{y}_a$, $\underset{\sim}{z}_a$ are principal axes of the beam at face a, where x_a is normal to the beam face. We define $\underset{\sim}{x}_a$, $\underset{\sim}{y}_a$, $\underset{\sim}{z}_a$ as mutually perpendicular unit vectors in these directions respectively. Quantities to be referred to the local beam face coordinate system will be written in terms of these unit vectors. For instance, the moment vector when written as $\underset{\sim}{M} = T\underset{\sim}{x}_a + M_y\underset{\sim}{y}_a + M_z\underset{\sim}{z}_a$ yields the torque T about the local x_a axis and the bending moments M_y and M_z about the local $\underset{\sim}{y}_a$ and $\underset{\sim}{z}_a$ axes, respectively.

In Figure 2-2(b) the quantities ϕ_x, ϕ_y, ϕ_z are rotations of the local beam face coordinate system about the global axes X, Y, Z. The angle θ involves beam bending deformation. Later we will be interested in the increments $\Delta\theta_y$ and $\Delta\theta_z$ which are incremental bending deformations of the beam about the local (y,z) axes due to the moment increments ΔM_y and ΔM_z, respectively.

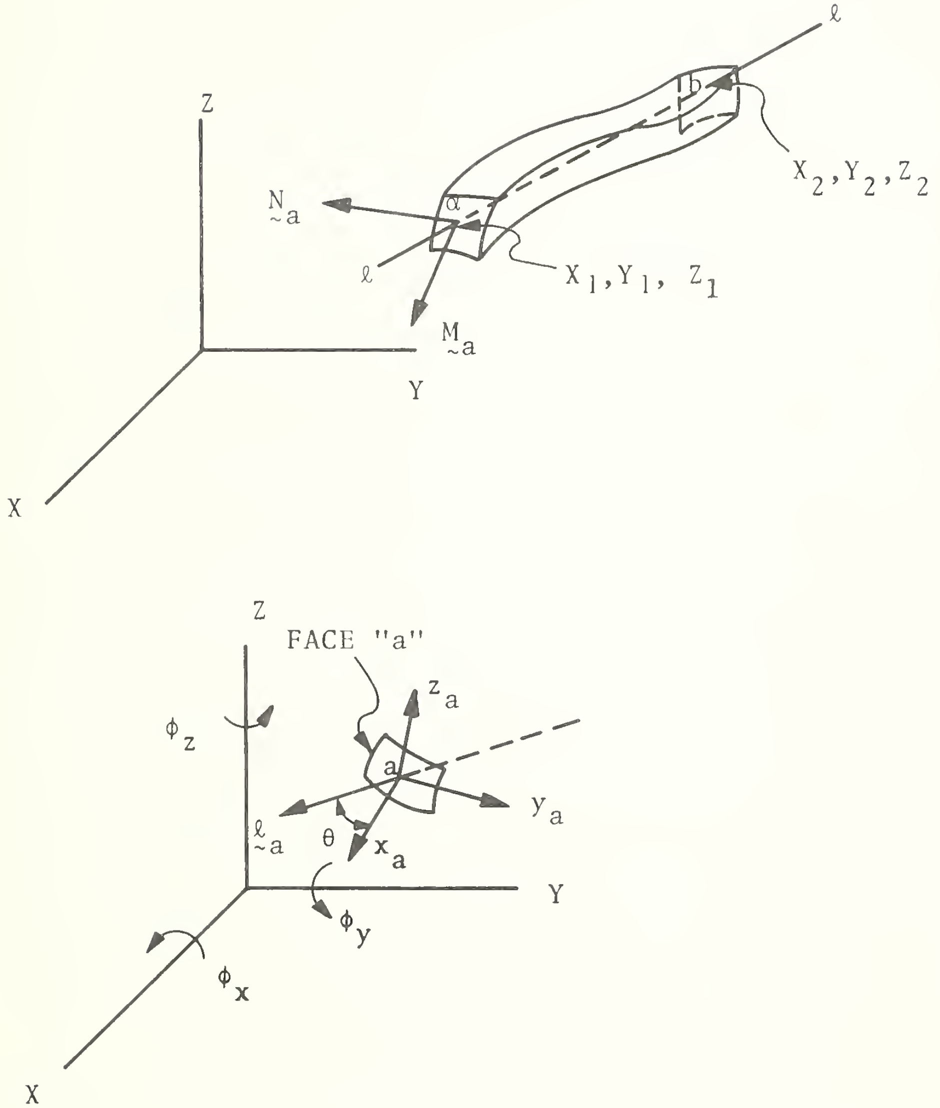

Figure 2-2 a,b. Beam Element Orientations

Associated with the torque T about the local x axis is the twisting deformation which is denoted by ψ in what follows. Also, associated with the axial force, N, acting on the beam element is the elongation δ.

2.1 STIFFNESS RELATIONS

With the notation just described, one can write the incremental stiffness relation for the ith beam element. In matrix form, this can be written as

$$\Delta \underset{\sim}{S}_i = \underset{\approx}{g}_i \Delta \underset{\sim}{R}_i \tag{2.1}$$

where

$\Delta \underset{\sim}{S}_i$ are increments in internal loads acting on the beam element

$\underset{\approx}{g}_i$ is the element stiffness matrix

$\Delta \underset{\sim}{R}_i$ are increments in the deformation quantities

so that for element, i, we have

$$\Delta \underset{\sim}{R}_i \quad \left\{ \begin{matrix} \Delta\theta_{ay} \\ \Delta\theta_{az} \\ \Delta\theta_{by} \\ \Delta\theta_{bz} \\ \Delta\delta_{\delta} \\ \Delta\psi \end{matrix} \right\}_i \qquad \Delta \underset{\sim}{S}_i = \left\{ \begin{matrix} \Delta M_{ay} \\ \Delta M_{az} \\ \Delta M_{by} \\ \Delta M_{bz} \\ \Delta N \\ \Delta T \end{matrix} \right\}_i \tag{2.2}$$

The beam element stiffness matrix, $\underset{\approx}{g}_i$, depends on whether the element nodes are elastic or plastic. The procedure in Reference 1 can be used to obtain the following values for $\underset{\approx}{g}_i$:

a. <u>For elastic behavior</u>

$$\underset{\sim}{g}_i = \begin{bmatrix} \frac{4EI_y}{\ell} & o & \frac{2EI_y}{\ell} & o & o & o \\ o & \frac{4EI_y}{\ell} & o & \frac{2EI_z}{\ell} & o & o \\ \frac{2EI_y}{\ell} & o & \frac{4EI_y}{\ell} & o & o & o \\ o & \frac{2EI_z}{\ell} & o & \frac{4EI_z}{\ell} & o & o \\ o & o & o & o & \frac{AE}{\ell} & o \\ o & o & o & o & o & \frac{GJ}{\ell} \end{bmatrix}_i \tag{2.3}$$

b. <u>For a plastic hinge at the "a" end</u>

$$\underset{\sim}{g}_i = \begin{bmatrix} o & o & o & o & o & o \\ o & o & o & o & o & o \\ o & o & \frac{3EI_y}{\ell} & o & o & o \\ o & o & o & \frac{3EI_z}{\ell} & o & o \\ o & o & o & o & \frac{AE}{\ell} & o \\ o & o & o & o & o & o \end{bmatrix}_i \tag{2.4}$$

c. <u>For a plastic hinge at the "b" end</u>

$$\underset{\sim}{g}_i = \begin{bmatrix} \frac{3EI_y}{\ell} & o & o & o & o & o \\ o & \frac{3EI_y}{\ell} & o & o & o & o \\ o & o & o & o & o & o \\ o & o & o & o & o & o \\ o & o & o & o & \frac{AE}{\ell} & o \\ o & o & o & o & o & o \end{bmatrix}_i \qquad (2.5)$$

d. <u>For a plastic hinge at both ends</u>

$$\underset{\sim}{g}_i = \begin{bmatrix} o & o & o & o & o & o \\ o & o & o & o & o & o \\ o & o & o & o & o & o \\ o & o & o & o & o & o \\ o & o & o & o & \frac{AE}{\ell} & o \\ o & o & o & o & o & o \end{bmatrix}_i \qquad (2.6)$$

We next define the G matrix which relates increments of the deformation quantities to the increments in internal loads for all elements of the frame, so that

$$\Delta\underset{\sim}{S} = \underset{\sim}{G}\Delta\underset{\sim}{R} \qquad (2.7)$$

where, for a frame having a total of n_B beam elements,

$$\Delta \underset{\sim}{S} = \begin{Bmatrix} \Delta \underset{\sim}{S}_1 \\ \Delta \underset{\sim}{S}_2 \\ \cdot \\ \cdot \\ \cdot \\ \Delta \underset{\sim}{S}_{n_B} \end{Bmatrix} ; \quad \Delta \underset{\sim}{R} = \begin{Bmatrix} \Delta \underset{\sim}{R}_1 \\ \Delta \underset{\sim}{R}_2 \\ \cdot \\ \cdot \\ \cdot \\ \Delta \underset{\sim}{R}_{n_B} \end{Bmatrix} ; \quad \underset{\sim}{G} = \begin{bmatrix} \underset{\sim}{g}_i & & & & \\ & \underset{\sim}{g}_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \underset{\sim}{g}_{n_a} \end{bmatrix} \tag{2.8}$$

Finally, it is necessary to relate the deformation quantities for an element to the displacements of the two nodes of the element, and rotations of the faces at these two nodes about the fixed (global) axes. These displacements and rotations comprise the total degrees of freedom for each element. There are six degrees of freedom at each node, and therefore, twelve degrees of freedom for each element. For n_B elements, we have 12 n_B equations of motion. Specifically, for purposes of solving the equations incrementally, we wish to express the increments in the deformation quantities, $\Delta \underset{\sim}{R}$, to the increments in displacement quantities, $\Delta \underset{\sim}{u}$, which are the changes in end point (nodal) locations and rotations. This is done in Section 3, where for a given element, i, a compatibility matrix $\underset{\sim}{C}_i$ is determined, so that the following relation holds for element, i.

$$\Delta \underset{\sim}{R} = \underset{\sim}{C}_i \Delta \underset{\sim}{u}_i = \left[C_{ij} \right]_{\substack{i \\ 6x12}} \Delta \underset{\sim}{u}_i \tag{2.9}$$

where

$$\Delta \underset{\sim}{R}_i^T = \left[\Delta\theta_{ay} \ \ \Delta\theta_{az} \ \ \Delta\theta_{by} \ \ \Delta\theta_{bz} \ \ \Delta\delta \ \ \Delta\psi \right]_i \tag{2.9a}$$

and

$$\underset{\sim}{\Delta u}_i^T = \left[\Delta X_1, \Delta Y_1, \Delta Z_1, \Delta X_2 \Delta Y_2 \Delta Z_2 \Delta\phi_{xa} \Delta\phi_{ya} \Delta\phi_{za} \Delta\phi_{xb} \Delta\phi_{yb} \Delta\phi_{zb} \right]_i \qquad (2.9b)$$

The superscript, T, denotes the transpose of a matrix quantity. By appropriately assembling the results for all elements of the frame, one arrives at the result

$$\Delta\underset{\sim}{R} = \underset{\sim}{C}\Delta\underset{\sim}{u} \qquad (2.10)$$

where

$$\Delta\underset{\sim}{R} = \begin{Bmatrix} \Delta\underset{\sim}{R}_1 \\ \Delta\underset{\sim}{R}_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \Delta\underset{\sim}{R}_{n_B} \end{Bmatrix} ; \quad \Delta\underset{\sim}{u} = \begin{Bmatrix} \Delta\underset{\sim}{u}_1 \\ \Delta\underset{\sim}{u}_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \Delta\underset{\sim}{u}_{n_B} \end{Bmatrix} ; \quad \underset{\sim}{C} = \begin{bmatrix} \underset{\sim}{C}_1 & & & \\ & \underset{\sim}{C}_2 & & \\ & & \ddots & \\ & & & \underset{\sim}{C}_{n_B} \end{bmatrix} \qquad (2.10a,b,c)$$

In Section 3, the $\underset{\sim}{C}_i$ matrix will be determined, but in that section, the subscript i will be dropped for convenience, since all work in Section 3 refers to an element.

