

ANALYSES FOR LATERAL DEFLECTION OF RAILROAD TRACK UNDER QUASI-STATIC LOADING

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ABSTRACT

This paper describes analyses to examine the lateral deflection of railroad track subjected to quasi-static loading. Rails are assumed to behave as beams in bending. Movement of the track in the lateral plane is constrained by idealized resistance characteristics, while movement in the vertical plane is resisted by a continuous, linear and elastic foundation. These analyses are based on solving the ordinary differential equations for beam deflections. In certain cases, convenient mathematical expressions may be used to represent idealized lateral resistance characteristics and derive closed-form equations to relate lateral force as a function of track lateral deflection. However, in general, the idealized lateral resistance characteristic may be nonlinear, in which case numerical methods are required to examine the lateral load versus track lateral deflection behavior. In these general cases, a Fourier series technique is used to solve the governing equations numerically.

The analysis of track lateral deflection subjected to quasi-static loads may be applied to examine track shift. For example, lateral resistance of track may be measured using Track Lateral Pull Tests (TLPT). The Fourier method is also used to examine the relationship between lateral and vertical wheel loads and track lateral shift.

INTRODUCTION

The structural capacity of railroad track to sustain train-induced loads has been a topic of research for several decades. Increasing trends toward higher train speeds and heavier axle loads have kept this topic on the forefront of research. More specifically, evaluating the load capacity of railroad track in terms of lateral strength is needed to retain track alignment and prevent occurrences of track buckling and track lateral shift.

The potentially severe consequences of track lateral shift were realized on April 6, 2004 when an Amtrak passenger train

travelling at about 78 miles per hour derailed near Flora, Mississippi [1]. The derailment resulted in one fatality, three serious injuries and 43 minor injuries. Equipment costs associated with accident were about \$7 million. Moreover, the National Transportation Safety Board (NTSB) determined that the probable cause of the accident was rail shift due to improper maintenance and inspection of the track.

This paper describes analyses to examine the mechanics of track lateral deflection. Both closed-form and numerical analyses are described and developed to determine the lateral deflection behavior of railroad track under quasi-static loading. In this context, quasi-static loading is specified as a single lateral load, which may be applied with or without a single vertical load. Moreover, these analyses may be used to provide a rational basis to evaluate track lateral strength.

TRACK VERTICAL AND LATERAL DEFLECTIONS

In the analyses described in this paper, the rail is assumed to behave as an infinite beam with the equivalent bending stiffness of two rails. Movement of the track is resisted by the ballast. Moreover, tie-ballast resistance against movement in the vertical and lateral directions is represented by idealized mathematical functions.

For instance, track vertical deflection is calculated based on the theory of beams on elastic foundation [2]. Under the assumption of linear elastic tie-ballast support in the vertical direction, the general differential equation for track vertical deflection is

$$EI_{yy} \frac{d^4 v}{dx^4} + P \frac{d^2 v}{dx^2} + k_v v = \frac{V}{2} \delta(0) \quad (1)$$

where E is the modulus of elasticity of rail steel (typically 30×10^6 psi), I_{yy} is the vertical bending inertia for two rails

(189.8 inch⁴ for 136 RE rail), v is the track vertical deflection, P is the rail longitudinal force¹, k_V is the track foundation modulus, V is the applied vertical load, and $\delta(0)$ is the Dirac delta function. Thus, x represents the longitudinal direction, y the lateral direction, and z the vertical direction (see Figure 1).

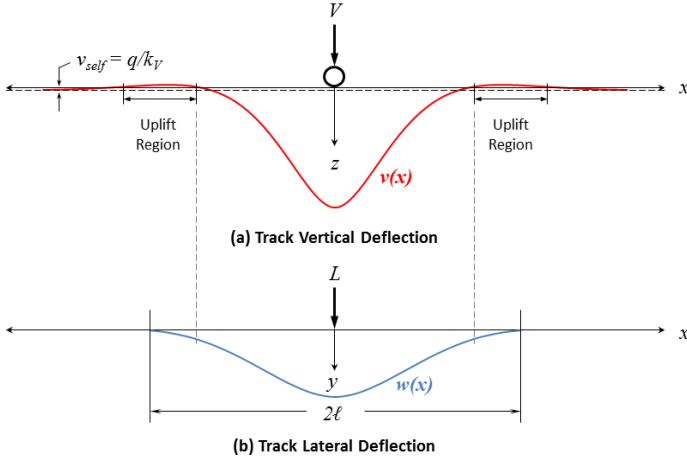


Figure 1: Track Vertical and Lateral Deflection Profiles

Similarly, the general differential equation for track lateral deflection is

$$EI_{zz} \frac{d^4 w}{dx^4} + P \frac{d^2 w}{dx^2} = \frac{L}{2} \delta(0) - F(w) - P \frac{d^2 w_0}{dx^2} \quad (2)$$

where I_{zz} is the lateral bending inertia for two rails (e.g. 29 inch⁴ for 136RE rail), w is the track lateral deflection, L is the applied lateral load, $F(w)$ is the track lateral resistance (which is described in detail in the next section of this paper), and w_0 is the initial lateral misalignment of the track. In the analyses described in remainder of this paper, the effects of rail longitudinal (i.e. thermal) force and initial lateral misalignment are neglected.