With regard to the equations of motion, the 12 degrees of freedom given in Equation (2.9b) will comprise the dependent variables for each element. The total number of dependent variables for the entire frame is then equal to the dimension of the vector $\Delta\underset{\sim}{u}$ in Equation (2.10b). We now wish to establish the relevant equations of motion, which are to be solved for the quantities $\Delta\underset{\sim}{u}$.

2.2 EQUATIONS OF MOTION

It is convenient to develop the equations of motion by first deriving the stiffness matrix for the associated statics problem and then using the concept of the d'Alembert force to account for inertias. For the statics problem, the appropriate stiffness relation to be used for our purposes can be written as

$$\Delta \underset{\sim}{P} = \underset{\sim}{K}\Delta \underset{\sim}{u} \tag{2.11}$$

where $\underset{\sim}{K}$ is defined as the stiffness matrix of the entire frame and the generalized displacements, $\Delta \underset{\sim}{u}$, are given by Equation (2.10b). The vector $\Delta \underset{\sim}{P}$ is the incremental generalized external loading vector, so that at each node the concentrated loads correspond to a particular degree of freedom in $\Delta \underset{\sim}{u}$. The stiffness matrix, $\underset{\sim}{K}$, is now derived by means of the principle of virtual work. If, for a virtual displacement pattern $\delta \underset{\sim}{u}$, we equate the internal virtual work to the external virtual work, this gives

$$\delta \underset{\sim}{R}^T \underset{\sim}{S} = \delta \underset{\sim}{u}^T \underset{\sim}{P} .$$

On using the variational form of Equation (2.10), this becomes

$$\delta \underset{\sim}{u}^T \underset{\sim}{C}^T \underset{\sim}{S} = \delta \underset{\sim}{u}^T \underset{\sim}{P} \tag{2.12}$$

For arbitrary $\delta \underset{\sim}{u}$, this implies that $\underset{\sim}{P} = \underset{\sim}{C}^T \underset{\sim}{S}$. On taking increments, get

$$\Delta \underset{\sim}{P} = \underset{\sim}{C}^T \Delta \underset{\sim}{S} \tag{2.12a}$$

Then, on using Equations (2.7) and (2.10) so that $\Delta \underset{\sim}{S} = \underset{\sim}{G}\underset{\sim}{C}\Delta \underset{\sim}{u}$, Equation (2.12a) becomes

$$\Delta \underset{\sim}{P} = \underset{\sim}{C}^T \underset{\sim}{G}\underset{\sim}{C}\Delta \underset{\sim}{u} \tag{2.13}$$

When Equation (2.13) is compared to Equation (2.11), it is clear that the stiffness matrix K is given by

$$\underset{\sim}{K} = \underset{\sim}{C}^T \underset{\sim}{G}\underset{\sim}{C} \tag{2.14}$$

Now, to provide internal force deformation relations for an elasto-plastic state in a typical time interval $\tau_p < t \leq \tau_{p+1}$, the internal forces $\underset{\sim}{S}(t)$ and deformations $\underset{\sim}{R}(t)$ are written as the sum of their value at τ_p plus an additional increment, so that

$$\underset{\sim}{S}(t) = \underset{\sim}{S}(\tau_p) + \Delta\underset{\sim}{S}(t) \tag{2.15}$$

$$\underset{\sim}{R}(\underset{\sim}{u}) = \underset{\sim}{R}(\underset{\sim}{u}(\tau_p)) + \Delta\underset{\sim}{R}(\underset{\sim}{u}) \tag{2.16}$$

where for an elasto-plastic material the deformation increment, $\Delta\underset{\sim}{R}$, consists of two parts (i.e., elastic $\Delta\underset{\sim}{e}$ and plastic $\Delta\underset{\sim}{r}$) so

$$\Delta\underset{\sim}{R} = \Delta\underset{\sim}{e} + \Delta\underset{\sim}{r} \tag{2.17}$$

where the $\Delta\underset{\sim}{r}$ vanish if the structure is in an elastic state.

At this point, the incremental equations of motion are derived by starting with the incremental statics result

$$\underset{\sim}{C}^T\Delta\underset{\sim}{S} = \Delta\underset{\sim}{P} \tag{2.18}$$

which is established from the intermediate virtual work result given by Equation (2.12a), by regarding $\delta\underset{\sim}{u}^T$ arbitrary at that stage and going through the same argument which led to Equation (2.14). In Equation (2.18) we have

$$\Delta\underset{\sim}{S} = \underset{\sim}{S}(t) - \underset{\sim}{S}(\tau_p) \tag{2.19}$$

$$\Delta\underset{\sim}{P} = \underset{\sim}{P}(t) - \underset{\sim}{P}(\tau_p) \tag{2.20}$$

On using Equations (2.18), (2.19), and (2.20), we can write

$$\underset{\sim}{C}^T\underset{\sim}{S}(t) - \underset{\sim}{C}^T\underset{\sim}{S}(\tau_p) = \underset{\sim}{P}(t) - \underset{\sim}{P}(\tau_p) \tag{2.21}$$

In Equation (2.21), since $\underset{\sim}{S}(\tau_p)$ and $\underset{\sim}{P}(\tau_p)$ are constants in the interval they cannot be related to anything that is a function of time, so we can deduce from Equation (2.21) that

$$\underset{\sim}{P}(\tau_p) = \underset{\sim}{C}^T\ \underset{\sim}{S}(\tau_p) \tag{2.22}$$

Equation (2.22) relates the external loads at the pth time step, $\underset{\sim}{P}(\tau_p)$, to the compatibility and internal force matrices $\underset{\sim}{C}$ and $\underset{\sim}{S}$ respectively, at the $t = \tau_p$. On using Equations (2.20) and (2.22), Equation (2.18) can be written as

$$\underset{\sim}{C}^T \Delta \underset{\sim}{S} = \underset{\sim}{P}(t) - \underset{\sim}{C}^T \underset{\sim}{S}(\tau_p) \tag{2.23}$$

Now, if we add a d'Alembert force, $-\underset{\sim}{M}\Delta\ddot{\underset{\sim}{u}} \equiv -\underset{\sim}{M}\ddot{\underset{\sim}{u}}$, (note that if $\underset{\sim}{u}(t) = \underset{\sim}{u}(\tau_p) + \Delta\underset{\sim}{u}(t)$, then $\ddot{\underset{\sim}{u}}(t) = \Delta\ddot{u}$, since $\underset{\sim}{u}(\tau_p)$ is constant) to the right hand side of Equation (2.23) the equation of dynamic equilibrium can be established. This equation of motion can then be written in the form

$$\underset{\sim}{M}\Delta\ddot{\underset{\sim}{u}} + \underset{\sim}{C}^T\Delta \underset{\sim}{S} = \underset{\sim}{P}(t) - \underset{\sim}{C}^T \underset{\sim}{S}(\tau_p) \tag{2.24}$$

where $\underset{\sim}{M}$ is a diagonal lumped mass matrix.

Now, at the initiation of the pth time increment (i.e., at $t = \tau_p^+$) the matrices $\underset{\sim}{G}$ and $\underset{\sim}{C}$ are updated (to account for elastic or plastic action as the case may be, and for geometry changes), and denoted by $\underset{\sim}{G}^{(p)}$ and $\underset{\sim}{C}^{(p)}$. Then, from Equations (2.7) and (2.10) we have

$$\Delta\underset{\sim}{S}(t) = \underset{\sim}{G}^{(p)}\Delta\underset{\sim}{R}(\underset{\sim}{u}) = \underset{\sim}{G}^{(p)}\underset{\sim}{C}^{(p)}\Delta\underset{\sim}{u} \tag{2.25}$$

Next, the vector $\underset{\sim}{Q}(\tau_p)$ is defined by

$$\underset{\sim}{Q}(\tau_p) = - [\underset{\sim}{C}^{(p)}]^T \underset{\sim}{S}(\tau_p) \ . \tag{2.26}$$

Using updated quantities as defined in Equations (2.25) and (2.26), Equation (2.24) becomes

$$\underset{\sim}{M}\Delta\ddot{\underset{\sim}{u}} + \underset{\sim}{K}^{(p)}\Delta\underset{\sim}{u} = \underset{\sim}{P}(t) + \underset{\sim}{Q}(\tau_p) \tag{2.27}$$

where the updated stiffness matrix is

$$\underset{\sim}{K}^{(p)} = \left[\underset{\sim}{C}^{(p)}\right]^T \underset{\sim}{G}^{(p)} \underset{\sim}{C}^{(p)} \tag{2.28}$$

3. THREE-DIMENSIONAL COMPATIBILITY MATRIX

As discussed in Section 2, the compatibility matrix, $\underset{\sim}{C}$, relates the increments in the local beam element deformation quantities, $\Delta\underset{\sim}{R}$, to the changes in nodal (beam element end points) locations and rotations, $\Delta\underset{\sim}{\ddot{u}}$, so that

$$\Delta\underset{\sim}{R} = \underset{\sim}{C}\Delta\underset{\sim}{u} \tag{3.1}$$

where

$$\underset{6\text{x}1}{\Delta R} = \begin{Bmatrix} \Delta\theta_{ay} \\ \Delta\theta_{az} \\ \Delta\theta_{by} \\ \Delta\theta_{bz} \\ \Delta\delta \\ \Delta\psi \end{Bmatrix} \quad ; \quad \underset{\sim}{C} = [C_{ij}]_{6\text{x}12} \quad ; \quad \Delta\underset{\sim}{u} = \begin{Bmatrix} \Delta X_1 \\ \Delta Y_1 \\ \Delta Z_1 \\ \Delta X_2 \\ \Delta Y_2 \\ \Delta Z_2 \\ \Delta\phi_{xa} \\ \Delta\phi_{ya} \\ \Delta\phi_{za} \\ \Delta\phi_{xb} \\ \Delta\phi_{yb} \\ \Delta\phi_{zb} \end{Bmatrix}.$$

The elements C_{ij} are determined in this section. The notation indicated in Figure 3-1 will be adopted in what follows.

In Figure 3-1, $\underset{\sim}{x}_a$, $\underset{\sim}{y}_a$, $\underset{\sim}{z}_a$ are mutually perpendicular unit vectors along principal axes at face "a", where $\underset{\sim}{x}_a$ is normal to face "a" (similar considerations apply to face "b"). The unit vector $\underset{\sim}{\ell}_a$ on face "a" is in line with the line joining the nodes at a and b. One task is to find the changes in bending components, $\Delta\theta_{ya}$, $\Delta\theta_{za}$, about the local y_a, z_a axes due to small rotations $\Delta\phi_x$, $\Delta\phi_y$, $\Delta\phi_z$ of the local beam face coordinate system, ($\underset{\sim}{x}_a$, $\underset{\sim}{y}_a$, $\underset{\sim}{z}_a$) about the fixed (global) axes, and small changes in locations of the beam element nodes, ΔX_1, ΔY_1, ΔZ_1, ΔX_2, ΔY_2, ΔZ_2 referred to global (X,Y,Z) axes. The subscripts 1 and 2 refer to nodes at face "a" and "b", respectively.