Figure 1 shows schematics for the vertical and lateral deflection profiles for track subjected to vertical and lateral loads. The vertical deflection profile also shows a component due to track self-weight. Moreover, the extent of the lateral deflection distribution, $2l$, is a function of lateral load.

LATERAL RESISTANCE CHARACTERIZATION

Lateral resistance is defined in Reference [3] as the reaction offered by the ballast to the rail-tie structure against lateral displacement of the structure. Previous experimental and

¹ Rail longitudinal force is associated with changes in rail temperature. The sign convention in equations (1) and (2) is such that the positive rail longitudinal force means compression.

analytical work to characterize track lateral resistance is reviewed in the following text.

Single Tie Push Tests

Extensive measurements of lateral resistance have been conducted by pushing a single tie with the rail fastening disconnected [4]. A portable device was designed and built specifically to perform this Single Tie Push Test (STPT). Figure 2 shows typical results from STPTs conducted on wood ties with lateral resistance levels labeled as strong, average, and weak. Typical STPT results conducted on concrete ties are shown in Figure 3. Both figures are reproductions from Reference [5].

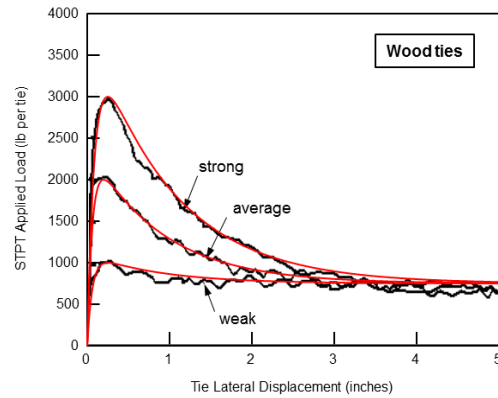


Figure 2: Typical STPT Results for Wood Ties [5]

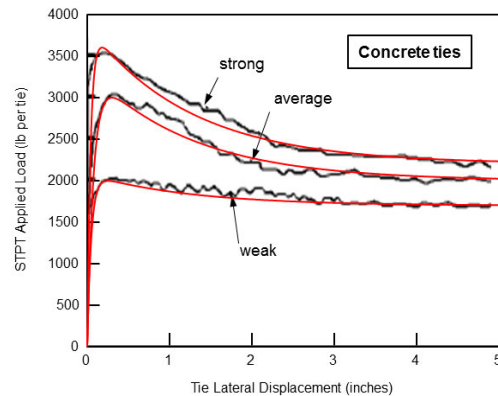


Figure 3: Typical STPT Results for Concrete Ties [5]

Moreover, the following observations are mentioned in Reference [5] regarding the STPT results:

- (1) The peak or maximum STPT load occurs at lateral displacements on the order of 0.25 to 0.5 inch.
- (2) The STPT load at which the resisting load levels off to a near constant value occurs at about 3 to 5 inches of tie displacement.
- (3) When the peak load is relatively low, the resistance is practically constant. When the peak load is relatively high, the resistance curve exhibits softening behavior after the peak.

Idealized Representations of Lateral Resistance

It is convenient to characterize lateral resistance in terms of an analytical function to facilitate the solution of the governing differential equation for track lateral deflection. Previous research has been conducted to characterize lateral resistance in convenient mathematical equations with physical interpretations. That is, the data from the STPT measurements [5] suggest that lateral resistance depends on the level of consolidation. For example, measurements conducted on freshly tamped track exhibit a constant resistance beyond a certain displacement. Figure 4 shows idealized lateral resistance characteristics for ties in tamped track. The simplest mathematical function to represent the lateral resistance is a constant:

$$F(w) = F_p \quad (3)$$

The figure also shows two other types of functions that include the initial rising portion of the resistance curve. These curves are characterized as bilinear and exponential functions:

$$F(w) = \begin{cases} F_p \frac{w}{w_p} & \text{if } w < w_p \\ F_p & \text{if } w \geq w_p \end{cases} \quad (4)$$

$$F(w) = F_p \left[1 - e^{-(2w/w_p)} \right] \quad (5)$$

The units of measure for F_p and F_L in the preceding equations are force per unit length (e.g. lb per inch), which correspond to the respective STPT load divided by the tie center spacing.

As the ballast becomes more consolidated from traffic accumulation, lateral resistance increases and the shape of the applied load versus tie displacement curve exhibits softening behavior. Two idealizations representing this type of behavior are shown in Figure 5, which are expressed mathematically as a piecewise linear function and an exponential decaying function:

$$F(w) = \begin{cases} F_p + (F_L - F_p) \left(\frac{w}{w_L} \right) & \text{if } w < w_L \\ F_L & \text{if } w \geq w_L \end{cases} \quad (6)$$

$$F(w) = F_L + (F_p - F_L) e^{-\left(\frac{4w}{w_L} \right)} \quad (7)$$

Equations (6) and (7), however, neglect the initial rising portion of the STPT load versus displacement curve.

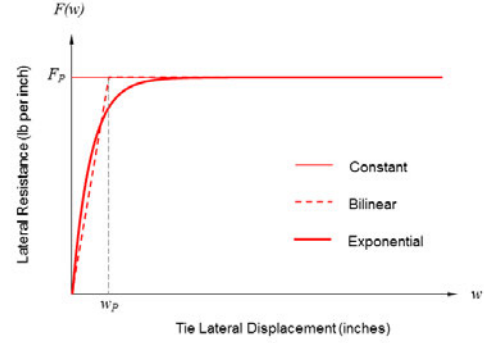


Figure 4: Idealized Lateral Resistance Characteristics for Tamped Track

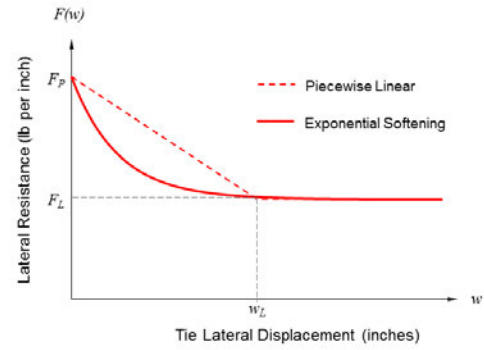


Figure 5: Idealized Lateral Resistance Characteristics for Consolidated Track

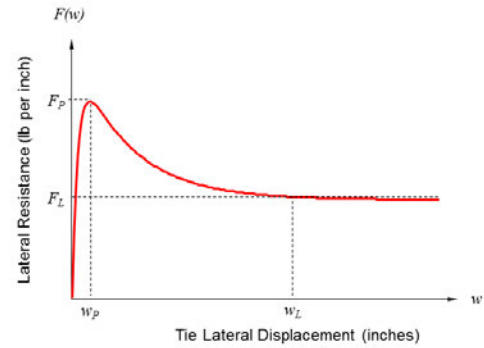


Figure 6: Full Nonlinear Lateral Resistance Characteristic

An alternative function is developed that includes two key features of the STPT lateral resistance characteristic observed in both wood and concrete ties; namely the initial rising portion of the curve and the softening behavior following the peak. Equation (8) is referred to as the full nonlinear characteristic for lateral resistance, and is shown schematically in Figure 6:

$$F(w) = \left(1 - e^{-4w/w_L} \right) \left[F_L + (F_p - F_L) e^{-4(w-w_p)/w_L} \right] \quad (8)$$

where μ_l and w_l are parameters that depend on w_p , w_L , F_p , and F_L . The following equations define the relationship among these parameters:

$$w_1 = w_p + \frac{1}{\mu_2} \ln \left[\frac{k + \frac{1}{e^{-\mu_l w_p} - 1}}{k - 1} \right] \quad (9)$$

$$w_p = -\frac{1}{\mu_1} \ln \left[1 - \frac{\mu_1 + \mu_2}{2k\mu_2} + \sqrt{\left(\frac{\mu_1 + \mu_2}{2k\mu_2} \right)^2 - \frac{\mu_1}{2k\mu_2}} \right] \quad (10)$$

$$k = \frac{F_L}{F_p} \quad (11)$$

$$\mu_2 = \frac{4}{w_L} \quad (12)$$

In these equations, μ_l characterizes the rate at which the initial part of the curve reaches the peak resistance, F_p and μ_2 describes the rate of softening to reach F_L .

Curve-fitting analysis is performed to approximate the STPT results with the full nonlinear characteristic. That is, the parameters for equation (8) are determined in the curve-fitting analysis, which are listed in Table 1 for wood ties and in Table 2 for concrete ties. The full nonlinear characteristics derived from this analysis are superimposed onto the STPT results in Figure 2 for wood ties and in Figure 3 for concrete ties. Moreover, the figures show that the full nonlinear characteristic for lateral resistance provides a reasonable representation of the STPT behavior.

Table 1: Parameters for Full Nonlinear Characteristic Applied to Wood Ties

	F_p (lb)	F_L (lb)	w_p (inches)	w_L (inches)	μ_l (1/inch)
Strong	3000	750	0.25	3.9	10.737
Average	2000	750	0.2	3.5	15.477
Weak	1000	750	0.25	3.5	15.946

Table 2: Parameters for Full Nonlinear Characteristic Applied to Concrete Ties

	F_p (lb)	F_L (lb)	w_p (inches)	w_L (inches)	μ_l (1/inch)
Strong	3600	2200	0.18	5.0	24.134
Average	3000	2000	0.3	5.0	12.85
Weak	2000	1700	0.25	5.0	20.441

Lateral Resistance under Vertical Loading

When a vertical load is applied to the track, the lateral resistance depends on the vertical reaction load on the tie, and therefore varies at each tie depending on the distance from the applied vertical load. Near the applied vertical load, the contribution of the base friction component to the total tie lateral resistance increases according to the following equation:

$$F(w, V) = F(w, 0) + \mu_f R_V \quad (13)$$

where μ_f is a coefficient of friction and R_V is the vertical ballast reaction force on the tie, which depends on the relative location between the tie and the applied vertical load. In addition, $F(w, 0)$ is the lateral resistance without vertical load. Limited data exist for the coefficient of friction μ_f related to conventional wood tie track. Data are needed to determine the coefficient of friction μ_f for high-speed track with concrete ties.