We will now show how the bending deformation angle increments, $\Delta\theta_{ya}$ and $\Delta\theta_{za}$ (which are caused by bending moment increments ΔM_{ay} and ΔM_{az}; see Figure 3-2) are related to $\Delta\ell_{az}$ and $\Delta\ell_{ay}$ which are incremental components of changes in the $\underset{\sim}{\ell}_a$ vector.

With regard to Figure 3-2, since $\underset{\sim}{x}_a$, $\underset{\sim}{y}_a$, $\underset{\sim}{z}_a$ are unit orthogonal base vectors of the local face "a" coordinate system, we can write

$$\underset{\sim}{\ell}_a = \ell_{ax}\,\underset{\sim}{x}_a + \ell_{ay}\,\underset{\sim}{y}_a + \ell_{az}\,\underset{\sim}{z}_a \tag{3.2}$$

Then

$$\underset{\sim}{1}\,\sin\theta_a = (\underset{\sim}{\ell}_a) \times (\underset{\sim}{x}_a) = (\ell_{ax}\underset{\sim}{x}_a + \ell_{ay}\,\underset{\sim}{y}_a + \underset{\sim}{\ell}_{az}\,\underset{\sim}{z}_a) \times (\underset{\sim}{x}_a)$$

$$= \ell_{ay}(\underset{\sim}{y}_a \times \underset{\sim}{x}_a) + \ell_{az}(\underset{\sim}{z}_a \times \underset{\sim}{x}_a)$$

But

$$\underset{\sim}{y}_a \times \underset{\sim}{x}_a = -\underset{\sim}{z}_a \quad \text{and} \quad \underset{\sim}{z}_a \times \underset{\sim}{x}_a = \underset{\sim}{y}_a$$

So that with respect to the local system

$$\underset{\sim}{1}\,\sin\theta_a = -\ell_{ay}\,\underset{\sim}{z}_a + \ell_{az}\,\underset{\sim}{y}_a \tag{3.3}$$

where $\underset{\sim}{1}$ is a unit vector in the $\underset{\sim}{z}_a\underset{\sim}{y}_a$ plane. Taking the differential of both sides of Equation (3.3) and setting $d\theta_a \cong \Delta\theta_a$, $d\ell_{ay} \cong \Delta\ell_{ay}$, and $d\ell_{az} \cong \Delta\ell_{az}$ we have

$$\cos\theta_a\Delta\underset{\sim}{\theta}_a = -\Delta\ell_{ay}\,\underset{\sim}{z}_a + \Delta\ell_a\,\underset{\sim}{y}_a \tag{3.4}$$

where $\Delta\underset{\sim}{\theta}_a = \underset{\sim}{1}\,\Delta\underset{\sim}{\theta}_a$. This result can be written in matrix form as follows

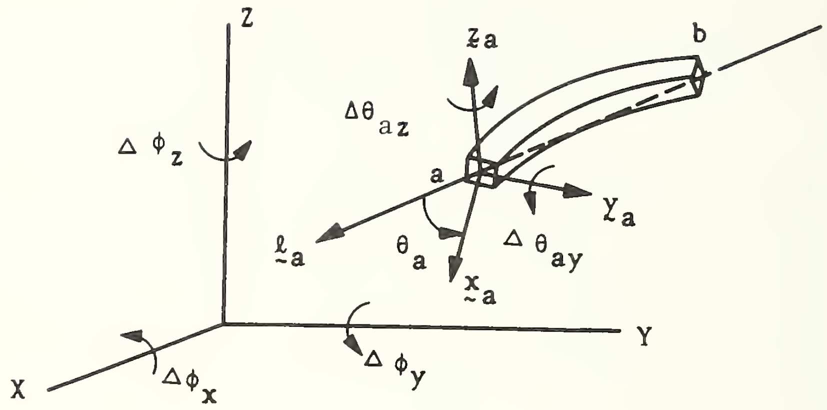

Figure 3-1. Face Local and Global Axes with Rotations

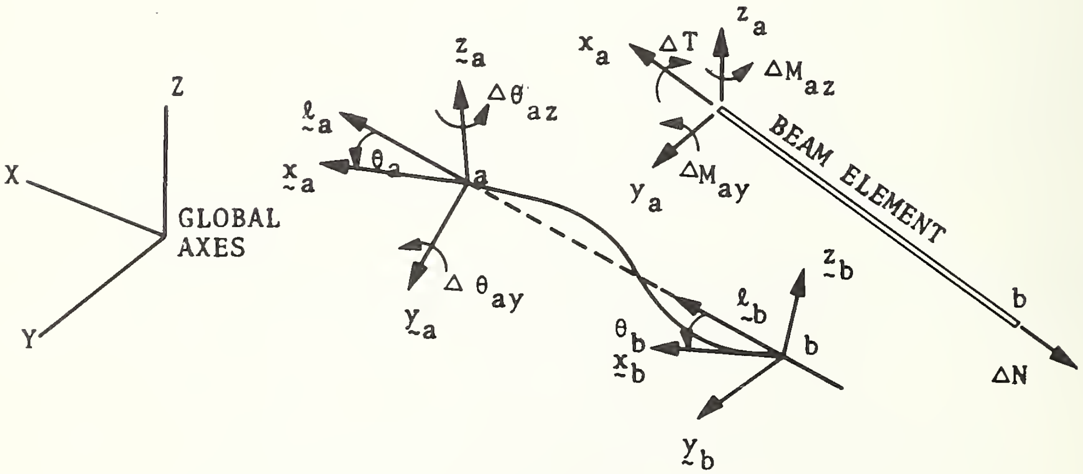

Figure 3-2. Beam Element Deformation

$$\underset{\sim}{\Delta\theta}_a = \begin{Bmatrix} \Delta\theta_{ay} \\ \\ \Delta\theta_{az} \end{Bmatrix} = \frac{1}{\cos\theta_a} \begin{Bmatrix} \Delta\ell_{az} \\ \\ -\Delta\ell_{ay} \end{Bmatrix} \tag{3.5}$$

Therefore, as was to be shown, the deformation angle increments are given by the z_a and y_a components of $\Delta\underset{\sim}{\ell}_a$ multiplied by the factor $1/\cos\theta_a$. Similar considerations are used at node b for $\Delta\underset{\sim}{\theta}_b$ (see Figure 3.2), and in this case, $\cos\theta_b$ would be involved.

The increments $\Delta\ell_{za}$ and $\Delta\ell_{ya}$ are defined so that $\Delta\underset{\sim}{\ell}_a \cong d\ell_a$. Therefore, the next step is to obtain the total differential of $\underset{\sim}{\ell}_a$ in terms of the quantities in the $\Delta\underset{\sim}{u}$ vector. The reason for this is to enable one to express some of the deformation quantities, namely $\Delta\theta_{ya}$ and $\Delta\theta_{za}$, in terms of $\Delta\underset{\sim}{u}$, and in this manner make contributions to the compatibility matrix $\underset{\sim}{C}$ defined in Equation (3.1).

In order to obtain $d\underset{\sim}{\ell}_a$ for our purposes, it is convenient to begin by defining the transformation $\ell_a = \underset{\sim}{T}\underset{\sim}{\ell}_r$ where the components of $\underset{\sim}{\ell}_r$ are with respect to the global (fixed) coordinate system, while $\underset{\sim}{\ell}_a$ is the unit vector coordinatized in the a-face frame x_a, y_a, z_a so that

$$\underset{\sim}{\ell}_a = \begin{Bmatrix} \ell_{ax} \\ \ell_{ay} \\ \ell_{az} \end{Bmatrix} \quad ; \quad \underset{\sim}{\ell}_r = \begin{Bmatrix} \ell_{rx} \\ \ell_{ry} \\ \ell_{rz} \end{Bmatrix} \tag{3.6}$$

and, T, which is the matirx of direction cosines, is given by

$$\underset{\sim}{T} = \begin{bmatrix} \cos(\hat{i},\underset{\sim}{x}_a) & \cos(\hat{j},\underset{\sim}{x}_a) & \cos(\hat{k},\underset{\sim}{x}_a) \\ \cos(\hat{i},\underset{\sim}{y}_a) & \cos(\hat{j},\underset{\sim}{y}_a) & \cos(\hat{k},\underset{\sim}{y}_a) \\ \cos(\hat{i},\underset{\sim}{z}_a) & \cos(\hat{j},\underset{\sim}{z}_a) & \cos(\hat{k},\underset{\sim}{z}_a) \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \tag{3.7}$$

Figure 3-3 shows some of the angles in the $\underset{\sim}{T}$ matrix, where $\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors in the global (X, Y, Z) directions. Other angles not shown in Figure 3-3, such as $(\hat{i}, y_a)$ which is the angle between the global X axis and the local $\underset{\sim}{y}_a$ axis, are defined in a similar manner.

Note that $\underset{\sim}{\ell}_r$ and $\underset{\sim}{\ell}_a$ are the same vector described in different coordinate systems. Now, in terms of the global coordinates of the beam element nodes (i.e., X_1, Y_1, Z_1 and X_2, Y_2, Z_2), one can express $\underset{\sim}{\ell}_r$ as follows

$$\underset{\sim}{\ell}_r = \frac{1}{|L|} \begin{Bmatrix} X_2 - X_1 \\ Y_2 - Y_1 \\ Z_2 - Z_1 \end{Bmatrix} \quad \text{where } |L| = [(X_2-X_1)^2+(Y_2-Y_1)^2+(Z_2-Z_1)^2]^{1/2}$$

So that

$$\underset{\sim}{\ell}_a = \frac{1}{|L|} \underset{\sim}{T} \begin{Bmatrix} X_2-X_1 \\ Y_2-Y_1 \\ Z_2-Z_1 \end{Bmatrix} . \tag{3.8}$$

The constant matrix $\underset{\sim}{D}$ is defined as

$$\underset{\sim}{D} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

Then $\underset{\sim}{\ell}_a$ can be written as

$$\underset{\sim}{\ell}_a = \frac{1}{|L|} \underset{\sim}{T} \; \underset{\sim}{D} \; \underset{\approx}{\bar{X}} \tag{3.9}$$

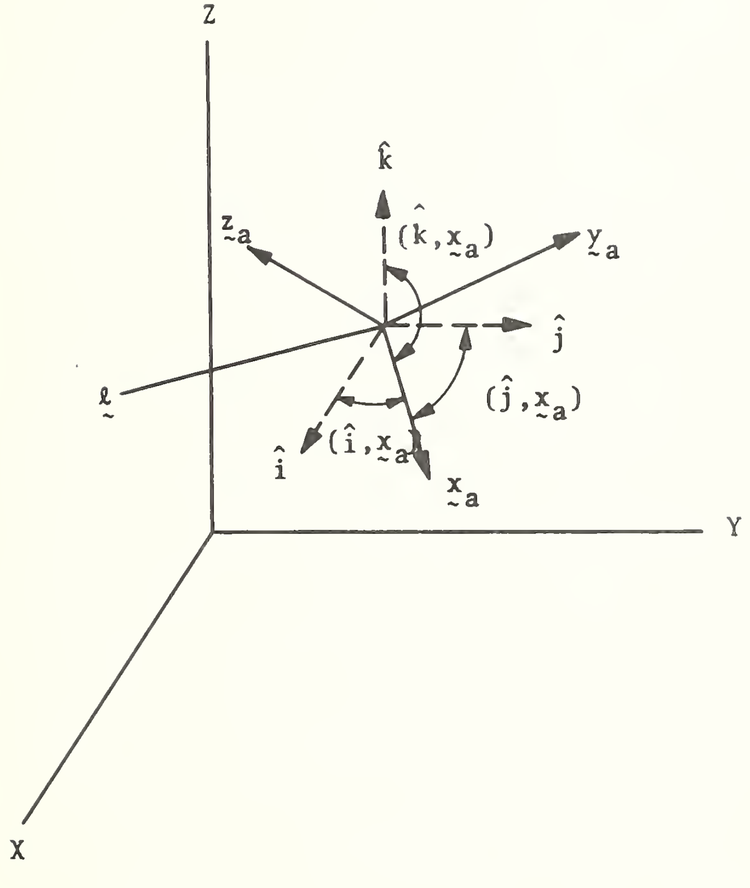

Figure 3-3. Direction Cosine Angles

where

$$\underset{\sim}{\overline{X}} = \begin{Bmatrix} X_1 \\ Y_1 \\ Z_1 \\ X_2 \\ Y_2 \\ Z_2 \end{Bmatrix} \tag{3.9a}$$

The next step is to take the total differential of $\underset{\sim}{\ell}_a$, as $\underset{\sim}{\ell}_a$ varies with the variations of the independent quantities which are the elements of the $\Delta\underset{\sim}{u}$ vector. This means that

$$d\underset{\sim}{\ell}_a = d\left(\frac{1}{|L|}\right) \underset{\sim}{T}\ \underset{\sim}{D}\ \underset{\sim}{\overline{X}} + \frac{1}{|L|}\ d\underset{\sim}{T}\ \underset{\sim}{D}\ \underset{\sim}{\overline{X}} + \frac{1}{|L|}\ \underset{\sim}{T}\ \underset{\sim}{D}\ d\underset{\sim}{\overline{X}} \tag{3.10}$$

We now evaluate the three terms on the right hand side of Equation (3.10). In regard to the first term, note that

$$d\left(\frac{1}{|L|}\right) = -\frac{1}{L^2}\, dL \cong -\frac{1}{L^2}\, \Delta L$$

represents a scalar change in length of the beam element. Since

$$L^2 = (X_2-X_1)^2 + (Y_2-Y_1)^2 + (Z_2-Z_1)^2$$

Then

$$2L\Delta L = 2(X_2-X_1)(\Delta X_2-\Delta X_1)+2(Y_2-Y_1)(\Delta Y_2-\Delta Y_1)+2(Z_2-Z_1)(\Delta Z_2-\Delta Z_1) \tag{3.11}$$

So

$$\Delta L = \frac{1}{|L|}\left[(X_2-X_1)(\Delta X_2-\Delta X_1)+(Y_2-Y_1)(\Delta Y_2-\Delta Y_1)+(Z_2-Z_1)(\Delta Z_2-\Delta Z_1)\right] \tag{3.12}$$

And,

$$d\left(\frac{1}{|L|}\right) = -\frac{1}{|L|^3}\,[X_2-X_1, Y_2-Y_1, Z_2-Z_1]\,\underset{\sim}{D}\begin{Bmatrix}\Delta X_1\\ \Delta Y_1\\ \Delta Z_1\\ \Delta X_2\\ \Delta Y_2\\ \Delta Z_2\end{Bmatrix} \tag{3.12a}$$

Or

$$d\left(\frac{1}{|L|}\right) = -\frac{1}{|L|^3}\,\underset{\approx}{\overline{X}}^T\,\underset{\sim}{D}^T\,\underset{\sim}{D}\,\Delta\,\underset{\approx}{\overline{X}} \tag{3.12b}$$

where $\underset{\approx}{\overline{X}}$ is defined in Equation (3.9a). Finally, the first term in the total differential of $\underset{\sim}{\ell}_a$ becomes

$$d\left(\frac{1}{|L|}\right)\underset{\sim}{T}\,\underset{\sim}{D}\,\underset{\sim}{X} = -\frac{1}{|L|^3}\,\underset{\approx}{\overline{X}}^T\,\underset{\sim}{D}^T\underset{\sim}{D}\Delta\,\underset{\approx}{\overline{X}}\,\underset{\sim}{T}\,\underset{\sim}{D}\,\underset{\approx}{\overline{X}} \tag{3.13}$$

Since $\underset{\approx}{\overline{X}}^T\underset{\sim}{D}^T\underset{\sim}{D}\Delta\underset{\approx}{\overline{X}}$ is a scalar, Equation (3.13) can be written as

$$d\left(\frac{1}{|L|}\right)\underset{\sim}{T}\,\underset{\sim}{D}\,\underset{\sim}{X} = -\frac{1}{|L|^3}\,\underset{\sim}{T}\,\underset{\sim}{D}\,\underset{\approx}{\overline{X}}\,\underset{\approx}{\overline{X}}^T\,\underset{\sim}{D}^T\,\underset{\sim}{D}\,\Delta\,\underset{\approx}{\overline{X}} \tag{3.14}$$

In order to evaluate the second term in Equation (3.10), we must determine $d\underset{\sim}{T}$ which indicates how the transformation matrix $\underset{\sim}{T}$ changes with rotations of the face or local coordinate system with respect to the fixed system. With regard to face "a", it is shown in Reference 8 that one can write

$$\Delta\underset{\sim}{T} \cong \underset{\sim}{T}\,\underset{\sim}{\Omega}_a \tag{3.15}$$

where

$$\underset{\sim}{\Omega}_a = \begin{bmatrix} o & \Delta\phi_{za} & -\Delta\phi_{ya} \\ -\Delta\phi_{za} & o & \Delta\phi_{xa} \\ \Delta\phi_{ya} & -\Delta\phi_{xa} & o \end{bmatrix} \tag{3.15a}$$

The second term in Equation (3.10) is then

$$\frac{1}{|L|} d\underset{\sim}{T} \underset{\sim}{D} \underset{\sim}{\overline{X}} \cong \frac{1}{|L|} \underset{\sim}{T} \underset{\sim}{\Omega}_a \underset{\sim}{D} \underset{\sim}{\overline{X}} . \tag{3.16}$$

This can be written in terms of $\underset{\sim}{\Phi}_a$ where

$$\underset{\sim}{\Phi}^T = [\Delta\phi_{xa} \; \Delta\phi_{ya} \; \Delta\phi_{za}] \tag{3.17}$$

So that

$$\frac{1}{|L|} d \; \underset{\sim}{T} \; \underset{\sim}{D} \; \underset{\sim}{\overline{X}} = \frac{1}{|L|} \underset{\sim}{T} \; \underset{\sim}{\overline{X}}^* \; \underset{\sim}{\Phi}_a \tag{3.18}$$

Where

$$\underset{\sim}{\overline{X}}^* = \begin{bmatrix} o & -(Z_2-Z_1) & (Y_2-Y_1) \\ (Z_2-Z_1) & o & (X_2-X_1) \\ -(Y_2-Y_1) & (X_2-X_1) & o \end{bmatrix} \tag{3.19}$$

Note that although $\underset{\sim}{\Phi}_a$ denotes quantities referred to face "a", similar considerations hold for face "b". Finally, the last term in Equation (3.10) is taken as

$$\frac{1}{|L|}\,\underset{\sim}{T}\,\underset{\sim}{D}\;\Delta\,\underset{\approx}{\overline{X}} = \frac{1}{|L|}\,\underset{\sim}{T}\,\underset{\sim}{D}\begin{Bmatrix}\Delta X_1\\ \Delta Y_1\\ \Delta Z_1\\ \Delta X_2\\ \Delta Y_2\\ \Delta Z_2\end{Bmatrix} \tag{3.20}$$

By combining the results in Equations (3.14), (3.16), and (3.20) $d\underset{\sim}{\ell}_a$ can be obtained, and if we set $d\underset{\sim}{\ell}_a = \Delta\underset{\sim}{\ell}_a$ where

$$\Delta\underset{\sim}{\ell}_a = \begin{Bmatrix}\Delta\ell_{ax}\\ \Delta\ell_{ay}\\ \Delta\ell_{az}\end{Bmatrix} \tag{3.21}$$

the quantities $\Delta\ell_{ay}$ and $\Delta\ell_{az}$ can be found in terms of $\Delta\underset{\sim}{u}$.

Another component of the beam deformation is the increment in the change in length of the beam element, $\Delta\delta$, which is given by the quantity ΔL in Equation (3.11). If we set $\Delta\delta = \Delta L$ and use previous notation, we can write,

$$\Delta\delta = \frac{1}{|L|}\,\underset{\approx}{\overline{X}}^T\,\underset{\sim}{D}^T\,\underset{\sim}{D}\;\Delta\,\underset{\approx}{\overline{X}} \tag{3.22}$$

where

$$\Delta\underset{\approx}{\overline{X}} \qquad \begin{Bmatrix}\Delta X_1\\ \Delta Y_1\\ \Delta Z_1\\ \Delta X_2\\ \Delta Y_2\\ \Delta Z_2\end{Bmatrix}$$

The expression for $\Delta\delta$ as given by Equation (3.22) is in the desired form, since it is given in terms of the changes in the locations of the beam element nodes, $\Delta\underset{\approx}{\overline{X}}$.

The remaining component of the deformation involves the twist angle ψ between the two faces of the beam element as shown in Figure 3-4.

In order to compute this angle, some preliminary considerations will first be indicated. If $\underset{\sim}{v}$ is an arbitrary vector then

$$\underset{\sim}{v}_a = \underset{\sim}{T}\,\underset{\sim}{v}_r$$

where $\underset{\sim}{v}_a$ contains components of $\underset{\sim}{v}$ with respect to "a" face local coordinates ($\underset{\sim}{x}_a$, $\underset{\sim}{y}_a$, $\underset{\sim}{z}_a$), and $\underset{\sim}{v}_r$ contains components of $\underset{\sim}{v}$ with respect to the global X, Y, Z axes. Similarly, we have

$$\underset{\sim}{v}_b = \underset{\sim}{S}\underset{\sim}{v}_r$$

where $\underset{\sim}{v}_b$ contains components of $\underset{\sim}{v}$ with respect to face "b" local coordinates ($\underset{\sim}{x}_b$, $\underset{\sim}{y}_b$, $\underset{\sim}{z}_b$). $\underset{\sim}{T}$ and $\underset{\sim}{S}$ are transformation matrices. From Equation (3.21) we have $\underset{\sim}{v}_r = \underset{\sim}{S}^T\underset{\sim}{v}_b$ so that on substitution into Equation (3.20) get

$$\underset{\sim}{v}_a = \underset{\sim}{T}\,\underset{\sim}{S}^T\,\underset{\sim}{v}_b \quad .$$

Now, the angle between the y-axes (or z-axes) of the two beam element face coordinate systems is an indication of the twist angle ψ. Next, select the $\underset{\sim}{y}_b$ axis (y-axis on b face) to be our vector, $\underset{\sim}{v}$. Then on applying Equation (3.22) get

$$\left(\underset{\sim}{y}_b\right)_a = \underset{\sim}{T}\,\underset{\sim}{S}^T\left(\underset{\sim}{y}_b\right)_b \qquad (3.23)$$

But $\left(\underset{\sim}{y}_b\right)_b = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$ since $\underset{\sim}{y}_b$ is a unit vector by definition.