Under the assumption of beam on elastic foundation, the vertical reaction force on the tie is calculated as the product of the track vertical foundation modulus and the track vertical deflection:

$$R_V(x) = k_v v(x) \quad (14)$$

Alternatively, track vertical deflections due to applied vertical loads may be determined on the basis of other assumptions regarding the tie-ballast support such as: tensionless foundation [6], cubic stiffening foundation [7], elastic half space [8], and elastic layer theory [9]. In principle, these alternative representations for tie-ballast support can be readily incorporated into the Fourier method described subsequently. In practice, the assumption of linear elastic foundation provides the most convenient mathematical functional form for the Fourier method.

At some distance away from the vertical load, the vertical deflection of the track is upward. Referring to Figure 1, upward track deflection or uplift occurs when the total track vertical deflection (i.e. deflection due to applied load plus deflection due to self-weight) is less than zero. Mathematically, this condition is expressed as:

$$v(x) + v_{self} < 0 \quad (15)$$

In the uplift region, the base or underside component no longer contributes to the total tie lateral resistance². The total tie lateral resistance in the uplift region is reduced compared to the lateral resistance without vertical load:

² Resistance of crossties against lateral movement consists of three frictional components: (1) base (i.e. bottom of tie) friction, (2) end shoulder friction, and (3) side friction [10]. The individual contribution to the total resistance to lateral displacement from these components is roughly 45 to 50 percent for the base or underside friction, 35 to 40 percent for the end shoulder friction, and 10 to 15 percent for the side resistance.

$$F(w, V) = F(w, 0) - \mu_f q \quad (16)$$

where q is the self-weight of the track. In the calculations presented in this paper, q is assumed to be 24 lb per inch.

SOLUTION METHODS

Depending on the assumed lateral resistance characteristic, a closed-form solution to the general differential equation for track lateral deflection can be derived. For example, closed-form equations are derived in Reference [3] for the case where the lateral resistance characteristic without vertical loading is represented mathematically by either a constant or a bilinear relation. The derivation can also be performed assuming a linear softening characteristic, i.e. only the initial part of the piecewise linear function listed in equation (6). The appendix to this paper contains a compendium of closed-form equations to calculate the load-deflection response of track subjected to lateral load for three idealized lateral resistance characteristics: constant, bilinear and linear softening. For each lateral resistance characteristic, the compendium lists a pair of parametric equations³, one for the lateral load and one for the track lateral deflection at the location where the lateral load is applied.

Alternatively, the differential equation for track lateral deflection can be solved numerically using a Fourier series technique. The Fourier method was used in previous work to examine track lateral stability under thermally-induced loading (i.e. track buckling) [11]. In the Fourier method, the track lateral deflection distribution is expressed in terms of an infinite sum:

$$w(x) = \sum_m^{1,3,5,\dots} A_m \cos\left(\frac{m\pi x}{2\ell}\right) \quad (17)$$

where ℓ is defined as the half-wavelength of the track lateral deflection distribution. In theory, the Fourier series is an infinite sum. In practice, eleven terms are computed to achieve convergence. The Fourier series in equation (17) satisfies two boundary conditions: $w'(0) = 0$ and $w(\ell) = 0$. That is, the slope at the origin (i.e. $x=0$) is equal to zero, and the lateral deflection at the end of the half-wavelength (i.e. at $x=\ell$) is equal to zero.

Furthermore, the right-hand side of equation (2) is also expressed in terms of a Fourier series (neglecting the effects of lateral misalignment and thermal load):

$$F(w) - \frac{L}{2} \delta(0) = \sum_m^{1,3,5,\dots} a_m \cos\left(\frac{m\pi x}{2\ell}\right) \quad (18)$$

Therefore,

$$a_m = \frac{2}{\ell} \int_0^\ell \left[F(w) - \frac{L}{2} \delta(0) \right] \cos\left(\frac{m\pi x}{2\ell}\right) dx = \frac{2}{\ell} \int_0^\ell F(w) \cos\left(\frac{m\pi x}{2\ell}\right) dx - \frac{L}{\ell} \quad (19)$$

$$A_m = \frac{-a_m}{EI \left[\left(\frac{m\pi}{2\ell}\right)^4 - P \left(\frac{m\pi}{2\ell}\right)^2 \right]} \quad (20)$$

The lateral load and the half-wavelength of the deflection distribution are related through the following equation which must be solved numerically:

$$\sum_m^{1,3,5,\dots} A_m \left(\frac{m\pi}{2\ell}\right) \sin\left(\frac{m\pi}{2}\right) = 0 \quad (21)$$

Equation (21) satisfies two boundary conditions at the end of the half wavelength (i.e. at $x = \ell$), that the first (i.e. slope) and third (i.e. shear) derivatives of the lateral deflection with respect to x are equal to zero. Moreover, the numerical calculations for the Fourier analysis described in this paper are implemented through a Fortran computer program.

APPLICATIONS

The utility of the Fourier method is demonstrated by applying the method to examine lateral load-deflection behavior in two cases in which lateral loads are applied to track while the loads and the track lateral deflection at the point of load application are monitored.