Equation (3.23) then becomes

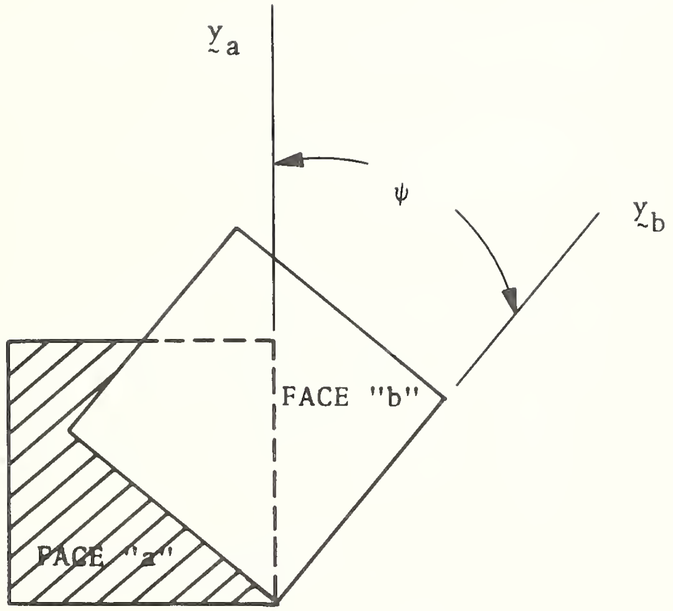

Figure 3-4. Twist Angle Between Beam Element End Faces

$$\left(\underset{\sim}{y}_b\right)_a = \underset{\sim}{T}\ \underset{\sim}{S}^T \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = \underset{\sim}{H} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} H_{12} \\ H_{22} \\ H_{32} \end{Bmatrix} = \begin{Bmatrix} y_{b_{a_x}} \\ y_{b_{a_y}} \\ y_{b_{a_z}} \end{Bmatrix} \tag{3.24}$$

Where

$$\underset{\sim}{H} = \underset{\sim}{T}\ \underset{\sim}{S}^T \tag{3.25}$$

Now,

$$|\sin\psi| = |\underset{\sim}{y}_a \times \left(\underset{\sim}{y}_b\right)_a| \tag{3.26}$$

Where

$$\left(\underset{\sim}{y}_b\right)_a = y_{b_{a_x}} \underset{\sim}{x}_a + y_{b_{a_y}} \underset{\sim}{y}_a + y_{b_{a_z}} \underset{\sim}{z}_a \tag{3.26a}$$

If $y_{b_{a_x}}$ is assumed small and can be neglected, then Equation (3.26) yields

$$|\sin\psi| = |y_{b_{a_z}} (\underset{\sim}{y}_a \times \underset{\sim}{z}_a)| = |y_{b_{a_z}} \underset{\sim}{x}_a| = y_{b_{a_z}} \tag{3.27}$$

So that $y_{b_{a_z}}$ is the magnitude of $\sin\psi$ which is also equal to H_{32} or the (3,2) term of $\underset{\sim}{T}\ \underset{\sim}{S}^T$ (as seen from Equations 3.24, 3.25 and 3.27). Hence,

$$\sin\psi = (3,2) \text{ term of } \underset{\sim}{T}\ \underset{\sim}{S}^T \tag{3.28}$$

However, since we are interested in the increment $\Delta\psi$ rather than $\sin\psi$ itself, we must take the total differential of $\underset{\sim}{H}$ or $\underset{\sim}{T}\ \underset{\sim}{S}^T$. If we take the differential of both sides of Equation (3.28), we get

$$\cos\psi d\psi = (3,2) \text{ term of } d(\underset{\sim}{T}\ \underset{\sim}{S}^T). \tag{3.29}$$

If in addition, we set $\Delta\psi \cong d\psi$ we have

$$\Delta\psi = \frac{1}{\cos\psi}\left[(3,2) \text{ term of } d(\underset{\sim}{T}\,\underset{\sim}{S}^T)\right]. \tag{3.30}$$

We now proceed to evaluate $d\underset{\sim}{H} = d(\underset{\sim}{T}\,\underset{\sim}{S}^T)$.

$$d\underset{\sim}{H} = d(\underset{\sim}{T}\,\underset{\sim}{S}^T) = (d\underset{\sim}{T})\,\underset{\sim}{S}^T + \underset{\sim}{T}\,d(\underset{\sim}{S}^T)$$

But,

$$d\underset{\sim}{T} = \underset{\sim}{T}\underset{\sim}{\Omega}_a \text{ and } d\underset{\sim}{S}^T = (d\underset{\sim}{S})^T = \underset{\sim}{\Omega}_b^T\,\underset{\sim}{S}^T$$

where $\underset{\sim}{\Omega}$ was defined in conjunction with Equation (3.11) Therefore

$$d(\underset{\sim}{T}\,\underset{\sim}{S}^T) = \underset{\sim}{T}\,\underset{\sim}{\Omega}_a\,\underset{\sim}{S}^T + \underset{\sim}{T}\,\underset{\sim}{\Omega}_b^T\,\underset{\sim}{S}^T \quad . \tag{3.31}$$

The (3,2) term of $d(\underset{\sim}{T}\,\underset{\sim}{S}^T)$ is obtained by performing the appropriate matrix multiplication indicated on the right hand side of Equation (3.31) and keeping track of those operations which contribute to the (3,2) term of $d(\underset{\sim}{T}\,\underset{\sim}{S}^T)$. The result can be written in the form

$$\begin{matrix}(3,2) \text{ term} \\ \text{of } d(\underset{\sim}{T}\,\underset{\sim}{S}^T)\end{matrix} = [A_1\ A_2\ A_3\ A_4\ A_5\ A_6] \begin{Bmatrix} \Delta\phi_{xa} \\ \Delta\phi_{ya} \\ \Delta\phi_{za} \\ \Delta\phi_{xb} \\ \Delta\phi_{yb} \\ \Delta\phi_{zb} \end{Bmatrix} \tag{3.32}$$

where

$$\begin{aligned}
A_1 &= T_{32}S_{23}-T_{33}S_{22} \\
A_2 &= T_{33}S_{21}-T_{31}S_{23} \\
A_3 &= T_{31}S_{22}-T_{32}S_{21} \\
A_4 &= T_{33}S_{22}-T_{32}S_{23} = -A_1 \\
A_5 &= -A_2 \\
A_6 &= -A_3.
\end{aligned}$$

Since we can now express all quantities in the $\Delta \underset{\sim}{R}$ vector in terms of those in $\Delta \underset{\sim}{u}$, the $\underset{\sim}{C}$ matrix in Equation (3.1), which is a 6 X 12 matrix can now be assembled from our previous results by adding all relevant contributions into their proper locations. The compatibility matrix, $\underset{\sim}{C}$ is denoted as follows:

$$\underset{\sim}{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdots & C_{1,11} & C_{1,12} \\ C_{21} & C_{22} & \cdots & & & C_{2,12} \\ \vdots & \vdots & & & & \\ \vdots & \vdots & & & & \\ C_{61} & \cdots & & & & C_{6,12} \end{bmatrix} \tag{3.33}$$

The elements of $\underset{\sim}{C}$ are given in the following section.

3.1 ELEMENTS OF $\underset{\sim}{C}$ MATRIX

The elements of the beam element compatibility matrix $\underset{\sim}{C}$ will now be presented. The following quantities are first defined for convenience:

$$\lambda_a = \frac{1}{\cos\theta_a}, \quad \lambda_b = \frac{1}{\cos\theta_b}, \quad \lambda_c = \frac{1}{\cos\psi} \tag{3.34}$$

$$X_{21} = X_2 - X_1, \; Y_{21} = Y_2 - Y_1, \; Z_{21} = Z_2 - Z_1 \tag{3.35}$$

$$|L| = (X_{21}^2 + Y_{21}^2 + Z_{21}^2)^{1/2} \tag{3.36}$$

The elements of $\underset{\sim}{C}$ are then given as follows:

$$C_{11} = \frac{\lambda_a}{|L|^3}\left(T_{31}X_{21}^2 + T_{32}Y_{21}X_{21} + T_{33}Z_{21}X_{21}\right) - \frac{T_{31}\lambda_a}{|L|}$$

$$C_{12} = \frac{\lambda_a}{|L|^3}\left(T_{31}X_{21}Y_{21} + T_{32}Y_{21}^2 + T_{33}Z_{21}Y_{21}\right) - \frac{T_{32}\lambda_a}{|L|}$$

$$C_{13} = \frac{\lambda_a}{|L|^3}\left(T_{31}X_{21}Z_{21} + T_{32}Y_{21}Z_{21} + T_{33}Z_{21}^2\right) - \frac{T_{33}\lambda_a}{|L|}$$

$$C_{14} = -C_{11}$$

$$C_{15} = -C_{12}$$

$$C_{16} = -C_{13}$$

$$C_{17} = \frac{\lambda_a}{|L|}(T_{32}Z_{21} - T_{33}Y_{21})$$

$$C_{18} = \frac{\lambda_a}{|L|}(T_{33}X_{21} - T_{31}Z_{21})$$

$$C_{19} = \frac{\lambda_a}{|L|}(T_{31}Y_{21} - T_{32}X_{21})$$

$$C_{1,10} = 0$$

$$C_{1,11} = 0$$

$$C_{1,12} = 0$$

$$C_{21} = -\frac{\lambda_a}{|L|^3}\left(T_{21}X_{21}^2 + T_{22}Y_{21}X_{21} + T_{23}Z_{21}X_{21}\right) + \frac{T_{21}\lambda_a}{|L|}$$

$$C_{22} = -\frac{\lambda_a}{|L|^3}\left(T_{21}X_{21}Y_{21} + T_{22}Y_{21}^2 + T_{23}Z_{21}Y_{21}\right) + \frac{T_{22}\lambda_a}{|L|}$$

$$C_{23} = -\frac{\lambda_a}{|L|^3}\left(T_{21}X_{21}Z_{21} + T_{22}Y_{21}Z_{21} + T_{23}Z_{21}^2\right) + \frac{T_{23}\lambda_a}{|L|}$$

$$C_{24} = -C_{21}$$

$$C_{25} = -C_{22}$$

$$C_{26} = -C_{23}$$

$$C_{27} = \frac{\lambda_a}{|L|}\ (T_{23}Y_{21}-T_{22}Z_{21})$$

$$C_{28} = \frac{\lambda_a}{|L|}\ (T_{21}Z_{21}-T_{23}X_{21})$$

$$C_{29} = \frac{\lambda_a}{|L|}\ (T_{22}X_{21}-T_{21}Y_{21})$$

$$C_{2,10} = 0$$

$$C_{2,11} = 0$$

$$C_{2,12} = 0$$

$$C_{31} = \frac{\lambda_b}{|L|^3}\left(S_{31}X_{21}^2 + S_{32}Y_{21}X_{21} + S_{33}Z_{21}X_{21}\right) - \frac{\lambda_b S_{31}}{|L|}$$

$$C_{32} = \frac{\lambda_b}{|L|^3}\left(S_{31}X_{21}Y_{21} + S_{32}Y_{21}^2 + S_{33}Z_{21}Y_{21}\right) - \frac{\lambda_b S_{32}}{|L|}$$

$$C_{33} = \frac{\lambda_b}{|L|^3}\left(S_{31}X_{21}Z_{21} + S_{32}Y_{21}Z_{21} + S_{33}Z_{21}^2\right) - \frac{\lambda_b S_{33}}{|L|}$$

$$C_{34} = -C_{31}$$

$$C_{35} = -C_{32}$$

$$C_{36} = -C_{33}$$

$$C_{37} = 0$$

$$C_{38} = 0$$

$$C_{39} = 0$$

$$C_{3,10} = \frac{\lambda_b}{|L|}\,(S_{32}Z_{21} - S_{33}Y_{21})$$

$$C_{3,11} = \frac{\lambda_b}{|L|}\,(S_{33}X_{21} - S_{31}Z_{21})$$

$$C_{3,12} = \frac{\lambda_b}{|L|}\,(S_{31}Y_{21} - S_{32}X_{21})$$

$$C_{41} = -\frac{\lambda_b}{|L|^3}\left(S_{21}X_{21}^2 + S_{22}Y_{21}X_{21} + S_{23}Z_{21}X_{21}\right) + \frac{\lambda_b S_{21}}{|L|}$$

$$C_{42} = -\frac{\lambda_b}{|L|^3}\left(S_{21}X_{21}Y_{21} + S_{22}Y_{21}^2 + S_{23}Z_{21}Y_{21}\right) + \frac{\lambda_b S_{22}}{|L|}$$

$$C_{43} = -\frac{\lambda_b}{|L|^3}\left(S_{21}X_{21}Z_{21} + S_{22}Y_{21}Z_{21} + S_{23}Z_{21}^2\right) + \frac{\lambda_b S_{23}}{|L|}$$

$$C_{44} = -C_{41}$$

$$C_{45} = -C_{42}$$

$$C_{46} = -C_{43}$$

$$C_{47} = 0$$

$$C_{48} = 0$$

$$C_{49} = 0$$

$$C_{4,10} = \frac{\lambda_b}{|L|} (S_{23}Y_{21} - S_{22}Z_{21})$$

$$C_{4,11} = \frac{\lambda_b}{|L|} (S_{21}Z_{21} - S_{23}X_{21})$$

$$C_{4,12} = \frac{\lambda_b}{|L|} (S_{22}X_{21} - S_{21}Y_{21})$$

$$C_{51} = - X_{21}/|L|$$

$$C_{52} = - Y_{21}/|L|$$

$$C_{53} = - Z_{21}/|L|$$

$$C_{54} = - C_{51}$$

$$C_{55} = -C_{52}$$

$$C_{56} = -C_{53}$$

$$C_{57} = C_{58} = C_{59} = C_{5,10} = C_{5,11} = C_{5,12} = 0$$

$$C_{61} = C_{62} = C_{63} = C_{64} = C_{65} = C_{66} = 0$$

$$C_{67} = \lambda_c(T_{32}S_{23} - T_{33}S_{22})$$

$$C_{68} = \lambda_c(T_{33}S_{21} - T_{31}S_{23})$$

$$C_{69} = \lambda_c \, (T_{31}S_{22}-T_{32}S_{21})$$

$$C_{6,10} = -C_{67}$$

$$C_{6,11} = -C_{68}$$

$$C_{6,12} = -C_{69}$$

where T_{ij} and S_{ij} are the elements of Transformation matrices $\underset{\sim}{T}$, $\underset{\sim}{S}$ defined in the text.

In order to determine λ_a, λ_b and λ_c, we need $\cos\theta_a$, $\cos\theta_b$ and $\cos\psi$. They are calculated as follows:

a. For $\cos\theta_a$, from Equations (3.2) and (3.8) of text,

$$\cos\theta_a = \underset{\sim}{\ell}_a \cdot \underset{\sim}{x}_a = \ell_{ax}$$

$$= \frac{1}{|L|} \, [T_{11}X_{21} + T_{12}Y_{21} + T_{13}Z_{21}]$$

Similarly

$$\cos\theta_b = \frac{1}{|L|} \, [S_{11}X_{21} + S_{12}Y_{21} + S_{13}Z_{21}]$$

b. For $\cos\psi$, from Equation (3.26a) of text,

$$\cos\psi = \underset{\sim}{y}_a \cdot (\underset{\sim}{y}_b)_a = y_{b_{a_y}} = (2,2) \text{ term of } \underset{\sim}{T}\,\underset{\sim}{S}^T$$

4. YIELD CRITERIA UNDER COMBINED LOADS

As the incremental solution of the equations of motion (Eq. 2.27) proceeds, the vector $\Delta \underset{\sim}{u}_i$, for element i, is calculated at each time step. The increments in internal force quantities, $\Delta \underset{\sim}{S}_i$, for element i can be determined by using Equations (2.1) and (2.9) to get

$$\Delta \underset{\sim}{S}_i(t) = \underset{\sim}{g}_i^{(p)} \underset{\sim}{C}_i^{(p)} \Delta \underset{\sim}{u}_i(t) \tag{4.1}$$

In Equation (4.1), $\underset{\sim}{g}_i^{(p)}$ is the element stiffness matrix which can take on any of the forms given by Equations (2.3) - (2.6), depending on whether elastic or plastic behavior exists. The superscript, p, refers to updated values at the pth time step. The cumulative values of the internal forces of element i at the pth time step are then given by

$$\underset{\sim}{S}_i(t) = \underset{\sim}{S}_i(t_p) + \Delta \underset{\sim}{S}_i(t) \tag{4.2}$$

where

$$\underset{\sim}{S}_i^T = [M_{ay}\ M_{az}\ M_{by}\ M_{bz}\ N\ T]\ . \tag{4.3}$$

At a given node of element i the bending moments M_y and M_z, the stress resultant, N, and the torque T are the quantities of interest in any yield cirterion, where they may take on critical values.

For the loading under discussion, it should be possible represent the yield criterion in the general form

$$f\left(\frac{M_z}{M_z^*}, \frac{M_y}{M_y^*} \cdot \frac{N}{N^*}, \frac{T}{T^*}\right) = C_o \tag{4.4}$$

where C_o is an appropriate constant, and the starred quantities represent fully plastic or yield values. The specific form of the function, f, will depend on the geometry of the element cross section, the particular loads which are assumed to be chiefly effective in yielding the member, and the kind of local deformation which is likely to occur (this is, of course, difficult to predict, since the orientation of the forces cannot be known a priori). Rather than attempt to develop a general criterion, it appears more reasonable to derive criteria which would be specialized for the particular configurations under study, and for which there may be experimental data available for possible empirical contribution to the form of f.

Some simple examples of possible yield criteria are presented here only for purposes of giving an exposition of what kind of criterion is being sought. For instance, for a rectangular cross section beam under combined bending in one direction (M_z only) and axial loading, N, bent its plane of symmetry (h - beam thickness, b - beam length), the following yield criterion can be dervied.[9]

$$\frac{M_z}{M_z^*} + \left(\frac{N}{N^*}\right)^2 = 1 \tag{4.5}$$

where $M_z^* = \sigma_y bh^2/4$, $N^* = \sigma_y bh$ and σ_y is the yield stress. For the same rectangular cross section, by using a method similar to the approach in Reference 9, one can derive a yield criterion where the two moments M_z and M_y are dominant, in the form

$$\frac{M_z}{M_z^*} + \frac{3}{4}\left(\frac{M_y}{M_y^*}\right)^2 = 1 \tag{4.6}$$

Results for thin walled sections involving M_z and M_y can also be obtained in a similar manner. Note that in these cases, including that of Equation (4.6), M_z^* and M_y^* would need to be defined appropriately.

Bounding methods can also be used to obtain simple lower bound yield loci which are approximately independent of cross section provided we deal with thin walled box sections or some types of I beams. For instance, by use of the convexity theorem and other arguments,[10] one may obtain the locus in T, M space which provides a safe combination of bending moment, M, about a "natural" or "preferred" axis, and torque, T. A pure torsion analysis provides two points on the yield locus ($\pm$ T*, o) while pure bending analysis gives the two points ($\pm$ M*, o) if we suppose no buckling in compression and no Bauschinger effect. The fully plastic moment M* is proportional to the yield strength, σ_y. Note that if we denote M_{yp} as the moment when the outer fiber just reaches the yield stress, then $M^* = C_1 M_{yp}$ where the constant C_1 is a shape factor which ranges between the values of one and two in most practical situations, and is closer to one for thin walled box sections. The four points ($\pm$ T*, o) and ($\pm$ M*, o) can be connected to form a quadrilateral as shown in Figure 4-1. The convexity theorem implies that the quadrilateral locus in Figure 4-1 is "safe" since it represents a curve closest to the origin without being concave anywhere. The quadrilateral, therefore, represents an "inner" bound on the true T, M yield locus, and applies whatever the cross section shape of the beam.

One can improve on this lower bound in T, M space by the following arguments. For the torsion problem assume shear stresses of magnitude k over the cross section are in equilibrium with T*; then lower proportional stresses of magnitude λk are in equilibrium with torque λT^*. Similarly, bending stresses, $\mu\sigma_y$ are in equilibrium with bending moment μM^* (note that this is approximate in the sense that M* is nearly equal to M_{yp} for thin walled sections). Therefore, if the combined stresses λk and $\mu\sigma_y$ under say the von-Mises criterion do not exceed yield, then the loads (λT^*, μM^*) will be "safe". The von-Mises condition for our case gives

$$\left(\frac{\tau}{k}\right)^2 + \left(\frac{\sigma}{\sigma_y}\right)^2 = 1 \qquad (4.7)$$

where τ is the shear stress and σ is a bending stress on a plane perpendicular to the beam axis. Here, $\tau = \lambda k$ and $\sigma = \mu\sigma_y$ so

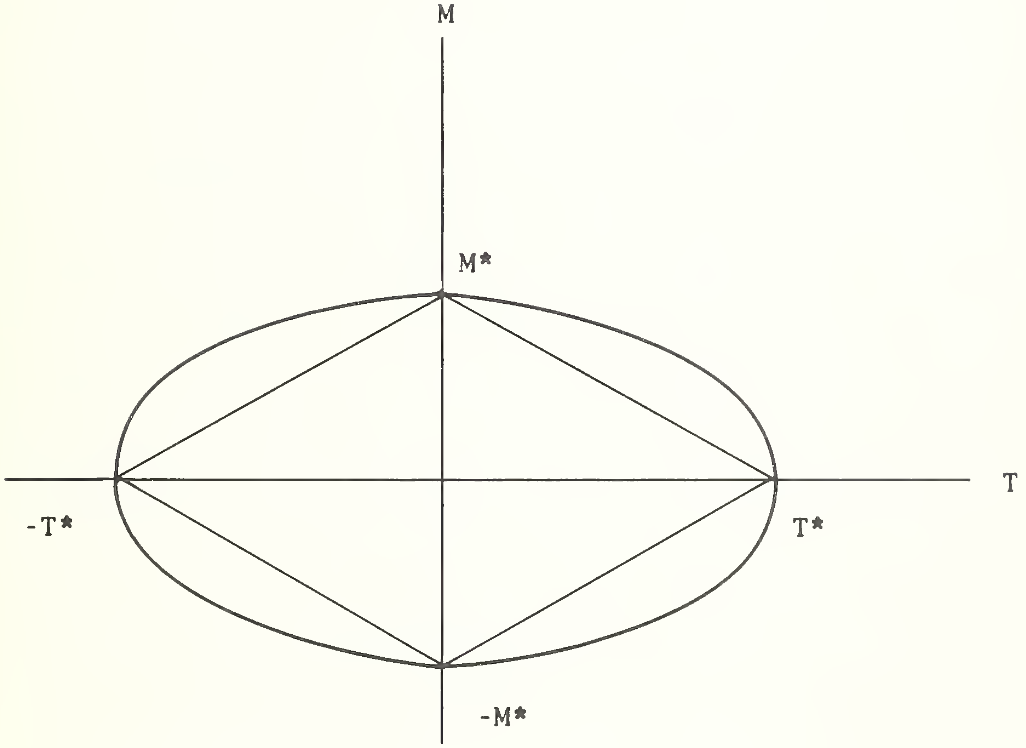

Figure 4-1. Yield Locus for Combined Bending and Torsion

Equation (4.7) gives

$$\lambda^2 + \mu^2 = 1 \tag{4.8}$$

and since by definition, $\lambda = T/T^*$ and $\mu = M/M^*$, Equation (4.8) gives

$$\left(\frac{T}{T^*}\right)^2 + \left(\frac{M}{M^*}\right)^2 = 1 \tag{4.9}$$

The ellipse represented by Equation (4.9) is shown in Figure 4-1, and encloses more area than the quadrilateral. Equation (4.9) is probably a good approximation for thin walled box beams, especially when the outline of the cross section is closer to being square. The magnitude of M could be given by $(M_y^2 + M_z^2)^{1/2}$ as an approximation for general unsymmetrical bending.