Track Lateral Pull Test

In the Track Lateral Pull Test (TLPT), a concentrated lateral load is applied to the rail at a point on a section of track. By applying sufficient amount of load, the ties on either side of the point of load application are mobilized to some extent. During this test, the lateral force and lateral deflection at the point of load application are measured. Given the tie lateral resistance as measured from the Single Tie Push Test (STPT), the question is: what is the expected load-deflection behavior in the TLPT? Figure 7 illustrates the distinction between these two types of lateral push/pull tests. The closed-form approach was applied in previous work [3] in which the lateral resistance characteristic was assumed to be either constant or bilinear. The closed-form results can now be used to corroborate the results from the Fourier method described in this paper. No initial lateral misalignment and no thermally-induced loading were assumed in these analyses.

³ Parametric equations are a set of equations expressing a set of quantities as explicit functions of independent variables called parameters.

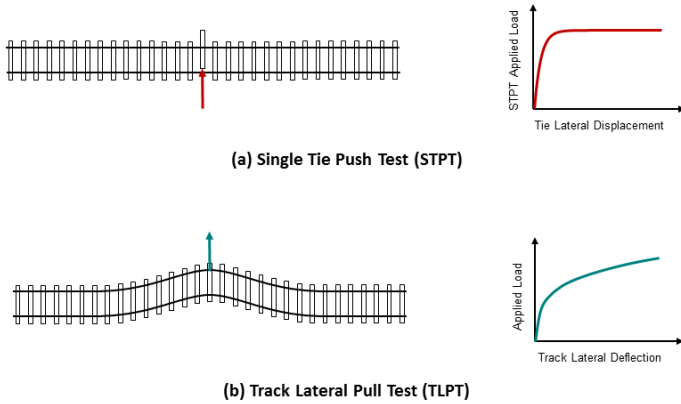


Figure 7: STPT and TLPT Depictions

Figure 8 shows TLPT data and the closed-form analysis as reported in [3]. The figure also shows the applied lateral load versus lateral deflection curve from applying the Fourier method, which practically coincides with the curve derived from the closed-form solution. The coincident results are unexpected given that the load-deflection curve from the closed-form solution assumes the bilinear lateral resistance characteristic defined in equation (4), and the result from the Fourier method assumes the exponential function given in equation (5).

Figure 9 shows a similar comparison for TLPT data on curved track and the corresponding solutions from the closed-form equations and the Fourier method, which are nearly identical. Although curvature is neglected in deriving the load-deflection curves from both the closed-form solution and the Fourier method, the agreement between test data and the respective analyses is good. Moreover, these comparisons provide verification of the Fourier analysis technique.

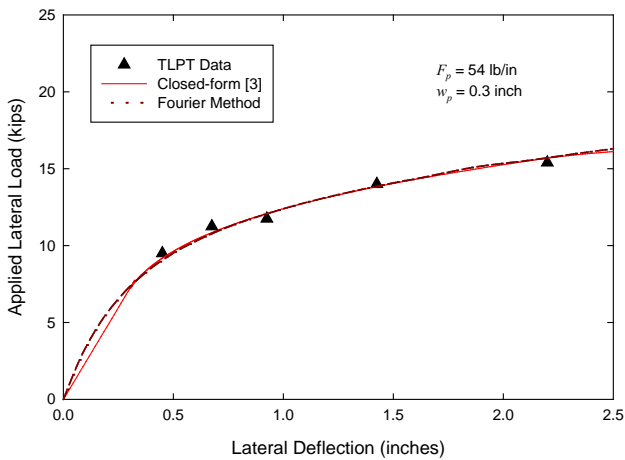


Figure 8: Comparison of TLPT Test Data with Analyses

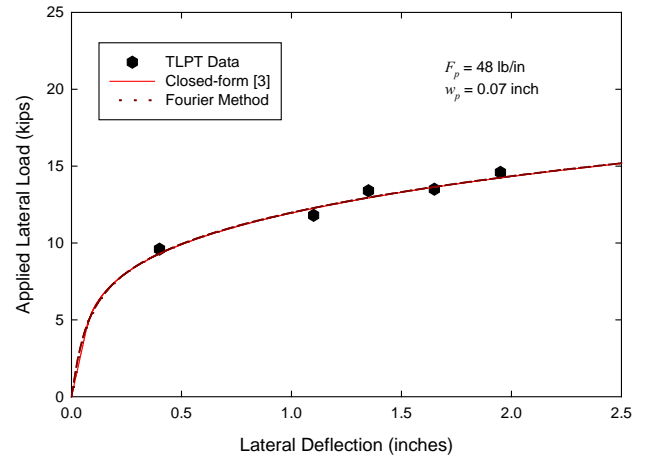


Figure 9: Comparison of TLPT Test Data with Analyses

Laboratory Tests of Track Shift

Track shift is defined in Reference [12] as the permanent lateral distortion of a track segment. The permanent lateral distortion can occur cumulatively under vehicle passes or suddenly under a single pass. Laboratory tests have been conducted to investigate track shift by measuring lateral load and track lateral deflections as the vertical load is varied [13]. The physical attributes of the track in these laboratory tests are summarized as follows: 136 RE rail, wood cross ties (7" by 9" by 9") with 19.5-inch center-to-center spacing, and limestone ballast (12 inches deep and 12-inch shoulders).