In a future documentation, which is concerned with computer assumptions would need to be made for the various types of beam members which exist, also taking into account the fact that the response depends heavily on the expected loading geometry and sequence.

In Section 2 of this report, which is concerned with computer implementation, Equation (4.9) will be used in the form

$$\left(\frac{T}{T^*}\right)^2 + \left(\frac{M_y}{M_y^*}\right)^2 + \left(\frac{M_z}{M_z^*}\right)^2 = 1 \quad . \tag{4.10}$$

For thin walled sections the phenomenon of local buckling, or crippling, is a very real possibility. How to take the gross effects of local buckling into account in a simple manner is still under investigation. As a rough approximation, rather than use the force deformation curve shown in Figure 4-2a, the local buckling effect may be incorporated into the analysis by use of the curve shown in Figure 4-2b, where the sudden drop in load is indicated in the otherwise simple elasto-plastic response. Loading and unloading paths will now be based on the reduced load capacity (i.e., lower yield strength). Reduced stiffness caused by a smaller effective cross sectional area may also be taken into account by smaller loading and unloading slopes, as shown by the dotted lines in Figure 4-2b.

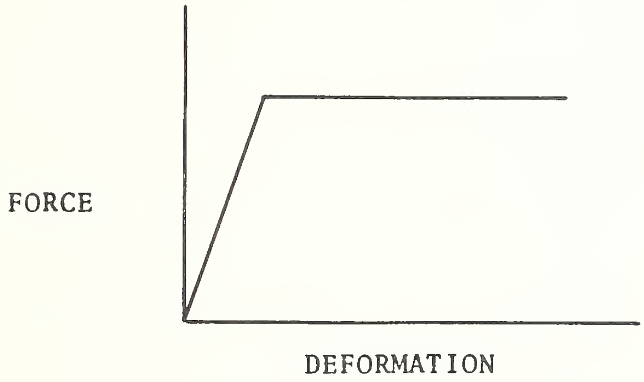

Figure 4-2a. Elasto-Plastic Response Curve

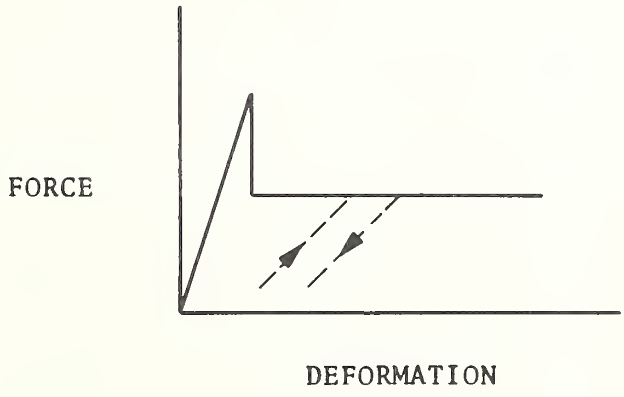

Figure 4-2b. Elasto-Plastic Plus Buckling Response Curve

5. SOLUTION OF THE EQUATIONS OF MOTION

In the present section, a technique for the numerical time integration of the equations of motion will be developed. Various numerical approaches to the calculation of the dynamic response of complex structural systems have been used in the past few years. The varying degrees of success of each method has been related to questions of accuracy and efficient implementation of digital computers (i.e., the respective algorithms of some methods lead to smaller computer running times than for other methods).

Two rather useful time integration schemes which have been used successfully to solve complex structural dynamics problems are the Houbolt method[11] and the Newmark Beta method.[12] Houbolt's method is based on the assumption of a cubic curve in the time coordinate for the displacements of the moving body, considering that four successive ordinates can be passed through by a cubic curve. It is designed to be self-starting and unconditionally stable. For sufficiently large time increments however, it does lead to some artificial damping in the response.[13]

The Newmark Beta method has been used successfully in a variety of complex structural response studies, which involved treatment of elasto-plastic behavior, yield hinges, and various dynamic loadings such as shock or impact, vibration and earthquake motion. Some of the first applications are given in Reference 14, although the method has been used often since then.[15] The technique is rather straightforward step by step approach, where the value of a parameter, β, can be selected to suit the requirements of the problem at hand, and also to give unconditionally stable results. The net effect of β is to change the form of the variation of acceleration in the time interval.

In the development to follow, the Newmark Beta method will be adopted. We shall be concerned with the incremental equation (2.27) which is written here for convenience. For the pth time step:

$$\underset{\sim}{M}\Delta\underset{\sim}{\ddot{u}} + \underset{\sim}{K}^{(p)} \Delta\underset{\sim}{u} = \underset{\sim}{P}^{(p)} + \underset{\sim}{Q}^{(p)} \tag{5.1}$$

where

$$\underset{\sim}{Q}^{(p)} = - [\underset{\sim}{C}^{(p)}]^T \underset{\sim}{S}^{(p)} \tag{5.2}$$

Since many degrees of freedom will have zero mass or inertia associated with them, Equation (5.1) is partitioned as follows

$$\underset{\sim}{M} \Delta \ddot{u}_{\sim\alpha} + K^{(p)}_{\sim\alpha\alpha} \Delta u_{\sim\alpha} + K^{(p)}_{\sim\alpha\beta} \Delta u_{\sim\beta} = P^{(p)}_{\sim\alpha} + Q^{(p)}_{\sim\alpha} = f^{(p)}_{\sim\alpha} \tag{5.3}$$

$$K^{(p)}_{\beta\alpha} \Delta u_{\sim\alpha} + K^{(p)}_{\sim\beta\beta} \Delta u_{\sim\beta} = P^{(p)}_{\sim\beta} + Q^{(p)}_{\sim\beta} = f^{(p)}_{\sim\beta} \tag{5.4}$$

where the $\Delta u_{\sim\alpha}$ involve degrees of freedom associated with masses, and the $\Delta u_{\sim\beta}$ involve degrees of freedom with no mass or inertias. Equation (5.4) is next used to express $\Delta u_{\sim\beta}$ in terms of $\Delta u_{\sim\alpha}$ as

$$\Delta u_{\sim\beta} = \left[K^{(p)}_{\sim\beta\beta} \right]^{-1} \left(f_{\sim\beta} - K^{(p)}_{\sim\beta\alpha} \Delta u_{\sim\alpha} \right) \tag{5.5}$$

When Equation (5.5) is substituted into Equation (5.3) and terms are combined appropriately, the following equation results.

$$M \ddot{\underset{\sim}{\xi}}^{(p)} + K^{(p)}_{\sim r} \ddot{\underset{\sim}{\xi}}^{(p)} = \underset{\sim}{f}^{(p)} \tag{5.6}$$

where

$$\underset{\sim}{M} = M_{\sim\alpha}$$

$$\xi^{(p)} = \Delta u^{(p)}_{\sim\alpha}$$

$$\underset{\sim}{f}^{(p)} = f^{(p)}_{\sim\alpha} - K^{(p)}_{\sim f} f^{(p)}_{\sim\beta}$$

$$\underset{\sim}{K}_f^{(p)} = \underset{\sim}{K}_{\alpha\beta}^{(p)} \left[\underset{\sim}{K}_{\beta\beta}^{(p)} \right]^{-1}$$

and what will be called the reduced stiffness matrix, $\underset{\sim}{K}_r$, is

$$\underset{\sim}{K}_r = \underset{\sim}{K}_{\alpha\alpha} - \underset{\sim}{K}_f^{(p)} \underset{\sim}{K}_{\beta\alpha}^{(p)} \tag{5.7}$$

The appropriate formulas relating the solution $\underset{\sim}{\xi}^{(p)}$ to $\underset{\sim}{\xi}^{(p-1)}$, $\underset{\sim}{\xi}^{(p-2)}$ and force quantities can be derived by using a technique described in Reference 15. This is accomplished here by first writing Equation (5.6) for different time steps as follows

$$\underset{\sim}{M} \, \ddot{\underset{\sim}{\xi}}^{(p)} + \underset{\sim}{K}_r^{(p)} \, \underset{\sim}{\xi}^{(p)} = \underset{\sim}{f}^{(p)} \tag{5.8}$$

$$\underset{\sim}{M} \, \ddot{\underset{\sim}{\xi}}^{(p+1)} + \underset{\sim}{K}_r^{(p+1)} \, \underset{\sim}{\xi}^{(p+1)} = \underset{\sim}{f}^{(p+1)} \tag{5.8a}$$

$$M \, \ddot{\underset{\sim}{\xi}}^{(p-1)} + \underset{\sim}{K}_r^{(p-1)} \underset{\sim}{\xi}^{(p-1)} = \underset{\sim}{f}^{(p-1)} \tag{5.8b}$$

The following relations for velocity and displacement originally suggested by Newmark[12] are written next in terms of incremental quantities, where the value of β can be anywhere between zero and 1/4.

$$\dot{\underset{\sim}{\xi}}^{(p+1)} = \dot{\underset{\sim}{\xi}}^{(p)} + \frac{h}{2} \left[\ddot{\underset{\sim}{\xi}}^{(p)} + \ddot{\underset{\sim}{\xi}}^{(p+1)} \right] \tag{5.9}$$

$$\underset{\sim}{\xi}^{(p+1)} = h \, \dot{\underset{\sim}{\xi}}^{(p)} + \left(\frac{1}{2} - \beta \right) h^2 \, \ddot{\underset{\sim}{\xi}}^{(p)} + \beta h^2 \, \ddot{\underset{\sim}{\xi}}^{(p+1)} \tag{5.10}$$

where h is the time increment and it is recalled that $\underset{\sim}{\xi}^{(p+1)}$ defined by

$$\underset{\sim}{\xi}^{(p+1)} = \underset{\sim}{u}_\alpha^{(p+1)} - \underset{\sim}{u}_\alpha^{(p)} \tag{5.10a}$$

Equations (5.9) and (5.10) can be used to eliminate $\dot{\underset{\sim}{\xi}}^{(p)}$ and

$\underset{\sim}{\xi}^{(p-1)}$ and obtain the relation,

$$\underset{\sim}{\xi}^{(p+1)} - \underset{\sim}{\xi}^{(p)} = h^2 \ddot{\underset{\sim}{\xi}}^{(p)} + \beta h^2 \left(\ddot{\underset{\sim}{\xi}}^{(p+1)} - 2\ddot{\underset{\sim}{\xi}}^{(p)} + \ddot{\underset{\sim}{\xi}}^{(p-1)}\right) \quad (5.11)$$