Figure 10 compares the laboratory test data and the load-deflection curves for different vertical loads, as calculated using the Fourier method assuming a constant lateral resistance characteristic. Different symbols in the figure represent the laboratory test data for different vertical loads; solid lines represent the results from the Fourier method. A constant value for the lateral resistance was assumed to be 86 lb per inch in the analyses. Qualitative agreement between the test data and the calculated load-deflection behavior is evident, but the analysis results generally overestimate the respective test data for different vertical loads, especially at low values of lateral deflection where the actual lateral resistance is less than the idealized value.

Figure 11 compares the test data and analyses assuming the exponential lateral resistance characteristic, as defined in equation (5), in which F_p is equal to 86 lb per inch and w_p is equal to 0.03 inch. Based purely upon inspection, the agreement between test data and analysis at each level of vertical load is better in Figure 11 than that shown in Figure 10 where the lateral resistance is assumed to be constant. Moreover, these results indicate that the initial rising portion of the lateral resistance curve should not be neglected if agreement between test data and analysis is required at low values of track lateral deflection.

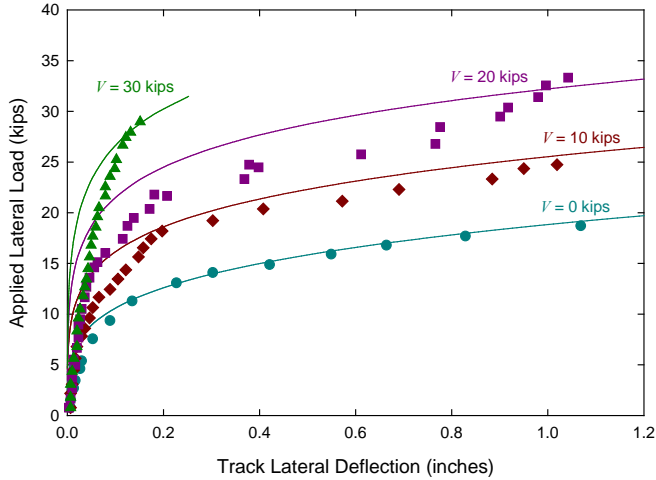


Figure 10: Comparison of Laboratory Test Data with Analyses Assuming Constant Lateral Resistance

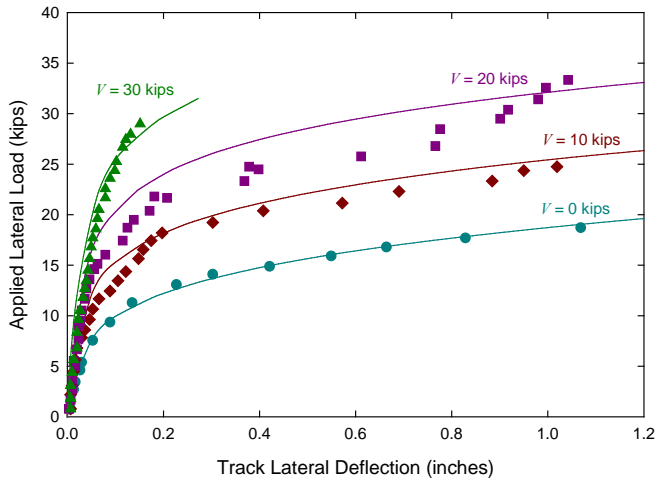


Figure 11: Comparison of Laboratory Test Data with Analyses Assuming Exponential Characteristic

Track vertical deflections were not measured in these laboratory tests, which would have been useful to compare with the analytical results and to confirm the assumption of linear elastic foundation support.

ESTIMATING LATERAL STRENGTH

Traditionally the evaluation of safety to prevent track shift entails the determination of allowable net axle lateral loads or NALs. An empirical equation for lateral strength of wood-tie track under vertical loads was developed by Prud'homme [14]:

$$L = 0.33V + 2.25 \quad (22)$$

where L is the lateral load (in kips) and V is the vertical load (in kips). Over the years, this equation has served as a guideline for vehicle qualification loads [12].

The reasonable agreement between test data and analysis in the preceding calculations for laboratory test data provides verification and confidence in the Fourier method. The method is now used to estimate allowable lateral loads to prevent track shift. The full nonlinear characteristic is assumed in applying the Fourier method to calculate the track lateral deflection under different levels of vertical load for strong wood and strong concrete ties. That is, equation (8) is used with its parameters defined in Table 1 and Table 2. The results from the Fourier analysis are shown in Figure 12 for strong wood ties and in Figure 13 for strong concrete ties.

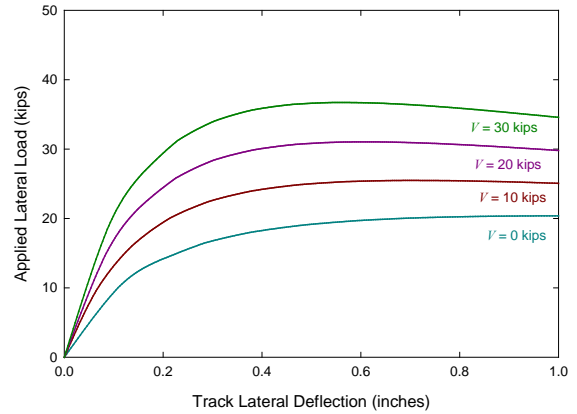


Figure 12: Track Lateral Deflections Estimated using Fourier Method for Strong Wood Ties

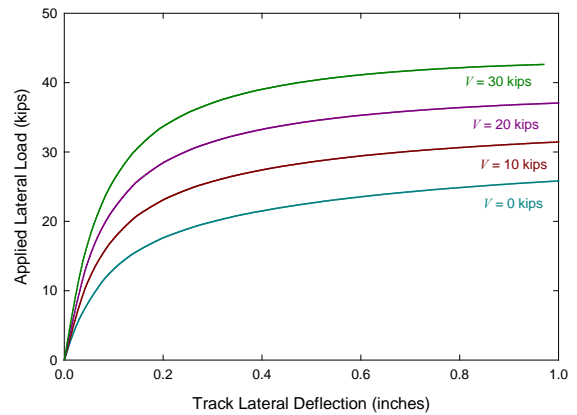


Figure 13: Estimated Track Lateral Deflections using Fourier Method for Strong Concrete Ties

Figure 14 is a schematic for using the lateral load-lateral deflection curves to estimate allowable lateral loads to mitigate the occurrence of track shift. In the schematic, a critical or limiting value for lateral deflection, w_C , is specified. Lateral loads corresponding to the critical lateral deflection are determined and cross-plotted as a function of vertical load, V . Figure 15 shows estimates for allowable lateral load as a function of vertical load from applying this procedure for the

respective curves obtained from the Fourier method for track with strong wood and strong concrete ties. In these results, the limiting value of lateral deflection is 0.05 inch. The figure also shows the lateral loads according to the Prud'homme limit. In addition, equations for the allowable lateral load are derived from the results from the Fourier method, and are displayed in the figure. The results shown in the figure suggest that the Prud'homme limit is restrictive, which is a conclusion that was reached in previous work [12].

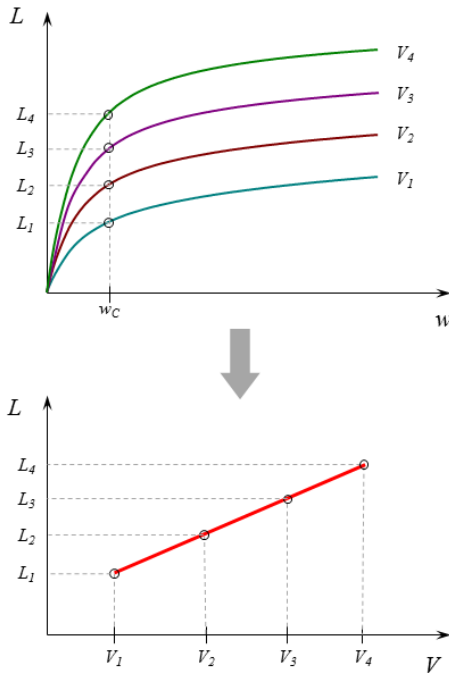


Figure 14: Schematic to Estimate Allowable Lateral Load

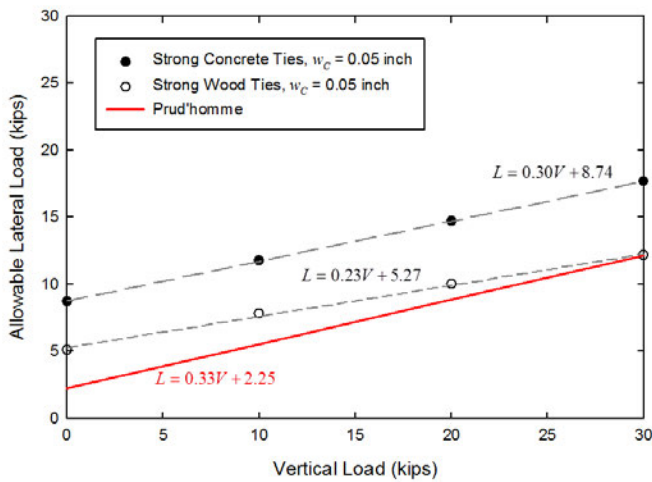


Figure 15: Allowable Lateral Load vs. Vertical Load

CONCLUDING REMARKS

This paper describes a numerical method based on Fourier analysis to examine lateral deflection of track subjected to vertical and lateral loads. Idealized mathematical functions are used to represent tie lateral resistance. The Fourier approach is shown to be in excellent agreement with closed-form solutions for track subjected to purely lateral load (i.e. no vertical load). The Fourier method is also used to examine track shift under vertical loading. Calculated results are compared to laboratory test data, and are also shown to give reasonable agreement.

Allowable lateral loads to mitigate track shift are estimated using results from the Fourier method. Compared to the estimated lateral loads, the empirical equation derived by Prud'homme [14] appears to be conservative. These estimated lateral loads, however, are based on static analyses. Previous work has been conducted to estimate the corresponding lateral loads based on dynamic analyses [15].