If both sides of Equation (5.11) are multiplied by $\underset{\sim}{M}$ and Equation (5.8) is used, one obtains

$$\begin{aligned}\underset{\sim}{M}\left(\underset{\sim}{\xi}^{(p+1)} - \underset{\sim}{\xi}^{(p)}\right) &= h^2 \underset{\sim}{f}^{(p)} - h^2 \underset{\sim}{K}_r^{(p)} \xi^{(p)} - \beta h^2 \left(2\underset{\sim}{f}^{(p)} - 2\underset{\sim}{K}_r^{(p)} \underset{\sim}{\xi}^{(p)}\right) \\ &+ \beta h^2 \left(\underset{\sim}{f}^{(p+1)} - \underset{\sim}{K}_r^{(p+1)} \underset{\sim}{\xi}^{(p+1)}\right) \\ &+ \beta h^2 \left(\underset{\sim}{f}^{(p-1)} - \underset{\sim}{K}_r^{(p-1)} \underset{\sim}{\xi}^{(p-1)}\right)\end{aligned} \quad (5.12)$$

The terms in Equation (5.12) can be grouped conveniently, so that for the pth time step, the following difference equation can be obtained

$$\underset{\sim}{D}^{(p)} \underset{\sim}{\xi}^{(p)} = \underset{\sim}{B}^{(p-1)} \underset{\sim}{\xi}^{(p-1)} - \beta h^2 \underset{\sim}{K}_r^{(p-2)} \underset{\sim}{\xi}^{(p-2)} + \beta h^2 \left[\underset{\sim}{f}^{(p)} + \left(\frac{1}{\beta} - 2\right) \underset{\sim}{f}^{(p-1)} + \underset{\sim}{f}^{(p-2)}\right] \quad (5.13)$$

where
$$\underset{\sim}{D}^{(p)} = \underset{\sim}{M} + \beta h^2 \underset{\sim}{K}_r{}^{(p)}$$

$$\underset{\sim}{B}^{(p)} = \underset{\sim}{M} - (1-2\beta)\, h^2 \underset{\sim}{K}_r^{(p)}$$

and the matrix $\underset{\sim}{K}_r^{(p)}$, has been calculated at the end of the previous time step. Note that the mass matrix, $\underset{\sim}{M}$, has been considered constant. This however, is not necessary, and if one wishes, one can vary $\underset{\sim}{M}$ and therefore use $\underset{\sim}{M}^{(p)}$. In fact with the right hand side of Equation (5.13) known, one regards (5.13) as a system of equations to calculate $\underset{\sim}{\xi}^{(p)} \equiv \Delta \underset{\sim}{u}_\alpha^{(p)}$ at the pth time step. Since we are using a matrix displacement method, the equations are banded, and rather efficient elimination procedures have been derived.[16]

With the increment $\left[\Delta \underset{\sim}{u}^{(p)}\right]^T = \left[\Delta \underset{\sim}{u}_\alpha, \Delta \underset{\sim}{u}_\beta\right]$ known, the displacement vector, $\underset{\sim}{u}$, at the end of the pth time interval is

$$\underset{\sim}{u}^{(p)} = \underset{\sim}{u}^{(p-1)} + \Delta \underset{\sim}{u}^{(p)} \quad . \tag{5.14}$$

Knowing $\underset{\sim}{u}^{(p)}$, the results of Section 3 can be used to update the compability matrix to $\underset{\sim}{C}^{(p+1)}$, for the next time step. Also, using appropriate relations in Section 2 one has

$$\Delta \underset{\sim}{S}^{(p)} = \underset{\sim}{G}^{(p)} \underset{\sim}{C}^{(p)} \Delta \underset{\sim}{u}^{(p)} \tag{5.15}$$

where $\Delta \underset{\sim}{u}^{(p)}$ is ordered to be consistent with appropriately defined matrices, and where $\underset{\sim}{G}^{(p)}$ was calculated in the previous time step. Using the result in Equation (5.15) the international forces to be used for the next time step, $\underset{\sim}{S}^{(p+1)}$, are

$$\underset{\sim}{S}^{(p+1)} = \underset{\sim}{S}^{(p)} + \Delta \underset{\sim}{S}^{(p)} \tag{5.16}$$

where $\underset{\sim}{S}^{(p)}$ was calculated at the previous time step. With the values of $\underset{\sim}{S}^{(p)}$, the yield criteria are used to update to $\underset{\sim}{G}^{(p+1)}$ for the next time step. We can therefore calculate $\underset{\sim}{C}^{(p+1)}$ and $\underset{\sim}{G}^{(p+1)}$ at the end of the pth time step and hence can determine

$$\underset{\sim}{K}^{(p+1)} = \left[\underset{\sim}{C}^{(p+1)}\right]^T \underset{\sim}{G}^{(p+1)} \underset{\sim}{C}^{(p+1)} \tag{5.17}$$

In these conditions the $\underset{\sim}{T}$ matrix (Eqs. 3.7, 3.15) must be updated as follows:

$$\underset{\sim}{T}^{(p+1)} = \underset{\sim}{T}^{(p)} + \Delta \underset{\sim}{T} = T^{(p)} + \underset{\sim}{T}\ \Omega_a \tag{5.17a}$$

and similarly for the $\underset{\sim}{S}$ matrix. Also, the philosophy of assembling $\underset{\sim}{K}^{(p)}$ is discussed at the end of this section.

5.1 STARTING PROCEDURE

As an initial value problem, the initial displacement, $\underset{\sim}{u}_\alpha^{(o)}$, initial velocity, $\dot{\underset{\sim}{u}}_\alpha^{(o)} \equiv \dot{\underset{\sim}{\xi}}^{(o)}$, and the time history of applied forces are necessary information. Since the difference equation, (5.13), contains terms for three consecutive time intervals, it is required to express the displacement $\underset{\sim}{u}_\alpha^{(1)}$ in terms of initial displacement and initial velocity in order to start the procedure. From Reference 15 we have in the present notation:

$$\underset{\sim}{D}^{(1)}\underset{\sim}{u}^{(1)} = \underset{\sim}{E}^{(1)}\underset{\sim}{u}^{(o)} + \underset{\sim}{M}h\dot{\xi}^{(o)} + \beta h^2 f^{(1)} + \left(\frac{1}{2} - \beta\right) h^2 f^{(o)} \qquad (5.18)$$

where $\underset{\sim}{E}^{(1)} = \underset{\sim}{M} - \left(\frac{1}{2} - \beta\right) h^2 \underset{\sim}{K}_r^{(1)} = \underset{\sim}{D}^{(1)} - \frac{1}{2} h^2 \underset{\sim}{K}_r^{(1)}$

$$\underset{\sim}{f}^{(1)} = \underset{\sim}{P}_\alpha^{(1)} - \underset{\sim}{K}^{(1)}\underset{\sim}{P}_\beta^{(1)} \; ; \; \underset{\sim}{f}^{(o)} = \underset{\sim}{P}_\alpha^{(o)} - \underset{\sim}{K}_f^{(o)}\underset{\sim}{P}_\beta^{(o)}$$

It is recalled that the quantity $\underset{\sim}{P}$ is a prescribed function and $\dot{\underset{\sim}{\xi}}^{(o)}$ is the initial velocity. If the definition for $D^{(1)}$ and $E^{(1)}$ are used together with the relation $\xi^{(1)} = \underset{\sim}{u}_\alpha^{(1)} - \underset{\sim}{u}_\alpha^{(o)} = \Delta\underset{\sim}{u}_\alpha^{(1)}$, then Equation (5.18) can be written in the form

$$\underset{\sim}{D}^{(1)}\underset{\sim}{\xi}^{(1)} = -\frac{1}{2}h^2\underset{\sim}{K}_r^{(1)}\underset{\sim}{u}^{(o)} + \underset{\sim}{M}h\dot{\underset{\sim}{\xi}}^{(o)} + \beta h^2\underset{\sim}{f}^{(1)} + \left(\frac{1}{2} - \beta\right) h^2\underset{\sim}{f}^{(o)} \qquad (5.19)$$

The matrix $K_r^{(1)}$ is given by

$$\underset{\sim}{K}_r^{(1)} = \left\{\left[\underset{\sim}{C}^{(1)}\right]^T \underset{\sim}{G}^{(1)}\underset{\sim}{C}^{(1)}\right\}_r \qquad (5.20)$$

where the matrices $C^{(1)}$ and $G^{(1)}$ represent initial values to be used for the first time step. The quantity $\underset{\sim}{\xi}^{(o)} \equiv \Delta\underset{\sim}{u}_\alpha^{(o)}$ taken as zero since $\underset{\sim}{u}^{(o)}$ itself is the prescribed initial displacement. Note also that $\underset{\sim}{S}^{(o)} = 0$.

The solution has now been started. The difference Equation (5.13) can now be used to continue the solution. For instance, Equation (5.13) can be used to obtain $\underset{\sim}{\xi}^{(2)}$ writing

$$\underset{\sim}{D}^{(2)}\underset{\sim}{\xi}^{(2)}=\underset{\sim}{B}^{(1)}\underset{\sim}{\xi}^{(1)}-\underset{\sim}{D}^{(o)}\underset{\sim}{\xi}^{(o)}+\beta h^2\left[\underset{\sim}{f}^{(2)}+\frac{1}{\beta}-2\quad\underset{\sim}{f}^{(1)}+\underset{\sim}{f}^{(o)}\right] \tag{5.21}$$

In order to calculate $\underset{\sim}{\xi}^{(2)}$, $\underset{\sim}{D}^{(2)}$ and $\underset{\sim}{f}^{(2)}$ are needed and should be calculated before the start of the second time step. We have

$$\underset{\sim}{f}^{(2)} = \underset{\sim}{f}_{\alpha}^{(2)} - \underset{\sim}{K}_{f}^{(2)}\underset{\sim}{f}_{\beta}^{(2)} \tag{5.22}$$

$$\underset{\sim}{D}^{(2)} = \underset{\sim}{M} + \beta\, h^2 \underset{\sim}{K}_{r}^{(2)} \tag{5.22a}$$

$$\underset{\sim}{K}_{r}^{(2)} = \left\{\left[\underset{\sim}{C}^{(2)}\right]^{T}\underset{\sim}{G}^{(2)}\underset{\sim}{C}^{(2)}\right\}_{r} \tag{5.22b}$$

The matrices $\underset{\sim}{C}^{(2)}$, $\underset{\sim}{G}^{(2)}$ and $\underset{\sim}{S}^{(2)}$ are needed also, and are calculated as follows.

From Equation (5.14)

$$\underset{\sim}{u}^{(1)} = \underset{\sim}{u}^{(o)} + \Delta\underset{\sim}{u}^{(1)} \tag{5.23}$$

Knowing $\underset{\sim}{u}^{(1)}$, the compatibility matrix $\underset{\sim}{C}^{(2)}$ can be obtained for the second time step. Also, from Equation (5.15) get

$$\Delta\underset{\sim}{S}^{(1)} = \underset{\sim}{G}^{(1)}\underset{\sim}{C}^{(1)}\Delta\underset{\sim}{u}^{(1)} \tag{5.24}$$

Use Equation (5.16) to write

$$\underset{\sim}{S}^{(2)} = \underset{\sim}{S}^{(1)} + \Delta\underset{\sim}{S}^{(1)} \tag{5.25}$$

where $\underset{\sim}{S}^{(1)}= 0$. With the values for the internal forces given by $\underset{\sim}{S}^{(2)}$, the yield criteria are used to decide on $\underset{\sim}{G}^{(2)}$ for the second time step. The matrix $\underset{\sim}{K}_{r}^{(2)}$ can then be determined using Equation (5.22c). The quantity, $\underset{\sim}{f}^{(2)}$ can also be determined using Equations (5.22a) and (5.25).

It is finally noted here that the development preceding and leading to Equation (5.17) is a formalism. The actual details of the implementation for programming purposes will use the approach where the stiffness matrix for each individual element is obtained first, and then all elements are assembled appropriately to obtain the overall system matrix $\underset{\sim}{K}^{(p)}$. The details will be given in a separate report.

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