NOMENCLATURE

a_m, A_m	Fourier coefficients
E	Modulus of elasticity
F_L	Level lateral resistance
F_p	Peak lateral resistance
$F(w)$	Lateral resistance
I_{yy}	Vertical bending inertia of rail about its centroid
I_{zz}	Lateral bending inertia of rail about its centroid
k	Ratio of level to peak lateral resistance = F_L/F_p
k_V	Track vertical foundation modulus
ℓ	Half-wavelength of lateral deflection distribution
L	Applied lateral load
P	Rail longitudinal force
q	Self-weight of track
R_V	Vertical reaction force on tie
V	Applied vertical load
v	Track vertical deflection due to applied vertical load
v_{self}	Track vertical deflection due to self-weight = q/k_V
w	Track lateral deflection due to applied lateral load
w_C	Critical or limiting value for track lateral deflection
w_0	Initial lateral imperfection
w_l	Parameter for lateral resistance characteristic
w_L	Track lateral deflection at "lower" lateral resistance
w_{max}	Maximum track lateral deflection
w_p	Track lateral deflection at peak lateral resistance
x	Coordinate in longitudinal direction
y	Coordinate in the lateral direction
z	Coordinate in the vertical direction
δ	Dirac delta function
λ, Λ	Characteristic wavelength
μ_1	Parameter for lateral resistance characteristic
μ_2	Parameter for lateral resistance characteristic
μ_f	Tie-ballast coefficient of friction

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This paper is dedicated to the memory of Dr. Gopal Samavedam, who passed away on April 16, 2011. The Fourier method described in this work is a technique the author learned from Gopal.

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APPENDIX

This appendix is a compendium of closed-form equations that can be used to calculate lateral load versus track lateral deflection behavior for different idealized lateral resistance characteristics. These equations do not account for the following effects: vertical load, longitudinal rail force (i.e. thermal loads), and initial lateral misalignment.

Constant Lateral Resistance Characteristic

In this case, the lateral resistance is represented mathematically by equation (3). The lateral load is related to the maximum track lateral deflection by the following equation:

$$L = \sqrt[4]{\frac{2048}{9} EI_{zz} F_p^3 w_{\max}}$$

Equivalently, the lateral load versus maximum track lateral deflection behavior can be calculated using the following parametric equations:

$$L = \frac{4}{3} F_p \ell$$

$$w_{\max} = \frac{F_p \ell^4}{72 EI_{zz}}$$

In these equations, the lateral load and maximum track lateral deflection are related parametrically by ℓ , which physically represents the half-wavelength of the track lateral deflection distribution, refer to Figure 1(b). The derivation of these equations is described in Reference [3].

Bilinear Lateral Resistance Characteristic

In this case, the lateral resistance is represented mathematically by the bilinear relation defined in equation (4) and shown schematically in Figure 4. The lateral load versus maximum lateral deflection behavior is calculated using the following pair of parametric equations:

$$L = \frac{2F_p \ell (2\lambda^2 \ell^2 + 6\lambda \ell + 3) + 24w_p EI_{zz} \lambda^3 (\lambda \ell + 1)}{3(\lambda \ell + 1)^2}$$

$$w_{\max} = w_p + \frac{\left[F_p \ell^2 (\lambda \ell + 3) + 24 w_p EI_{zz} \lambda^2 (\lambda \ell + 1) \right] (\lambda \ell + 3) \ell^2}{72 EI_{zz} (\lambda \ell + 1)^2}$$

The physical interpretation of the parameter ℓ is the half-length of the deflection distribution in which the deflection is greater than w_p . These equations also depend on λ , which is defined as

$$\lambda = \sqrt[4]{\frac{F_p}{4 EI_{zz} w_p}}$$

These equations apply only for cases where w_{\max} is greater than or equal to w_p . The derivation of these equations is also given in Reference [3].

Linear Softening Lateral Resistance Characteristic

The lateral resistance in this case is represented mathematically by only the initial portion of the piecewise linear function given in equation (6):

$$F(w) = F_p - (F_p - F_L) \left(\frac{w}{w_L} \right)$$

The following parametric equations can be derived to calculate lateral load and maximum track lateral deflection for cases where w_{\max} is less than or equal to w_L :

$$L = \frac{2F_p}{\Lambda} \left[\frac{\cosh(\Lambda \ell) \sin(\Lambda \ell) - \sinh(\Lambda \ell) \cos(\Lambda \ell)}{\cosh(\Lambda \ell) - \cos(\Lambda \ell)} \right]$$

$$w_{\max} = \frac{F_p w_L}{F_p - F_L} \left[1 - \frac{\sinh(\Lambda \ell) \sin(\Lambda \ell)}{\cosh(\Lambda \ell) - \cos(\Lambda \ell)} \right]$$

The parameter ℓ physically represents the half-wavelength of the deflection distribution. In these equations, Λ is defined as

$$\Lambda = \sqrt[4]{\frac{F_p - F_L}{EI_{zz} w_L}}$$

The derivation of these equations is similar to that in the previous two cases presented in this appendix. These equations, however, are valid only for cases where w_{\max} is less than or equal to w_L , refer to Figure 5. For cases where w_{\max} is greater than w_L , the calculation of lateral load as a function of maximum track lateral deflection requires a more rigorous derivation that must be carried out numerically.